Chin. Ann. Math.
34B(1), 2013, 99–112
DOI: 10.1007/s11401-012-0756-6

Chinese Annals of Mathematics, Series B

Sharp Interpolation Inequalities on the Sphere: New Methods and Consequences*

Jean DOLBEAULT[1] Maria J. ESTEBAN[1]
Michal KOWALCZYK[2] Michael LOSS[3]

(*In honor of the scientific heritage of Jacques-Louis Lions*)

Abstract This paper is devoted to various considerations on a family of sharp interpolation inequalities on the sphere, which in dimension greater than 1 interpolate between Poincaré, logarithmic Sobolev and critical Sobolev (Onofri in dimension two) inequalities. The connection between optimal constants and spectral properties of the Laplace-Beltrami operator on the sphere is emphasized. The authors address a series of related observations and give proofs based on symmetrization and the ultraspherical setting.

Keywords Sobolev inequality, Interpolation, Gagliardo-Nirenberg inequality, Logarithmic Sobolev inequality, Heat equation
2000 MR Subject Classification 26D10, 46E35, 58E35

1 Introduction

The following interpolation inequality holds on the sphere:

$$\frac{p-2}{d}\int_{\mathbb{S}^d}|\nabla u|^2\mathrm{d}\mu+\int_{\mathbb{S}^d}|u|^2\mathrm{d}\mu\geq\Big(\int_{\mathbb{S}^d}|u|^p\mathrm{d}\mu\Big)^{\frac{2}{p}},\quad\forall\,u\in\mathrm{H}^1(\mathbb{S}^d,\mathrm{d}\mu)\tag{1.1}$$

for any $p\in(2,2^*]$ with $2^*=\frac{2d}{d-2}$ if $d\geq3$, and for any $p\in(2,\infty)$ if $d=2$. In (1.1), $\mathrm{d}\mu$ is the uniform probability measure on the d-dimensional sphere, that is, the measure induced by Lebesgue's measure on $\mathbb{S}^d\subset\mathbb{R}^{d+1}$, up to a normalization factor such that $\mu(\mathbb{S}^d)=1$.

Such an inequality was established by Bidaut-Véron and Véron [21] in the more general context of compact manifolds with uniformly positive Ricci curvature. Their method is based on the Bochner-Lichnerowicz-Weitzenböck formula and the study of the set of solutions to an elliptic equation, which is seen as a bifurcation problem and contains the Euler-Lagrange equation associated to the optimality case in (1.1). Later, in [12], Beckner gave an alternative proof based on Legendre's duality, the Funk-Hecke formula, proved in [27, 31], and the expression of

Manuscript received October 10, 2012.
[1]Ceremade, Université Paris-Dauphine, Place de Lattre de Tassigny, 75775 Paris Cédex 16, France. E-mail: dolbeaul@ceremade.dauphine.fr esteban@ceremade.dauphine.fr
[2]Departamento de Ingeniería Matemática and Centro de Modelamiento Matemático (UMI 2807 CNRS), Universidad de Chile, Casilla 170 Correo 3, Santiago, Chile. E-mail: kowalczy@dim.uchile.cl
[3]Skiles Building, Georgia Institute of Technology, Atlanta GA 30332-0160, USA. E-mail: loss@math.gatech.edu
*Project supported by ANR grants CBDif and NoNAP, the ECOS project (No. C11E07), the Chilean research grants Fondecyt (No. 1090103), Fondo Basal CMM-Chile, Project Anillo ACT-125 CAPDE and the National Science Foundation (No. DMS-0901304).

some optimal constants found by Lieb [33]. Bakry, Bentaleb and Fahlaoui in a series of papers based on the carré du champ method and mostly devoted to the ultraspherical operator showed a result which turns out to give yet another proof, which is anyway very close to the method of [21]. Their computations allow to slightly extend the range of the parameter p (see [7–8, 14–20] and [34, 37] for earlier related works).

In all computations based on the Bochner-Lichnerowicz-Weitzenböck formula, the choice of exponents in the computations appears somewhat mysterious. The seed for such computations can be found in [28]. Our purpose is on one hand to give alternative proofs, at least for some ranges of the parameter p, which do not rely on such a technical choice. On the other hand, we also aim at simplifying the existing proofs (see Section 3.2).

Inequality (1.1) is remarkable for several reasons as follows:

(1) It is optimal in the sense that 1 is the optimal constant. By Hölder's inequality, we know that $\|u\|_{\mathrm{L}^2(\mathbb{S}^d)} \le \|u\|_{\mathrm{L}^p(\mathbb{S}^d)}$, so that the equality case can only be achieved by functions, which are constants a.e. Of course, the main issue is to prove that the constant $\frac{p-2}{d}$ is optimal, which is one of the classical issues of the so-called A-B problem, for which we primarily refer to [30].

(2) If $d \ge 3$, the case $p = 2^*$ corresponds to the Sobolev's inequality. Using the stereographic projection as in [33], we easily recover Sobolev's inequality in the Euclidean space $\mathbb{R}^d$ with the optimal constant and obtain a simple characterization of the extremal functions found by Aubin and Talenti [5, 36–37].

(3) In the limit $p \to 2$, one obtains the logarithmic Sobolev inequality on the sphere, while by taking $p \to \infty$ if $d = 2$, one recovers Onofri's inequality (see [25] and Corollary 2.1 below).

Exponents are not restricted to $p > 2$. Consider indeed the functional

$$\mathcal{Q}_p[u] := \frac{p-2}{d}\,\frac{\int_{\mathbb{S}^d} |\nabla u|^2 \mathrm{d}\mu}{\left(\int_{\mathbb{S}^d} |u|^p \mathrm{d}\mu\right)^{\frac{2}{p}} - \int_{\mathbb{S}^d} |u|^2 \mathrm{d}\mu}$$

for $p \in [1,2) \cup (2, 2^*]$ if $d \ge 3$, or $p \in [1,2) \cup (2,\infty)$ if $d = 2$, and

$$\mathcal{Q}_2[u] := \frac{2}{d}\,\frac{\int_{\mathbb{S}^d} |\nabla u|^2 \mathrm{d}\mu}{\int_{\mathbb{S}^d} |u|^2 \log\left(\frac{|u|^2}{\int_{\mathbb{S}^d}|u|^2\mathrm{d}\mu}\right) \mathrm{d}\mu}$$

for any $d \ge 1$. Because $\mathrm{d}\mu$ is a probability measure, $\left(\int_{\mathbb{S}^d} |u|^p \mathrm{d}\mu\right)^{\frac{2}{p}} - \int_{\mathbb{S}^d} |u|^2 \mathrm{d}\mu$ is nonnegative if $p > 2$, nonpositive if $p \in [1,2)$, and equal to zero if and only if u is constant a.e. Denote by $\mathcal{A}$ the set of $\mathrm{H}^1(\mathbb{S}^d, \mathrm{d}\mu)$ functions, which are not a.e. constants. Consider the infimum

$$\mathcal{I}_p := \inf_{u \in \mathcal{A}} \mathcal{Q}_p[u]\,. \tag{1.2}$$

With these notations, we can state a slight result more general than the one of (1.1), which goes as follows and also covers the range $p \in [1,2]$.

