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TWO PROBLEMS IN NONLINEAR PDEs: EXISTENCE IN SUPERCRITICAL ELLIPTIC EQUATIONS AND SYMMETRY FOR A HYPO-ELLIPTIC OPERATOR

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Resumen

En este trabajo se aborda el problema de encontrar soluciones regulares para algunas EDPs elípticas e hipo-elípticas no lineales y estudiar sus propiedades cualitativas.

En una primera etapa, se considera la ecuación

$$-\Delta u = \lambda \mathrm{e}^u,$$

$\lambda > 0$, en un dominio exterior con condición de Dirichlet nula. Un esquema de reducción finito-dimensional permite encontrar infinitas soluciones regulares cuando λ es suficientemente pequeño.

En la segunda parte se estudia la existencia de soluciones de la ecuación no local

$$(-\Delta)^s u = u^p, \; u > 0,$$

en un dominio acotado y suave, con condición de Dirichlet nula; donde $s > 0$ y $p := (N+2s)/(N-2s) \pm \varepsilon$ es cercano al exponente crítico ($\varepsilon > 0$ pequeño). Para hallar soluciones, se utiliza un esquema de reducción finito-dimensional en espacios de funciones adecuados, donde el término principal de la función reducida se expresa a partir de las funciones de Green y de Robin del dominio. La existencia de soluciones dependerá de la existencia de puntos críticos de este término principal y de una condición de no degeneración.

Por último, se considera un problema no local en el grupo de Heisenberg $\mathbb{H}$. En particular, se buscan propiedades de rigidez para soluciones estables de

$$(-\Delta_{\mathbb{H}})^s v = f(v) \quad \text{en } \mathbb{H},$$

$s \in (0,1)$. Como paso fundamental, se prueba una desigualdad del tipo Poincaré en conexión con un problema elíptico degenerado en $\mathbb{R}^4_+$. Esta desigualdad se usará en un procedimiento de extensión para dar un criterio bajo el cual los conjuntos de nivel de las soluciones del problema anterior son superficies mínimas en $\mathbb{H}$, es decir, tienen $\mathbb{H}$-curvatura media nula.

Abstract

This work is devoted to nonlinear PDEs. The aim is to find regular solutions to some elliptic and hypo-elliptic PDEs and study their qualitative properties.

The first part deals with the supercritical problem

$$-\Delta u = \lambda \mathrm{e}^u,$$

$\lambda > 0$, in an exterior domain under zero Dirichlet condition. A finite-dimensional reduction scheme provides the existence of infinitely many regular solutions whenever λ is sufficiently small.

The second part is focused on the existence of bubbling solutions for the non-local equation

$$(-\Delta)^s u = u^p,\ u > 0,$$

in a bounded, smooth domain under zero Dirichlet condition; where $0 < s < 1$ and $p := (N+2s)/(N-2s) \pm \varepsilon$ is close to the *critical exponent* ($\varepsilon > 0$ small). To this end, a finite-dimensional reduction scheme in suitable functional spaces is used, where the main part of the reduced function is given in terms of the Green's and Robin's functions of the domain. The existence of solutions depends on the existence of critical points of such a main term together with a non-degeneracy condition.

In the third part, a non-local entire problem in the Heisenberg group $\mathbb{H}$ is studied. The main interests are rigidity properties for stable solutions of

$$(-\Delta_{\mathbb{H}})^s v = f(v) \quad \text{in } \mathbb{H},$$

$s \in (0,1)$. A Poincaré-type inequality in connection with a degenerate elliptic equation in $\mathbb{R}^4_+$ is provided. Through an extension (or "lifting") procedure, this inequality will be then used to give a criterion under which the level sets of the above solutions are minimal surfaces in $\mathbb{H}$, i.e. they have vanishing mean $\mathbb{H}$-curvature.

A todas las personas que me acompañaron en este viaje...
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Contents

Introduction

This document compile the work done along three years of a jontly supervised doctoral thesis between the *Universidad de Chile* and the *Aix-Marseille Université*. The work is composed by three chapters mainly based on two papers [33, 53] and a preprint under review [1].

Two main subjects are considered: elliptic and hypo-elliptic PDEs. In the first chapter a supercritical elliptic problem with exponential nonlinearity is studied. This problem is settled in an exterior domain under zero Dirichlet condition. The second chapter is focused on a non-local problem which involves the fractional Laplacian and nonlinearities with power-like growth, it is settled in bounded domains under zero Dirichlet condition. Finally, the third chapter is about a non-local entire problem in the Heisenberg group, a related hypo-elliptic PDE shall be studied as well.

In the next paragraphs, the properties of interest in this work are explained. A briefly discussion of the results shall be done as well, leaving the precise framework for the subsequence chapters.

Nonlinear elliptic PDEs

A basic model of nonlinear elliptic boundary problem is the Lane-Emden-Fowler equation,

$$\begin{cases} \Delta u + u^p = 0, \quad u > 0, \text{ in } \Omega, \\ \qquad\quad u = 0, \quad \text{on } \partial\Omega, \end{cases} \tag{L-E-F}$$

where Ω is a domain with smooth boundary in $\mathbb{R}^N$, $N \geq 3$ and $p > 1$. First formulated by Lane, an astrophysicist, in the mid 19th century, the role of this and related elliptic PDEs has been broad outside and inside mathematics. The solution set of this problem may be surprisingly complex; much has been learned over the last decades, particularly thanks to the development of techniques from the calculus of variations (see Struwe [71]). Nevertheless, various basic issues are not yet fully understood.

An important characteristic of this problem is the role played by the *critical exponent* $p = (N+2)/(N-2)$ in the solvability question. When Ω is bounded and p is subcritical, problem (L-E-F) can be solved using variational methods, see [71]. However, something quite different happens in the supercritical case: existence fails in general, as Pohozaev discovered in 1965 [62]. This has been a widely open matter, particularly since variational machinery no longer applies, at least in its naturally adapted way for subcritical or critical problems.

[1] Joint work with Manuel del Pino, Juan Dávila and Yannick Sire.

A supercritical elliptic problem in exterior domains

Let $\mathcal{D}$ be a bounded, smooth domain in $\mathbb{R}^N$, $N \geq 3$. Chapter 1 deals with the problem of finding classical solutions to

$$-\Delta u = \lambda e^u \quad \text{in } \mathbb{R}^N \setminus \overline{\mathcal{D}}, \tag{1a}$$
$$u = 0 \quad \text{on } \partial\mathcal{D}, \tag{1b}$$

where $\lambda > 0$ is a parameter.

In contrast with problem (L-E-F), the nonlinearity in (1a) is of exponential type. Nevertheless, both problems share some important properties as it'll be shown in Chapter 1 (see [29, 31]). Problem (1a)-(1b) is supercritical in the sense that variational methods cannot be applied, at least in its naturally adapted way for subcritical or critical problems.

The exponential nonlinearity in (1a) appears, among others phenomena, in thermal self-ignition models (see Bebernes and Eberly [5]). The full model describes the reaction process in a combustible material during what is referred to as the ignition period. u represents a dimensionless temperature inside a cylindrical vessel (which walls are ideally conducting) when the system has reached an intermediate asymptotic steady state. The balance between diffusion and reaction is quantified by the parameter $\lambda > 0$. This parameter is sometimes referred to as the Frank-Kamenetskii constant, see [46].

In the case $\mathcal{D} = B_1(0)$, the unit ball, problem (1a)-(1b) has a family of radial solutions, for $\lambda > 0$ small, obtained by scaling an entire solution of (1a) in $\mathbb{R}^N$. Actually, through a phase plane analysis, it can be proven that this family of solutions is much richer. The purpose of Chapter 1 is to show that part of this family of solutions still exists for general exterior domains $\mathbb{R}^N \setminus \overline{\mathcal{D}}$. In particular, the following existence result is proven.

Theorem (Dávila and López [33])**.** *There exist $\Lambda > 0$ such that, for $0 < \lambda < \Lambda$, there is a family of solutions $\{u_{\lambda,\alpha}\}_{\alpha>0}$ to problem (1a)–(1b) with the following asymptotic behavior:*

$$u_{\lambda,\alpha}(x) = -2\log|x| - \log\left(\frac{\lambda}{2(N-2)}\right) + O(|x|^{-\beta}) \quad \text{as } |x| \to +\infty,$$

where $\beta = \beta(N)$ is a positive number.

The proof of this existence result makes use of a finite-dimensional reduction scheme in appropriate weighted spaces. Similar phenomenon to the one presented in this work for the exponential nonlinearity was detected for a supercritical equation in exterior domains by Dávila, del Pino, Musso and Wei [29, 31] and for a supercritical Schrödinger equation in entire space, with a rapidly decaying potential (see [30]).

Non-local elliptic problems with power-like nonlinearity

In Chapter 2, the existence of bubbling solutions for a nonlocal version of problem (L-E-F) is studied, namely

$$\begin{cases} (-\Delta)^s u = u^p, \ u > 0 \quad \text{in } \Omega, \\ \qquad\qquad u = 0, \quad \text{on } \partial\Omega, \end{cases} \tag{2}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $0 < s < 1$, $N > 2s$ and $p = (N+2s)/(N-2s) \pm \varepsilon$ is close to the *critical exponent* ($\varepsilon > 0$ small). This problem is studied in both subcritical and supercritical regime. The precise definition of the Fractional Laplacian is given in Chapter 2.

Fractional operators have been studied in connection with different phenomena such as optimization (see Duvaut and Lions [37]), flame propagation (see Caffarelli, Roquejoffre and Sire [18]); and finance (see Cont and Tankov [28]). It is also worthwhile to mention the thin obstacle problem and phase transition problems (Caffarelli, Salsa and Silvestre [19] and Sire, Valdinoci [68]). From a probabilistic point of view, the standard fractional Laplacian is the infinitesimal generator of a Levy process (see Bertoin [6]).

For the usual Laplacian, problem (2) was extensively studied when the exponent p approaches critical from below, namely $p = (N+2s)/(N-2s) - \varepsilon$, see Brezis and Peletier [10], Rey [63, 64, 65], Han [50] and Bahri, Li and Rey [4]. In the latter reference, bubbling solutions are found for $N \geq 4$, concentrating around nondegenerate critical points of certain object which involve the Green's and Robins's function of Ω. On the other hand, the supercritical case $p = (N+2s)/(N-2s) + \varepsilon$ was studied by del Pino, Felmer and Musso [35, 36]. They showed a concentration phenomena for bubbling solutions to this problem when the domain satisfy certain "topological condition", for instance a domain exhibiting multiple holes.

For the fractional framework, several authors studied nonlinear problems of the form $(-\Delta)^s u = f(u)$ for a certain function $f : \mathbb{R}^N \to \mathbb{R}$. Among others, it is worthwhile to mention the work by Cabré and Tan [16] and Tan [72] when $s = 1/2$. They established the existence of positive solutions for equations having the subcritical growth, their regularity and symmetry properties. See also [9]. For the subcritical case, Choi, Kim and Lee [27] recently developed a nonlocal version of the results by Han [50] and Rey [64], mentioned in the previous paragraph. They also proved a concentration phenomena for (2) in the subcritical case.

The aim in Chapter 2 is to study the bubbling concentration phenomena for problem (2) in both regimes: subcritical and supercritical. This is done through a finite-dimensional reduction scheme in suitable weighted functional spaces, where the main part of the reduced functional is given in terms of the Green's and Robin's functions of the domain. It will be proven that the existence of solutions depends on the existence of critical points of such a main term together with a non-degeneracy condition. In contrast with [27], where a reduction scheme was carried out with a variational approach in fractional Sobolev spaces, this work extensively makes use of weighted Hölder spaces to circumvent the lack of Sobolev's embedding. In the subcritical case, our concentrating results constitute a different proof of the corresponding by [27]. On the other hand, in the supercritical regime new bubbling concentration properties for nonlocal problems of type (2) are found. In Section 2, the following results are proven.

Theorem. *Assume $0 < s < 1$, $N > 2s$, and Ω be a smooth bounded domain in $\mathbb{R}^N$. Suppose that Ψ in (2.6) has a stable critical set $\mathcal{A}$ (see Definition 2.1). Then, there exists a point $\xi = (\xi_1, \ldots, \xi_m) \in \mathcal{A}$ and a family of solutions of problem (2) which blow up and concentrate at each point ξ_i, $i = 1, \ldots, m$, as ε tends to zero.*

Theorem. *Assume $0 < s < 1$, $N > 2s$, and Ω be a smooth bounded domain in $\mathbb{R}^N$. Then, if $p = (N+2s)/(N-2s) - \varepsilon$, there exists a point $\xi \in \Omega$ and a family of solutions of problem (2) which concentrate at the point ξ as ε tends to zero. Moreover, ξ is a critical point of the Robin's function $\varphi(x) = H(x, x)$, see (2.5).*

Non-local phase transitions in the Heisenberg group

Chapter 3 is focused on rigidity properties for stable solutions (see Definition 3.6) of non-local equations of the type

$$(-\Delta_{\mathbb{H}})^s v = f(v) \quad \text{in } \mathbb{H}, \tag{3}$$

where $s \in (0,1)$, $f \in C^{1,\gamma}(\mathbb{R})$, $\gamma > \max\{0, 1-2s\}$ and $\mathbb{H}$ is the Heisenberg group. The aim of this chapter is to give a geometric insight of the phase transition for equation (3). Following Sire and Valdinoci [68], a geometric proof of rigidity properties for fractional boundary reactions in $\mathbb{H}$ is provided.

The relation between entire stable solutions and minimal surfaces, as performed in this work, is inspired by a famous conjecture of De Giorgi [34] (in the Euclidean setting) and in it is in the spirit of the proof of Bernstein theorem given by Giusti [49]. Similar De Giorgi-type results (in the Euclidean setting) have been proven by Cabré and Solà-Morales [15] for the square root of the Laplacian, and later generalized by Cabré and Sire [12] for arbitrary roots. Sire and Valdinoci [68] gave a proof of analogous rigidity properties for phase transitions driven by fractional Laplacians. Unlike the method in [12, 15], which require a Liouville-type result, the proof in [68] relies heavily on a Poincaré-type inequality which involves the geometry of the level sets of u. Last technique is inspired on the work by Sternberg and Zumbrun [69, 70].

In Chapter 2, a Poincaré-type inequality will be found through a suitable development of some techniques for level set analysis. This follows the ideas of Sternberg and Zumbrun [69, 70]; Farina [38]; Farina, Sciunzi and Valdinoci [39]; Sire and Valdinoci [68]. Some properly modified computations by Ferrari and Valdinoci [41] are needed in order to understand the complicated geometry of the Heisenberg group. This inequality, together with an "abstract" formulation of a technique recently introduced by Caffarelli and Silvestre [20], is used to study symmetry properties of (3).

On the other hand, sub-Laplacians in Carnot groups, as $\Delta_{\mathbb{H}}$ (see (3.17)), exhibit strong analogies with classical Laplace operators in the Euclidean space (Harnack inequality, maximum principle, existence and estimates of the fundamental solution). Following [20], a construction of a $\Delta_{\mathbb{H}}$-harmonic "lifting" operator $v = v(x) \mapsto u = u(x,y)$ from $\mathbb{H}$ to $\mathbb{H} \times \mathbb{R}^+$ can be carried out by means of the spectral resolution of $-\Delta_{\mathbb{H}}$ in $L^2(\mathbb{H})$, in such a way that v is the trace of the normal derivative of u on $\{y = 0\}$ (see Ferrari and Franchi [40] and the references therein).

In Chapter 3 the following Poincaré-type inequality is proven, the precise meaning of the terms in this result will be given there. Here $\widehat{\mathbb{H}} := \mathbb{H} \times \mathbb{R}_+$ and $B_R^+ := B_R \cap \mathbb{R}_+$.

Theorem (López and Sire [53])**.** *Let $u \in C^2(\widehat{\mathbb{H}})$ be a bounded and stable weak solution of a degenerate elliptic problem in $\widehat{\mathbb{H}}$, see (3.4). Assume furthermore that for all $R > 0$,*

$$|\nabla_{\mathbb{H}} u| \in L^\infty(\overline{B_R^+}).$$

Then, for any $\phi \in C_0^\infty(\mathbb{R}^4)$, we have

$$\int_{\widehat{\mathbb{H}}} y^a |\nabla_{\mathbb{H}} u|^2 |\nabla_{\widehat{\mathbb{H}}} \phi|^2$$

$$\geq \int_{\mathcal{R}^4_+} y^a \left(|Hu|^2 - \langle (Hu)^2 \nu_{x,y}, \nu_{x,y} \rangle_{\mathbb{H}} - 2(TYuXu - TXuYu) \right) \phi^2$$

$$= \int_{\mathcal{R}^4_+} y^a |\nabla_{\mathbb{H}} u|^2 \left[h^2_{x,y} + \left(p_{x,y} + \frac{\langle Huv_{x,y}, \nu_{x,y} \rangle_{\mathbb{H}}}{|\nabla_{\mathbb{H}} u|} \right)^2 + 2\langle T\nu_{x,y}, v_{x,y} \rangle_{\mathbb{H}} \right] \phi^2.$$

This result may be interpreted in two ways:

i) to think that some interesting geometric objects which describe u, such as its intrinsic Hessian and the curvature of its level sets, are bounded by an energy term. These quantities involved in the inequality are weighted by an arbitrary test function ϕ, and

ii) thinking that the inequality bounds a suitably weighted L^2-norm of its gradient. The weights here are given by the stable solution u. This interpretation sees the inequality as a Sobolev-Poincaré inequality.

As a consequence of the previous theorem, the following rigidity result is deduced:

Theorem (Rigidity result)**.** *Let $v \in C^{2,\sigma}(\mathbb{H})$, $\sigma \in (0, 2s)$, be a bounded stable solution of problem (3). Assume also that the "harmonic lifting" of v to $\widehat{\mathbb{H}}$ (see Ch. 2, Sec. 3), which we denote by u, satisfies (3.13) and (3.14). Then, the level sets of v in the vicinity of non-characteristic points are minimal surfaces in the Heisenberg group (i.e., the curvature h vanishes identically).*

Chapter 1

A supercritical problem in exterior domains

Let $\mathcal{D}$ be a bounded, smooth domain in $\mathbb{R}^N$, $N \geq 3$. In this chapter we consider the problem of finding classical solutions of

$$-\Delta u = \lambda \mathrm{e}^u \quad \text{in } \mathbb{R}^N \setminus \overline{\mathcal{D}}, \tag{1.1}$$
$$u = 0 \quad \text{on } \partial\mathcal{D}, \tag{1.2}$$

where $\lambda > 0$ is a parameter. This chapter is mainly based on the paper [33].

1.1 Introduction and main results

Véron and Matano [73, Theorem 3.1] considered Eq. (1.1) in dimension 3 with $\mathcal{D} = B_1(0)$, the unit ball, and a non-homogeneous boundary condition $u = \phi$ on $\partial B_1(0)$. They proved that if ϕ is close enough to $2w - \log(\lambda/2)$ where w is a smooth solution of

$$\Delta_{S^2} w + \mathrm{e}^{2w} - 1 = 0 \tag{1.3}$$

on the sphere S^2, then there is a solution with the asymptotic behavior

$$u(x) = -2\log|x| - \log(\lambda/2) + 2w(x/|x|) + o(1)$$

as $|x| \to +\infty$. This result is based on the understanding of the solutions of (1.3) obtained in [23] (see also [7]).

In all dimensions $N \geq 3$ and still with $\mathcal{D} = B_1(0)$ we can also describe many radial solutions of (1.1)–(1.2). To fix ideas, we denote by U the unique radial solution of

$$-\Delta U = \lambda_0 \mathrm{e}^U \quad \text{in } \mathbb{R}^N, \tag{1.4}$$
$$U(0) = 0, \tag{1.5}$$

where $\lambda_0 := 2(N-2)$. This solution can be constructed by solving the initial value problem (1.4) with $U(0) = U'(0) = 0$, and can be proved to be defined for all $r > 0$. For any $\alpha > 0$ the function

$$U_\alpha = U(\alpha r) + 2\log\alpha \tag{1.6}$$

also satisfies (1.4). Note that for $\lambda > 0$, $u = U_\alpha - \log(\frac{\lambda}{\lambda_0})$ satisfies

$$-\Delta u = \lambda e^u \quad \text{in } \mathbb{R}^N.$$

The boundary condition (1.2) is fulfilled if $0 = U(\alpha) + 2\log\alpha - \log(\frac{\lambda}{\lambda_0})$.

Regarding $\alpha > 0$ as a parameter we find a family of solutions of (1.1)–(1.2) of the form $\lambda_\alpha = \lambda_0 \alpha^2 e^{U(\alpha)}$ and $u_\alpha(r) = U_\alpha(r) - U_\alpha(1)$. As $\alpha \to 0$ we see that $\lambda_\alpha \to 0$, while $\lambda_\alpha \to \lambda_0$ as $\alpha \to +\infty$, which follows from the asymptotic behavior of $U(r) = -2\log(r) + o(1)$ as $r \to +\infty$. Let us point out that the family of solutions $(\lambda_\alpha, u_\alpha)$ with $\alpha > 0$ also describes all classical solutions of the problem

$$-\Delta u = \lambda e^u \quad \text{in } B_1(0), \quad u = 0 \quad \text{on } \partial B_1(0),$$

with $\lambda > 0$, which was studied in dimension 3 in [48] and later in all higher dimensions in [51].

Still in the case $\mathcal{D} = B_1(0)$ one can see that the set of classical solutions of (1.1)–(1.2) is much richer than the family given by $(\lambda_\alpha, u_\alpha)$ with $\alpha > 0$. To see this it is convenient to work with Emden-Fowler change of variables

$$v(s) = u(r), \; r = e^s, \tag{1.7}$$

which transform (1.1) into

$$v'' + (N-2)v' = -\lambda e^{v+2s} \quad \text{in } \mathbb{R}, \tag{1.8}$$

and then define

$$v_1 = \lambda e^{v+2s}, \quad v_2 = v'. \tag{1.9}$$

This transforms (1.8) into the autonomous system

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}' = \begin{pmatrix} v_1(v_2+2) \\ -v_1 - (N-2)v_2 \end{pmatrix}. \tag{1.10}$$

This system has two stationary points: $(0,0)$ and $(2(N-2), -2)$, the first being a saddle point and the second an spiral or an asymptotically stable node depending on the dimension.

The solution U of (1.4)–(1.5) corresponds to a heteroclinic orbit which connects the equilibria $(0,0)$ and $(2(N-2),-2)$ in the phase plane (v_1, v_2). Take any point $P = (\lambda, \beta)$ in this orbit. Then for any $\tilde{P} = (\lambda, \tilde{\beta})$ sufficiently close to P the solution of (1.10) with initial condition $\tilde{P}$ at time $s = 0$ will be defined for all positive s and will converge to $(2(N-2),-2)$ as $s \to +\infty$. Under the previous change of variables this will be a solution of (1.1)–(1.2) associated to the same parameter λ. Note that $\lambda = \lambda_\alpha$ for some $\alpha > 0$. The previous discussion shows that together with the *special* solution $(\lambda_\alpha, u_\alpha)$ there is a continuum of other solutions of (1.1)–(1.2) with the same λ_α, and all share the behavior $u(r) = -2\log(r) - \log(\frac{\lambda}{2(N-2)}) + o(1)$ as $r \to +\infty$.