Theorem 1.1 *With the above notations, $\mathcal{I}_p = 1$ for any $p \in [1, 2^*]$ if $d \ge 3$, or any $p \in [1,\infty)$ if $d = 1, 2$.*

As already explained above, in the case $(2, 2^*]$, the above theorem was proved first in [21, Corollary 6.2], and then in [12], by using the previous results of Lieb [33] and the Funk-Hecke formula (see [27, 31]). The case $p = 2$ was covered in [12]. The whole range $p \in [1, 2^*]$ was covered in the case of the ultraspherical operator in [19–20]. Here we give alternative proofs for various ranges of p, which are less technical and interesting in themselves, as well as some extensions.

Notice that the case $p=1$ can be written as

$$\int_{\mathbb{S}^d} |\nabla u|^2 \mathrm{d}\mu \geq d\Big[\int_{\mathbb{S}^d} |u|^2 \mathrm{d}\mu - \Big(\int_{\mathbb{S}^d} |u| \mathrm{d}\mu\Big)^2\Big], \quad \forall\, u \in \mathrm{H}^1(\mathbb{S}^d, \mathrm{d}\mu),$$

which is equivalent to the usual Poincaré inequality

$$\int_{\mathbb{S}^d} |\nabla u|^2 \mathrm{d}\mu \geq d \int_{\mathbb{S}^d} |u-\overline{u}|^2 \mathrm{d}\mu, \quad \forall\, u \in \mathrm{H}^1(\mathbb{S}^d, \mathrm{d}\mu) \quad \text{with } \overline{u} = \int_{\mathbb{S}^d} u \mathrm{d}\mu\,.$$

See Remark 2.1 for more details. The case $p=2$ provides the logarithmic Sobolev inequality on the sphere. It holds as a consequence of the inequality for $p \neq 2$ (see Corollary 1.1).

For $p \neq 2$, the existence of a minimizer of

$$u \mapsto \int_{\mathbb{S}^d} |\nabla u|^2 \mathrm{d}\mu + \frac{\mathrm{d}\mathcal{I}_p}{p-2}[\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} - \|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)}]$$

in $\big\{u \in \mathrm{H}^1(\mathbb{S}^d, \mathrm{d}\mu) : \int_{\mathbb{S}^d} |u|^p \mathrm{d}\mu = 1\big\}$ is easily achieved by variational methods, and will be taken for granted. The compactness for either $p \in [1,2)$ or $2 < p < 2^*$ is indeed classical, while the case $p = 2^*$, $d \geq 3$ can be studied by concentration-compactness methods. If a function $u \in \mathrm{H}^1(\mathbb{S}^d, \mathrm{d}\mu)$ is optimal for (1.1) with $p \neq 2$, then it solves the Euler-Lagrange equation

$$-\Delta_{\mathbb{S}^d} u = \frac{\mathrm{d}\,\mathcal{I}_p}{p-2}[\|u\|^{2-p}_{\mathrm{L}^p(\mathbb{S}^d)}\, u^{p-1} - u], \tag{1.3}$$

where $\Delta_{\mathbb{S}^d}$ denotes the Laplace-Beltrami operator on the sphere $\mathbb{S}^d$.

In any case, it is possible to normalize the $\mathrm{L}^p(\mathbb{S}^d)$-norm of u to 1 without restriction because of the zero homogeneity of $\mathcal{Q}_p$. It turns out that the optimality case is achieved by the constant function, with value $u \equiv 1$ if we assume $\int_{\mathbb{S}^d} |u|^p \mathrm{d}\mu = 1$, in which case the inequality degenerates because both sides are equal to 0. This explains why the dimension d shows up here: the sequence $(u_n)_{n\in\mathbb{N}}$, satisfying

$$u_n(x) = 1 + \frac{1}{n}\, v(x)$$

with $v \in \mathrm{H}^1(\mathbb{S}^d, \mathrm{d}\mu)$, such that $\int_{\mathbb{S}^d} v \mathrm{d}\mu = 0$, is indeed minimizing if and only if

$$\int_{\mathbb{S}^d} |\nabla v|^2 \mathrm{d}\mu \geq d \int_{\mathbb{S}^d} |v|^2 \mathrm{d}\mu,$$

and the equality case is achieved if v is an optimal function for the above Poincaré inequality, i.e., a function associated to the first non-zero eigenvalue of the Laplace-Beltrami operator $-\Delta_{\mathbb{S}^d}$ on the sphere $\mathbb{S}^d$. Up to a rotation, this means

$$v(\xi) = \xi_d, \quad \forall\, \xi = (\xi_0, \xi_1, \cdots, \xi_d) \in \mathbb{S}^d \subset \mathbb{R}^{d+1},$$

since $-\Delta_{\mathbb{S}^d} v = \mathrm{d}v$. Recall that the corresponding eigenspace of $-\Delta_{\mathbb{S}^d}$ is d-dimensional and is generated by the composition of v with an arbitrary rotation.

1.1 The logarithmic Sobolev inequality

As the first classical consequence of (1.2), we have a logarithmic Sobolev inequality. This result is rather classical. Related forms of the result can be found, for instance, in [9] or in [3].

Corollary 1.1 *Let $d \geq 1$. For any $u \in \mathrm{H}^1(\mathbb{S}^d, \mathrm{d}\mu) \setminus \{0\}$, we have*

$$\int_{\mathbb{S}^d} |u|^2 \, \log\Big(\frac{|u|^2}{\int_{\mathbb{S}^d} |u|^2 \mathrm{d}\mu}\Big) \mathrm{d}\mu \leq \frac{2}{d} \int_{\mathbb{S}^d} |\nabla u|^2 \mathrm{d}\mu.$$

Moreover, the constant $\frac{2}{d}$ is sharp.

Proof The inequality is achieved by taking the limit as $p \to 2$ in (1.2). To see that the constant $\frac{2}{d}$ is sharp, we can observe that

$$\lim_{\varepsilon \to 0} \int_{\mathbb{S}^d} |1 + \varepsilon\, v|^2 \, \log\Big(\frac{|1+\varepsilon\, v|^2}{\int_{\mathbb{S}^d} |1+\varepsilon\, v|^2 \mathrm{d}\mu}\Big) \mathrm{d}\mu = 2 \int_{\mathbb{S}^d} |v - \overline{v}|^2 \mathrm{d}\mu$$

with $\overline{v} = \int_{\mathbb{S}^d} v \mathrm{d}\mu$. The result follows by taking $v(\xi) = \xi_d$.

2 Extensions

2.1 Onofri's inequality

In the case of dimension $d = 2$, (1.1) holds for any $p > 2$, and we recover Onofri's inequality by taking the limit $p \to \infty$. This result is standard in the literature (see for instance [12]). For completeness, let us give a statement and a short proof.

Corollary 2.1 *Let $d = 1$ or $d = 2$. For any $v \in \mathrm{H}^1(\mathbb{S}^d, \mathrm{d}\mu)$, we have*

$$\int_{\mathbb{S}^d} \mathrm{e}^{v - \overline{v}} \mathrm{d}\mu \leq \mathrm{e}^{\frac{1}{2d} \int_{\mathbb{S}^d} |\nabla v|^2 \mathrm{d}\mu},$$

where $\overline{v} = \int_{\mathbb{S}^d} v \mathrm{d}\mu$ is the average of v. Moreover, the constant $\frac{1}{2d}$ in the right-hand side is sharp.

Proof In dimension $d = 1$ or $d = 2$, (1.1) holds for any $p > 2$. Take $u = 1 + \frac{v}{p}$ and consider the limit as $p \to \infty$. We observe that

$$\int_{\mathbb{S}^d} |\nabla u|^2 \mathrm{d}\mu = \frac{1}{p^2} \int_{\mathbb{S}^d} |\nabla v|^2 \mathrm{d}\mu \quad \text{and} \quad \lim_{p \to \infty} \int_{\mathbb{S}^d} |u|^p \mathrm{d}\mu = \int_{\mathbb{S}^d} \mathrm{e}^v \mathrm{d}\mu,$$

so that

$$\Big(\int_{\mathbb{S}^d} |u|^p \mathrm{d}\mu\Big)^{\frac{2}{p}} - 1 \sim \frac{2}{p} \, \log\Big(\int_{\mathbb{S}^d} \mathrm{e}^v \mathrm{d}\mu\Big) \quad \text{and} \quad \int_{\mathbb{S}^d} |u|^2 \mathrm{d}\mu - 1 \sim \frac{2}{p} \int_{\mathbb{S}^d} v \mathrm{d}\mu\ .$$

The conclusion holds by passing to the limit $p \to \infty$ in (1.1). Optimality is once more achieved by considering $v = \varepsilon\, v_1$, $v_1(\xi) = \xi_d$, $d = 1$ and Taylor expanding both sides of the inequality in terms of $\varepsilon > 0$ small enough. Notice indeed that $-\Delta_{\mathbb{S}^d} v_1 = \lambda_1\, v_1$ with $\lambda_1 = d$, so that

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} = \varepsilon^2 \, \|\nabla v_1\|^2_{\mathrm{L}^2(\mathbb{S}^d)} = \varepsilon^2 d \, \|v_1\|^2_{\mathrm{L}^2(\mathbb{S}^d)},$$

$\int_{\mathbb{S}^d} v_1 \mathrm{d}\mu = \overline{v}_1 = 0$, and

$$\int_{\mathbb{S}^d} \mathrm{e}^{v - \overline{v}} \mathrm{d}\mu - 1 \sim \frac{\varepsilon^2}{2} \int_{\mathbb{S}^d} |v - \overline{v}|^2 \mathrm{d}\mu = \frac{1}{2}\, \varepsilon^2 \, \|v_1\|^2_{\mathrm{L}^2(\mathbb{S}^d)}.$$

2.2 Interpolation and a spectral approach for $p \in (1,2)$

In [10], Beckner gave a method to prove interpolation inequalities between the logarithmic Sobolev and the Poincaré inequalities in the case of a Gaussian measure. Here we shall prove that the method extends to the case of the sphere and therefore provides another family of interpolating inequalities, in a new range: $p \in [1,2)$, again with optimal constants. For further considerations on inequalities that interpolate between the Poincaré and the logarithmic Sobolev inequalities, we refer to [1–2, 9–10, 23–24, 27, 33] and the references therein.

Our purpose is to extend (1.1) written as

$$\frac{1}{d}\int_{\mathbb{S}^d}|\nabla u|^2 \mathrm{d}\mu \ge \frac{\left(\int_{\mathbb{S}^d}|u|^p \mathrm{d}\mu\right)^{\frac{2}{p}} - \int_{\mathbb{S}^d}|u|^2 \mathrm{d}\mu}{p-2}, \quad \forall\, u \in \mathrm{H}^1(\mathbb{S}^d, \mathrm{d}\mu) \tag{2.1}$$

to the case $p \in [1,2)$. Let us start with a remark.

Remark 2.1 At least for any nonnegative function v, using the fact that μ is a probability measure on $\mathbb{S}^d$, we may notice that

$$\int_{\mathbb{S}^d}|v-\overline{v}|^2 \mathrm{d}\mu = \int_{\mathbb{S}^d}|v|^2 \mathrm{d}\mu - \left(\int_{\mathbb{S}^d} v\,\mathrm{d}\mu\right)^2$$

can be rewritten as

$$\int_{\mathbb{S}^d}|v-\overline{v}|^2 \mathrm{d}\mu = \frac{\int_{\mathbb{S}^d}|v|^2 \mathrm{d}\mu - \left(\int_{\mathbb{S}^d}|v|^p \mathrm{d}\mu\right)^{\frac{2}{p}}}{2-p}$$

for $p=1$. Hence this extends (1.1) to the case $q=1$. However, as already noticed for instance in [1], the inequality

$$\int_{\mathbb{S}^d}|v|^2 \mathrm{d}\mu - \left(\int_{\mathbb{S}^d}|v|\mathrm{d}\mu\right)^2 \le \frac{1}{d}\int_{\mathbb{S}^d}|\nabla v|^2 \mathrm{d}\mu$$

also means that, for any $c \in \mathbb{R}$,

$$\int_{\mathbb{S}^d}|v+c|^2 \mathrm{d}\mu - \left(\int_{\mathbb{S}^d}|v+c|\mathrm{d}\mu\right)^2 \le \frac{1}{d}\int_{\mathbb{S}^d}|\nabla v|^2 \mathrm{d}\mu.$$

If v is bounded from below a.e. with respect to μ and $c > -\operatorname*{ess\,inf}_{\mu} v$, so that $v+c>0$ μ a.e., and the left-hand side is

$$\int_{\mathbb{S}^d}|v+c|^2 \mathrm{d}\mu - \left(\int_{\mathbb{S}^d}|v+c|\mathrm{d}\mu\right)^2 = c^2 + 2\,c\int_{\mathbb{S}^d} v\,\mathrm{d}\mu + \int_{\mathbb{S}^d}|v|^2 \mathrm{d}\mu - \left(c + \int_{\mathbb{S}^d} v\,\mathrm{d}\mu\right)^2 = \int_{\mathbb{S}^d}|v-\overline{v}|^2 \mathrm{d}\mu,$$

so that the inequality is the usual Poincaré inequality. By density, we recover that (2.1) written for $p=1$ exactly amounts to Poincaré inequality written not only for $|v|$, but also for any $v \in \mathrm{H}^1(\mathbb{S}^d, \mathrm{d}\mu)$.

Next, using the method introduced by Beckner [10] in the case of a Gaussian measure, we are in the position to prove (2.1) for any $p \in (1,2)$, knowing that the inequality holds for $p=1$ and $p=2$.

Proposition 2.1 *Inequality* (2.1) *holds for any* $p \in (1,2)$ *and any* $d \ge 1$*. Moreover,* d *is the optimal constant.*

Proof Optimality can be checked by Taylor expanding $u = 1 + \varepsilon\, v$ at order two in terms of $\varepsilon > 0$ as in the case $p=2$ (the logarithmic Sobolev inequality). To establish the inequality itself, we may proceed in two steps.

Step 1 (Nelson's Hypercontractivity Result) Although the result can be established by direct methods, we follow here the strategy of Gross [29], which proves the equivalence of the optimal hypercontractivity result and the optimal logarithmic Sobolev inequality.