The purpose in this chapter is to show that part of the family of solutions described in the preceding paragraph in the radial setting still exists for general exterior domains $\mathbb{R}^N \setminus \overline{\mathcal{D}}$. For the precise statement of our result, we need to distinguish between the cases $N \geq 4$ and $N = 3$.

Theorem 1.1. *Assume $N \geq 4$. Let $\alpha > 0$ and $\xi \in \mathbb{R}^N$. Then for sufficiently small $\lambda > 0$ there is a solution u_λ to problem (1.1)–(1.2) such that*

$$u_\lambda \to U_\alpha(\xi)(1 - \varphi_0(x)) \quad \text{as } \lambda \to 0^+,$$

uniformly on bounded subsets of $\mathbb{R}^N \setminus \mathcal{D}$, where φ_0 is the Newtonian potential of the layer $\partial\mathcal{D}$ (see (1.19)), and has the asymptotic behavior

$$u_\lambda(x) = -2\log|x| - \log\left(\frac{\lambda}{2(N-2)}\right) + O(|x|^{-\beta}) \quad \text{as } |x| \to +\infty,$$

where β is a positive number (see (1.25)).

The analysis of the radial case suggests looking for a solution u_λ close to a rescaled and translated form of U_α. It turns out that $U_\alpha(\xi + \sqrt{\lambda/\lambda_0}x)$, where $\xi \in \mathbb{R}^N$ is fixed arbitrarily, is a good approximation. This leads us to construct an inverse of the linearized operator $-\Delta u + \lambda_0 \mathrm{e}^{U_\alpha}$ in $\mathbb{R}^N \setminus (\xi + \sqrt{\lambda/\lambda_0}\mathcal{D})$. This set approaches $\mathbb{R}^N$ as $\lambda \to 0^+$, so such an inverse is constructed as a small perturbation of an inverse of this operator in entire space. This inverse indeed exists for $N \geq 4$ and adding a lower order correction to the initial approximation yields the desired solution. In dimension 3, however, the linearized operator is not surjective, having a range orthogonal to the generator of translations. Thus for $N = 3$ we find a family of solutions provided ξ is adjusted properly. This explains the next result.

Theorem 1.2. *Let $\alpha > 0$. Then there exist $\Lambda, Z > 0$ such that for $0 < \lambda < \Lambda$ there are $\xi_\lambda \in \mathbb{R}^3$ with $|\xi_\lambda| < Z$ and a solution u_λ to problem (1.1)–(1.2) such that*

$$u_\lambda(x) - U_\alpha(\xi_\lambda)(1 - \varphi_0(x)) \to 0 \quad \text{as } \lambda \to 0^+,$$

uniformly on bounded subsets of $\mathbb{R}^3 \setminus \mathcal{D}$ (see (1.19) for φ_0). Moreover, u_λ has the behavior

$$u_\lambda(x) = -2\log|x| - \log\left(\frac{\lambda}{2(N-2)}\right) + O(|x|^{-\beta}) \quad \text{as } |x| \to +\infty,$$

where $\beta \in (0, 1/2)$.

In summary, the difference between the cases $N = 3$ and $N \geq 4$ is that in the former case, the solutions found constitute a one-parameter family only dependent on $\alpha > 0$, while in the latter case is an $(N+1)$-dimensional family depending on $\alpha > 0$ and ξ.

Theorem 1.2 is similar to the one of Véron and Matano [73, Theorem 3.1] where we choose to work with the solution $w = 0$ of (1.3), but we obtain existence for a general bounded smooth domain $\mathcal{D}$.

Similar phenomenon to the one presented in this work for the exponential nonlinearity was detected for a supercritical equation in exterior domains in [29, 31] and for a supercritical Schrödinger equation in entire space, with a rapidly decaying potential in [30].

1.2 Preliminaries

Let us make some additional comments on (1.4)–(1.5) and system (1.10). As we mentioned, the solution U to (1.4)–(1.5) can be obtained using the Picard fixed point theorem applied to the equivalent integral equation

$$u(r) = -\lambda \int_0^r \int_0^s \left(\frac{t}{s}\right)^{N-1} \mathrm{e}^{u(t)}\, \mathrm{d}t\, \mathrm{d}s$$

on a maximal interval $(0, T)$. We can show that in fact $T = +\infty$ by observing that v, defined in (1.7), remains bounded in $(-\infty, \log T)$. Indeed, note that the Lyapunov function

$$\mathcal{L}(v) = \frac{(v' + 2)^2}{2} + \lambda e^{v+2s} - 2(N-2)(v + 2s)$$

decreases.

1.2.1 Phase plane analysis

Recall that the system (1.10) has two stationary points $(0, 0)$ and $(\lambda_0, -2)$, where $\lambda_0 = 2(N-2)$. If we linearize around the second point we have the associated eigenvalues

$$\mu_{\pm} = -\frac{N-2}{2} \pm \frac{1}{2}\sqrt{(N-2)(N-10)}. \tag{1.11}$$

If $3 \le N \le 9$, (1.11) gives complex values and $(\lambda_0, -2)$ is a spiral point. If $N \ge 10$, (1.11) gives negative eigenvalues and $(\lambda_0, -2)$ is an asymptotically stable node.

We can get an expression of U and U' in terms of v_1 and v_2, namely

$$e^{U(r)} = \frac{v_1(\log r)}{\lambda_0} r^{-2}, \quad U'(r) = r^{-1} v_2(\log r), \tag{1.12}$$

which implies that $U(r) = -2\log(r) + o(1)$ as $r \to +\infty$. This behavior is actually common to all the radial solutions of (1.4).

In dimensions $3 \le N \le 9$ the behavior of $U(r)$ is oscillatory around the singular solution $-2\log(r)$. Instead, if $N \ge 10$ then $U(r) < -2\log(r)$ for all $r > 0$, since it can be shown that $v_1(s) < \lambda_0$ for all $s \in \mathbb{R}$, as the next result shows.

Claim 1.3. *Suppose that $N \ge 10$. Then*

$$0 < v_1(s) < \lambda_0, \quad -2 < v_2(s) < 0, \quad \textit{for all } s \in \mathbb{R}.$$

Proof. Associated to the eigenvalue μ_+ in (1.11) we have the eigenvector $\xi_1 = (\mu_-, 1)$. To prove the claim it is enough to show that the curve $(v_1(s), v_2(s))$, $s \in \mathbb{R}$, is between the lines $v_2 = -2$ and $v_1 = \lambda_0 + 2\mu_- + \mu_- v_2$ (i.e. the line passing through the point $(\lambda_0, -2)$ in the direction ξ_1).

First suppose, by contradiction, that there exists $s_0 \in \mathbb{R}$ such that $0 < v_1(s_0) < \lambda_0$ and $v_2(s_0) = -2$. Choose s_0 as the smallest one with this property. Recalling (1.9)–(1.10) and using the minimality condition of s_0, we see that

$$v_1'(s_0) = 0 \quad \text{and} \quad v_2'(s_0) \le 0.$$

But, from (1.10), we have that $v_2'(s_0) = -v_1(s_0) + 2(N-2) > 0$, which contradicts the previous. So, if $0 < v_1(s) < \lambda_0$ then $v_2(s) > -2$.

On the other hand, suppose, by contradiction, that the curve $(v_1(s), v_2(s))$ crosses the line previously described. So that, there exists $s_0 \in \mathbb{R}$ with

$$v_1(s_0) = \mu_- v_2(s_0) + \lambda_0 + 2\mu_-, \tag{1.13}$$

$$\frac{1}{\mu_-} \le \frac{v_2'(s_0)}{v_1'(s_0)}. \tag{1.14}$$

Of course here we are choosing the smallest point where the curves cross each other. Last inequality and (1.10) yield

$$\frac{1}{\mu_-}v_1(v_2+2) \leq -v_1 - (N-2)v_2.$$

Replacing v_1 of (1.13) in the last inequality, we have

$$\mu_- v_2^2 + [\mu_-^2 + (N-2)\mu_- + \lambda_0 + 4\mu_-]v_2 + 2[\mu_-^2 + (N-2)\mu_- + \lambda_0] + 4\mu_- \geq 0.$$

Recalling that μ_- satisfies

$$\mu_-^2 + (N-2)\mu_- + \lambda_0 = 0,$$

we deduce that

$$v_2^2 + 4v_2 + 4 \leq 0.$$

This contradicts the previous claim. The proof is complete. □

1.2.2 Asymptotic behavior

Regarding the function U in (1.4)–(1.5), we'll need a further analysis of its asymptotic behavior. In this respect, we have the following result.

Claim 1.4. *Let U be the only radial solution to (1.4)–(1.5) and v be as in (1.7). Then:*

i) if $3 \leq N \leq 9$, $v(s) = -2s + O(\mathrm{e}^{-\frac{N-2}{2}s})$ as $s \to +\infty$;

ii) if $N = 10$, there exist $a \in \mathbb{R}$ and $b < 0$ such that

$$v(s) = -2s + a\mathrm{e}^{-4s} + bs\mathrm{e}^{-4s} + o(s\mathrm{e}^{-4s}) \quad \text{as } s \to +\infty;$$

iii) if $N > 10$, there exist $a \in \mathbb{R}$ and $b < 0$ such that

$$v(s) = -2s + a\mathrm{e}^{\mu_- s} + bs\mathrm{e}^{\mu_+ s} + o(s\mathrm{e}^{\mu_+ s}) \quad \text{as } s \to +\infty.$$

We shall prove the case of resonance ii), the other cases can be handled similarly.
A preliminary analysis of the autonomous system (1.10) suggests to consider

$$w = v + 2s, \quad s \in \mathbb{R},$$

which satisfies

$$w'' + 8w' + 16w = -16(\mathrm{e}^w - 1 - w) \quad \text{in } \mathbb{R}. \tag{1.15}$$

This implies that there exist constants $a, b \in \mathbb{R}$ such that

$$w = a\mathrm{e}^{-4s} + bs\mathrm{e}^{-4s} + w_p, \tag{1.16}$$

where w_p is a particular solution of the non-homogeneous equation (1.15) and a, b are numbers depending on w_p.

Following the variation of parameters method, we look for a solution of (1.15) of the form $w_p = u_1 \mathrm{e}^{-4s} + u_2 s\mathrm{e}^{-4s}$, where

$$u_1' = \frac{\begin{vmatrix} 0 & s\mathrm{e}^{-4s} \\ -16(\mathrm{e}^w - 1 - w) & \mathrm{e}^{-4s} - 4s\mathrm{e}^{-4s} \end{vmatrix}}{\begin{vmatrix} \mathrm{e}^{-4s} & s\mathrm{e}^{-4s} \\ -4\mathrm{e}^{-4s} & \mathrm{e}^{-4s} - 4s\mathrm{e}^{-4s} \end{vmatrix}} = 16s\mathrm{e}^{4s}(\mathrm{e}^w - 1 - w)$$

and

$$u_2' = \frac{\begin{vmatrix} \mathrm{e}^{-4s} & 0 \\ -4\mathrm{e}^{-4s} & -16(\mathrm{e}^w - 1 - w) \end{vmatrix}}{\begin{vmatrix} \mathrm{e}^{-4s} & s\mathrm{e}^{-4s} \\ -4\mathrm{e}^{-4s} & \mathrm{e}^{-4s} - 4s\mathrm{e}^{-4s} \end{vmatrix}} = -16\mathrm{e}^{4s}(\mathrm{e}^w - 1 - w).$$

Considering the expected asymptotic behavior, we choose the particular solution

$$w_p = -\int_s^{+\infty} 16t\mathrm{e}^{4t}(\mathrm{e}^w - 1 - w)\,\mathrm{d}t\,\mathrm{e}^{-4s} + \int_s^{+\infty} 16\mathrm{e}^{4t}(\mathrm{e}^w - 1 - w)\,\mathrm{d}t\,s\mathrm{e}^{-4s}$$

Claim 1.3 and the asymptotic behavior, as $s \to -\infty$, of w in (1.16) implies that $b \leq 0$. Moreover, $b \neq 0$. Indeed, arguing by contradiction, suppose $b = 0$, i.e.,

$$w = a\mathrm{e}^{-4s} + w_p.$$

Multiplying by e^{4s} and differentiating both sides we have

$$(w\mathrm{e}^{4s})' = (w_p\mathrm{e}^{4s})'.$$

We can compute right hand side directly from the expression of w_p to obtain that, for all $s \leq 0$,

$$(w_p\mathrm{e}^{4s})' = \int_s^{+\infty} 16\mathrm{e}^{4t}(\mathrm{e}^w - 1 - w)\,\mathrm{d}t \geq \int_0^{+\infty} 16\mathrm{e}^{4t}(\mathrm{e}^w - 1 - w)\,\mathrm{d}t > 0.$$

On the other hand,

$$(w\mathrm{e}^{4s})' = w'\mathrm{e}^{4s} + 4w\mathrm{e}^{4s} \to 0 \quad \text{as } s \to -\infty$$

(this limit is zero due to the continuity of U and its derivative around the origin). This contradiction implies $b \neq 0$, the proof of the claim is complete.

1.3 The Method

The construction that we describe next is motivated by [29] and [31]. Let us consider the change of variables

$$\tilde{u} = u\left(\sqrt{\frac{\lambda_0}{\lambda}}(x - \xi)\right),$$

which transform (1.1)–(1.2) into the equivalent problem

$$\begin{cases} -\Delta\tilde{u} = \lambda_0 \mathrm{e}^{\tilde{u}} & \text{in } \mathbb{R}^N \setminus \overline{\mathcal{D}}_{\lambda,\xi}, \\ \tilde{u} = 0 & \text{on } \partial\mathcal{D}_{\lambda,\xi}, \end{cases} \tag{1.17}$$

where $\mathcal{D}_{\lambda,\xi}$ is the shrinking domain

$$\mathcal{D}_{\lambda,\xi} = \xi + \sqrt{\frac{\lambda}{\lambda_0}}\mathcal{D}.$$

The closer λ is taken from zero, the "closer" $\mathbb{R}^N \setminus \overline{\mathcal{D}}_{\lambda,\xi}$ is to $\mathbb{R}^N$, so it is natural to seek for a solution $\widetilde{u}$ in the form of a small perturbation of U_α in (1.6). We need a correction so that the boundary condition is satisfied.

Let φ_λ be the solution of the problem

$$\begin{cases} \Delta\varphi_\lambda = 0 & \text{in } \mathbb{R}^N \setminus \overline{\mathcal{D}}_{\lambda,\xi}, \\ \varphi_\lambda(x) = U_\alpha(x) & \text{on } \partial\mathcal{D}_{\lambda,\xi}, \\ \lim\limits_{|x|\to+\infty} \varphi_\lambda(x) = 0. \end{cases} \tag{1.18}$$

In the same way, consider φ_0 the function such that

$$\begin{cases} \Delta\varphi_0 = 0 & \text{in } \mathbb{R}^N \setminus \overline{\mathcal{D}}, \\ \varphi(x) = 1 & \text{on } \partial\mathcal{D}, \\ \lim\limits_{|x|\to+\infty} \varphi_0(x) = 0. \end{cases} \tag{1.19}$$

By the maximum principle,

$$\varphi_\lambda(x) = (U_\alpha(\xi) + O(\sqrt{\lambda}))\varphi_0\left(\sqrt{\frac{\lambda_0}{\lambda}}(x-\xi)\right).$$

We also note that

$$f_0 := \lim_{|x|\to+\infty} |x|^{N-2}\varphi_0(x) = \frac{1}{(N-2)|S^{N-1}|}\int_{\mathbb{R}^N\setminus\mathcal{D}} |\nabla\varphi_0|^2 \, \mathrm{d}x, \tag{1.20}$$

which in particular implies

$$|\varphi_\lambda(x)| \le C\lambda^{(N-2)/2}|x-\xi|^{2-N} \quad \text{for all } x \in \mathbb{R}^N \setminus \mathcal{D}_{\lambda,\xi}. \tag{1.21}$$

The number $\int_{\mathbb{R}^N\setminus\mathcal{D}} |\nabla\varphi_0|^2 \, \mathrm{d}x$ is the *Newtonian capacity* of $\mathcal{D}$.

Thus we look for a solution to problem (1.17) of the form

$$\widetilde{u} = U_\alpha - \varphi_\lambda + \phi,$$

which yields the following equation for ϕ

$$\begin{cases} \Delta\phi + \lambda_0 \mathrm{e}^{U_\alpha}\phi = M(\phi) + E_\lambda & \text{in } \mathbb{R}^N \setminus \overline{\mathcal{D}}_{\lambda,\xi}, \\ \phi = 0 & \text{on } \partial\mathcal{D}_{\lambda,\xi}, \end{cases} \tag{1.22}$$

where

$$E_\lambda = \lambda_0 \mathrm{e}^{U_\alpha}\varphi_\lambda, \quad M(\phi) = -\lambda_0\mathrm{e}^{U_\alpha}(\mathrm{e}^{\phi-\varphi_\lambda} - 1 - \phi + \varphi_\lambda). \tag{1.23}$$

Remark 1.5. We emphasize that, until here, this scheme applies to all dimensions $N \ge 3$. However, if $N = 3$, in order to solve (1.22) we need to choose a special ξ, depending on λ. This will be done in Section 7.

For the time being let us consider $N \geq 4$. In order to solve (1.22) it is first necessary to construct a bounded right inverse of the linearization of (1.1) around U_α in the whole of $\mathbb{R}^N$, this construction is carried out in Section 4. The method has previously been used by Mazzeo and Pacard (see [56]) where solutions with prescribed singular set for subcritical problems are constructed (see [59] and [60]). For small $\lambda > 0$ a similar solvability property is established for the linearized operator around U_α in the exterior domain $\mathbb{R}^N \setminus \overline{\mathcal{D}}_{\lambda,\xi}$. In Section 5 we construct such a bounded right inverse, namely a solution for the linear problem

$$\begin{cases} \Delta\phi + \lambda_0 e^{U_\alpha}\phi = h & \text{in } \mathbb{R}^N \setminus \overline{\mathcal{D}}_{\lambda,\xi}, \\ \phi = 0 & \text{on } \partial\mathcal{D}_{\lambda,\xi}, \end{cases} \tag{1.24}$$

for norms on functions ϕ and h defined on $\mathbb{R}^N \setminus \overline{\mathcal{D}}_{\lambda,\xi}$ given as follows. For given $0 < \sigma < 2$ and

$$0 < \beta < \begin{cases} 1, & 4 \leq N \leq 9, \\ \min\{\mu_0^-, 1\} & N \geq 10, \end{cases} \tag{1.25}$$

where

$$\mu_0^- = \frac{N-2}{2} - \frac{1}{2}\sqrt{(N-2)(N-10)},$$

we consider the norms

$$\|\phi\|_{*,\xi} = \|\phi\|_{L^\infty(B_1(\xi))} + \sup_{|x-\xi|\geq 1} |x-\xi|^\beta |\phi(x)|, \tag{1.26}$$

$$\|h\|_{**,\xi} = \sup_{|x-\xi|\leq 1} |x-\xi|^\sigma |h(x)| + \sup_{|x-\xi|\geq 1} |x-\xi|^{2+\beta} |h(x)|. \tag{1.27}$$

In this context there is continuity, as the next result states.

Proposition 1.6. *Assume $N \geq 4$. Then given numbers $\alpha > 0$ and $Z > 0$, there exist positive constants C, Λ such that for any $|\xi| \leq Z$ and any $0 < \lambda < \Lambda$ the following holds:*
For any h with $\|h\|_{**,\xi} < +\infty$, there exists a solution of problem (1.24)

$$\phi = \Psi_\lambda(h),$$

which defines a linear operator of h such that

$$\|\phi\|_{*,\xi} \leq C\|h\|_{**,\xi}.$$

In Section 6 we use this result and the contraction mapping principle to solve (1.22).

1.4 The operator $\Delta + \lambda_0 e^{U_\alpha}$ in $\mathbb{R}^N$

Let U_α be a radial solution of (1.4). In this section we study the linear equation

$$\Delta\phi + \lambda_0 e^{U_\alpha}\phi = h, \quad \text{in } \mathbb{R}^N. \tag{1.28}$$

The main result concerns with solvability of this equation and estimates for the solution in the weighted L^∞ norms given by (1.26) and (1.27). The main result in this section is the following.

Proposition 1.7. *Assume $N \geq 4$. Then given $\alpha > 0$ and $Z > 0$, there exists $C > 0$ such that for any $|\xi| \leq Z$ the following holds: For any h with $\|h\|_{**,\xi} < +\infty$, there exists a solution of* (1.28)

$$\phi = \Psi(h),$$

which defines a linear operator of h such that

$$\|\phi\|_{*,\xi} \leq C\|h\|_{**,\xi}. \tag{1.29}$$

To prove this result we first consider $\xi = 0$. We denote the corresponding norms by $\|\|_*$ and $\|\|_{**}$.

1.4.1 A Right Inverse

In this subsection we consider $N \geq 4$ as well as $N = 3$, pointing out the main differences between both cases.

The linear operator in (1.28) is of regular singular type and it is well known that it is Fredholm on weighted spaces provided the weight does not equal one of indicial roots (see for instance [55, 56, 57]). We include the main points of the argument and omit some technical computations.

Let us write h as

$$h(x) = \sum_{k=0}^{\infty} h_k(r)\Theta_k(\theta),\ r > 0,\ \theta \in S^{N-1},$$

where $\Theta_k,\ k \geq 0$ are the eigenfunctions of the Laplace-Beltrami operator $-\Delta_{S^{N-1}}$ on the sphere S^{N-1}, normalized so that they constitute an orthonormal system in $L^2(S^{N-1})$. We take Θ_0 to be a positive constant, associated to the eigenvalue 0 and $\Theta_i,\ 1 \leq i \leq N$ is an appropriate multiple of $x_i/|x|$ which has eigenvalue $\lambda_i = N-1,\ 1 \leq i \leq N$. In general, λ_k denotes the eigenvalue associated to Θ_k, we repeat eigenvalues according to their multiplicity and we arrange them in a non-decreasing sequence. We recall that the set of eigenvalues is given by $\{i(N-2+i)\}_{i\geq 0}$.

We look for a solution ϕ to (1.28) of the form

$$\phi(x) = \sum_{k=0}^{\infty} \phi_k(r)\Theta_k(\theta),\ x = r\theta.$$

Therefore, ϕ satisfies (1.28) if and only if

$$\phi_k'' + \frac{N-1}{r}\phi_k' + \left(2(N-2)e^{U_\alpha} - \frac{\lambda_k}{r^2}\right)\phi_k = h_k, \tag{1.30}$$

for all $r > 0$, for all $k \geq 0$.

To construct solutions of this ODE we need to consider two linearly independent solutions $z_{1,k},\ z_{2,k}$ of the homogeneous equation

$$\phi_k'' + \frac{N-1}{r}\phi_k' + \left(2(N-2)e^{U_\alpha} - \frac{\lambda_k}{r^2}\right)\phi_k = 0, \quad r > 0. \tag{1.31}$$

Once these generators are identified, the general solution of the equation can be written through the variation of parameters formula as

$$\phi_k(r) = z_{1,k}(r)\int z_{2,k}h_k r^{N-1}\,dr - z_{2,k}(r)\int z_{1,k}h_k r^{N-1}\,dr,$$

where the symbol $\int$ designates arbitrary antiderivatives, which will be specify later.