Consider the heat equation of $\mathbb{S}^d$, namely,

$$\frac{\partial f}{\partial t} = \Delta_{\mathbb{S}^d} f$$

with the initial data $f(t=0,\,\cdot\,)=u \in L^{\frac{2}{p}}(\mathbb{S}^d)$ for some $p \in (1,2]$, and let $F(t) := \|f(t,\,\cdot\,)\|_{\mathrm{L}^{p(t)}(\mathbb{S}^d)}$. The key computation goes as follows:

$$\begin{aligned}\frac{F'}{F} &= \frac{\mathrm{d}}{\mathrm{d}t}\log F(t) = \frac{\mathrm{d}}{\mathrm{d}t}\Big[\frac{1}{p(t)}\,\log\Big(\int_{\mathbb{S}^d}|f(t,\,\cdot\,)|^{p(t)}\mathrm{d}\mu\Big)\Big]\\ &= \frac{p'}{p^2\,F^p}\Big[\int_{\mathbb{S}^d} v^2\log\Big(\frac{v^2}{\int_{\mathbb{S}^d} v^2\mathrm{d}\mu}\Big)\mathrm{d}\mu + 4\,\frac{p-1}{p'}\int_{\mathbb{S}^d}|\nabla v|^2\mathrm{d}\mu\Big]\end{aligned}$$

with $v := |f|^{\frac{p(t)}{2}}$. Assuming that $4\,\frac{p-1}{p'} = \frac{2}{d}$, that is,

$$\frac{p'}{p-1} = 2d,$$

we find that

$$\log\Big(\frac{p(t)-1}{p-1}\Big) = 2dt,$$

if we require that $p(0) = p < 2$. Let $t_* > 0$ satisfy $p(t_*) = 2$. As a consequence of the above computation, we have

$$\|f(t_*,\,\cdot\,)\|_{\mathrm{L}^2(\mathbb{S}^d)} \le \|u\|_{\mathrm{L}^{\frac{2}{p}}(\mathbb{S}^d)}, \quad \text{if } \frac{1}{p-1} = \mathrm{e}^{2dt_*}. \tag{2.2}$$

Step 2 (Spectral Decomposition) Let $u = \sum_{k\in\mathbb{N}} u_k$ be a decomposition of the initial datum on the eigenspaces of $-\Delta_{\mathbb{S}^d}$, and denote by $\lambda_k = k\,(d+k-1)$ the ordered sequence of the eigenvalues: $-\Delta_{\mathbb{S}^d} u_k = \lambda_k\,u_k$ (see for instance [20]). Let $a_k = \|u_k\|^2_{\mathrm{L}^2(\mathbb{S}^d)}$. As a straightforward consequence of this decomposition, we know that $\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} = \sum_{k\in\mathbb{N}} a_k$, $\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} = \sum_{k\in\mathbb{N}} \lambda_k\,a_k$ and

$$\|f(t_*,\,\cdot\,)\|^2_{\mathrm{L}^2(\mathbb{S}^d)} = \sum_{k\in\mathbb{N}} a_k\,\mathrm{e}^{-2\,\lambda_k\,t_*}.$$

Using (2.2), it follows that

$$\frac{\big(\int_{\mathbb{S}^d}|u|^p\mathrm{d}\mu\big)^{\frac{2}{p}} - \int_{\mathbb{S}^d}|u|^2\mathrm{d}\mu}{p-2} \le \frac{\big(\int_{\mathbb{S}^d}|u|^2\mathrm{d}\mu\big) - \int_{\mathbb{S}^d}|f(t_*,\,\cdot\,)|^2\mathrm{d}\mu}{2-p} = \frac{1}{2-p}\sum_{k\in\mathbb{N}^*}\lambda_k\,a_k\,\frac{1-\mathrm{e}^{-2\,\lambda_k\,t_*}}{\lambda_k}.$$

Notice that $\lambda_0 = 0$ so that the term corresponding to $k=0$ can be omitted in the series. Since $\lambda \mapsto \frac{1-\mathrm{e}^{-2\,\lambda\,t_*}}{\lambda}$ is decreasing, we can bound $\frac{1-\mathrm{e}^{-2\,\lambda_k\,t_*}}{\lambda_k}$ from above by $\frac{1-\mathrm{e}^{-2\,\lambda_1\,t_*}}{\lambda_1}$ for any $k \ge 1$. This proves that

$$\frac{\big(\int_{\mathbb{S}^d}|u|^p\mathrm{d}\mu\big)^{\frac{2}{p}} - \int_{\mathbb{S}^d}|u|^2\mathrm{d}\mu}{p-2} \le \frac{1-\mathrm{e}^{-2\,\lambda_1\,t_*}}{(2-p)\,\lambda_1}\sum_{k\in\mathbb{N}^*}\lambda_k\,a_k = \frac{1-\mathrm{e}^{-2\,\lambda_1\,t_*}}{(2-p)\,\lambda_1}\,\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}.$$

The conclusion follows easily if we notice that $\lambda_1 = d$ and $\mathrm{e}^{-2\,\lambda_1\,t_*} = p-1$, so that

$$\frac{1-\mathrm{e}^{-2\,\lambda_1\,t_*}}{(2-p)\,\lambda_1} = \frac{1}{d}.$$

The optimality of this constant can be checked as in the case $p > 2$ by a Taylor expansion of $u = 1 + \varepsilon\, v$ at order two in terms of $\varepsilon > 0$ small enough.

3 Symmetrization and the Ultraspherical Framework

3.1 A reduction to the ultraspherical framework

We denote by $(\xi_0, \xi_1, \cdots, \xi_d)$ the coordinates of an arbitrary point $\xi \in \mathbb{S}^d$ with $\sum\limits_{i=0}^{d} |\xi_i|^2 = 1$. The following symmetry result is a kind of folklore in the literature, and we can see [5, 33, 11] for various related results.

Lemma 3.1 *Up to a rotation, any minimizer of* (1.2) *depends only on* ξ_d.

Proof Let u be a minimizer for $\mathcal{Q}_p$. By writing u in (1.1) in spherical coordinates $\theta \in [0,\pi]$, $\varphi_1, \varphi_2, \cdots, \varphi_{d-1} \in [0, 2\pi)$ and using decreasing rearrangements (see, for instance, [24]), it is not difficult to prove that among optimal functions, there is one which depends only on θ. Moreover, the equality in the rearrangement inequality means that u has to depend on only one coordinate, i.e., $\xi_d = \sin\theta$.