It is helpful to recall the reduction of order method: If one solution $z_{1,k}$ to (1.31) is known, a second linearly independent solution can be found in any interval where $z_{1,k}$ does not vanish as

$$z_{2,k}(r) = z_{1,k}(r) \int z_{1,k}(r)^{-2} r^{1-N}\,\mathrm{d}r.$$

One can find the asymptotic behavior of any solution z of (1.31) as $r \to 0$ and as $r \to +\infty$ by examining the indicial roots of the associated Euler equations. We recall (1.12) to get, as $r \to +\infty$, the limiting equation of (1.31)

$$r^2\phi_k'' + (N-1)r\phi_k' + (2(N-2) - \lambda_k)\phi_k = 0, \quad k \geq 0.$$

As $r \to 0$ the limiting equation is given by

$$r^2\phi_k'' + (N-1)r\phi_k' - \lambda_k\phi_k = 0.$$

In this way, the behavior will be ruled by $z(r) \sim r^{-\mu}$, where μ satisfies

$$\mu^2 - (N-2)\mu - \lambda_k = 0.$$

Equation (1.30) can be solved for each k separately:

Case $k = 0$. Since $\lambda_0 = 0$, Eq. (1.30) is the radial form of the linear problem (1.28). As $r \to +\infty$ the limiting equation is

$$r^2\phi_0'' + (N-1)r\phi_0' + 2(N-2)\phi_0 = 0.$$

The indicial roots of the associated Euler equations are

$$\mu_0^{\pm} = \frac{N-2}{2} \pm \frac{1}{2}\sqrt{(N-2)(N-10)}. \tag{1.32}$$

As $r \to 0^+$, the indicial roots are

$$\mu_1 = 0 \quad \text{and} \quad \mu_2 = N-2. \tag{1.33}$$

Since Eq. (1.4) is invariant under the transformation $\alpha \mapsto U(\alpha r) + 2\log\alpha$, we see by differentiation in α (recall (1.9)) that the function

$$z_{1,0} = v_2(\log r) + 2$$

satisfies (1.31). By Claim 1.4 in Section 2, the asymptotic behavior of $z_{1,0}$, as $r \to +\infty$, depends on the dimension in the following way:

i) if $4 \leq N \leq 9$, then $z_{1,0} = O(r^{-\frac{N-2}{2}})$ as $r \to +\infty$ and $z_{1,0}(r) = O(1)$ as $r \to 0^+$;

ii) if $N = 10$, there exists $c > 0$ such that $z_{1,0} = cr^{-4}\log r(1 + o(1))$ as $r \to +\infty$ and $z_{1,0}(r) = O(1)$ as $r \to 0^+$;

iii) if $N > 10$, there exists $c > 0$ such that $z_{1,0} = cr^{-\mu_0^-}(1 + o(1))$ as $r \to +\infty$ and $z_{1,0}(r) = O(1)$ as $r \to 0^+$.

Let's construct a second solution to (1.31) for each dimension separately. If $4 \leq N \leq 9$, define $z_{2,0}$ for small $r > 0$ by

$$z_{2,0}(r) = z_{1,0}(r) \int_{r_0}^{r} z_{1,0}^{-2} s^{1-N} \, \mathrm{d}s,$$

where r_0 is small so that $z_{1,0} > 0$ in $(0, r_0)$ (which is possible because $z_{1,0} \sim 1$ near to 0). Then $z_{2,0}$ is extended to $(0, +\infty)$ so that it is a solution to the homogeneous equation (1.31) in this interval. By (1.32) and (1.33), $z_{2,0} = O(r^{-\frac{N-2}{2}})$ as $r \to +\infty$ and $z_{2,0} \sim r^{2-N}$ as $r \to 0^+$. We define

$$\phi_0(r) = z_{1,0}(r) \int_1^r z_{2,0} h_0 s^{N-1} \, \mathrm{d}s - z_{2,0}(r) \int_0^r z_{1,0} h_0 s^{N-1} \, \mathrm{d}s.$$

ϕ_0 depends linearly on h_0 and is a solution of (1.30). We omit a calculation to verify that

$$\|\phi_0\|_* \leq C_0 \|h_0\|_{**}.$$

If $N \geq 10$, the strategy is the same as previously, but this time is more convenient to rewrite the variation of parameters formula in the form

$$\phi_0(r) = -z_{1,0}(r) \int_0^r z_{1,0}(s)^{-2} s^{1-N} \int_0^s z_{1,0}(\tau) h_0(\tau) \tau^{N-1} \, \mathrm{d}\tau \mathrm{d}s, \;\; r > 0,$$

This formula is well defined because $z_{1,0} > 0$ (see Claim 1.3 in Section 2). Again, a straightforward calculation shows that ϕ_0 satisfies

$$\|\phi_0\|_* \leq C_0 \|h_0\|_{**}.$$

Case $k = 1, \ldots, N$. In this case as $r \to +\infty$ eq. (1.31) becomes

$$r^2 \phi_k'' + (N-1) r \phi_k' + (N-3) \phi_k = 0.$$

The indicial roots of the associated Euler equations are

$$\mu_k^+ = N - 3 \quad \text{and} \quad \mu_k^- = 1. \tag{1.34}$$

As $r \to 0^+$, the indicial roots are

$$\mu_1 = -1 \quad \text{and} \quad \mu_2 = N - 1. \tag{1.35}$$

Similarly to the case $k = 0$ we have a solution to (1.31), namely $z_{1,k}(r) = -U_\alpha'(r)$ which is positive in all $(0, +\infty)$. Using (1.12) we find that

$$z_{1,k} = -r^{-1} v_2(\log r).$$

About the behavior of $z_{1,k}$, by (1.12) we deduce that there exist constants $c_\infty, c_0 > 0$ such that $z_{1,k} = c_\infty r^{-1}(1 + o(1))$ as $r \to +\infty$ and $z_{1,k}(r) = c_0 r (1 + o(r))$ as $r \to 0^+$. With it we can build a solution to (1.30)

$$\phi_k(r) = -z_{1,k}(r) \int_0^r z_{1,k}(s)^{-2} s^{1-N} \int_0^s z_{1,k}(\tau) h_k(\tau) \tau^{N-1} \, \mathrm{d}\tau \mathrm{d}s. \tag{1.36}$$

We omit a calculation to show that ϕ_k satisfies

$$\|\phi_k\|_* \leq C_k \|h_k\|_{**}.$$

Case $k > N$. Define

$$L_k \phi = \phi'' + \frac{N-1}{r}\phi' + \left(2(N-2)\mathrm{e}^{U_\alpha} - \frac{\lambda_k}{r^2}\right)\phi = 0.$$

This operator satisfies the maximum principle in any interval of the form $(\delta, 1/\delta)$, $\delta > 0$. Indeed, the positive function $z = -U'_\alpha$ is a supersolution, because

$$L_k z = \frac{N-1-\lambda_k}{r^2} z < 0 \quad \text{in } (0, +\infty),$$

since $\{\lambda_k\}_k$ is an increasing sequence. To prove the solvability of (1.30) in the appropriate space we observe that

$$\rho(r) = \pm \frac{C_k \|h_k\|_{**}}{r^{\sigma-2} + r^{\beta}},$$

(for some suitable large C_k) provides sub and supersolutions to $L_k \phi = h_k$. Then the method of sub and supersolutions shows that ϕ_k, founded in this way, satisfies

$$\|\phi_k\|_* \leq C_k \|h_k\|_{**}.$$

Remark 1.8 (Case $N = 3$).

i) Fourier mode $k = 0$: is handled exactly as in dimensions $4 \leq N \leq 9$.

ii) Fourier modes $k = 1, 2, 3$: due to (1.34) some functions in a subspace of solutions to the homogeneous equation (1.31) don't have decay at infinity, as we require. So, in order to solve the non-homogeneous equation (1.30), we have to impose an orthogonality condition on h_k, $k = 1, 2, 3$. If we look at (1.36), we find out that such an orthogonality condition is

$$\int_0^\infty z_{1,k}(\tau) h_k(\tau) \tau^2 \, \mathrm{d}\tau = 0, \quad k = 1, 2, 3. \tag{1.37}$$

If so, it follows easily from (1.36) that ϕ_k satisfies

$$\|\phi_k\|_* \leq C_k \|h_k\|_{**}.$$

iii) Fourier modes $k > 3$: the method previously used for higher dimensions works also.

1.4.2 Continuity

The previous construction implies that given an integer $m > 0$, if $\|h\|_{**} < +\infty$ and $h_k = 0$, for all $k \geq m$ then there exists a solution ϕ to (1.28) that depends linearly with respect to h and

$$\|\phi\|_* \leq C_m \|h\|_{**},$$

where C_m may depend only in m. We shall show that C_m can be chosen independently of m using a blow-up argument that has been previously used by [17, 29, 30, 31, 56].

Suppose, by contradiction, that there is a sequence of functions h_j such that $\|h_j\|_{**} < +\infty$, each h_j has only finitely many non-trivial Fourier modes and that the solution $\phi_j \neq 0$ satisfies

$$\|\phi_j\|_* \geq C_j \|h_j\|_{**},$$

where $C_j \to +\infty$ as $j \to \infty$ (no confusion should arise between ϕ_j, h_j and the associated Fourier modes). Replacing ϕ_j by $\phi_j/\|\phi_j\|_*$ we may assume that $\|\phi_j\|_* = 1$ and $\|h_j\|_{**} \to 0$ as $j \to \infty$. We may also assume that the Fourier modes associated to $\lambda_0 = 0$ and $\lambda_1 = ... = \lambda_N = N-1$ are zero.

Along a subsequence (which we write the same) we must have

$$\sup_{x>1} |x|^\beta |\phi_j(x)| \geq \frac{1}{2} \tag{1.38}$$

or

$$\|\phi_j(x)\|_{L^\infty(B_1(0))} \geq \frac{1}{2}. \tag{1.39}$$

Assume first that (1.38) occurs and let $x_j \in \mathbb{R}^N$ with $|x_j| > 1$ be such that

$$|x_j|^\beta |\phi_j(x_j)| > \frac{1}{4}.$$

Along a new sequence (denote by the same) $x_j \to x_0$ or $x_j \to +\infty$.

If $x_j \to x_0$ then $x_0 \geq 1$ and by standard elliptic estimates $\phi_j \to \phi$ uniformly on compacts sets of $\mathbb{R}^N$. Thus ϕ is a solution to (1.28) with right hand side equal to zero that also satisfies $\|\phi\|_* < +\infty$ and is such that the Fourier modes $\phi_0, ..., \phi_N$ are zero. But the unique solution to this problem is $\phi = 0$, contradicting $\phi(x_0) \neq 0$.

If $|x_j| \to +\infty$, consider $\widetilde{\phi}_j(y) = |x_j|^\beta \phi_j(|x_j|y)$. Then $\widetilde{\phi}_j$ satisfies

$$\Delta \widetilde{\phi}_j + \lambda_0 e^{U_\alpha(|x_j|y)} |x_j|^2 \widetilde{\phi}_j = \widetilde{h}_j \quad \text{in } \mathbb{R}^N,$$

where $\widetilde{h}_j = |x_j|^{\beta+2} h_j(|x_j|y)$. But since $\|\phi_j\|_* = 1$ we have

$$|\widetilde{\phi}_j(y)| \leq |y|^{-\beta}, \quad |y| > \frac{1}{|x_j|}. \tag{1.40}$$

So $\widetilde{\phi}_j$ is uniformly bounded on compact sets of $\mathbb{R}^N \setminus \{0\}$. Similarly, for $|y| > 1/|x_j|$

$$|\widetilde{h}_j(y)| \leq \|h_j\|_* |y|^{-\beta-2}$$

and hence $\widetilde{h}_j \to 0$ uniformly on compact sets of $\mathbb{R}^N \setminus \{0\}$ as $j \to \infty$. By elliptic estimates $\widetilde{\phi}_j \to \phi$ uniformly on compact sets of $\mathbb{R}^N \setminus \{0\}$ and ϕ solves

$$\Delta \phi + \lambda_0 |y|^{-2} \phi = 0 \quad \text{in } \mathbb{R}^N \setminus \{0\}.$$

From (1.40) we deduce the bound

$$|\phi(y)| \leq |y|^{-\beta}, \quad |y| > 0. \tag{1.41}$$

Expanding ϕ as

$$\phi(x) = \sum_{k=N+1}^{\infty} \phi_k(r) \Theta_k(\theta),$$

where ϕ_k denotes the Fourier modes of ϕ (recall that we assumed at the beginning that the first $N+1$ of these modes were zero), we see that ϕ_k has to be a solution to

$$\phi_k'' + \frac{N-1}{r}\phi_k' + \frac{2(N-2)-\lambda_k}{r^2}\phi_k = 0, \quad \forall r>0,\ \forall k>N+1.$$

The solutions of this equation are linear combinations of $r^{-\mu_k^{\pm}}$, where

$$\mu_k^{\pm} = \frac{N-2}{2} \pm \frac{1}{2}\sqrt{(N-2)(N-10)-4\lambda_k}, \quad k>N+1.$$

It's easy to check that $\mu_k^- < 0$ and $\beta < \mu_k^+$. Thus, ϕ_k cannot have a bound of the form (1.41) unless it is identically zero. This is a contradiction because $\widetilde{\phi}_j(x_j/|x_j|) \geq 1/4$ for all j.

The analysis of the case (1.39) is similar. By density, for any h with $\|h\|_{**} < +\infty$ a solution ϕ of (1.28) can be constructed and it satisfies $\|\phi\|_* \leq C\|h\|_{**}$. This proves Proposition 1.7 in the case $\xi = 0$.

1.4.3 Proof of Proposition 1.7

Let η be a smooth cut-off function such that

$$\eta(x) = 0 \quad \text{for all } |x-\xi| \leq \delta,$$
$$\eta(x) = 1 \quad \text{for all } |x-\xi| \geq 2\delta,$$

where $\delta > 0$ is small. We shall solve

$$\Delta\phi_2 + \lambda_0 e^{U_\alpha}(1-\eta)\phi_2 = (1-\eta)h \quad \text{in } \mathbb{R}^N,$$
$$\lim_{|x|\to+\infty} \phi_2(x) = 0.$$

Note that for $\delta > 0$ sufficiently small but fixed the operator $\Delta + \lambda_0 e^{U_\alpha}(1-\eta)$ is coercive, hence there exists a solution to this problem and we have the estimates

$$|\phi_2(x)| \leq C\|h\|_{**,\xi} \quad \text{for all } |x-\xi| \leq 1, \tag{1.42}$$
$$|\phi_2(x)| \leq C\|h\|_{**,\xi}(1+|x|)^{2-N} \quad \text{for all } |x-\xi| \geq 1. \tag{1.43}$$

According to the above arguments, we can solve the equation

$$\Delta\phi_1 + \lambda_0 e^{U_\alpha}\phi_1 = -\lambda_0 e^{U_\alpha}\eta\phi_2 + \eta h \quad \text{in } \mathbb{R}^N, \tag{1.44}$$

provided the right hand side has finite $\|\ \|_{**}$ norm. But, since $\eta\phi_2 = 0$ for $|x-\xi| \leq \delta$, (1.42) and (1.43) imply that

$$\|\lambda_0 e^{U_\alpha}\eta\phi_2\|_{**} \leq C\|h\|_{**,\xi}.$$

Thus, there exists a solution ϕ_1 to (1.44), such that

$$\|\phi_1\|_* \leq C\|h\|_{**,\xi}.$$

Note that the norms $\|\|_*$ and $\|\|_{*,\xi}$ are equivalent, as directly can be checked from their definitions. Then there exists $C > 0$ (which might depends on Z) such that

$$\|\phi_1\|_{*,\xi} \leq C\|h\|_{**,\xi}. \tag{1.45}$$

Define $\phi = \phi_1 + \phi_2$, which is a solution to (1.28). Then from (1.42), (1.43) and (1.45) we see that (1.29) holds, and the proof is complete. □

1.5 Proof of Proposition 1.6

We shall use the operator constructed in the previous section in order to prove Proposition 1.6. We fix $Z > 0$ large and work with $|\xi| \le Z$. The estimates depend on ξ only through Z. We assume that $0 \in \mathcal{D}$. Let $0 < R_0 < R_1$ be fixed such that $2R_0 < R_1$ and $\mathcal{D} \subset B_{R_0}$. Let $\rho \in C^\infty(\mathbb{R}^N),\ 0 \le \rho \le 1$ be such that

$$\rho(x) = 0 \quad \text{for } |x| \le 1, \quad \rho(x) = 1 \quad \text{for } |x| \ge 2$$

and set

$$\eta_\lambda(x) = \rho\left(\frac{\lambda_0^{1/2}}{R_0\lambda^{1/2}}(x-\xi)\right), \quad \zeta_\lambda(x) = \rho\left(\frac{\lambda_0^{1/2}}{R_1\lambda^{1/2}}(x-\xi)\right).$$

We look for a solution to (1.24) of the form

$$\phi = \eta_\lambda \varphi + \psi.$$

We need then to solve the system of equations

$$\begin{cases} \Delta\psi + (1-\zeta_\lambda)\lambda_0 e^{U_\alpha}\psi \\ \qquad = -2\nabla\eta_\lambda\nabla\varphi - \varphi\Delta\eta_\lambda + (1-\zeta_\lambda)h \quad \text{in } \mathbb{R}^N \setminus \overline{\mathcal{D}}_{\lambda,\xi}, \\ \psi = 0 \quad \text{on } \partial\mathcal{D}_{\lambda,\xi}, \quad \lim_{|x|\to+\infty} \psi(x) = 0; \end{cases} \tag{1.46}$$

$$\Delta\varphi + \lambda_0 e^{U_\alpha}\varphi = -\lambda_0 e^{U_\alpha}\zeta_\lambda\psi + \zeta_\lambda h, \quad \text{in } \mathbb{R}^N; \tag{1.47}$$

where φ, ψ are the unknowns.

Proposition 1.6 will be proved using a fixed point argument. We assume $\|h\|_{**,\xi} < +\infty$. Let

$$E_\lambda = B_{2\sqrt{\frac{\lambda}{\lambda_0}}R_0}(\xi) \setminus B_{\sqrt{\frac{\lambda}{\lambda_0}}R_0}(\xi)$$

and consider the Banach space

$$X = \{\varphi /\ \varphi :\ \mathbb{R}^N \longrightarrow \mathbb{R} \text{ is Lipschitz continuous in } E_\lambda \text{ with } \|\varphi\|_{*,\xi} < +\infty\}$$

with the norm

$$\|\varphi\|_X = \|\varphi\|_{*,\xi} + \lambda^{1/2}\|\nabla\varphi\|_{L^\infty(E_\lambda)}.$$

Given $\varphi \in X$ we first note that (1.46) has a solution for suitable small λ because $\|(1-\zeta_\lambda)\lambda_0 e^{U_\alpha}\|_{L^{N/2}(\mathbb{R}^N\setminus\overline{\mathcal{D}}_{\lambda,\xi})} \to 0$ as $\lambda \to 0^+$. Let $\psi(\varphi)$ denote this solution, which is clearly linear in φ. As we shall see, $|\psi| \le C/|x|^{N-2}$ for large $|x|$, which implies that the right hand side of (1.47) has a finite $\|\ \|_{**,\xi}$. Then, by Proposition 1.7, Eq. (1.47) has a solution $\overline{\varphi}$ such that $\|\overline{\varphi}\|_{*,\xi} < +\infty$. Set $F(\varphi) = \overline{\varphi}$.

For $\varphi \in X$ we will fist prove the estimate

$$|\psi(x)| \le C\lambda^{(N-2)/2}(\|h\|_{**,\xi} + \|\varphi\|_X)|x-\xi|^{2-N}, \tag{1.48}$$

for all $x \in \mathbb{R}^N \setminus \overline{\mathcal{D}}_{\lambda,\xi}$. Indeed, let $\widetilde{\psi}(z) = \psi\left(\xi + \sqrt{\dfrac{\lambda}{\lambda_0}}z\right),\ z \in \mathbb{R}^N \setminus \mathcal{D}$. Then

$$\begin{cases} \Delta\widetilde{\psi} + \lambda(1-\rho(z/R_1))e^{U_\alpha}\widetilde{\psi} = g \quad \text{in } \mathbb{R}^N \setminus \overline{\mathcal{D}}, \\ \widetilde{\psi} = 0 \quad \text{on } \partial\mathcal{D}, \quad \lim_{|x|\to+\infty} \widetilde{\psi}(x) = 0, \end{cases} \tag{1.49}$$

where

$$g = -2\frac{\lambda^{1/2}}{R_0\lambda_0^{1/2}}\nabla\rho\left(\frac{z}{R_0}\right)\nabla\varphi\left(\xi+\sqrt{\frac{\lambda}{\lambda_0}}z\right) - \frac{1}{R_0^2}\Delta\rho\left(\frac{z}{R_0}\right)\varphi\left(\xi+\sqrt{\frac{\lambda}{\lambda_0}}z\right) + \frac{\lambda}{\lambda_0}\left(1-\rho\left(\frac{z}{R_1}\right)\right)h\left(\xi+\sqrt{\frac{\lambda}{\lambda_0}}z\right).$$

Then the support of g is contained in the ball B_{2R_1} and we can estimate for all $z \in \mathbb{R}^N \setminus \mathcal{D}$, $|z| \le 2R_1$,

$$2\frac{\lambda^{1/2}}{R_0\lambda_0^{1/2}}\left|\nabla\rho\left(\frac{z}{R_0}\right)\nabla\varphi\left(\xi+\sqrt{\frac{\lambda}{\lambda_0}}z\right)\right| \le C\|\varphi\|_X \tag{1.50}$$

$$\frac{1}{R_0^2}\left|\Delta\rho\left(\frac{z}{R_0}\right)\varphi\left(\xi+\sqrt{\frac{\lambda}{\lambda_0}}z\right)\right| \le C\|\varphi\|_X \tag{1.51}$$

$$\frac{\lambda}{\lambda_0}\left|\left(1-\rho\left(\frac{z}{R_1}\right)\right)h\left(\xi+\sqrt{\frac{\lambda}{\lambda_0}}z\right)\right| \le C\lambda^{1-\sigma/2}\|h\|_{**,\xi}. \tag{1.52}$$

Since $0 \in \mathcal{D}$ and $\sigma < 2$, we see from (1.50)–(1.52) that

$$|g(z)| \le C(\|\varphi\|_X + \|h\|_{**,\xi})\chi_{2R_1}.$$

This estimate and (1.49) yield

$$|\widetilde{\psi}(z)| \le C(\|\varphi\|_X + \|h\|_{**,\xi})|z|^{2-N} \quad \text{for all } z \in \mathbb{R}^N \setminus \mathcal{D}$$

which implies (1.48).