Let us observe that the problem on the sphere can be reduced to a problem involving the ultraspherical operator as follows:

(1) Using Lemma 3.1, we know that (1.1) is equivalent to

$$\frac{p-2}{d}\int_0^\pi |v'(\theta)|^2 \mathrm{d}\sigma + \int_0^\pi |v(\theta)|^2 \mathrm{d}\sigma \ge \Big(\int_0^\pi |v(\theta)|^p \mathrm{d}\sigma\Big)^{\frac{2}{p}}$$

for any function $v \in \mathrm{H}^1([0,\pi], \mathrm{d}\sigma)$, where

$$\mathrm{d}\sigma(\theta) := \frac{(\sin\theta)^{d-1}}{Z_d}\mathrm{d}\theta \quad \text{with } Z_d := \sqrt{\pi}\,\frac{\Gamma(\frac{d}{2})}{\Gamma(\frac{d+1}{2})}.$$

(2) The change of variables $x = \cos\theta$ and $v(\theta) = f(x)$ allows to rewrite the inequality as

$$\frac{p-2}{d}\int_{-1}^1 |f'|^2\,\nu \mathrm{d}\nu_d + \int_{-1}^1 |f|^2 \mathrm{d}\nu_d \ge \Big(\int_{-1}^1 |f|^p \mathrm{d}\nu_d\Big)^{\frac{2}{p}},$$

where $\mathrm{d}\nu_d$ is the probability measure defined by

$$\nu_d(x)\mathrm{d}x = \mathrm{d}\nu_d(x) := Z_d^{-1}\,\nu^{\frac{d}{2}-1}\mathrm{d}x \quad \text{with } \nu(x) := 1-x^2,\ Z_d = \sqrt{\pi}\,\frac{\Gamma(\frac{d}{2})}{\Gamma(\frac{d+1}{2})}\ .$$

We also want to prove the result in the case $p < 2$, to obtain the counterpart of Theorem 1.1 in the ultraspherical setting. On $[-1,1]$, consider the probability measure $\mathrm{d}\nu_d$, and define

$$\nu(x) := 1 - x^2\,,$$

so that $\mathrm{d}\nu_d = Z_d^{-1}\,\nu^{\frac{d}{2}-1}\mathrm{d}x$. We consider the space $\mathrm{L}^2((-1,1), \mathrm{d}\nu_d)$ with the scalar product

$$\langle f_1, f_2\rangle = \int_{-1}^1 f_1\, f_2 \mathrm{d}\nu_d,$$

and use the notation

$$\|f\|_p = \Big(\int_{-1}^{1} f^p \mathrm{d}\nu_d \Big)^{\frac{1}{p}}.$$

On $\mathrm{L}^2((-1,1), \mathrm{d}\nu_d)$, we define the self-adjoint ultraspherical operator by

$$\mathcal{L}\, f := (1-x^2)\, f'' - d\, x\, f' = \nu\, f'' + \frac{d}{2}\, \nu'\, f',$$

which satisfies the identity

$$\langle f_1, \mathcal{L}\, f_2 \rangle = - \int_{-1}^{1} f_1'\, f_2'\, \nu \mathrm{d}\nu_d.$$

Then the result goes as follows.

Proposition 3.1 *Let $p \in [1, 2^*]$, $d \geq 1$. Then we have*

$$- \langle f, \mathcal{L}\, f \rangle = \int_{-1}^{1} |f'|^2\, \nu \mathrm{d}\nu_d \geq d\, \frac{\|f\|_p^2 - \|f\|_2^2}{p-2}, \quad \forall\, f \in \mathrm{H}^1([-1,1], \mathrm{d}\nu_d), \tag{3.1}$$

if $p \neq 2$; and

$$- \langle f, \mathcal{L}\, f \rangle = \frac{d}{2} \int_{-1}^{1} |f|^2\, \log\Big(\frac{|f|^2}{\|f\|_2^2} \Big) \mathrm{d}\nu_d,$$

if $p = 2$.

We may notice that the proof in [21] requires $d \geq 2$, while the case $d = 1$ is also covered in [12]. In [20], the restriction $d \geq 2$ was removed by Bentaleb et al. Our proof is inspired by [21] and also [14, 17], but it is a simplification (in the particular case of the ultraspherical operator) in the sense that only integration by parts and elementary estimates are used.

3.2 A proof of Proposition 3.1

Let us start with some preliminary observations. The operator $\mathcal{L}$ does not commute with the derivation, but we have the relation

$$\Big[\frac{\partial}{\partial x}, \mathcal{L} \Big]\, u = (\mathcal{L}\, u)' - \mathcal{L}\, u' = -2\, x\, u'' - d\, u'.$$

As a consequence, we obtain

$$\langle \mathcal{L}\, u, \mathcal{L}\, u \rangle = - \int_{-1}^{1} u'\, (\mathcal{L}\, u)'\, \nu \mathrm{d}\nu_d = - \int_{-1}^{1} u'\, \mathcal{L}\, u'\, \nu \mathrm{d}\nu_d + \int_{-1}^{1} u'\, (2\, x\, u'' + d\, u')\, \nu \mathrm{d}\nu_d,$$

$$\langle \mathcal{L}\, u, \mathcal{L}\, u \rangle = \int_{-1}^{1} |u''|^2\, \nu^2 \mathrm{d}\nu_d - d\, \langle u, \mathcal{L}\, u \rangle$$

and

$$\int_{-1}^{1} (\mathcal{L}\, u)^2 \mathrm{d}\nu_d = \langle \mathcal{L}\, u, \mathcal{L}\, u \rangle = \int_{-1}^{1} |u''|^2\, \nu^2 \mathrm{d}\nu_d + d \int_{-1}^{1} |u'|^2\, \nu \mathrm{d}\nu_d. \tag{3.2}$$

On the other hand, a few integrations by parts show that

$$\Big\langle \frac{|u'|^2}{u}\, \nu \mathcal{L}\, u \Big\rangle = \frac{d}{d+2} \int_{-1}^{1} \frac{|u'|^4}{u^2}\, \nu^2 \mathrm{d}\nu_d - 2\, \frac{d-1}{d+2} \int_{-1}^{1} \frac{|u'|^2\, u''}{u}\, \nu^2 \mathrm{d}\nu_d, \tag{3.3}$$

where we have used the fact that $\nu\,\nu'\,\nu_d = \frac{2}{d+2}\,(\nu^2\,\nu_d)'$.

Let $p \in (1,2)\cup(2,2^*)$. In $H^1([-1,1],\mathrm{d}\nu_d)$, now consider a minimizer f for the functional

$$f \mapsto \int_{-1}^1 |f'|^2\,\nu\mathrm{d}\nu_d - d\,\frac{\|f\|_p^2-\|f\|_2^2}{p-2} =: \mathcal{G}[f],$$

made of the difference of the two sides in (3.1). The existence of such a minimizer can be proved by classical minimization and compactness arguments. Up to a multiplication by a constant, f satisfies the Euler-Lagrange equation

$$-\frac{p-2}{\mathrm{d}}\,\mathcal{L}\,f + f = f^{p-1}.$$

Let β be a real number to be fixed later and define u by $f=u^\beta$, such that

$$\mathcal{L}\,f = \beta\,u^{\beta-1}\left(\mathcal{L}\,u + (\beta-1)\,\frac{|u'|^2}{u}\,\nu\right).$$

Then u is a solution to

$$-\mathcal{L}\,u - (\beta-1)\,\frac{|u'|^2}{u}\,\nu + \lambda\,u = \lambda\,u^{1+\beta\,(p-2)} \quad \text{with } \lambda := \frac{d}{(p-2)\,\beta}.$$

If we multiply the equation for u by $\frac{|u'|^2}{u}\,\nu$ and integrate, we get

$$-\int_{-1}^1 \mathcal{L}\,u\,\frac{|u'|^2}{u}\,\nu\mathrm{d}\nu_d - (\beta-1)\int_{-1}^1\frac{|u'|^4}{u^2}\,\nu^2\mathrm{d}\nu_d + \lambda\int_{-1}^1|u'|^2\,\nu\mathrm{d}\nu_d = \lambda\int_{-1}^1 u^{\beta\,(p-2)}\,|u'|^2\,\nu\mathrm{d}\nu_d.$$

If we multiply the equation for u by $-\mathcal{L}\,u$ and integrate, we get

$$\int_{-1}^1(\mathcal{L}\,u)^2\mathrm{d}\nu_d + (\beta-1)\int_{-1}^1\mathcal{L}\,u\,\frac{|u'|^2}{u}\,\nu\mathrm{d}\nu_d + \lambda\int_{-1}^1|u'|^2\,\nu\mathrm{d}\nu_d = (\lambda+d)\int_{-1}^1 u^{\beta\,(p-2)}\,|u'|^2\,\nu\mathrm{d}\nu_d.$$

Collecting terms, we find that

$$\int_{-1}^1(\mathcal{L}\,u)^2\mathrm{d}\nu_d + \left(\beta+\frac{d}{\lambda}\right)\int_{-1}^1\mathcal{L}\,u\,\frac{|u'|^2}{u}\,\nu\mathrm{d}\nu_d + (\beta-1)\left(1+\frac{d}{\lambda}\right)\int_{-1}^1\frac{|u'|^4}{u^2}\,\nu^2\mathrm{d}\nu_d - d\int_{-1}^1|u'|^2\,\nu\mathrm{d}\nu_d = 0.$$

Using (3.2)–(3.3), we get

$$\begin{aligned}&\int_{-1}^1|u''|^2\,\nu^2\mathrm{d}\nu_d + \left(\beta+\frac{d}{\lambda}\right)\left[\frac{d}{d+2}\int_{-1}^1\frac{|u'|^4}{u^2}\,\nu^2\mathrm{d}\nu_d - 2\,\frac{d-1}{d+2}\int_{-1}^1\frac{|u'|^2\,u''}{u}\,\nu^2\mathrm{d}\nu_d\right]\\ &+ (\beta-1)\left(1+\frac{d}{\lambda}\right)\int_{-1}^1\frac{|u'|^4}{u^2}\,\nu^2\mathrm{d}\nu_d = 0,\end{aligned}$$

that is,

$$\mathsf{a}\int_{-1}^1|u''|^2\,\nu^2\mathrm{d}\nu_d + 2\,\mathsf{b}\int_{-1}^1\frac{|u'|^2\,u''}{u}\,\nu^2\mathrm{d}\nu_d + \mathsf{c}\int_{-1}^1\frac{|u'|^4}{u^2}\,\nu^2\mathrm{d}\nu_d = 0, \tag{3.4}$$

where

$$\begin{aligned}\mathsf{a} &= 1,\\ \mathsf{b} &= -\left(\beta+\frac{d}{\lambda}\right)\frac{d-1}{d+2},\\ \mathsf{c} &= \left(\beta+\frac{d}{\lambda}\right)\frac{d}{d+2} + (\beta-1)\left(1+\frac{d}{\lambda}\right).\end{aligned}$$

Using $\frac{d}{\lambda} = (p-2)\,\beta$, we observe that the reduced discriminant

$$\delta = \mathsf{b}^2 - \mathsf{a}\,\mathsf{c} < 0$$

can be written as

$$\delta = A\,\beta^2 + B\,\beta + 1 \quad \text{with } A = (p-1)^2\,\frac{(d-1)^2}{(d+2)^2} - p + 2 \text{ and } B = p - 3 - \frac{d\,(p-1)}{d+2}.$$

If $p < 2^*$, $B^2 - 4\,A$ is positive, and therefore it is possible to find β, such that $\delta < 0$.

Hence, if $p < 2^*$, we have shown that $\mathcal{G}[f]$ is positive unless the three integrals in (3.4) are equal to 0, that is, u is constant. It follows that $\mathcal{G}[f] = 0$, which proves (3.1) if $p \in (1,2) \cup (2,2^*)$. The cases $p = 1$, $p = 2$ (see Corollary 1.1) and $p = 2^*$ can be proved as limit cases. This completes the proof of Proposition 3.1.

4 A Proof Based on a Flow in the Ultraspherical Setting

Inequality (3.1) can be rewritten for $g = f^p$, i.e., $f = g^\alpha$ with $\alpha = \frac{1}{p}$, as

$$-\,\langle f, \mathcal{L}\,f\rangle = -\,\langle g^\alpha, \mathcal{L}\,g^\alpha\rangle =: \mathcal{I}[g] \geq d\,\frac{\|g\|_1^{2\,\alpha} - \|g^{2\,\alpha}\|_1}{p-2} =: \mathcal{F}[g].$$

4.1 Flow

Consider the flow associated to $\mathcal{L}$, that is,

$$\frac{\partial g}{\partial t} = \mathcal{L}\,g, \tag{4.1}$$

and observe that

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\|g\|_1 = 0, \quad \frac{\mathrm{d}}{\mathrm{d}t}\,\|g^{2\,\alpha}\|_1 = -\,2\,(p-2)\,\langle f, \mathcal{L}\,f\rangle = 2\,(p-2)\int_{-1}^{1} |f'|^2\;\nu\,\mathrm{d}\nu_d,$$

which finally gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}[g(t,\cdot)] = -\frac{d}{p-2}\,\frac{\mathrm{d}}{\mathrm{d}t}\,\|g^{2\,\alpha}\|_1 = -\,2d\,\mathcal{I}[g(t,\cdot)].$$

4.2 Method

If (3.1) holds, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}[g(t,\cdot)] \leq -\,2d\,\mathcal{F}[g(t,\cdot)], \tag{4.2}$$

and thus we prove

$$\mathcal{F}[g(t,\cdot)] \leq \mathcal{F}[g(0,\cdot)]\,\mathrm{e}^{-\,2dt}, \quad \forall\, t \geq 0.$$

This estimate is actually equivalent to (3.1) as shown by estimating $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}[g(t,\cdot)]$ at $t = 0$.

The method based on the Bakry-Emery approach amounts to establishing first that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{I}[g(t,\cdot)] \leq -\,2d\,\mathcal{I}[g(t,\cdot)] \tag{4.3}$$

and proving (4.2) by integrating the estimates on $t \in [0, \infty)$. Since

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathcal{F}[g(t,\,\cdot\,)] - \mathcal{I}[g(t,\,\cdot\,)]) \geq 0$$

and $\lim_{t\to\infty}(\mathcal{F}[g(t,\,\cdot\,)] - \mathcal{I}[g(t,\,\cdot\,)]) = 0$, this means that

$$\mathcal{F}[g(t,\,\cdot\,)] - \mathcal{I}[g(t,\,\cdot\,)] \leq 0, \quad \forall\, t \geq 0,$$

which is precisely (3.1) written for $f(t,\,\cdot\,)$ for any $t \geq 0$ and in particular for any initial value $f(0,\,\cdot\,)$.