Recall that $\varphi \in X$, $\psi = \psi(\varphi)$ is the solution to (1.46) and we use the notation $\overline{\varphi} = F(\varphi)$. By Proposition 1.7 we have

$$\|\overline{\varphi}\|_{*,\xi} \le C(\|\lambda_0 e^{U_\alpha}\zeta_\lambda\psi\|_{**,\xi} + \|\zeta_\lambda h\|_{**,\xi}). \tag{1.53}$$

Using (1.48) we can estimate $\|\lambda_0 e^{U_\alpha}\zeta_\lambda\psi\|_{**,\xi}$. We have

$$\begin{aligned} &\sup\nolimits_{|x-\xi|\le 1}|x-\xi|^\sigma e^{U_\alpha}\zeta_\lambda|\psi| \\ &\quad\le C\lambda^{(N-2)/2}(\|h\|_{**,\xi} + \|\varphi\|_X)\sup_{\sqrt{\lambda/\lambda_0}R_1\le|x-\xi|\le 1}|x-\xi|^{2-N+\sigma} \\ &\quad\le C\lambda^{\sigma/2}(\|h\|_{**,\xi} + \|\varphi\|_X). \end{aligned} \tag{1.54}$$

On the other hand

$$\begin{aligned} &\sup_{|x-\xi|\ge 1}|x-\xi|^{2+\beta}e^{U_\alpha}\zeta_\lambda|\psi| \\ &\qquad\le C\lambda^{(N-2)/2}(\|h\|_{**,\xi} + \|\varphi\|_X)\sup_{|x-\xi|\ge 1}|x-\xi|^{2-N+\beta} \\ &\qquad\le C\lambda^{(N-2)/2}(\|h\|_{**,\xi} + \|\varphi\|_X). \end{aligned} \tag{1.55}$$

We deduce from (1.54) and (1.55) that

$$\|\lambda_0 e^{U_\alpha}\zeta_\lambda\psi\|_{**,\xi} \le C\lambda^{\sigma/2}(\|h\|_{**,\xi} + \|\varphi\|_X). \tag{1.56}$$

Therefore, from (1.53) and (1.56), we find that

$$\|\overline{\varphi}\|_{*,\xi} \le C(\lambda^{\sigma/2}\|\varphi\|_X + \|h\|_{**,\xi}).$$

Using a scaling argument and elliptic estimates we can prove

$$\sup_{E_\lambda} |\nabla\overline{\varphi}| \le C\lambda^{-1/2}\|\overline{\varphi}\|_{*,\xi}$$

and hence

$$\|F(\varphi)\|_X \le C(\lambda^{\sigma/2}\|\varphi\|_X + \|h\|_{**,\xi}).$$

Since F is affine, this estimate shows that F has a unique fix point φ in X for $\lambda > 0$ suitable small, and the fix point satisfies

$$\|\varphi\|_X \le C\|h\|_{**,\xi}.$$

□

1.6 Proof of Theorem 1.1

In this section we prove Theorem 1.1 by using a fixed point argument to solve problem (1.22). In particular, we prove the following result:

Proposition 1.9. *Assume $N \ge 4$. Then given $\alpha > 0$ and $Z > 0$, there are positive numbers Λ, C such that for any $|\xi| < Z$ and any $0 < \lambda < \Lambda$, there exists $\phi_{\lambda,\xi}$ solution to problem (1.22) such that*

$$\|\phi_{\lambda,\xi}\|_{*,\xi} \le C\lambda^{\sigma/2} \quad \text{for all } 0 < \lambda < \Lambda,\ |\xi| < Z.$$

Proof. There is not loss of generality in assuming $0 \in \mathcal{D}$. Fix $\delta > 0$ such that $B_\delta(0) \subset \mathcal{D}$. We first estimate $\|E_\lambda\|_{**,\xi}$ and $\|M(\phi)\|_{**,\xi}$ in (1.23). In particular we have

$$\|E_\lambda\|_{**,\xi} \le C\lambda^{\sigma/2} \tag{1.57}$$

$$\|M(\phi)\|_{**,\xi} \le C(\|\phi\|_{*,\xi}^2 + \lambda^{\sigma/2})\mathrm{e}^{\|\phi\|_{*,\xi}}. \tag{1.58}$$

In fact, by (1.21)

$$\begin{aligned}\sup_{|x-\xi|\le 1,\ x\notin\mathcal{D}_{\lambda,\xi}} |x-\xi|^\sigma|\varphi_\lambda(x)|\lambda_0\mathrm{e}^{U_\alpha} &\le C\lambda^{(N-2)/2} \sup_{\delta\sqrt{\lambda/\lambda_0}\le|x-\xi|\le 1} |x-\xi|^{\sigma+2-N}\\ &\le C\lambda^{\sigma/2},\end{aligned}$$

and

$$\begin{aligned}\sup_{|x-\xi|\ge 1} |x-\xi|^{2+\beta}|\varphi_\lambda(x)|\lambda_0\mathrm{e}^{U_\alpha} &\le C\lambda^{(N-2)/2} \sup_{|x-\xi|\ge 1} |x-\xi|^{\beta+2-N}\\ &\le C\lambda^{(N-2)/2},\end{aligned}$$

which yields (1.57).

For (1.58), by the definition of M and the identity $\mathrm{e}^\varepsilon = 1 + \varepsilon + \int_0^\varepsilon \mathrm{e}^t(\varepsilon - t)\,\mathrm{d}t$, valid for all $\varepsilon \in \mathbb{R}$, we have

$$|M(\phi)| \le C\mathrm{e}^{U_\alpha}(\phi^2 + \varphi_\lambda^2)\mathrm{e}^{|\phi|+|\varphi_\lambda|}.$$

Additionally,

$$\sup_{|x-\xi|\le 1,\ x\notin\mathcal{D}_{\lambda,\xi}} |x-\xi|^{\sigma}\mathrm{e}^{U_\alpha}\phi^2 \le C\|\phi\|^2_{*,\xi}$$

and

$$\begin{aligned}\sup_{|x-\xi|\le 1,\ x\notin\mathcal{D}_{\lambda,\xi}} |x-\xi|^{\sigma}\mathrm{e}^{U_\alpha}\varphi_\lambda^2 &\le C\lambda^{N-2} \sup_{\delta\sqrt{\lambda/\lambda_0}\le|x-\xi|\le 1} |x-\xi|^{\sigma+4-2N}\\ &\le C\lambda^{\sigma/2}.\end{aligned}$$

Note also that, by (1.21),

$$|\varphi_\lambda(x)| \le C\delta^{2-N} \quad \text{for all } x\notin\mathcal{D}_{\lambda,\xi},\ \lambda>0.$$

These inequalities yield

$$\sup_{|x-\xi|\le 1,\ x\notin\mathcal{D}_{\lambda,\xi}} |x-\xi|^{\sigma}|M(\phi)| \le C(\|\phi\|^2_{*,\xi}+\lambda^{\sigma/2})\mathrm{e}^{\|\phi\|_{*,\xi}} \tag{1.59}$$

On the other hand

$$\begin{aligned}\sup_{|x-\xi|\ge 1} |x-\xi|^{2+\beta}\mathrm{e}^{U_\alpha}\phi^2 &\le C\|\phi\|^2_{*,\xi} \sup_{|x-\xi|\ge 1}|x-\xi|^{-\beta}\\ &\le C\|\phi\|^2_{*,\xi}\end{aligned}$$

and

$$\begin{aligned}\sup_{|x-\xi|\ge 1} |x-\xi|^{2+\beta}\mathrm{e}^{U_\alpha}\varphi_\lambda^2 &\le C\lambda^{N-2} \sup_{|x-\xi|\ge 1}|x-\xi|^{\beta+4-2N}\\ &\le C\lambda^{N-2}.\end{aligned}$$

Then

$$\sup_{|x-\xi|\ge 1} |x-\xi|^{2+\beta}|M(\phi)| \le C(\|\phi\|^2_{*,\xi}+\lambda^{N-2})\mathrm{e}^{\|\phi\|_{*,\xi}}. \tag{1.60}$$

Combining (1.59) with (1.60) we obtain (1.58).

Now let us focus on the fixed point argument. We define for small $\rho>0$

$$\mathcal{F} = \{\phi:\ \mathbb{R}^N\setminus\mathcal{D}_{\lambda,\xi} \longrightarrow \mathbb{R}\ /\ \|\phi\|_{*,\xi}\le\rho\}$$

and the operator $\overline{\phi}=\mathcal{A}(\phi)$ where $\overline{\phi}$ is the solution of Proposition 1.6 to

$$\begin{cases} \Delta\overline{\phi}+\lambda_0\mathrm{e}^{U_\alpha}\overline{\phi} = M(\phi)+E_\lambda & \text{in } \mathbb{R}^N\setminus\overline{\mathcal{D}}_{\lambda,\xi},\\ \phi=0 & \text{on } \partial\mathcal{D}_{\lambda,\xi}, \end{cases} \tag{1.61}$$

where M and E_λ are given by (1.23). We prove that choosing $\rho>0$ small enough, $\mathcal{A}$ has a fixed point in $\mathcal{F}$. From Proposition 1.6 we have the estimate

$$\|\mathcal{A}(\phi)\|_{*,\xi} \le C(\|M(\phi)\|_{**,\xi}+\|E_\lambda\|_{**,\xi})$$

and, by (1.57) and (1.58),

$$\|\mathcal{A}\|_{*,\xi} \le C(\rho^2\mathrm{e}^{\rho}+\lambda^{\sigma/2}\mathrm{e}^{\rho}+\lambda^{\sigma/2}) \le \rho,$$

if $\rho > 0$ is fixed suitable small and then one consider $\lambda \to 0^+$. This proves $\mathcal{A}(\mathcal{F}) \subset \mathcal{F}$.

Now let us take ϕ_1 and ϕ_2 in $\mathcal{F}$. Then

$$\|\mathcal{A}(\phi_1) - \mathcal{A}(\phi_2)\|_{*,\xi} \leq C\|M(\phi_1) - M(\phi_2)\|_{**,\xi}. \tag{1.62}$$

To estimate the right hand side, consider $\overline{\phi} \in (\phi_1, \phi_2) \cup (\phi_2, \phi_1)$ such that

$$M(\phi_1) - M(\phi_2) = M'(\overline{\phi})(\phi_1 - \phi_2).$$

Directly from the definition of M, we compute

$$M'(\phi) = -\lambda_0 e^{U_\alpha}(e^{\phi - \varphi_\lambda} - 1).$$

Indeed, note that

$$|M'(\phi)| \leq C e^{U_\alpha}(|\phi| + |\varphi_\lambda|)e^{|\phi| + |\varphi_\lambda|}.$$

Therefore,

$$|M(\phi_1) - M(\phi_2)| \leq C e^{U_\alpha}(|\overline{\phi}| + |\varphi_\lambda|)e^{|\overline{\phi}|}|\phi_1 - \phi_2|.$$

Similarly to (1.57) and (1.58),

$$\sup_{|x-\xi| \leq 1,\ x \notin \mathcal{D}_{\lambda,\xi}} |x - \xi|^\sigma e^{U_\alpha}|\overline{\phi}|e^{|\overline{\phi}|}|\phi_1 - \phi_2| \leq C\rho e^\rho \|\phi_1 - \phi_2\|_{*,\xi}$$

and

$$\begin{aligned}
\sup_{|x-\xi| \leq 1,\ x \notin \mathcal{D}_{\lambda,\xi}} &|x - \xi|^\sigma e^{U_\alpha}|\varphi_\lambda|e^{|\overline{\phi}|}|\phi_1 - \phi_2| \\
&\leq C e^\rho \lambda^{(N-2)/2} \sup_{\delta\sqrt{\lambda/\lambda_0} \leq |x-\xi| \leq 1} |x - \xi|^{\sigma+2-N}\|\phi_1 - \phi_2\|_{*,\xi} \\
&\leq C e^\rho \lambda^{\sigma/2}\|\phi_1 - \phi_2\|_{*,\xi}.
\end{aligned}$$

These inequalities yield

$$\sup_{|x-\xi| \leq 1,\ x \notin \mathcal{D}_{\lambda,\xi}} |x - \xi|^\sigma |M(\phi_1) - M(\phi_2)| \leq C(\rho + \lambda^{\sigma/2})e^\rho \|\phi_1 - \phi_2\|_{*,\xi}. \tag{1.63}$$

On the other hand

$$\sup_{|x-\xi| \geq 1} |x - \xi|^{2+\beta} e^{U_\alpha}|\overline{\phi}|e^{|\overline{\phi}|}|\phi_1 - \phi_2| \leq C\rho e^\rho \|\phi_1 - \phi_2\|_{*,\xi}$$

and

$$\begin{aligned}
\sup_{|x-\xi| \geq 1} &|x - \xi|^{2+\beta} e^{U_\alpha}|\varphi_\lambda|e^{|\overline{\phi}|}|\phi_1 - \phi_2| \\
&\leq C\lambda^{(N-2)/2} e^\rho \sup_{|x-\xi| \geq 1} |x - \xi|^{2-N}\|\phi_1 - \phi_2\|_{*,\xi} \\
&= C\lambda^{(N-2)/2} e^\rho \|\phi_1 - \phi_2\|_{*,\xi}.
\end{aligned}$$

Then

$$\sup_{|x-\xi| \geq 1} |x - \xi|^{2+\beta}|M(\phi_1) - M(\phi_2)| \leq C(\rho + \lambda^{(N-2)/2})e^\rho \|\phi_1 - \phi_2\|_{*,\xi}. \tag{1.64}$$

Combining (1.63) with (1.64) we obtain

$$\|M(\phi_1) - M(\phi_2)\|_{**,\xi} \le C(\rho + \lambda^{\sigma/2})\mathrm{e}^{\rho}\|\phi_1 - \phi_2\|_{*,\xi}. \tag{1.65}$$

Gathering (1.62) and (1.65) we conclude that $\mathcal{A}$ is a contraction mapping in $\mathcal{F}$ provided $\rho > 0$ is fixed suitable small, and hence it has unique fixed point in this set. Moreover, from the previous steps we deduce the estimate

$$\|\phi_{\lambda,\xi}\|_{*,\xi} \le C\lambda^{\sigma/2} \quad \text{for all } 0 < \lambda < \Lambda,$$

which is the desired conclusion. □

1.7 The case $N = 3$

In this section, we show the modifications needed in Theorem 1.1 and its proof for the low dimension case, so we consider without mentioning $N = 3$.

We use again the norms defined in (1.26)–(1.27), but this time $\beta \in (0, 1/2)$. As we pointed out in Remark 1.8, the problem

$$\Delta\phi - \lambda_0 \mathrm{e}^{U_\alpha}\phi = h \quad \text{in } \mathbb{R}^3, \quad \|h\|_{**,\xi} < +\infty,$$

may not be solvable for $\|\phi\|_{*,\xi} < +\infty$, unless h satisfies the orthogonality conditions

$$\int_{\mathbb{R}^3} h\frac{\partial U_\alpha}{\partial x_i}\,\mathrm{d}x = 0, \quad i = 1, 2, 3, \tag{1.66}$$

(note that these conditions are equivalent to those in (1.37)).

Therefore, problem (1.22) may not be solvable in the required space unless ξ would be chosen in a very special way. So, in low dimension we consider instead the projected problem

$$\begin{cases} \Delta\phi + \lambda_0 \mathrm{e}^{U_\alpha}\phi = M(\phi) + E_\lambda + \displaystyle\sum_{i=1}^{3} c_i\Phi_i & \text{in } \mathbb{R}^3 \setminus \overline{\mathcal{D}}_{\lambda,\xi}, \\ \phi = 0 & \text{on } \partial\mathcal{D}_{\lambda,\xi}, \end{cases} \tag{1.67}$$

where c_i's are constants, which are part of the unknown, and

$$\Phi_i(x) = \eta(x)\frac{\partial U_\alpha}{\partial x_i}(x), \quad i = 1, 2, 3.$$

η is a fixed radial cut-off function, i.e. $\eta \in C^\infty(\mathbb{R}^3)$, $\eta(x) = \eta(|x|)$, $0 \le \eta \le 1$ and

$$\eta(x) = 1 \text{ for } |x| \le 1, \quad \eta(x) = 0 \text{ for } |x| \ge 2.$$

The only purpose of η is to make Φ_i "sufficiently" integrable in $\mathbb{R}^3$.

We handle problem (1.67) using a similar scheme to problem (1.22). Through an application of the Banach fixed point theorem in a suitable L^∞ space, we prove that (1.67) is solvable in the form $\phi = \phi(\lambda, \xi)$, $c_i = c_i(\lambda, \xi)$, where the dependence on the parameters is continuous. We then obtain a solution of problem (1.22) if

$$c_i(\lambda, \xi) = 0 \quad \text{for all } i = 1, 2, 3.$$

We will show that for each sufficiently small λ there is indeed a point ξ such that this system of equations is satisfied.

Similarly to higher dimensions, the use of contraction mapping principle is based on the construction of a bounded inverse for the linear problem

$$\begin{cases} \Delta\phi + \lambda_0 \mathrm{e}^{U_\alpha}\phi = h + \sum_{i=1}^{3} c_i \Phi_i & \text{in } \mathbb{R}^3 \setminus \overline{\mathcal{D}}_{\lambda,\xi}, \\ \phi = 0 & \text{on } \partial\mathcal{D}_{\lambda,\xi}, \end{cases} \tag{1.68}$$

We have this analogous result to Proposition 1.6.

Proposition 1.10. *Let us consider numbers $\alpha > 0$ and $Z > 0$. Then there exist positive constants C, Λ such that for any $|\xi| \leq Z$ and any $0 < \lambda < \Lambda$ the following holds: For any h with $\|h\|_{**,\xi} < +\infty$, there exists a solution of problem* (1.68)

$$(\phi, c_1, c_2, c_3) = \Psi_\lambda(h),$$

which defines a linear operator of h such that

$$\|\phi\|_{*,\xi} + \max_{i=1,2,3} |c_i| \leq C\|h\|_{**,\xi}.$$

As we did in Section 4, we first consider the version of problem (1.68) in entire space,

$$\Delta\phi + \lambda_0 \mathrm{e}^{U_\alpha}\phi = h + \sum_{i=1}^{3} c_i \Phi_i \quad \text{in } \mathbb{R}^3. \tag{1.69}$$

The corresponding result to Proposition 1.7 is the following.

Proposition 1.11. *Let $\alpha > 0$ and $Z > 0$. Then there exists a $C > 0$ such that for any $|\xi| \leq Z$ the following holds: For any h with $\|h\|_{**,\xi} < +\infty$, there exists a solution of* (1.69)

$$(\phi, c_1, c_2, c_3) = \Psi(h),$$

which defines a linear operator of h such that

$$\|\phi\|_{*,\xi} + \max_{i=1,2,3} |c_i| \leq C\|h\|_{**,\xi}. \tag{1.70}$$

We observe that the numbers c_i are explicit functions of h. Indeed, if ϕ solves (1.69) with the bound (1.70) then two integrations by parts again Φ_i yield

$$c_i = -\frac{\int_{\mathbb{R}^3} h\Phi_i \,\mathrm{d}x}{\int_{\mathbb{R}^3} \eta \left|\frac{\partial U_\alpha}{\partial x_i}\right|^2 \mathrm{d}x}, \quad i = 1, 2, 3. \tag{1.71}$$

This expression allows us to estimate $|c_i|$ in terms of $\|h\|_{**,\xi}$.

The scheme of the proof of Proposition 1.11 is analogous to the one used in Proposition 1.7. We first consider $\xi = 0$ and write h in its Fourier modes. Then we treat each Fourier mode of Eq. (1.69) separately. For Fourier modes $k = 1, 2, 3$, we have to take care of choosing c_i according to (1.71); in this way, orthogonality conditions (1.66), and then (1.37), will be satisfied. The estimates for $|c_i|$, $i = 1, 2, 3$, in (1.70) are obtained using (1.71). The blow-up method used to prove the continuity of the operator, as well as the gluing argument are similar to Section 4, we omit the details.

Likewise, we can prove Proposition 1.10 from Proposition 1.11 using a similar scheme to Section 5.

1.7.1 Proof of Theorem 1.2

Using a similar scheme to Section 6, from Proposition 1.10 we can prove the existence of solutions to problem (1.67) in low dimension, we omit the details. In particular, we have

Proposition 1.12. *Let us consider $\alpha > 0$ and $Z > 0$. Then there are positive numbers Λ, C such that for any $|\xi| < Z$ and any $0 < \lambda < \Lambda$ there exist $\phi_{\lambda,\xi}$, $c_1(\lambda,\xi)$, $c_2(\lambda,\xi)$, $c_3(\lambda,\xi)$ solution to problem (1.67) such that*

$$\|\phi_{\lambda,\xi}\|_{*,\xi} + \max_{i=1,2,3} |c_i(\lambda,\xi)| \le C\lambda^{\gamma} \quad \textit{for all } 0 < \lambda < \Lambda,\ |\xi| < Z, \tag{1.72}$$

where

$$\gamma = 1/2 \min\{\sigma, 1\}.$$

Next we make a remark on how to recognize when $c_i = 0$ in Eq. (1.68).

Lemma 1.13. *There is $\varepsilon_0 > 0$ small such that if $\lambda < \varepsilon_0$ and ϕ is a solution to (1.68) such that $\|\phi\|_{*,\xi} < +\infty$, $\|h\|_{**,\xi} < +\infty$, then $c_i = 0$ for all $i = 1, 2, 3$ if and only if*

$$\int_{\partial\mathcal{D}_{\lambda,\xi}} \frac{\partial\phi}{\partial n}\frac{\partial U_\alpha}{\partial x_i}\,\mathrm{dS}(x) + \int_{\mathbb{R}^3\setminus\mathcal{D}_{\lambda,\xi}} h\frac{\partial U_\alpha}{\partial x_i}\,\mathrm{d}x = 0 \quad \textit{for all } i = 1, 2, 3.$$

Proof. Since $\partial_{x_j} U_\alpha$ satisfies the linear homogeneous equation in $\mathbb{R}^3$, multiplying (1.68) by this function and integrating by parts in $B_R(0) \setminus \mathcal{D}_{\lambda,\xi}$, where R is large, we obtain

$$\begin{aligned}\int_{\partial(B_R(0)\setminus\mathcal{D}_{\lambda,\xi})} &\left(\frac{\partial\phi}{\partial n}\frac{\partial U_\alpha}{\partial x_j} - \phi\frac{\partial}{\partial n}\frac{\partial U_\alpha}{\partial x_j}\right)\mathrm{dS}(x)\\ &= \int_{B_R(0)\setminus\mathcal{D}_{\lambda,\xi}} \left(h + \sum_{i=1}^{3} c_i\Phi_i\right)\frac{\partial U_\alpha}{\partial x_j}\,\mathrm{d}x.\end{aligned} \tag{1.73}$$

Since $\|\phi\|_{*,\xi} < +\infty$ we see that

$$|\phi(x)| \le C|x|^{-\beta} \quad \text{for all } |x| \ge R'.$$

A scaling argument and elliptic estimates show that

$$|\nabla\phi(x)| \le C|x|^{-\beta-1} \quad \text{for all } |x| \ge R',$$

where $R' > 0$ is a large fixed number. Thus

$$\left|\frac{\partial\phi}{\partial n}\frac{\partial U_\alpha}{\partial x_j} - \phi\frac{\partial}{\partial n}\frac{\partial U_\alpha}{\partial x_j}\right| \le C|x|^{-\beta-2} \quad \text{for all } |x| \ge R',$$

and hence

$$\lim_{R\to+\infty} \int_{\partial B_R(0)} \left(\frac{\partial\phi}{\partial n}\frac{\partial U_\alpha}{\partial x_j} - \phi\frac{\partial}{\partial n}\frac{\partial U_\alpha}{\partial x_j}\right)\mathrm{dS}(x) = 0.$$

Letting $R \to +\infty$ in (1.73) yields

$$\sum_{i=1}^{3} c_i \int_{\mathbb{R}^3\setminus\mathcal{D}_{\lambda,\xi}} \Phi_i\frac{\partial U_\alpha}{\partial x_j}\,\mathrm{d}x = -\int_{\mathbb{R}^3\setminus\mathcal{D}_{\lambda,\xi}} h\frac{\partial U_\alpha}{\partial x_j}\,\mathrm{d}x - \int_{\partial\mathcal{D}_{\lambda,\xi}} \frac{\partial\phi}{\partial n}\frac{\partial U_\alpha}{\partial x_j}\,\mathrm{dS}(x).$$

For $\lambda > 0$ sufficiently small the matrix with entries $\int_{\mathbb{R}^3\setminus\mathcal{D}_{\lambda,\xi}} \Phi_i\frac{\partial U_\alpha}{\partial x_j}\,\mathrm{d}x$ is close to $\int_{\mathbb{R}^3} \Phi_i\frac{\partial U_\alpha}{\partial x_j}\,\mathrm{d}x$ which is invertible. The lemma follows. □

Seeking $c_i = 0$. Finally we show how to find a $\xi = \xi(\lambda)$, $\lambda > 0$ small, such that

$$c_i(\lambda, \xi) = 0 \quad \text{for all } i = 1, 2, 3,$$

and thereby prove Theorem 1.2.

Let us assume $0 \in \mathcal{D}$ and $\sigma \in (1, 2)$. We have found a solution $\phi_{\lambda,\xi}$, $c_1(\lambda, \xi)$, $c_2(\lambda, \xi)$, $c_3(\lambda, \xi)$ to (1.67). By the previous lemma, for all λ small $c_1 = c_2 = c_3 = 0$ if and only if

$$\int_{\mathbb{R}^3 \setminus \mathcal{D}_{\lambda,\xi}} (E_\lambda + M(\phi_{\lambda,\xi})) \frac{\partial U_\alpha}{\partial x_i} \, \mathrm{d}x + \int_{\partial \mathcal{D}_{\lambda,\xi}} \frac{\partial \phi_{\lambda,\xi}}{\partial n} \frac{\partial U_\alpha}{\partial x_i} \, \mathrm{dS}(x) = 0 \tag{1.74}$$

for all $i = 1, 2, 3$.

Let us define

$$G_j(\xi) = \int_{\mathbb{R}^3 \setminus \mathcal{D}_{\lambda,\xi}} (E_\lambda + M(\phi_{\lambda,\xi})) \frac{\partial U_\alpha}{\partial x_i} \, \mathrm{d}x + \int_{\partial \mathcal{D}_{\lambda,\xi}} \frac{\partial \phi_{\lambda,\xi}}{\partial n} \frac{\partial U_\alpha}{\partial x_i} \, \mathrm{dS}(x). \tag{1.75}$$

Using local uniqueness, the fixed point characterization of ϕ_λ and elliptic estimates, one can prove that the functions G_j are continuous; we omit the details.

Recalling the definition of f_0 in (1.20), we claim that

$$G_j(\xi) = f_0 \lambda^{1/2} \int_{\mathbb{R}^3} |x - \xi|^{-1} \mathrm{e}^{U_\alpha} \frac{\partial U_\alpha}{\partial x_j} \, \mathrm{d}x + o(\lambda^{1/2}) \tag{1.76}$$

uniformly on compacts sets of $\mathbb{R}^3$. Then, for λ small $G_j(\xi) \sim f_0 \lambda^{1/2} \frac{\partial U_\alpha}{\partial x_j}(\xi)$, so we can expect that there exists ξ annulling the functions G_j, $j = 1, 2, 3$.

We first observe that

$$\int_{\mathbb{R}^3 \setminus \mathcal{D}_{\lambda,\xi}} M(\phi_{\lambda,\xi}) \frac{\partial U_\alpha}{\partial x_i} \, \mathrm{d}x = O(\lambda^{\sigma/2}). \tag{1.77}$$

Indeed,

$$\int_{\mathbb{R}^3 \setminus \mathcal{D}_{\lambda,\xi}} \left| M(\phi_{\lambda,\xi}) \frac{\partial U_\alpha}{\partial x_i} \right| \mathrm{d}x = \int_{B_1(\xi) \setminus \mathcal{D}_{\lambda,\xi}} \cdots \, \mathrm{d}x + \int_{\mathbb{R}^3 \setminus B_1(\xi)} \cdots \, \mathrm{d}x;$$

by (1.59) and (1.72),

$$\begin{aligned} \int_{B_1(\xi) \setminus \mathcal{D}_{\lambda,\xi}} \left| M(\phi_{\lambda,\xi}) \frac{\partial U_\alpha}{\partial x_i} \right| \mathrm{d}x &\leq C(\|\phi\|_{*,\xi}^2 + \lambda^{\sigma/2}) \mathrm{e}^{\|\phi\|_{*,\xi}} \int_{B_1(\xi) \setminus \mathcal{D}_{\lambda,\xi}} |x - \xi|^{-\sigma} \, \mathrm{d}x \\ &\leq C(\lambda + \lambda^{\sigma/2}) \mathrm{e}^{\lambda^{1/2}}. \end{aligned}$$

Likewise, (1.60) and (1.72) yield

$$\begin{aligned} \int_{\mathbb{R}^3 \setminus B_1(\xi)} \left| M(\phi_{\lambda,\xi}) \frac{\partial U_\alpha}{\partial x_i} \right| \mathrm{d}x &\leq C(\|\phi\|_{*,\xi}^2 + \lambda) \mathrm{e}^{\|\phi\|_{*,\xi}} \int_{\mathbb{R}^3 \setminus B_1(\xi)} |x - \xi|^{-3-\beta} \, \mathrm{d}x \\ &\leq C \lambda \mathrm{e}^{\lambda^{1/2}}, \end{aligned}$$

These inequalities prove (1.77).

On the other hand, we need to estimate the boundary integral of (1.75). We claim that

$$\left|\frac{\partial \phi_{\lambda,\xi}}{\partial n}(x)\right| = O(\lambda^{-1/2}), \quad \text{uniformly for } x \in \partial\mathcal{D}_{\lambda,\xi}. \tag{1.78}$$

In fact,

$$\widetilde{\phi}_{\lambda,\xi}(y) = \phi_{\lambda,\xi}\left(\xi + \sqrt{\frac{\lambda}{\lambda_0}}y\right) \quad \text{for all } y \in \mathbb{R}^3 \setminus \mathcal{D}.$$

By (1.72), we have

$$|\widetilde{\phi}_{\lambda,\xi}(y)| \le C\lambda^{1/2} \quad \text{for all } |y| \le \sqrt{\frac{\lambda_0}{\lambda}}.$$

Likewise, by (1.72) and the definition of the norm $\|\|_{*,\xi}$, we see that $\phi_{\lambda,\xi}$ is uniformly bounded. Furthermore, (1.67) implies that $|\Delta\phi_{\lambda,\xi}| \le C$ in $\mathbb{R}^N \setminus \mathcal{D}_{\lambda,\xi}$, thereby $\widetilde{\phi}_{\lambda,\xi}$ satisfies

$$|\Delta\widetilde{\phi}_{\lambda,\xi}| \le C\lambda.$$

By elliptic estimates

$$\sup_{\partial\mathcal{D}} |\nabla\widetilde{\phi}_{\lambda,\xi}| \le C\lambda^{1/2},$$

which proves (1.78). Using the last inequality we derive

$$\int_{\partial\mathcal{D}_{\lambda,\xi}} \frac{\partial\phi_{\lambda,\xi}}{\partial n}\frac{\partial U_\alpha}{\partial x_i}\,\mathrm{dS}(x) = O(\lambda).$$

This fact together (1.77) prove the claim made in (1.76).

Finally, let us consider the vector field

$$G(\xi) = (G_1(\xi), G_2(\xi), G_3(\xi)).$$

G is continuous and, thanks to (1.76),

$$G(\xi)\cdot\xi < 0 \quad \text{for all } |\xi| = R,$$

for any fixed small $R > 0$. Using this and degree theory we obtain the existence of ξ such that $c_1 = c_2 = c_3 = 0$. Which is the desired conclusion. □

Chapter 2

Bubbling solutions for nonlocal elliptic problems

This chapter is focused on the existence of bubbling solutions for the problem with nonlocal equation

$$\begin{cases} (-\Delta)^s u = u^p, \ u > 0 \quad \text{in } \Omega, \\ \qquad\qquad u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $0 < s < 1$, $N > 2s$ and $p = (N+2s)/(N-2s) \pm \varepsilon$ ($\varepsilon > 0$ small).

2.1 Introduction

For the usual Laplacian, problem (2.1) was extensively studied when the exponent p approaches critical from below, namely $p = (N+2s)/(N-2s) - \varepsilon$, see Brezis and Peletier [10], Rey [63, 64, 65], Han [50] and Bahri, Li and Rey [4]. In the latter reference, bubbling solutions are found for $N \geq 4$, concentrating around nondegenerate critical points of certain object which involve the Green's and Robins's function of Ω. On the other hand, the supercritical case $p = (N+2s)/(N-2s) + \varepsilon$ was studied by del Pino, Felmer and Musso [35, 36], in particular they showed a concentration phenomena for bubbling solutions to this problem when the domain satisfy certain "topological condition", for instance a domain exhibiting multiple holes.

In the fractional framework, several authors studied nonlinear problems of the form $(-\Delta)^s u = f(u)$ for a certain function $f : \mathbb{R}^N \to \mathbb{R}$. Among others, it is worthwhile to mention the work by Cabré and Tan [16] and Tan [72] when $s = 1/2$. They established the existence of positive solutions for equations having the subcritical growth, their regularity and symmetry properties. See also [9]. Recently, and for the subcritical case, Choi, Kim and Lee [27] developed a nonlocal analog of the results by Han [50] and Rey [64] mentioned in the previous paragraph. They also proved Theorem 2.2 below in the case $p = p^* - \varepsilon$. With a new framework in the spirit of [35, 36], we'll be able to generalize the work by Choi, Kim and Lee, and consider both subcritical and supercritical case.

Throughout this Chapter, $p^* := (N+2s)/(N-2s)$ represents the critical exponent. For this exponent, the corresponding equation in $\mathbb{R}^N$

$$(-\Delta)^s u = u^{p^*} \tag{2.2}$$

has an explicit family of solutions of the form

$$w_{\lambda,\xi}(x) = \lambda^{-\frac{N-2s}{2}} w(\lambda^{-1}(x-\xi))$$

with $\xi \in \mathbb{R}^N$ and $\lambda > 0$, where

$$w(x) = \frac{a_{N,s}}{(1+|x|^2)^{\frac{N-2s}{2}}}$$

and $a_{N,s} > 0$ (see [24] for classification results).

We construct solutions of (2.1) that concentrate at certain points in Ω as $\varepsilon \to 0$. These concentration points are determined by the critical points of a map which involves the Green's function of the operator $(-\Delta)^s$ and its regular part. Let G denote the Green's function for $(-\Delta)^s$ in Ω, that is, for any $\xi \in \Omega$, $G(\cdot,\xi)$ satisfies

$$\begin{aligned} (-\Delta)^s G(\cdot,\xi) = \delta_\xi(\cdot) \quad &\text{in } \Omega, \\ G(\cdot,\xi) = 0 \quad &\text{on } \Omega, \end{aligned} \tag{2.3}$$

where δ_ξ denotes the Dirac mass at the point ξ. In the entire space, we denote the Green's function by Γ, which satisfies

$$\begin{aligned} (-\Delta)^s \Gamma(x,\xi) = \delta_\xi(x) \quad \text{for all } x \in \mathbb{R}^N, \\ \lim_{|x|\to\infty} \Gamma(x,\xi) = 0, \end{aligned}$$

for each fixed $\xi \in \mathbb{R}^N$. The function Γ is explicitly given by

$$\Gamma(x,\xi) = \frac{b_{N,s}}{|x-\xi|^{N-2s}}, \tag{2.4}$$

where $b = b(N,s) > 0$. We also define the regular part of the Green's function G of Ω by

$$H(x,\xi) = G(x,\xi) - \Gamma(x,\xi) \quad \text{for } x,\xi \in \Omega,\ x \neq \xi. \tag{2.5}$$

Its diagonal is usually called the Robin's function of the domain.

Given $m \in \mathbb{N}$, the following function will provide to be very important for constructing solutions of (2.1):

$$\Psi(\xi,\Lambda) = \frac{1}{2}\left\{\sum_{i=1}^m H(\xi_i,\xi_i)\Lambda_i^2 - 2\sum_{i<j} G(\xi_i,\xi_j)\Lambda_i\Lambda_j\right\} \pm \log(\Lambda_1 \cdots \Lambda_m), \tag{2.6}$$

$\xi_1,\ldots,\xi_m \in \Omega$ and $\Lambda_1,\ldots,\Lambda_m > 0$ (see (2.30)).

We recall the definition of stable critical sets introduced by Y.Y. Li [52].

Definition 2.1 (Stable critical set)**.** Suppose that $\Omega \subset \mathbb{R}^N$ is a domain and Ψ is a $C^1(\Omega)$. We say that a bounded set $\mathcal{A} \subset \Omega$ of critical points of Ψ is a stable critical set if there is a number $\delta > 0$ such that $\|\Psi - \Phi\|_{L^\infty(\mathcal{A})} + \|\nabla(\Psi - \Phi)\|_{L^\infty(\mathcal{A})} < \delta$ for some $\Phi \in C^1(\Omega)$ implies the existence of a critical point of Φ in $\mathcal{A}$.

Our main result in this chapter is the following:

Theorem 2.2. *Assume $0 < s < 1$, $N > 2s$, and Ω be a smooth bounded domain in $\mathbb{R}^N$. Suppose that Ψ in (2.6) has a stable critical set $\mathcal{A}$. Then, there exists a point $\xi = (\xi_1, \ldots, \xi_m) \in \mathcal{A}$ such and a family of solutions of (2.1) which blow up and concentrate at each point ξ_i, $i = 1, \ldots, m$, as ε tends to zero.*

We shall also show the following result in the subcritical case and with $m = 1$, i.e. a single bubble.

Theorem 2.3. *Assume $0 < s < 1$, $N > 2s$, and Ω be a smooth bounded domain in $\mathbb{R}^N$. Then, if $p = p^* - \varepsilon$, there exists a point $\xi \in \Omega$ and a family of solutions of problem (2.1) which concentrate at the point ξ as ε tends to zero. Moreover, ξ is a critical point of the Robin's function $\varphi(x) = H(x, x)$.*

The chapter is organized as follows: In Section 2 we recall the definition and the basic properties of the fractional Laplacian in bounded domains and in the whole $\mathbb{R}^N$. In Section 3 we shall develop the analytical tools toward the main results. We study the linearization around specials entire solution of (2.2), an initial approximation shall be done as well as. Finally, Section 4 and 5 contains the reduction to a finite dimensional functional and its relation with the original problem (2.1); these sections contain the final tools to proof Theorem 2.2 and 2.3 in Section 6.

2.2 Preliminaries

In this section we recall some basic properties of the fractional Laplacian. The notation used throughout this chapter is settled down as well.

In the entire space, the operator $(-\Delta)^s$ in $\mathbb{R}^N$, $0 < s < 1$, is defined through Fourier transform $\mathcal{F}$, by

$$\mathcal{F}[(-\Delta)^s u](\zeta) = |\zeta|^{2s}\mathcal{F}[u](\zeta).$$

On a bounded domain Ω, we define $(-\Delta)^s$ through the spectral decomposition of $(-\Delta)$ in $H_0^1(\Omega)$:

$$(-\Delta)^s = \sum_{i=1}^{\infty} \mu_i^s P_i$$

where $\{\mu_i, \phi_i\}_{i=1}^{\infty}$ are the eigenvalues and corresponding eigenvectors of $-\Delta$ on $H_0^1(\Omega)$ and P_i is the orthogonal projection on the eigenspace corresponding to μ_i. The fractional Laplacian is well defined in the fractional Sobolev space $H_0^s(\Omega)$,

$$H_0^s(\Omega) = \{u = \sum_{i=1}^{\infty} a_i\phi_i \in L^2(\Omega) : \sum_{i=1}^{\infty} a_i^2\mu_i^2 < \infty\},$$

which is a Hilbert space endowed with the following inner product

$$\left\langle \sum_{i=1}^{\infty} a_i\phi_i, \sum_{i=1}^{\infty} b_i\phi_i \right\rangle_{H_0^s(\Omega)} = \sum_{i=1}^{\infty} a_i b_i \mu_i^s,$$

and we have the following expression for this inner product:

$$< u, v >_{H_0^s(\Omega)} = \int_\Omega (-\Delta)^{s/2}u(-\Delta)^{s/2}v = \int_\Omega (-\Delta)^s uv, \quad u, v \in H_0^s(\Omega). \tag{2.7}$$

We will often work with an equivalent definition based on an appropriate extension problem introduced by Caffarelli and Silvestre [20]. This problem is set in $\Omega \times (0,\infty)$ and it will be convenient to use the following notation: $x \in \mathbb{R}^N$, $y > 0$, and $X = (x,y) \in \mathbb{R}^N_+ := \mathbb{R}^N \times (0,\infty)$; likewise, we denote by $\mathcal{C}$ the cylinder $\Omega \times (0,\infty)$ and by $\partial_L \mathcal{C}$ its lateral boundary, i.e. $\partial\Omega \times (0,\infty)$. The ambient space $H^s_{0,L}(\mathcal{C})$ is defined as the completion of

$$C^s_{0,L}(\mathcal{C}) := \{U \in C^\infty(\overline{\mathcal{C}}) : U = 0 \text{ on } \partial_L \mathcal{C}\}$$

with respect to the norm

$$\|U\|_{\mathcal{C}} = \left(\int_{\mathcal{C}} y^{1-2s} |\nabla U|^2\right)^{1/2}. \tag{2.8}$$

This is a Hilbert space endowed with the following inner product

$$\langle U, V\rangle = \int_{\mathcal{C}} y^{1-2s} \nabla U \cdot \nabla V \quad \text{for all } U, V \in H^\infty_{0,L}(\mathcal{C}).$$

In the entire space, we denote by $\mathcal{D}^s(\mathbb{R}^{N+1}_+)$ the completion of $C^\infty_0(\overline{\mathbb{R}^{N+1}_+})$ with respect to the norm $\|\cdot\|_{\mathbb{R}^{N+1}_+}$ defined as in (2.8). We point out that if Ω is a smooth bounded domain then

$$H^s_0(\Omega) = \{u = \operatorname{tr}|_{\Omega\times\{0\}} U : U \in H^s_{0,L}(\mathcal{C})\}.$$

The extension problem is the following: given $u \in H^s_0(\Omega)$, we solve

$$\begin{cases} \operatorname{div}(y^{1-2s}\nabla U) = 0 & \text{in } \mathcal{C}, \\ U = 0 & \text{on } \partial_L\mathcal{C}, \\ U = u & \text{on } \Omega, \end{cases} \tag{2.9}$$

for $U \in H^s_{0,L}(\mathcal{C})$, where divergence and ∇ are operators acting on all variables $X = (x,y)$. Then, up to a multiplicative constant,

$$(-\Delta)^s u = -\lim_{y\to 0} y^{1-2s}\partial_y U, \tag{2.10}$$

where $c = c(N,s) > 0$ (see [20] and [21] for the entire and bounded domain case, respectively).

Regarding this extension procedure, the Green's function defined in (2.3) can be seen, up to a positive constant, as the trace of the solution G for the following extended Dirichlet-Neumann problem

$$\begin{cases} \operatorname{div}(y^{1-2s}\nabla G(\cdot,\xi)) = 0 & \text{in } \mathcal{C}, \\ G(\cdot,\xi) = 0 & \text{on } \partial_L\mathcal{C}, \\ -\lim_{y\to 0} y^{1-2s}\partial_y G(\cdot,\xi) = \delta_\xi(\cdot) & \text{on } \Omega, \end{cases} \tag{2.11}$$

$\xi \in \Omega$ (we denote the Green's function, as well as its extension, by G). Moreover, we have the following representation formula

$$U(z) = \int_\Omega G(z,\xi)(-\Delta)^s u(\xi)\,\mathrm{d}\xi \quad \text{for all } z \in \mathcal{C}, \tag{2.12}$$

where $u = \operatorname{tr}|_{\Omega\times\{0\}} U$. Likewise, the regular part of the Green's function defined in (2.5) can be extended in $U \in H^s_{0,L}(\mathcal{C})$ as the unique solution of

$$\begin{cases} \operatorname{div}(y^{1-2s}\nabla H(z,\xi)) = 0, & z \in \mathcal{C}, \\ H(z,\xi) = \Gamma(z-\xi), & z \in \partial_L\mathcal{C}, \\ \lim_{y\to 0} y^{1-2s}\partial_y H(z,\xi) = 0, & z \in \Omega, \end{cases} \tag{2.13}$$

$\xi \in \Omega$ (we denote the regular part of the Green's function, as well as its extension, by H).

In the next sections, given a a function $u \in H_0^s(\Omega)$, when we speak of its s-harmonic extension to $\Omega \times (0, \infty)$ we will always refer to the solution of (2.9). This extension process depends on the domain, and we include the possibility that the domain is $\mathbb{R}^N$, in which case U can be written as a convolution of u and an explicit kernel

$$U(x, y) = \int_{\mathbb{R}^N} P(x - t, y) u(t)\, \mathrm{d}t \tag{2.14}$$

where

$$P(x, y) = C_{N,s} \frac{y^{2s}}{(|x|^2 + y^2)^{\frac{N+2s}{2}}}$$

(see [20]). Then, the s-harmonic extension of the fundamental solution (2.4) to $\mathbb{R}^N_+ := \mathbb{R}^N \times (0, \infty)$ is given just by

$$\Gamma(z_1, z_2) = \frac{c}{|z_1 - z_2|^{N-2s}} \quad \text{for } z_1, z_2 \in \mathbb{R}^N_+,\ z_1 \neq z_2.$$

2.3 Initial approximation and reduced energy

Let Ω be a bounded domain with smooth boundary in $\mathbb{R}^N$. It will be convenient to work with the enlarged domain

$$\Omega_\varepsilon = \varepsilon^{-\frac{1}{N-2s}} \Omega.$$

Then the change of variables

$$v(x) = \varepsilon^{\frac{1}{2+\varepsilon(N-2/2)}} u(\varepsilon^{\frac{1}{N-2s}} x), \quad x \in \Omega_\varepsilon,$$

transforms equation (2.1) into

$$\begin{cases} (-\Delta)^s v = v^{p^* \pm \varepsilon},\ v > 0 & \text{in } \Omega_\varepsilon \\ \qquad v = 0 & \text{on } \partial\Omega_\varepsilon. \end{cases} \tag{2.15}$$

For the sake of generality, we will develop an initial approximation with concentration at m uniformly separated points in Ω that stay away from the boundary. Let us fix a small $\delta > 0$ and work with $\xi_1, \ldots, \xi_m \in \Omega$ such that

$$|\xi_i - \xi_j| \geq \delta \quad \text{for all } i \neq j \quad \text{and} \quad \operatorname{dist}(\xi_i, \partial\Omega) \geq \delta \quad \text{for all } i. \tag{2.16}$$

Let us write

$$\xi_i' = \varepsilon^{-\frac{1}{N-2s}} \xi_i \in \Omega_\varepsilon. \tag{2.17}$$

Given points as above and $\lambda_1, \ldots, \lambda_m > 0$ we consider the following family of solutions to (2.15):

$$w_i(x) = w_{\lambda_i, \xi_i'}(x) = a_{N,s} \left(\frac{\lambda_i}{\lambda_i^2 + |x - \xi_i'|^2} \right)^{\frac{N-2s}{2}}. \tag{2.18}$$

We will restrict the parameters λ_i so that

$$\lambda_i \in (\delta, \delta^{-1}) \quad \text{for all } i \in \{1, \ldots, m\}. \tag{2.19}$$

Let W_i denote the s-harmonic extension of w_i to $\mathbb{R}^{N+1}_+$ given by the formula (2.14), so that W_i satisfies

$$\begin{cases} \operatorname{div}(y^{1-2s}\nabla W_i) = 0 \quad \text{in } \mathbb{R}^{N+1}_+, \\ W_i = w_i \quad \text{on } \mathbb{R}^N. \end{cases} \tag{2.20}$$

We introduce the function v_i to be the $H^s_0(\Omega_\varepsilon)$-projection of w_i, namely the unique solution of the equation

$$\begin{cases} (-\Delta)^s v_i = w_i^{p^*} \quad \text{in } \Omega_\varepsilon \\ v_i = 0 \quad \text{on } \partial\Omega_\varepsilon. \end{cases} \tag{2.21}$$

The functions v_i can be expressed as

$$v_i = w_i - \varphi_i \quad \text{in } \Omega_\varepsilon,$$

where φ_i is the trace on Ω_ε of the unique solution Φ_i of

$$\begin{cases} \operatorname{div}(y^{1-2s}\nabla \Phi_i) = 0 & \text{in } \mathcal{C}_\varepsilon, \\ \Phi_i = W_i & \text{on } \partial_L\mathcal{C}_\varepsilon, \\ \lim_{y\to 0} y^{1-2s}\partial_y \Phi_i = 0 & \text{on } \Omega_\varepsilon, \end{cases} \tag{2.22}$$

where $\mathcal{C}_\varepsilon$ is the enlarged cylinder $\Omega_\varepsilon \times (0,\infty)$ and $\partial_L\mathcal{C}_\varepsilon$ its lateral boundary.