The equation for $g = f^p$ can be rewritten in terms of f as

$$\frac{\partial f}{\partial t} = \mathcal{L}\, f + (p-1)\,\frac{|f'|^2}{f}\,\nu.$$

Hence, we have

$$-\frac{1}{2}\,\frac{\mathrm{d}}{\mathrm{d}t}\int_{-1}^{1}|f'|^2\,\nu \mathrm{d}\nu_d = \frac{1}{2}\,\frac{\mathrm{d}}{\mathrm{d}t}\,\langle f, \mathcal{L}\, f\rangle = \langle \mathcal{L}\, f, \mathcal{L}\, f\rangle + (p-1)\left\langle \frac{|f'|^2}{f}\nu, \mathcal{L}\, f\right\rangle.$$

4.3 An inequality for the Fisher information

Instead of proving (3.1), we will established the following stronger inequality, for any $p \in (2, 2^\sharp]$, where $2^\sharp := \frac{2\mathrm{d}^2+1}{(d-1)^2}$:

$$\langle \mathcal{L}\, f, \mathcal{L}\, f\rangle + (p-1)\left\langle \frac{|f'|^2}{f}\,\nu, \mathcal{L}\, f\right\rangle + d\,\langle f, \mathcal{L}\, f\rangle \geq 0. \tag{4.4}$$

Notice that (3.1) holds under the restriction $p \in (2, 2^\sharp]$, which is stronger than $p \in (2, 2^*]$. We do not know whether the exponent $2^\sharp$ in (4.4) is sharp or not.

4.4 Proof of (4.4)

Using (3.2)–(3.3) with $u = f$, we find that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_{-1}^{1}|f'|^2\,\nu \mathrm{d}\nu_d + 2d\int_{-1}^{1}|f'|^2\,\nu \mathrm{d}\nu_d\\ =&-2\int_{-1}^{1}\Big(|f''|^2 + (p-1)\,\frac{d}{d+2}\frac{|f'|^4}{f^2} - 2\,(p-1)\,\frac{d-1}{d+2}\frac{|f'|^2\,f''}{f}\Big)\,\nu^2 \mathrm{d}\nu_d\,.\end{aligned}$$

The right-hand side is nonpositive, if

$$|f''|^2 + (p-1)\,\frac{d}{d+2}\frac{|f'|^4}{f^2} - 2\,(p-1)\,\frac{d-1}{d+2}\frac{|f'|^2\,f''}{f}$$

is pointwise nonnegative, which is granted if

$$\Big[(p-1)\,\frac{d-1}{d+2}\Big]^2 \leq (p-1)\,\frac{d}{d+2},$$

a condition which is exactly equivalent to $p \leq 2^\sharp$.

4.5 An improved inequality

For any $p \in (2, 2^\sharp)$, we can write that

$$\begin{aligned} &|f''|^2 + (p-1)\,\frac{d}{d+2}\,\frac{|f'|^4}{f^2} - 2\,(p-1)\,\frac{d-1}{d+2}\,\frac{|f'|^2\,f''}{f} \\ &= \alpha\,|f''|^2 + \frac{p-1}{d+2}\left|\frac{d-1}{\sqrt{d}}\,f'' - \sqrt{d}\,\frac{|f'|^2}{f}\right|^2 \ge \alpha\,|f''|^2, \end{aligned}$$

where

$$\alpha := 1 - (p-1)\,\frac{(d-1)^2}{d\,(d+2)}$$

is positive. Now, using the Poincaré inequality

$$\int_{-1}^{1} |f''|^2 \mathrm{d}\nu_{d+4} \ge (d+2) \int_{-1}^{1} |f' - \overline{f'}|^2 \mathrm{d}\nu_{d+2},$$

where

$$\overline{f'} := \int_{-1}^{1} f' \mathrm{d}\nu_{d+2} = -d \int_{-1}^{1} x\,f \mathrm{d}\nu_d,$$

we obtain an improved form of (4.4), namely,

$$\langle \mathcal{L}\,f, \mathcal{L}\,f\rangle + (p-1)\Big\langle \frac{|f'|^2}{f}\nu, \mathcal{L}\,f\Big\rangle + [d + \alpha\,(d+2)]\,\langle f, \mathcal{L}\,f\rangle \ge 0,$$

if we can guarantee that $\overline{f'} \equiv 0$ along the evolution determined by (4.1). This is the case if we assume that $f(x) = f(-x)$ for any $x \in [-1, 1]$. Under this condition, we find that

$$\int_{-1}^{1} |f'|^2\,\nu \mathrm{d}\nu_d \ge [d + \alpha\,(d+2)]\,\frac{\|f\|_p^2 - \|f\|_2^2}{p-2}.$$

As a consequence, we also have

$$\int_{\mathbb{S}^d} |\nabla u|^2 \mathrm{d}\mu + \int_{\mathbb{S}^d} |u|^2 \mathrm{d}\mu \ge \frac{d + \alpha\,(d+2)}{p-2}\Big(\int_{\mathbb{S}^d} |u|^p \mathrm{d}\mu\Big)^{\frac{2}{p}}$$

for any $u \in \mathrm{H}^1(\mathbb{S}^d, \mathrm{d}\mu)$, such that, using spherical coordinates,

$$u(\theta, \varphi_1, \varphi_2, \cdots, \varphi_{d-1}) = u(\pi-\theta, \varphi_1, \varphi_2, \cdots, \varphi_{d-1}), \quad \forall (\theta, \varphi_1, \varphi_2, \cdots, \varphi_{d-1}) \in [0, \pi] \times [0, 2\pi)^{d-1}\,.$$

4.6 One more remark

The computation is exactly the same if $p \in (1, 2)$, and henceforth we also prove the result in such a case. The case $p = 1$ is the limit case corresponding to the Poincaré inequality

$$\int_{-1}^{1} |f'|^2 \mathrm{d}\nu_{d+2} \ge d\Big(\int_{-1}^{1} |f|^2 \mathrm{d}\nu_d - \Big|\int_{-1}^{1} f \mathrm{d}\nu_d\Big|^2\Big)$$

and arises as a straightforward consequence of the spectral properties of $\mathcal{L}$. The case $p = 2$ is achieved as a limiting case. It gives rise to the logarithmic Sobolev inequality (see, for instance, [34]).

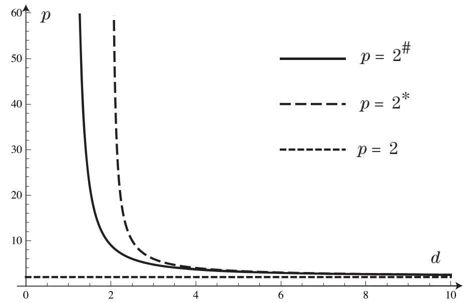

Figure 1 Plot of $d \mapsto 2^\sharp = \frac{2d^2+1}{(d-1)^2}$ and $d \mapsto 2^* = \frac{2d}{d-2}$.

4.7 Limitation of the method

The limitation $p \le 2^\sharp$ comes from the pointwise condition

$$h := |f''|^2 + (p-1)\,\frac{d}{d+2}\,\frac{|f'|^4}{f^2} - 2\,(p-1)\,\frac{d-1}{d+2}\,\frac{|f'|^2\,f''}{f} \ge 0.$$

Can we find special test functions f, such that this quantity can be made negative? Which are admissible, such that $h\,\nu^2$ is integrable? Notice that at $p = 2^\sharp$, we have that $f(x) = |x|^{1-d}$, such that $h \equiv 0$, but such a function or functions obtained by slightly changing the exponent, are not admissible for larger values of p.