We look for a solution of problem (2.15) of the form

$$v = \bar{v} + \phi, \tag{2.23}$$

where

$$\bar{v} = \sum_{i=1}^m v_i$$

and $\{\xi_i, \lambda_i\}_{i=1}^m$ are suitable points and scalars which made the term ϕ of small order all over Ω_ε. As we note in the previous section (see (2.9) and (2.10)), solutions of (2.15) are close related to those of

$$\begin{cases} \operatorname{div}(y^{1-2s}\nabla V) = 0 & \text{in } \mathcal{C}_\varepsilon, \\ V > 0 & \text{in } \mathcal{C}_\varepsilon, \\ V = 0 & \text{on } \partial_L\mathcal{C}_\varepsilon, \\ -\lim_{y\to 0} y^{1-2s}\partial_y V = v^{p^*\pm\varepsilon} & \text{on } \Omega_\varepsilon. \end{cases} \tag{2.24}$$

These functions correspond, in turn, to stationary points of the energy functional

$$I_{\pm\varepsilon}(V) = \frac{1}{2}\int_{\mathcal{C}_\varepsilon} y^{1-2s}|\nabla V|^2 - \frac{1}{p^*+1\pm\varepsilon}\int_{\Omega_\varepsilon} |V|^{p^*+1\pm\varepsilon}. \tag{2.25}$$

We remark that in the subcritical case these functionals are well defined and C^1 in the Hilbert space $H^1_{0L}(\mathcal{C}_\varepsilon, y^{1-2s}dxdy)$ defined as the completion of $C^\infty_c(\mathcal{C}_\varepsilon)$ with respect to the norm

$$\left(\int_{\mathcal{C}_\varepsilon} y^{1-2s}|\nabla V|^2 \, \mathrm{d}x\mathrm{d}y\right)^{1/2}.$$

If a solution of the form (2.23) exists, we should have $I_{\pm\varepsilon}(V) \sim I_{\pm\varepsilon}(\bar{V})$, where V and $\bar{V}$ denote the s-harmonic extension of v and $\bar{v}$, respectively. Then the corresponding points (ξ, λ) in the definition of $\bar{v}$ are also "approximately stationary" for the finite dimensional functional $(\xi, \lambda) \mapsto I_{\pm\varepsilon}(\bar{V})$. It is then necessary to understand the structure of this functional and to find critical points of it which survive small perturbations.

In order to understand the previous energy functional, we first analyze the behavior of φ_i and v_i as ε tends to 0, we'll need the following maximum principle.

Lemma 2.4 (Maximum principle). *Suppose that U is a weak solution of the problem*

$$\begin{cases} \operatorname{div}(y^{1-2s}\nabla U) = 0 & \text{in } \mathcal{C}, \\ U = g & \text{on } \partial_L \Omega, \\ \lim_{y\to 0} y^{1-2s}\partial_y U = 0 & \text{on } \Omega, \end{cases}$$

for some function $g : \partial_L\Omega \to \mathbb{R}$. Then

$$\sup_{z\in\mathcal{C}} |U(z)| \leq \sup_{z\in\partial_L\mathcal{C}} |g(z)|.$$

Proof. Let $\bar{U} = \sup_{z\in\partial_L\mathcal{C}} g(z)$, and consider the function $V(z) = \bar{U} - U(z)$ which satisfies

$$\begin{cases} \operatorname{div}(y^{1-2s}\nabla V) = 0 & \text{in } \mathcal{C}, \\ V \geq 0 & \text{on } \partial_L \Omega, \\ \lim_{y\to 0} y^{1-2s}\partial_y V = 0 & \text{on } \Omega. \end{cases}$$

Note that $V^+ = 0$ on $\partial_L\Omega$. Then, we deduce that

$$0 = \int_{\mathcal{C}} y^{1-2s}\nabla V \cdot \nabla V^+ = -\int_{\mathcal{C}} y^{1-2s}|\nabla V^+|^2.$$

It implies that $V^+ = 0$, and then $U \leq \bar{U}$ in $\mathcal{C}$. By a similar argument, we can deduce that $\inf_{z\in\partial_L\mathcal{C}} g(z) \leq U$ in $\mathcal{C}$, which completes the proof. □

Lemma 2.5 (Expansion of φ_i and v_i). *Assume that $\xi_1, \ldots, \xi_m$ satisfy (2.16) and $\lambda_1, \ldots, \lambda_m$ satisfy (2.19). Then*

$$\varphi_i(\varepsilon^{-\frac{1}{N-2s}}x) = \alpha\lambda_i^{\frac{N-2s}{2}} H(x, \xi_i)\varepsilon + o(\varepsilon), \tag{2.26}$$

uniformly for $x \in \Omega$. And, away from $x = \xi_i$,

$$v_i(\varepsilon^{-\frac{1}{N-2s}}x) = \alpha\lambda_i^{\frac{N-2s}{2}} G(x, \xi_i)\varepsilon + o(\varepsilon), \tag{2.27}$$

uniformly for x on each compact subset of Ω. Here $\alpha = \alpha(N, s) = \int_{\mathbb{R}^N} w^{p^*}$ and G, H are respectively the Green's function of the fractional Laplacian with Dirichlet boundary condition on Ω and its regular part.

Proof. Using (2.12), the function W_i in (2.20) can be written as

$$W_i(z) = \int_{\mathbb{R}^N} \Gamma(z, \tau) w_i^{p^*}(\tau)\, \mathrm{d}\tau.$$

Then, recalling the definition of w_i in (2.18), we have that

$$W_i(z) = \lambda_i^{-\frac{N+2s}{2}} \int_{\mathbb{R}^N} \Gamma(z, \tau) w^{p^*}(\lambda_i^{-1}(\tau - \xi_i'))\, \mathrm{d}\tau \quad \text{for all } z = (x, y) \in \mathbb{R}^{N+1}_+.$$

Regarding (2.22), let us now consider the functions $\Phi_{i,\varepsilon}(z) = \Phi_i(\varepsilon^{-\frac{1}{N-2s}}z)$ and $H_{i,\varepsilon}(z) = \alpha\lambda_i^{\frac{N-2s}{2}} H(x,\xi_i)\varepsilon$ defined in $\mathcal{C}$ (H is the extended function in (2.13)). Using the previous identity, we have that,

$$\begin{aligned}\Phi_{i,\varepsilon}(z) = W_i(\varepsilon^{-\frac{1}{N-2s}}z) &= \lambda_i^{-\frac{N+2s}{2}} \int_{\mathbb{R}^N} \Gamma(\varepsilon^{-\frac{1}{N-2s}}z,\tau) w^{p^*}(\lambda_i^{-1}(\tau-\xi_i'))\,\mathrm{d}\tau \\ &= \lambda_i^{\frac{N-2s}{2}} \int_{\mathbb{R}^N} \Gamma(\varepsilon^{-\frac{1}{N-2s}}z,\xi_i'+\lambda_i\tau) w^{p^*}(\tau)\,\mathrm{d}\tau \\ &= \lambda_i^{\frac{N-2s}{2}} \varepsilon \int_{\mathbb{R}^N} \Gamma(z,\xi_i+\lambda_i\varepsilon^{\frac{1}{N-2s}}\tau) w^{p^*}(\tau)\,\mathrm{d}\tau \\ &= \alpha\lambda_i^{\frac{N-2s}{2}}\Gamma(z,\xi_i)\varepsilon + o(\varepsilon),\end{aligned}$$

uniformly for $z \in \partial_L\Omega$ (recall also that ξ_i remains far from $\partial\Omega$). Therefore,

$$\sup_{z\in\partial_L\Omega} |\Phi_{i,\varepsilon}(z) - H_{i,\varepsilon}(z)| = o(\varepsilon)$$

By the previous lemma, we deduce that

$$\sup_{z\in\Omega} |\Phi_{i,\varepsilon}(z) - H_{i,\varepsilon}(z)| = o(\varepsilon).$$

This establishes (2.26). A similar argument can be used to state (2.27), we omit the details. □

With these estimates on hand let us focus on the energy functional. As the points ξ_i are taken far apart from each other and far away from the boundary, we have the fist approximation

$$I_{\pm\varepsilon}(\bar{V}) \sim \sum_{i=1}^{m} I_{\pm\varepsilon}(W_i) \sim mC_N$$

where

$$C_N = \frac{1}{2}\int_{\mathbb{R}^{N+1}_+} y^{1-2s}|\nabla W|^2 - \frac{1}{p+1}\int_{\mathbb{R}^N} |w|^{p^*+1}$$

and W is the s-harmonic extension of w. To work out a more precise expansion, it will be convenient to recast the variables λ_i into the Λ_i's given by

$$\lambda_i = (a_N\Lambda_i)^{\frac{1}{N-2s}} \tag{2.28}$$

with

$$a_N = \frac{\int_{\mathbb{R}^N} w^{p^*+1}}{(p^*+1)(\int_{\mathbb{R}^N} w^{p^*})^2}$$

Lemma 2.6 (Expansion of the energy)**.** *The following expansion of the energy holds:*

$$I_{\pm\varepsilon}(\bar{V}) = mC_N + [\gamma_N + \rho_N\Psi(\xi,\Lambda)]\varepsilon + o(\varepsilon) \tag{2.29}$$

uniformly with respect to (ξ,Λ) *satisfying* (2.16), (2.19) *and* (2.28)*. Here*

$$\Psi(\xi,\Lambda) = \frac{1}{2}\left\{\sum_{i=1}^{m} H(\xi_i,\xi_i)\Lambda_i^2 - 2\sum_{i<j} G(\xi_i,\xi_j)\Lambda_i\Lambda_j\right\} \pm \log(\Lambda_1\cdots\Lambda_m) \tag{2.30}$$

$$\gamma_N = \left\{\pm\frac{m}{p^*+1}\rho_N \pm \frac{m}{2}\rho_N\log a_N \mp \frac{m}{p^*+1}\int_{\mathbb{R}^N} w^{p^*+1}\log w\right\} \tag{2.31}$$

and

$$\rho_N = \frac{\int_{\mathbb{R}^N} w^{p^*+1}}{p^*+1}.$$

Proof. Consider the energy functional

$$I_0(V) = \frac{1}{2}\int_{\mathcal{C}_\varepsilon} y^{1-2s}|\nabla V|^2 - \frac{1}{p+1}\int_{\Omega_\varepsilon} |V|^{p^*+1}.$$

In order to prove (2.29), let us first estimate $I_0(\bar{V})$. Recall that $\bar{v} = \sum_{i=1}^m v_i$, and then $\bar{V} = \sum_{i=1}^m V_i$ where $\bar{V}$ and V_i represent the s-harmonic extension of $\bar{v}$ and v_i, respectively. We have

$$\begin{aligned} I_0(\bar{V}) &= I_0(\sum_{i=1}^m V_i) \\ &= \sum_{i=1}^m \frac{1}{2}\int_{\mathcal{C}_\varepsilon} y^{1-2s}|\nabla V_i|^2 - \frac{1}{p^*+1}\int_{\Omega_\varepsilon} v_i^{p^*+1} \\ &+ \sum_{i\neq j}\int_{\mathcal{C}_\varepsilon} y^{1-2s}\nabla V_i \nabla V_j - \frac{1}{p^*+1}\left[\int_{\Omega_\varepsilon}\left(\sum_{i=1}^m v_i\right)^{p^*+1} - \sum_{i=1}^m v_i^{p^*+1}\right]. \end{aligned} \tag{2.32}$$

Now, recall that, by (2.21), V_i satisfies up to a constant

$$\begin{cases} \operatorname{div}(y^{1-2s}\nabla V_i) = 0 & \text{in } \mathcal{C}_\varepsilon, \\ V_i = 0 & \text{on } \partial_L \mathcal{C}_\varepsilon, \\ -\lim_{y\to 0} y^{1-2s}\partial_y V_i = w_i^{p^*} & \text{on } \Omega_\varepsilon. \end{cases} \tag{2.33}$$

Integrating by parts, we deduce that

$$\int_{\mathcal{C}_\varepsilon} y^{1-2s}|\nabla V_i|^2 = \int_{\Omega_\varepsilon} w_i^{p^*} v_i = \int_{\Omega_\varepsilon} w_i^{p^*+1} - w_i^{p^*}\varphi_i.$$

Thus, recalling the definition of w_i given in (2.18) and then using the previous lemma, we have that

$$\begin{aligned} \int_{\mathcal{C}_\varepsilon} y^{1-2s}|\nabla V_i|^2 &= \int_{\mathbb{R}^N} w^{p^*+1} - a_N\left(\int_{\mathbb{R}^N} w^{p^*}\right)^2 H(\xi_i,\xi_i)\Lambda_i^2\varepsilon + o(\varepsilon) \\ &= \int_{\mathbb{R}^{N+1}_+} y^{1-2s}|\nabla W|^2 - a_N\left(\int_{\mathbb{R}^N} w^{p^*}\right)^2 H(\xi_i,\xi_i)\Lambda_i^2\varepsilon + o(\varepsilon), \end{aligned} \tag{2.34}$$

where the last equality is due to W is the s-harmonic extension of w, which satisfies equation (2.2).

By a similar argument, we see that

$$\int_{\mathcal{C}_\varepsilon} y^{1-2s}\nabla V_i \nabla V_j = a_N\left(\int_{\mathbb{R}^N} w^{p^*}\right)^2 G(\xi_i,\xi_j)\Lambda_i\Lambda_j\varepsilon + o(\varepsilon), \tag{2.35}$$

$$\int_{\Omega_\varepsilon} v_i^{p^*+1} = \int_{\mathbb{R}^N} w^{p^*+1} - (p^*+1)a_N\left(\int_{\mathbb{R}^N} w^{p^*}\right)^2 H(\xi_i,\xi_i)\Lambda_i^2\varepsilon + o(\varepsilon) \tag{2.36}$$

and

$$\begin{aligned}\frac{1}{p^*+1}&\left[\int_{\Omega_\varepsilon}\left(\sum_{i=1}^m v_i\right)^{p^*+1}-\sum_{i=1}^m v_i^{p^*+1}\right]\\ &=2a_N\left(\int_{\mathbb{R}^N}w^{p^*}\right)^2 G(\xi_i,\xi_j)\Lambda_i\Lambda_j\varepsilon+o(\varepsilon)\quad\text{for all } i\neq j.\end{aligned}\tag{2.37}$$

Putting (2.34)–(2.37) in (2.32), we conclude that

$$I_0(\bar V)=mC_N+\frac{\omega_N}{2}\left\{\sum_{i=1}^m H(\xi_i,\xi_i)\Lambda_i^2-2\sum_{i<j}G(\xi_i,\xi_j)\Lambda_i\Lambda_j\right\}\varepsilon+o(\varepsilon).$$

On the other hand,

$$I_{\pm\varepsilon}(\bar V)-I_0(\bar V)=\pm\frac{\varepsilon}{(p^*+1)^2}\int_{\Omega_\varepsilon}\bar V^{p^*+1}\mp\frac{\varepsilon}{p^*+1}\int_{\Omega_\varepsilon}\bar V^{p^*+1}\log\bar V+o(\varepsilon).$$

The right hand side can be computed as in [35, Lemma 2.1] and [36], it gives us the following expansion

$$\begin{aligned}I_{\pm\varepsilon}(\bar V)-&I_0(\bar V)\\ &=\left[\pm\frac{m}{(p^*+1)^2}\int_{\mathbb{R}^N}w^{p^*+1}\pm\frac{m}{2(p^*+1)}\log a_N\left(\int_{\mathbb{R}^N}w^{p^*+1}\right)\right.\\ &\quad\left.\pm\frac{\int_{\mathbb{R}^N}w^{p^*+1}}{p^*+1}\log(\Lambda_1\cdots\Lambda_m)\mp\frac{m}{p^*+1}\int_{\mathbb{R}^N}w^{p^*+1}\log w\right]\varepsilon+o(\varepsilon),\end{aligned}$$

which concludes the proof. □

2.4 The finite-dimensional reduction

Given $\delta>0$, consider pints $\xi_i'\in\Omega_\varepsilon$, and numbers $\Lambda_i>0$, for $i=1,\dots,m$, such that

$$|\xi_i'-\xi_j'|\geq\varepsilon^{-\frac{1}{N-2s}}\delta,\quad \operatorname{dist}(\xi_i',\partial\Omega_\varepsilon)>\varepsilon^{-\frac{1}{N-2s}}\delta,\quad \delta<\Lambda_i<\delta^{-1}.\tag{2.38}$$

As we mention in the previous section, we look for solutions to problem (2.15) of the form $v=\bar v+\phi$ (see (2.23)). So that, in this section we consider the intermediate problem of finding ϕ and c_{ij} such that

$$\begin{cases}(-\Delta)^s(\bar v+\phi)=(\bar v+\phi)^{p^*\pm\varepsilon}+\sum\limits_{i,j}c_{ij}w^{p^*-1}z_{ij} & \text{in } \Omega_\varepsilon,\\ \phi=0 & \text{on } \partial\Omega_\varepsilon,\\ \displaystyle\int_{\Omega_\varepsilon}\phi w^{p^*-1}z_{ij}=0 & \text{for all } i,j\end{cases}\tag{2.39}$$

where the ξ_i's and the λ_i's satisfy (2.16) and (2.19), respectively; and z_{ij} are defined as follows: consider the functions

$$\bar z_{ij}=\partial_{x_j}w_{\lambda_i,\xi_i'},\quad 1\leq i\leq m,\ 1\leq j\leq N,\tag{2.40}$$

$$\bar z_{i0}=\frac{N-2s}{2}w_{\lambda_i,\xi_i'}+(x-\xi_i')\cdot\nabla w_{\lambda_i,\xi_i'},\quad 1\leq i\leq m\tag{2.41}$$

and then define the z_{ij}'s in (2.39) to their respective $H_0^s(\Omega_\varepsilon)$-projection, i.e. the unique solutions of

$$\begin{cases} (-\Delta)^s z_{ij} = (-\Delta)^s \bar{z}_{ij} & \text{in } \Omega_\varepsilon, \\ z_{ij} = 0 & \text{on } \partial\Omega_\varepsilon. \end{cases}$$

Remark 2.7. i) In order to find solutions of (2.15), we have to solve (2.39) and then find points ξ_i and scalars Λ_i such that the associated c_{ij} are all zero.

ii) Observe that for $\phi \in L^\infty(\mathbb{R}^N)$ the integral

$$\int_{\mathbb{R}^N} w^{p^*-1} z_{ij} \phi$$

is well defined because $w^{p^*-1}(x) \leq C(1+|x|)^{-4s}$ and $|z_{ij}(x)| \leq C(1+|x|)^{-N+2s}$.

iii) The role of the functions $\bar{z}_{ij}$ will be clarified in Proposition 2.9.

The first equation of (2.39) can be rewritten in the following form:

$$(-\Delta)^s \phi - (p^* \pm \varepsilon)\bar{v}^{p^*-1\pm\varepsilon}\phi = R_\varepsilon + M_\varepsilon(\phi) + \sum_{i,j} c_{ij} w^{p^*-1} z_{ij}$$

where

$$R_\varepsilon = \bar{v}^{p^*\pm\varepsilon} - \sum_{i=1}^m w_i^{p^*},$$
$$M_\varepsilon(\phi) = (\bar{v}+\phi)_+^{p^*\pm\varepsilon} - \bar{v}^{p^*\pm\varepsilon} - (p^*\pm\varepsilon)\bar{v}^{p^*-1\pm\varepsilon}.$$

Then we need to understand the following linear problem: given $h \in C^\alpha(\bar{\Omega}_\varepsilon)$, find a function ϕ such that for certain constants c_{ij}, $i = 1, \ldots, m$, $j = 0, \ldots, N$ one has

$$\begin{cases} (-\Delta)^s \phi - (p^* \pm \varepsilon)\bar{v}^{p^*-1\pm\varepsilon}\phi = h + \sum_{i,j} c_{ij} w^{p^*-1} z_{ij} & \text{in } \Omega_\varepsilon, \\ \phi = 0 & \text{on } \partial\Omega_\varepsilon, \\ \int_{\Omega_\varepsilon} \phi w^{p^*-1} z_{ij} = 0 & \text{for all } i, j. \end{cases} \tag{2.42}$$

To solve this problem, we consider appropriate weighted L^∞-norms: For a given $\alpha \geq 0$, let us define the following norm of a function $h : \Omega_\varepsilon \to \mathbb{R}$

$$\|h\|_\alpha = \sup_{x \in \Omega_\varepsilon} \frac{|h(x)|}{\sum_{i=1}^m (1+|x-\xi_i'|)^{-\alpha}}.$$

With these norms, we have the following a priori estimate for bounded solutions of (2.42).

Lemma 2.8. *Let $\alpha > 2s$. Assume that the points $\{\xi_i\}_{i=1}^m \subset \Omega$ and the scalars $\{\lambda_i\}_{i=1}^m$ satisfy (2.16) and (2.19), respectively. Assume also that $\phi \in L^\infty(\Omega_\varepsilon)$ is a solution of (2.42) for a function $h \in C^\alpha(\bar{\Omega}_\varepsilon)$. Then there is C such that for $\varepsilon > 0$ sufficiently small*

$$\|\phi\|_{L^\infty(\Omega_\varepsilon)} \leq C\|h\|_\alpha \tag{2.43}$$

and

$$|c_{ij}| \leq C\|h\|_\alpha. \tag{2.44}$$

From now on, we denote by C a generic constant which is independent of ε and $\{\xi_i\}_{i=1}^m$, $\{\lambda_i\}_{i=1}^m$ satisfying (2.16) and (2.19), respectively. The proof of this lemma is based on the following non-degeneracy property of the solutions $w_{\lambda,\xi}$ (see [32]).

Proposition 2.9. *Any bounded solution ϕ of equation*

$$(-\Delta)^s\phi = p^* w_{\lambda,\xi'}^{p^*-1}\phi \quad \textit{in } \mathbb{R}^N$$

is a linear combinations of the functions

$$\frac{N-2s}{2}w_{\lambda,\xi'} + (x-\xi')\cdot\nabla w_{\lambda,\xi'}, \quad \partial_{x_j}w_{\lambda,\xi'}, \quad 1\le j\le N. \tag{2.45}$$

We will also need the following convolution estimate.

Lemma 2.10. *For $2s<\alpha<N$ there is C such that*

$$\|(1+|x|)^{\alpha-2s}(\Gamma * h)\|_{L^\infty(\mathbb{R}^N)} \le \|(1+|x|)^\alpha h\|_{L^\infty(\mathbb{R}^N)},$$

where Γ is defined in (2.4).

Proof of Lemma 2.8. Let us first estimate the constants c_{ij}. Testing the first equation in (2.42) against z_{lk} and then integrating by parts twice, we deduce that

$$\sum_{i,j} c_{ij}\int_{\Omega_\varepsilon} w^{p^*-1}z_{ij}z_{lk} = \int_{\Omega_\varepsilon}[(-\Delta)^s z_{lk} - (p^*\pm\varepsilon)\bar v^{p^*-1\pm\varepsilon}z_{lk}]\phi - \int_{\Omega_\varepsilon} hz_{lk}, \quad \varepsilon>0.$$

This defines a linear system in the c_{ij}'s which is almost diagonal as ε approaches to zero, indeed, for $k=1,\dots,N$

$$\int_{\Omega_\varepsilon} w^{p^*-1}z_{ij}z_{lk} = \delta_{il}\delta_{jk}\int_{\mathbb{R}^N} w_{\lambda_i,0}^{p^*-1}\left(\frac{\partial w_{\lambda_i,0}}{\partial x_k}\right)^2 + o(1)$$

and for $k=0$

$$\int_{\Omega_\varepsilon} w^{p^*-1}z_{ij}z_{l0} = \delta_{il}\delta_{j0}\int_{\mathbb{R}^N} w_{\lambda_i,0}^{p^*-1}\left(\frac{N-2s}{2}w_{\lambda_i,0} + x\cdot\nabla w_{\lambda_i,0}\right)^2 + o(1).$$

On the other hand, we deduce that, for $l=1,\dots,m$,

$$\int_{\Omega_\varepsilon}[(-\Delta)^s z_{lk} - (p^*\pm\varepsilon)\bar v^{p^*-1\pm\varepsilon}z_{lk}]\phi = o(1)\|\phi\|_{L^\infty(\Omega_\varepsilon)},$$

after noticing that $(-\Delta)^s\bar z_{lk} - p^* w_{\lambda_l,0}^{p^*-1}\bar z_{lk} = 0$ (recall the definition of $\bar z_{lk}$ in (2.40) and (2.41)), and then applying the dominated convergence theorem. Also, it is easy to see that

$$\left|\int_{\Omega_{\varepsilon_n}} hz_{lk}\right| \le C\|h\|_\alpha.$$

Therefore, the constants c_{ij} satisfy the estimate

$$|c_{ij}| \le C\|h\|_\alpha + o(1)\|\phi\|_{L^\infty(\Omega_\varepsilon)} \quad \text{as } \varepsilon\to 0. \tag{2.46}$$

let us prove (2.43). We proceed by contradiction, assuming there are sequences $\varepsilon_n \to 0$, $\phi_n \in L^\infty(\Omega_{\varepsilon_n})$, which is a solution of (2.42) for some h_n, and such that

$$\|\phi_n\|_{L^\infty(\Omega_{\varepsilon_n})} = 1, \quad \|h_n\|_\alpha \to 0 \quad \text{as } n \to \infty.$$

Let $\xi'_{i,n}$, denote the sequence of associated points, scaled according to (2.17), and let $\lambda_{i,n}$ be the sequence of parameters. Observe that, by (2.46),

$$|c_{ij}| \le C\|h_n\|_\alpha + o(1)\|\phi_n\|_{L^\infty(\Omega_{\varepsilon_n})} = o(1) \quad \text{as } n \to \infty. \tag{2.47}$$

We shall prove that

$$\lim_{n\to\infty} \|\phi_n\|_\gamma = 0, \tag{2.48}$$

for $\gamma = \min\{\alpha, \beta\} - 2s$ and β is any number such that $2s < \beta < 4s$. In particular $\|\phi_n\|_{L^\infty(\Omega_{\varepsilon_n})} \to 0$ as $n \to +\infty$, which is a contradiction.

To show (2.48), we first prove that for any $R > 0$,

$$\phi_n \to 0 \quad \text{uniformly on } B_R(\xi'_{i,n}). \tag{2.49}$$

Suppose that this is not true and translate the system of coordinates so that $\xi'_{i,n} = 0$. Then there is some point $x_n \in B_R(0)$ such that

$$|\phi_n(x_n)| \ge \frac{1}{2}. \tag{2.50}$$

By passing to a subsequence we can assume that ϕ_n converges uniformly on compact sets of $\mathbb{R}^N$ to a bounded solution ϕ of the problem

$$(-\Delta)^s \phi = p^* w_{\lambda,0}^{p^*-1} \phi \quad \text{in } \mathbb{R}^N$$

for some $\lambda > 0$ (recall that $\lambda_{i,n}$ stay bounded and bounded away from zero by (2.19)). By Proposition 2.9, ϕ is a linear combination of the z_{ij}'s. We can take the limit in the third equation of (2.42) and use the Lebesgue dominated convergence theorem to find that ϕ satisfies

$$\int_{\mathbb{R}^N} w_{\lambda,0}^{p^*-1} z_{ij} \phi = 0 \quad \text{for all } 1 \le i \le m, 0 \le j \le N,$$

and we deduce from this that $\phi \equiv 0$. But because of (2.50) there must be a point x such that $|\phi(x)| \ge \frac{1}{2}$, which is a contradiction.

We claim that for any $2s \le \beta < 4s$

$$\lim_{n\to\infty} \|(p^* \pm \varepsilon_n)\bar{v}^{p^*-1\pm\varepsilon_n} \phi_n\|_\beta = 0. \tag{2.51}$$

Indeed, observe that $0 < (p^* \pm \varepsilon_n)\bar{v}^{p^*-1\pm\varepsilon_n} \le C\sum_{i=1}^m (1 + |x - \xi'_{i,n}|)^{-4s}$, so

$$\|(p^* \pm \varepsilon_n)\bar{v}^{p^*-1\pm\varepsilon_n} \phi_n\|_\beta \le C \sup_{x\in\Omega_{\varepsilon_n}} \left(\frac{\sum_{i=1}^m (1 + |x - \xi'_{i,n}|)^{-4s}}{\sum_{i=1}^m (1 + |x - \xi'_{i,n}|)^{-\beta}} |\phi_n(x)| \right).$$

Let $\bar{\varepsilon} > 0$ be given. Then there exists $R > 0$ large so that

$$\frac{\sum_{i=1}^m (1 + |x - \xi'_{i,n}|)^{-4s}}{\sum_{i=1}^m (1 + |x - \xi'_{i,n}|)^{-\beta}} \le \bar{\varepsilon} \quad \forall x \in \Omega_{\varepsilon_n} \setminus \cup_{i=1}^m B_R(\xi'_{i,n}).$$

By (2.49), there is n_0 such that for all $n \geq n_0$

$$\sup_{B_R(\xi'_{i,n})} |\phi_n| \leq \bar{\varepsilon}.$$

It follows that for $n \geq n_0$,

$$\sup_{x\in\Omega_{\varepsilon_n}} \frac{\sum_{i=1}^m (1+|x-\xi'_{i,n}|)^{-4s}}{\sum_{i=1}^m (1+|x-\xi'_{i,n}|)^{-\beta}} |\phi_n(x)| \leq \bar{\varepsilon},$$

and this proves (2.51).

Indeed, let

$$f_{i,n}(x) = \frac{(1+|x-\xi'_i|)^{-\beta}}{\sum_{j=1}^m (1+|x-\xi'_j|)^{-\beta}} |(p^* \pm \varepsilon_n)\bar{v}^{p^*-1\pm\varepsilon_n}\phi_n(x)|,$$
$$h_{i,n}(x) = \frac{(1+|x-\xi'_i|)^{-\alpha}}{\sum_{j=1}^m (1+|x-\xi'_j|)^{-\alpha}} |h_n(x)|,$$
$$t_{i,n}(x) = \frac{(1+|x-\xi'_i|)^{-\alpha}}{\sum_{j=1}^m (1+|x-\xi'_j|)^{-\alpha}} |\sum_{l,k} c_{l,k} w^{p^*-1} z_{lk}|,$$

and observe that $\sum_i f_{i,n} = |(p^* \pm \varepsilon_n)\bar{v}^{p^*-1\pm\varepsilon_n}\phi_n|$, $\sum_i h_{i,n} = |h_n|$ and $\sum_i t_{i,n} = \sum_{l,k} c_{l,k} w^{p^*-1} z_{lk}$. We extend the functions $f_{i,n}$, $h_{i,n}$ and $t_{i,n}$ by zero outside Ω_{ε_n}. Let $\psi_{i,n}$ be the solution to

$$(-\Delta)^s \psi_{i,n} = f_{i,n} + h_{i,n} + t_{i,n} \quad \text{in } \mathbb{R}^N,$$

with $\psi_{i,n}(x) \to 0$ as $|x| \to \infty$, obtained by convolution with Γ.

Let $\psi_n = \sum_i \psi_{i,n}$ and observe that ψ_n satisfies

$$(-\Delta)^s \psi_n = g_n \quad \text{in } \mathbb{R}^N$$

where

$$g_n(x) = \begin{cases} |(p^* \pm \varepsilon_n)\bar{v}(x)^{p^*-1\pm\varepsilon_n}\phi_n(x)| + |h_n(x)| + |\sum_{l,k} c_{l,k} w(x)^{p^*-1} z_{lk}(x)| & \text{if } x \in \Omega_{\varepsilon_n}, \\ 0 & \text{if } x \notin \Omega_{\varepsilon_n}. \end{cases}$$

Using the maximum principle for the extended problem in $\Omega_{\varepsilon_n} \times (0,\infty)$, Lemma 2.4, we find

$$|\phi_n| \leq \psi_n \quad \text{in } \Omega_{\varepsilon_n}. \tag{2.52}$$

Therefore we can get weighted L^∞ estimates for ϕ_n by establishing these estimates for ψ_n.

Note that centering at $\xi'_{i,n} = 0$,

$$\|(1+|x|)^\alpha h_{i,n}\|_{L^\infty} \leq \|h_n\|_\alpha$$

and therefore, by the previous lemma,

$$\|(1+|x|)^{\alpha-2s}\Gamma * h_{i,n}\|_{L^\infty(\mathbb{R}^N)} \leq \|h_n\|_\alpha. \tag{2.53}$$

Similarly,

$$\|(1+|x|)^{\alpha-2s}\Gamma * t_{i,n}\|_{L^\infty(\mathbb{R}^N)} \leq \sum_{l,k} |c_{lk}| \|w^{p^*-1} z_{lk}\|_\alpha. \tag{2.54}$$

Finally, if $2s < \beta < 4s$, using again the previous lemma we find that

$$\|(1+|x|)^{\beta-2s}\Gamma * f_{i,n}\|_{L^\infty(\mathbb{R}^N)} \le \|(1+|x|)^{\beta} f_{i,n}\|_{L^\infty(\mathbb{R}^N)} \le \|(p^* \pm \varepsilon_n)\bar{v}^{p^*-1\pm\varepsilon_n}\phi_n\|_\beta. \quad (2.55)$$

Hence, by (2.53)–(2.55)

$$\|\psi_n\|_{\gamma,\mathbb{R}^N} \le C(\|(p^* \pm \varepsilon_n)\bar{v}^{p^*-1\pm\varepsilon_n}\phi_n\|_\beta + \|h_n\|_\alpha + \sum_{l,k}|c_{lk}|\|w^{p^*-1}z_{lk}\|_\alpha),$$

where $\gamma = \min\{\beta - 2s, \alpha - 2s\}$. Using (2.52) we get

$$\|\phi_n\|_\gamma \le C(\|(p^* \pm \varepsilon_n)\bar{v}^{p^*-1\pm\varepsilon_n}\phi_n\|_\beta + \|h_n\|_\alpha + \sum_{l,k}|c_{lk}|\|w^{p^*-1}z_{lk}\|_\alpha).$$

But $\|(p^* \pm \varepsilon_n)\bar{v}^{p^*-1\pm\varepsilon_n}\phi_n\|_\beta + \|h_n\|_\alpha + \sum_{l,k}|c_{lk}|\|w^{p^*-1}z_{lk}\|_\alpha \to 0$ as $n \to \infty$ by (2.47) and (2.51). This proves (2.48).

Finally, (2.44) is a consequence of (2.43) and (2.46).

□

As a consequence of Lemma 2.8, we deduce the following proposition.

Proposition 2.11. *Let $\alpha \in (2s, 4s)$ and assume constrains (2.16) and (2.19) hold. Then, there is $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ and all $h \in C^\alpha(\bar{\Omega}_\varepsilon)$, problem (2.42) admits a unique solution $\phi = L_\varepsilon(h)$. Moreover, for certain constant $C > 0$,*

$$\|L_\varepsilon(h)\|_{\alpha-2s} \le C\|h\|_\alpha \quad (2.56)$$

and

$$|c_{ij}| \le C\|h\|_\alpha. \quad (2.57)$$

Proof. Let us consider the space

$$H = \{\phi \in H_0^s(\Omega_\varepsilon) : \int_{\Omega_\varepsilon} \phi w^{p^*-1} z_{ij} = 0 \quad \forall i, j\}$$

endowed with the usual inner product $<\phi, \psi> = \int_{\Omega_\varepsilon}(-\Delta)^{s/2}\phi(-\Delta)^{s/2}\psi$, as in (2.7). The weak formulation of problem (2.42) is the following: Find $\phi \in H$ such that

$$<\phi, \psi> = \int_{\Omega_\varepsilon}(p \pm \varepsilon)\bar{v}^{p^*-1\pm\varepsilon}\phi\psi + \int_{\Omega_\varepsilon}(h + \sum_{i,j} c_{ij} w^{p^*-1} z_{ij})\psi \quad \text{for all } \psi \in H.$$

With the aid of the Riesz's representation theorem, this equation takes the form

$$\phi = \mathcal{F}_\varepsilon(\phi) + \tilde{h} \quad (2.58)$$

where $\mathcal{F}_\varepsilon$ and $\tilde{h}$ are operators defined in $L^2(\Omega_\varepsilon)$ by

$$\mathcal{F}_\varepsilon = (-\Delta)^{-s} \circ l_1,$$
$$\tilde{h} = (-\Delta)^{-s} \circ l_2;$$

l_1 and l_2 are the functions defined in $L^2(\Omega_\varepsilon)$ given by

$$l_1(\psi) = \int_{\Omega_\varepsilon} (p^* \pm \varepsilon)\bar{v}^{p^*-1\pm\varepsilon}\phi,$$
$$l_2(\psi) = \int_{\Omega_\varepsilon} (h + \sum_{i,j} c_{ij} w^{p^*-1} z_{ij})\psi.$$

$(-\Delta)^{-s}$ represents the inverse of the fractional Laplacian operator.

Using compact embeddings in fractional Sobolev spaces, we deduce that $\mathcal{F}_\varepsilon$ is compact (see for instance [1, Ch. VII] and [58]). Fredholm's alternative guarantees unique solvability of this problem for any h provided that the homogeneous equation

$$\phi = \mathcal{F}_\varepsilon(\phi)$$

has only the zero solution in H. Lemma 2.8 guarantees that this is true provided that $\varepsilon > 0$ is small enough.

Finally, estimate (2.56) is a consequence of (2.43) and a simple argument by contradiction. $\square$

It is important for later purposes to understand the differentiability of $L\varepsilon$ on the variables ξ_i and Λ_i. To this end, given $\alpha \in (2s, 4s)$, we define the space

$$L^\infty_\alpha(\Omega_\varepsilon) = \{h \in L^\infty(\Omega_\varepsilon) : \|h\|_\alpha < \infty\},$$

and consider the map

$$(\xi', \Lambda, h) \mapsto S(\xi', \Lambda, h) \equiv L_\varepsilon(h),$$

as a map with values in $L^\infty_\alpha \cap H^s_0(\Omega_\varepsilon)$.

The proof of the next results are similar to that found in [36] for the case $m = 2$ (see also [35]). We omit the details.

Proposition 2.12. *Under the conditions of the previous proposition, the map S is of class C^1 and*

$$\|\nabla_{\xi',\Lambda} S(\xi', \Lambda, h)\|_{\alpha-2s} \leq C\|h\|_\alpha.$$

Proposition 2.13. *Assume the conditions of Proposition 2.11 are satisfied. Then, there is a constant $C > 0$ such that, for all $\varepsilon > 0$ small enough, there exists a unique solution*

$$\phi = \phi(\xi', \Lambda) = \tilde{\phi} + \psi$$

to problem (2.39) with ψ defined by $\psi = L_\varepsilon(R_\varepsilon)$ and for points ξ', Λ satisfying (2.38). Moreover, the map $(\xi', \Lambda) \mapsto \tilde{\psi}(\xi', \Lambda)$ is of class C^1 for the $\|\cdot\|_{\alpha-2s}$-norm and

$$\|\tilde{\phi}\|_{\alpha-2s} \leq C\varepsilon^{\min\{p^*,2\}}, \tag{2.59}$$
$$\|\nabla_{\xi',\Lambda}\tilde{\phi}\|_{\alpha-2s} \leq C\varepsilon^{\min\{p^*,2\}}. \tag{2.60}$$

2.5 The reduced functional

Let us consider points (ξ', Λ) which satisfy constrains (2.38) for some $\delta > 0$, and recall that $\xi' = \varepsilon^{-\frac{1}{N-2s}}$. Let $\phi(x) = \phi(\xi', \Lambda)(x)$ be the unique solution of (2.39) given by Proposition 2.13. Let Φ the s-harmonic extension of ϕ (recall (2.9)) and consider the functional

$$\mathcal{I}(\xi, \Lambda) = I_{\pm\varepsilon}(\bar{V} + \Phi),$$

where I_ε is defined in (2.25). The definition of Φ yields that

$$\mathcal{I}'(\bar{V} + \Phi)[\Theta] = 0 \quad \text{for all } \Phi \in H^s_{0,L}(\Omega_\varepsilon),$$

and such that

$$\int_{\Omega_\varepsilon} \theta w^{p^*-1} z_{ij},$$

where $\theta = \mathrm{tr}\,|_{\Omega_\varepsilon \times \{0\}} \Theta$.

It is easy to check that

$$\partial_{x_j} v_i = z_{ij} + o(1), \quad \partial_{\Lambda_j} v_i = z_{i0} + o(1),$$

as $\varepsilon \to 0$. The last part of Proposition 2.13 gives the validity of the following result, see [36, Sec. 6] for details.

Lemma 2.14. *$v = \bar{v} + \phi$ is a solution of problem (2.15) if and only if (ξ, Λ) is a critical point of $\mathcal{I}$.*

Next step is then to give an asymptotic estimate for $\mathcal{I}(\xi, \Lambda)$. As we expected, this functional and $I_\varepsilon(\bar{V})$ coincide up to order $o(\varepsilon)$. The steps to proof this result are basically contained in [35, Sec. 4] and [36, Sec. 6], we omit the details.

Proposition 2.15. *We have the expansion*

$$\mathcal{I}(\xi, \Lambda) = mC_N + [\gamma_N + \rho_N \Psi(\xi, \Lambda)]\varepsilon + o(\varepsilon), \tag{2.61}$$

where $o(\varepsilon) \to 0$ as $\varepsilon \to 0$ in the uniform C^1-sense with respect to (ξ, Λ) satisfying (2.16) and (2.19). The constants in (2.61) are those in Lemma 2.6 and

$$\Psi(\xi, \Lambda) = \frac{1}{2}\left\{\sum_{i=1}^m H(\xi_i, \xi_i)\Lambda_i^2 - 2\sum_{i<j} G(\xi_i, \xi_j)\Lambda_i\Lambda_j\right\} \pm \log(\Lambda_1 \cdots \Lambda_m).$$

2.6 Proof of Theorem 2.2 and Theorem 2.3

In this section we prove the main theorems of this chapter. Let us firs note that Theorem 2.2 is a direct consequence of the previous proposition, Lemma 2.14 and the stability of the set $\mathcal{A}$ of critical points of Ψ.

Concerning Theorem 2.3, let us suppose that $p = p^* - \varepsilon$ in (2.15) and $m = 1$, that is, we consider the subcritical case and study the concentration phenomena for just one bubble. In this case the function Ψ in (2.30) takes the form

$$\Psi(\xi, \Lambda) = \frac{1}{2} H_\Omega(\xi, \xi)\Lambda^2 - \log \Lambda, \quad \xi \in \Omega, \Lambda > 0.$$

Thanks to the coercivity of Ψ in Λ, in order to find a critical point of $\Psi(\xi,\Lambda)$, we have to find one to $H_\Omega(\xi,\xi)$. it is called the Robin's function of the domain Ω, and we denote it by

$$\varphi(\xi) = H_\Omega(\xi,\xi), \quad \xi \in \Omega.$$

The next result provide the existence of a critical point ξ_0 of the Robin's function. Moreover, ξ_0 is a minimum of φ in Ω and then (ξ_0,Λ) is a stable critical point of Ψ, in the sense of Definition 2.1, for a suitable $\Lambda > 0$. Before the precise statement of the result, let us review a fractional version of the Kelvin transform (see Appendix A in [66]).

Lemma 2.16 (Fractional Kelvin transform)**.** *Let u be a smooth bounded function in $\mathbb{R}^N\setminus\{0\}$. Let $x \mapsto x/|x|^2$ be the inversion with respect to the unit sphere. Define $u^*(x) = |x|^{2s-N}u(x^*)$. Then,*

$$(-\Delta)^s u^*(x) = |x|^{-2N-s}(-\Delta)^s u(x^*),$$

for all $x \neq 0$.

Recall also the following identity

$$|x^* - y^*| = \frac{|x-y|}{|x||y|}. \tag{2.62}$$

Lemma 2.17. *Given $\xi \in \Omega$, we define the function $d(\xi) := \operatorname{dist}(\xi,\partial\Omega)$. Then, there exists positive constants c_1 and c_2 such that,*

$$c_1 d(\xi)^{2s-N} \leq \varphi(\xi) \leq c_2 d(\xi)^{2s-N}, \quad \xi \in \Omega, \tag{2.63}$$

and then φ has a minimum value in Ω.

Proof. Let $\xi^0 = (\xi^0_1,\ldots,\xi^0_N) \in \partial\Omega$, and consider the ball $B := B_{1/2}(1/2,0,\ldots,0) \subset \mathbb{R}^N$. After a rearrange of variables, we can assume that $\xi^0_1 = 1$ and $B \subset \Omega^c$. We shall use the Green's function of a semi-space in $\mathbb{R}^{N+1}_+$, namely $S_- = \{\xi = (\xi_1,\ldots,\xi_N,y) : \xi_1 < 1, y > 0\}$. The Kelvin transform is then used to bound the Green's function of Ω, which we denote by G_Ω.

Notice that the Green's function of a semi-plane like S_- (recall the definition of fractional Green's function in (2.11)) is given by

$$G_{S_-}(Z,Y) = \Gamma(Z-Y) - \Gamma(Z-\bar{Y}), \quad Z,Y \in \bar{S}_-, Z \neq Y,$$

where $\bar{Y}$ is the reflection of Y with respect to the plane ∂S_-.