By proving that there is contraction of $\mathcal{I}$ along the flow, we look for a condition which is stronger than one of asking that there is contraction of $\mathcal{F}$ along the flow. It is therefore possible that the limitation $p \le 2^\sharp$ is intrinsic to the method.

References

[1] Arnold, A., Bartier, J. P. and Dolbeault, J., Interpolation between logarithmic Sobolev and Poincaré inequalities, *Commun. Math. Sci.*, **5**, 2007, 971–979.

[2] Arnold, A. and Dolbeault, J., Refined convex Sobolev inequalities, *J. Funct. Anal.*, **225**, 2005, 337–351.

[3] Arnold, A., Markowich, P., Toscani, G., et al., On convex Sobolev inequalities and the rate of convergence to equilibrium for Fokker-Planck type equations, *Comm. Part. Diff. Eq.*, **26**, 2001, 43–100.

[4] Aubin, T., Problèmes isopérimétriques et espaces de Sobolev, *J. Diff. Geom.*, **11**, 1976, 573–598.

[5] Baernstein, A. and Taylor, B. A., Spherical rearrangements, subharmonic functions, and $*$-functions in n-space, *Duke Math. J.*, **43**, 1976, 245–268.

[6] Bakry, D., Une suite d'inégalités remarquables pour les opérateurs ultrasphériques, *C. R. Acad. Sci. Paris Sér. I Math.*, **318**, 1994, 161–164.

[7] Bakry, D. and Bentaleb, A., Extension of Bochner-Lichnérowicz formula on spheres, *Ann. Fac. Sci. Toulouse Math.*, **14**(6), 2005, 161–183.

[8] Bakry, D. and Émery, M., Hypercontractivité de semi-groupes de diffusion, *C. R. Acad. Sci. Paris Sér. I Math.*, **299**, 1984, 775–778.

[9] Bakry, D. and Émery, M., Diffusions hypercontractives, séminaire de probabilités, XIX, 1983/1984, Lecture Notes in Math., Vol. 1123, Springer-Verlag, Berlin, 1985, 177–206.

[10] Beckner, W., A generalized Poincaré inequality for Gaussian measures, *Proc. Amer. Math. Soc.*, **105**, 1989, 397–400.

[11] Beckner, W., Sobolev inequalities, the Poisson semigroup, and analysis on the sphere S^n, *Proc. Nat. Acad. Sci. U. S. A.*, **89**, 1992, 4816–4819.

[12] Beckner, W., Sharp Sobolev inequalities on the sphere and the Moser-Trudinger inequality, *Ann. of Math.*, **138**(2), 1993, 213–242.

[13] Bentaleb, A., Développement de la moyenne d'une fonction pour la mesure ultrasphérique, *C. R. Acad. Sci. Paris Sér. I Math.*, **317**, 1993, 781–784.

[14] Bentaleb, A., Inégalité de Sobolev pour l'opérateur ultrasphérique, *C. R. Acad. Sci. Paris Sér. I Math.*, **317**, 1993, 187–190.

[15] Bentaleb, A., Sur l'hypercontractivité des semi-groupes ultrasphériques, séminaire de probabilités, XXXIII, Lecture Notes in Math., Vol. 1709, Springer-Verlag, Berlin, 1999, 410–414.

[16] Bentaleb, A., L'hypercontractivité des semi-groupes de Gegenbauer multidimensionnels — famille d'inégalités sur le cercle, *Int. J. Math. Game Theory Algebra*, **12**, 2002, 259–273.

[17] Bentaleb, A., Sur les fonctions extrémales des inégalités de Sobolev des opérateurs de diffusion, séminaire de probabilités, XXXVI, Lecture Notes in Math., Vol. 1801, Springer-Verlag, Berlin, 2003, 230–250.

[18] Bentaleb, A. and Fahlaoui, S., Integral inequalities related to the Tchebychev semigroup, *Semigroup Forum*, **79**, 2009, 473–479.

[19] Bentaleb, A. and Fahlaoui, S., A family of integral inequalities on the circle $\mathbf{S}^1$, *Proc. Japan Acad. Ser. A Math. Sci.*, **86**, 2010, 55–59.

[20] Berger, M., Gauduchon, P. and Mazet, E., Le spectre d'une variété riemannienne, Lecture Notes in Mathematics, Vol. 194, Springer-Verlag, Berlin, 1971.

[21] Bidaut-Véron, M. F. and Véron, L., Nonlinear elliptic equations on compact Riemannian manifolds and asymptotics of Emden equations, *Invent. Math.*, **106**, 1991, 489–539.

[22] Bolley, F. and Gentil, I., Phi-entropy inequalities and Fokker-Planck equations, Progress in Analysis and Its Applications, World Sci. Publ., Hackensack, NJ, 2010, 463–469.

[23] Bolley, F. and Gentil, I., Phi-entropy inequalities for diffusion semigroups, *J. Math. Pures Appl.*, **93**(9), 2010, 449–473.

[24] Brock, F., A general rearrangement inequality à la Hardy-Littlewood, *J. Inequal. Appl.*, **5**, 2000, 309–320.

[25] Carlen, E. and Loss, M., Computing symmetries, the logarithmic HLS inequality and Onofri's inequality on S^n, *Geom. Funct. Anal.*, **2**, 1992, 90–104.

[26] Chafaï, D., Entropies, convexity, and functional inequalities: on Φ-entropies and Φ-Sobolev inequalities, *J. Math. Kyoto Univ.*, **44**, 2004, 325–363.

[27] Funk, P., Beiträge zur Theorie der Kegelfunktionen., *Math. Ann.*, **77**, 1915, 136–162.

[28] Gidas, B. and Spruck, J., Global and local behavior of positive solutions of nonlinear elliptic equations, *Comm. Pure Appl. Math.*, **34**, 1981, 525–598.

[29] Gross, L., Logarithmic Sobolev inequalities, *Amer. J. Math.*, **97**, 1975, 1061–1083.

[30] Hebey, E., Nonlinear analysis on manifolds: Sobolev spaces and inequalities, Courant Lecture Notes in Mathematics, Vol. 5, New York University Courant Institute of Mathematical Sciences, New York, 1999.

[31] Hecke, E., Über orthogonal-invariante Integralgleichungen., *Math. Ann.*, **78**, 1917, 398–404.

[32] Latała, R. and Oleszkiewicz, K., Between Sobolev and Poincaré, Geometric aspects of functional analysis, Lecture Notes in Math., Vol. 1745, Springer-Verlag, Berlin, 2000, 147–168.

[33] Lieb, E. H., Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities, *Ann. of Math.*, **118**(2), 1983, 349–374.

[34] Mueller, C. E. and Weissler, F. B., Hypercontractivity for the heat semigroup for ultraspherical polynomials and on the n-sphere, *J. Funct. Anal.*, **48**, 1982, 252–283.

[35] Rosen, G., Minimum value for c in the Sobolev inequality $\phi^3\| \leq c\nabla\phi\|^3$, *SIAM J. Appl. Math.*, **21**, 1971, 30–32.

[36] Talenti, G., Best constant in Sobolev inequality, *Ann. Mat. Pura Appl.*, **110**(4), 1976, 353–372.

[37] Weissler, F. B., Logarithmic Sobolev inequalities and hypercontractive estimates on the circle, *J. Funct. Anal.*, **37**, 1980, 218–234.