Concerning the Kelvin inversion with respect the $N+1$-dimensional unit sphere, observe that $\Omega \times \mathbb{R}_+ \subset B^c \times \mathbb{R}^+ \ \subset S_-{}^*$. Then, we consider the Kelvin transform of the Green's function of the semi-space S^- and define

$$F(Z,\xi) = |\xi|^{2s-N}|Z|^{2s-N}\left[\Gamma(Z^*-\xi^*) - \Gamma(Z^*-\bar{\xi}^*)\right], \quad Z \in B^c \times \mathbb{R}_+, \xi \in B^c.$$

It is easy to check that $F(Z,\xi) \geq 0$ in $\partial\Omega \times \mathbb{R}_+$. On the other hand, using (2.62), function F can be written as

$$F(Z,\xi) = \Gamma(Z-\xi) - c\left|Z|\bar{\xi}^*| - \frac{\bar{\xi}^*}{|\bar{\xi}^*|}\right|^{2s-N},$$

then, F satisfies up to a positive constant

$$\begin{cases} \operatorname{div}(y^{1-2s}\nabla F(\cdot,\xi)) = 0 & \text{in } \Omega\times\mathbb{R}_+, \\ F(\cdot,\xi) \geq 0 & \text{on } \partial\Omega\times\mathbb{R}_+, \\ -\lim_{y\to 0} y^{1-2s}\partial_y F(\cdot,\xi) = \delta_\xi(\cdot) & \text{on } \Omega, \end{cases} \tag{2.64}$$

Therefore, using a minor variant of the maximum principle, Lemma 2.4, we deduce that $G_\Omega \leq F$ in Ω. This implies that

$$H_\Omega(Z,\xi) \geq \tilde{c}\left|Z|\bar{\xi}^*| - \frac{\bar{\xi}^*}{|\bar{\xi}^*|}\right|^{2s-N} \quad \text{for all } Z\in\Omega\times\mathbb{R}_+, \xi\in\Omega.$$

As $\xi^0 \in \partial\Omega$ is arbitrary, we conclude that in a neighborhood of $\partial\Omega$ there exist a constant $c_1 > 0$ such that $\varphi(\xi) = H_\Omega(\xi,\xi) \geq c_1 d(\xi)^{2s-N}$. The smoothness of H_Ω allows us to extend this inequality to the whole domain Ω.

The other inequality in (2.63) can be proven by a similar argument using an interior ball instead.

□

Theorem 2.3 is a direct consequence of previous lemma and Theorem 2.2.

Chapter 3

Non-local phase transitions in the Heisenberg group

In this chapter we study rigidity properties for stable solutions (see Definition 3.6) of non-local equations of the type

$$(-\Delta_{\mathbb{H}})^s v = f(v) \quad \text{in } \mathbb{H}, \tag{3.1}$$

where $s \in (0,1)$, $f \in C^{1,\gamma}(\mathbb{R})$, $\gamma > \max\{0, 1-2s\}$ and $\mathbb{H}$ is the Heisenberg group (see Section 2). We want to give a geometric insight of the phase transition for Eq. (3.1). Following the ideas in [68], we give a geometric proof of rigidity properties for fractional boundary reactions in $\mathbb{H}$. This chapter is mainly based on the paper [53].

3.1 Introduction

The relation between entire stable solutions and minimal surfaces, as performed in this work, is inspired by a famous conjecture of De Giorgi [34] (in the Euclidean setting) and in it is in the spirit of the proof of Bernstein theorem given by Giusti [49]. Similar De Giorgi-type results (in the Euclidean setting) have been proven by Cabré and Solà-Morales [15] for the square root of the Laplacian, and later generalized by Cabré and Sire [12] for arbitrary roots. Sire and Valdinoci [68] gave a proof of analogous rigidity properties for phase transitions driven by fractional Laplacians. Unlike the method in [12, 15], which require a Liouville-type result, the proof in [68] relies heavily on a Poincaré-type inequality which involves the geometry of the level sets of u. Last technique is inspired on the work by Sternberg and Zumbrun [69, 70].

In Chapter 2, a Poincaré-type inequality will be found through a suitable development of some techniques for level set analysis. This follows the ideas of Sternberg and Zumbrun [69, 70]; Farina [38]; Farina, Sciunzi and Valdinoci [39]; Sire and Valdinoci [68]. Some properly modified computations by Ferrari and Valdinoci [41] are needed in order to understand the complicated geometry of the Heisenberg group. This inequality, together with an “abstract" formulation of a technique recently introduced by Caffarelli and Silvestre [20], is used to study rigidity properties of solutions to (3.1).

The standard fractional Laplacian is a non-local operator. This fact does not allow to apply local PDE techniques to treat nonlinear problems for $(-\Delta)^s$. To overcome this difficulty, Caffarelli and Silvestre showed in [20] that any fractional power of the Laplacian can be determined as an operator that maps a Dirichlet boundary condition to a Neumann-type condition via an extension problem. More precisely, let us consider the boundary reaction problem for $u = u(x, y)$, $x \in \mathbb{R}^N$ and $y > 0$,

$$\begin{cases} \operatorname{div}(y^a \nabla u) = 0 & \text{in } \mathbb{R}^N \times (0, \infty), \\ -y^a u_y = f(u) & \text{on } \mathbb{R}^N \times \{0\}, \end{cases} \tag{3.2}$$

where $a = 1 - 2s$. It is proved in [20] that, up to a normalizing factor, the Dirichlet-to-Neumann operator $\Gamma_a : u|_{\partial \mathbb{R}^{N+1}_+} \mapsto -y^a u_y|_{\partial \mathbb{R}^{N+1}_+}$ is precisely $(-\Delta)^s$ and then that $u(x, 0)$ is a solution of

$$(-\Delta)^s u(x, 0) = f(u(x, 0)). \tag{3.3}$$

On the other hand, sub-Laplacians in Carnot groups (i.e. simply connected stratified nilpotent Lie groups) exhibit strong analogies with classical Laplace operators in the Euclidean space (Harnack inequality, maximum principle, existence and estimates of the fundamental solution). Following [20], a construction of a $\Delta_{\mathbb{H}}$-harmonic "lifting" operator $v = v(x) \mapsto u = u(x, y)$ from $\mathbb{H}$ to $\mathbb{H} \times \mathbb{R}^+$ can be carried out by means of the spectral resolution of $-\Delta_{\mathbb{H}}$ in $L^2(\mathbb{H})$ in such a way that v is the trace of the normal derivative of u on $\{y = 0\}$ (see Ferrari and Franchi [40] and the references therein).

For the time being, we leave the precise framework for Section 2, instead we discuss the main results.

Let us define $\widehat{\mathbb{H}} := \mathbb{H} \times (0, +\infty)$. As in the Euclidean case, the study of the non-local equation (3.1) is related to the analysis of the following degenerate elliptic problem (see Section 2 for details):

$$\begin{cases} \operatorname{div}_{\widehat{\mathbb{H}}}(y^a \nabla_{\widehat{\mathbb{H}}} u) = 0 & \text{in } \mathbb{H} \times (0, \infty), \\ -y^a u_y = f(u) & \text{on } \mathbb{H} \times \{0\}. \end{cases} \tag{3.4}$$

Definition 3.1 (Functional framework)**.** (I) *Notion of weak solution*: (3.4) may be understood in the *weak sense*, namely supposing that $u \in L^\infty_{\mathrm{loc}}(\overline{\mathbb{R}^4_+})$ with

$$y^a |\nabla_{\widehat{\mathbb{H}}} u|^2 \in L^1(B_R^+) \tag{3.5}$$

for any $R > 0$, and that

$$\int_{\widehat{\mathbb{H}}} y^a \langle \nabla_{\widehat{\mathbb{H}}} u, \nabla_{\widehat{\mathbb{H}}} \xi \rangle_{\widehat{\mathbb{H}}} = \int_{\mathbb{H}} f(u) \xi \tag{3.6}$$

for all $\xi : \mathbb{R}^4_+ \to \mathbb{R}$ bounded, locally Lipschitz, which vanishes on $\mathbb{R}^4_+ \setminus B_R$ and such that

$$y^a |\nabla_{\widehat{\mathbb{H}}} \xi|^2 \in L^1(B_R^+). \tag{3.7}$$

We use here the notation $\mathbb{R}^4_+ = \mathbb{R}^3 \times (0, \infty)$ and $B_R^+ := B_R \cap \mathbb{R}_4^+$.

(II) *Notion of stability*: Let u be a weak solution of (3.4), u is *stable* if

$$\int_{\widehat{\mathbb{H}}} y^a |\nabla_{\widehat{\mathbb{H}}} \xi|^2 - \int_{\mathbb{H}} f'(u) \xi^2 \geq 0 \tag{3.8}$$

for any ξ as above. This condition is natural in the calculus of variation framework, in particular it says that the second variation of the associated functional has a sign, as it happens for local minima, for instance.

For the precise statement of our geometric result, we introduce the following notation: fixed $y > 0$ and $c \in \mathbb{R}$, we look at the level set

$$S := \{x \in \mathbb{R}^3 \text{ s.t. } u(x,y) = c\},$$

and we consider the regular points of S, i.e.

$$L := \{x \in S \text{ s.t. } \nabla_{\mathbb{H}} u(x,y) \neq 0\}. \tag{3.9}$$

Although S and L depend on $y \in (0,+\infty)$, we do not make it explicit in the notation.

We also define

$$\mathcal{R}^4_+ := \{(x,y) \in \mathbb{H} \times (0,+\infty) \text{ s.t. } \nabla_{\mathbb{H}} u(x,y) \neq 0\}.$$

Since L is a smooth manifold, given $x \in L$, we denote:

(i) $\nu_{x,y}$ be the *intrinsic normal* along L,

(ii) $v_{x,y}$ be the *intrinsic unit tangent* along L,

(iii) $h_{x,y}$ be the *the intrinsic mean curvature* along L,

(iv) $p_{x,y}$ be the *imaginary curvature* along L,

(see Definition 3.8 for details).

In this framework, we can state our geometric formula; see Section 2 for the definition of the vector fields X, Y, T and the Hessian matrix H.

Theorem 3.2. *Let $u \in C^2(\widehat{\mathbb{H}})$ be a bounded and stable weak solution of (3.4). Assume furthermore that for all $R > 0$,*

$$|\nabla_{\mathbb{H}} u| \in L^\infty(\overline{B^+_R}). \tag{3.10}$$

Then, for any $\phi \in C^\infty_0(\mathbb{R}^4)$, we have

$$\begin{aligned}\int_{\widehat{\mathbb{H}}} & y^a |\nabla_{\mathbb{H}} u|^2 |\nabla_{\widehat{\mathbb{H}}} \phi|^2 \\ & \geq \int_{\mathcal{R}^4_+} y^a \Big(|Hu|^2 - \langle (Hu)^2 \nu_{x,y}, \nu_{x,y} \rangle_{\mathbb{H}} - 2(TYuXu - TXuYu) \Big) \phi^2 \\ & = \int_{\mathcal{R}^4_+} y^a |\nabla_{\mathbb{H}} u|^2 \left[h^2_{x,y} + \left(p_{x,y} + \frac{\langle Huv_{x,y}, \nu_{x,y} \rangle_{\mathbb{H}}}{|\nabla_{\mathbb{H}} u|} \right)^2 + 2\langle T\nu_{x,y}, v_{x,y} \rangle_{\mathbb{H}} \right] \phi^2. \end{aligned} \tag{3.11}$$

Remark 3.3. We observe that (3.11) may be interpreted in two ways:

(i) One way is to think that some interesting geometric objects which describe u, such as its intrinsic Hessian and the curvature of its level sets, are bounded by an energy term. These quantities involved in the inequality are weighted by a test function ϕ which can be chosen as we wish.

(ii) Another point of view consists in thinking that (3.11) bounds a suitably weighted L^2-norm of its gradient. The weights here are given by the stable solution u. So, this interpretation sees (3.11) as a Sobolev-Poincaré inequality.

The result in Theorem 3.2 has been inspired by [69, 70]; in particular, they obtained a similar inequality for stable solutions of the Allen-Cahn equation, and symmetry results for possibly singular or degenerated models have been obtained in [38, 39]. Actually, the study of geometric inequalities for semilinear equations goes back to [69, 70], where uniformly elliptic PDEs in the Euclidean space were taken into account, and further important developments have been performed in [38]. Recently, in [68] has been proved a similar inequality to (3.11) in the Euclidean setting. Related geometric inequalities also played an important role in [11].

The next theorem is a rigidity result. For the precise statement of it, let us define the following suitably weighted energy:

$$\eta(\tau) = \int_{B(0,\tau)} 4y^a |\nabla_{\mathbb{H}} u(x_1, x_2, x_3, y)|^2 (x_1^2 + x_2^2 + y^2)\, \mathrm{d}(x_1, x_2, x_3, y). \tag{3.12}$$

In this expression, $B(0,\tau)$ represents a ball in $\widehat{\mathbb{H}}$ with a gauge norm that will be defined in Section 4 (see (3.43) and (3.44)). No confusion should arise with the Euclidean ball.

Theorem 3.4. *Let the assumptions of the previous theorem hold. Suppose also that*

$$\langle T\nu_{x,y}, v_{x,y}\rangle_{\mathbb{H}} \geq 0 \quad \textit{for all } x \in \mathbb{H}, y > 0; \tag{3.13}$$

and η, previously defined, satisfies the growth

$$\liminf_{R\to+\infty} \frac{\int_{\sqrt{R}}^{R} \frac{\eta(\tau)}{\tau^5}\, \mathrm{d}\tau + \frac{\eta(R)}{R^4}}{\log^2 R} = 0. \tag{3.14}$$

Then, the level sets of u intersected with L (recall (3.9)) are minimal surfaces in the Heisenberg group (i.e., the curvature $h_{x,y}$ vanishes identically) and on such surfaces the following holds

$$p_{x,y} = -\frac{\langle Huv_{x,y}, \nu_{x,y}\rangle_{\mathbb{H}}}{|\nabla_{\mathbb{H}} u|}. \tag{3.15}$$

Remark 3.5. (i) We observe that (3.14) may be seen as a condition on the growth of a suitably weighted energy η.

(ii) Notice also that if, for any R large enough,

$$\eta(R) \leq CR^4,$$

for some constant $C > 0$, then (3.14) is satisfied.

Before stating the rigidity result, let us precise the notion of stable solution for Eq. (3.1):

Definition 3.6. A bounded solution $v \in C^2(\mathbb{H})$ of (3.1) is stable if for all $\varphi \in W^{s,2}_{\mathbb{H}}(\mathbb{H})$ we have

$$\int_{\mathbb{H}} |(-\Delta_{\mathbb{H}})^{\frac{s}{2}} \varphi|^2 - \int_{\mathbb{H}} f'(v)\varphi^2 \geq 0. \tag{3.16}$$

For the precise definition of the space $W^{s,2}_{\mathbb{H}}(\mathbb{H})$ and the fractional operator $(-\Delta_{\mathbb{H}})^{\frac{s}{2}}$, we refer the reader to Section 2.

Throughout this chapter, $C^\alpha(\mathbb{H})$ denotes the set of Hölder continuous functions with respect to the norm ρ defined in the next section (see (3.21)). Our rigidity result is the following:

Theorem 3.7. *Let $v \in C^{2,\sigma}(\mathbb{H})$, $\sigma \in (0, 2s)$, be a bounded stable solution of Eq. (3.1). Assume also that the "harmonic lifting" of v to $\widehat{\mathbb{H}}$ (see Subsection 2.3), which we denote by u, satisfies (3.13) and (3.14). Then, the level sets of v in the vicinity of non-characteristic points are minimal surfaces in the Heisenberg group (i.e., the curvature h vanishes identically).*

The chapter is organized as follows: In Section 2 we recall the definition and the basic properties of the Heisenberg group, as well as the precise definition of the fractional sub-Laplacian involved in Eq. (3.1); we also discuss some regularity properties related to the degenerate elliptic problem (3.4). In Section 3 we shall develop the analytical tools toward (3.11), in particular one part of this inequality will be given in Theorem 3.19; the geometry of the Heisenberg group will be fundamental in the proof of Theorem 3.2 at the end of Section 3 (see [41, Section 2]). Finally, Section 4 contains the application to the stable solutions in the entire space; we prove Theorem 3.4 and Theorem 3.7.

3.2 Preliminaries

Let us briefly recall the definition and the basic properties of the Heisenberg group, so we will be able to precise the meaning of the fractional sub-Laplacian operator involved in (3.1).

3.2.1 The Heisenberg group

Let $\mathbb{H}$ be the Heisenberg group, namely $\mathbb{R}^3$ endowed with the following non-commutative law: for every $(x_1, x_2, x_3),\ (y_1, y_2, y_3) \in \mathbb{R}^3$

$$(x_1, x_2, x_3) \circ (y_1, y_2, y_3) = (x_1 + y_1, x_2 + y_2, x_3 + y_3 + 2(x_2y_1 - x_1y_2)).$$

We shall denote $X = (1, 0, 2x_2)$ and $Y = (0, 1, -2x_1)$. With the same notation we denote the two vector fields $X = \frac{\partial}{\partial x_1} + 2x_2\frac{\partial}{x_3}$ and $Y = \frac{\partial}{\partial x_2} - 2x_1\frac{\partial}{x_3}$ generating the algebra. We denote also by

$$T := [X, Y] = -4\frac{\partial}{\partial x_3}.$$

In particular, on each fiber $\mathcal{H}_P = \operatorname{span}\{X, Y\}$ an internal product is given as follows: for every $U,\ V \in \mathcal{H}_P$, with $U = \alpha_1 X + \beta_1 Y$ and $V = \alpha_2 X + \beta_2 Y$, we have

$$\langle U, V\rangle_{\mathbb{H}} = \alpha_1\alpha_2 + \beta_1\beta_2.$$

This internal product makes the vectors X and Y orthonormal on $\mathcal{H}_P$. We shall denote the norm on $\mathcal{H}_P$ for every $U \in \mathcal{H}_P$ as

$$|U|_{\mathbb{H}} = \sqrt{\langle U, X\rangle_{\mathbb{H}}^2 + \langle U, Y\rangle_{\mathbb{H}}^2}.$$

No confusion should arise between the Euclidean objects $\langle\cdot,\cdot\rangle$ and $|\cdot|$ and the ones on the fibers in the Heisenberg group respectively denoted by $\langle\cdot,\cdot\rangle_{\mathbb{H}}$ and $|\cdot|_{\mathbb{H}}$.

For a smooth function u, we denote $\nabla_{\mathbb{H}} u(P) = (Xu(P), Yu(P))$, where $Xu(P)$ and $Yu(P)$ are the coordinates of the vector $\nabla_{\mathbb{H}} u(P)$ with respect to the basis given by X and Y at P. The vector $\nabla_{\mathbb{H}} u$ is called the *intrinsic gradient* of u.

Definition 3.8. (I) We remind that a point $P \in \Sigma$ is *characteristic* for the C^1 level set Σ of u when the fiber in P coincides with the Euclidean tangent space Σ at P, namely $\mathcal{H}_P = T_P\Sigma$. In particular if $\nabla_{\mathbb{H}}u(P) \neq 0$, then P is not characteristic.

(II) Whenever $P \in \{u = k\} \cap \{\nabla_{\mathbb{H}}u \neq 0\}$, one can consider the smooth surface $\{u = k\}$ and define

$$\nu = \frac{\nabla_{\mathbb{H}}u(P)}{|\nabla_{\mathbb{H}}u(P)|}.$$

Usually, such ν is called the *intrinsic normal*. Associated with ν, to any non-characteristic point $P \in \{u = k\}$, there exists the so called *intrinsic unit tangent* direction to the level set $\{u = k\}$ at P defined as

$$v = \frac{(Yu(P), -Xu(P))}{|\nabla_{\mathbb{H}}u|},$$

where the above coordinates are given with respect to the (X, Y)-frame. We observe that ν and v are orthonormal in $\mathbb{H}$.

(III) The *intrinsic mean curvature* h, in a non-characteristic point $P \in \Sigma$ of the level surface given by u, is defined as

$$h = \mathrm{div}_{\mathbb{H}}\nu(P),$$

while the *imaginary curvature* p at the point $P \in \Sigma$ of the level surface Σ, given by u, is defined as

$$p = -\frac{Tu(P)}{|\nabla_{\mathbb{H}}u(P)|}.$$

Remark 3.9. For the notion of *intrinsic mean curvature* we refer to [2, 3, 22, 25, 47, 61], while for the notion of *imaginary curvature* and its geometric meaning we refer to [2, 3].

The Kohn-Laplace operator on $\mathbb{H}$ is defined by

$$\Delta_{\mathbb{H}}u = X^2u + Y^2u. \tag{3.17}$$

Since a divergence operator is defined on each fiber, we can write

$$\Delta_{\mathbb{H}}u = \mathrm{div}_{\mathbb{H}}(\nabla_{\mathbb{H}}u) = X(Xu) + Y(Yu).$$

With regards to problem (3.4), we define $\widehat{\mathbb{H}} := \mathbb{H} \times \mathbb{R}_+$ and given u and $h = (h_1, h_2, h_3)$ we denote

$$\nabla_{\widehat{\mathbb{H}}}u = (Xu, Yu, u_y), \quad \mathrm{div}_{\widehat{\mathbb{H}}}\, h = Xh_1 + Yh_2 + \partial_y h_3.$$

We define the horizontal intrinsic Hessian matrix as

$$Hu = \begin{bmatrix} XXu & YXu \\ XYu & YYu \end{bmatrix}.$$

Its norm is given by

$$|Hu| = \sqrt{(XXu)^2 + (YXu)^2 + (XYu)^2 + (YYu)^2}.$$

As usual, we set

$$(Hu)^2 = (Hu)(Hu)^T.$$

For any $\lambda > 0$, the dilatation $\delta_\lambda : \mathbb{H} \to \mathbb{H}$ is defined as

$$\delta_\lambda(x_1, x_2, x_3) = (\lambda x_1, \lambda x_2, \lambda^2 x_3). \tag{3.18}$$

Through this chapter, by $\mathbb{H}$-homogeneity we mean homogeneity with respect to group dilatations δ_λ.

The Haar measure of $\mathbb{H} = (\mathbb{R}^3, \cdot)$ is the Lebesgue measure $\mathcal{L}^3$ in $\mathbb{R}^3$. If $A \subset \mathbb{H}$ is $\mathcal{L}^3$-measurable, we write also $|A| := \mathcal{L}^3(A)$. Moreover, if $m \geq 0$, we denote by $\mathcal{H}^m$ the m-dimensional Hausdorff measure obtained from the Euclidean distance in $\mathbb{R}^3 \simeq \mathbb{H}$.

Definition 3.10 (Carnot-Carathéodory distance)**.** An absolutely continuous curve $\gamma : [0, T] \to \mathbb{H}$ is a *sub-unit curve* whit respect to X and Y if it is an *horizontal curve*, i.e., if there real measurable functions $c_1(s), c_2(s)$, $s \in [0, T]$ such that

$$\dot\gamma(s) = c_1(s)X(\gamma(s)) + c_2(s)Y(\gamma(s)), \quad \text{for a.e. } s \in [0, T],$$

and if, in addition,

$$c_1^2 + c_2^2 \leq 1.$$

If $x, y \in \mathbb{H}$, their Carnot-Carathéodory distance (cc-distance) $d_c(x, y)$ is defined as follows:

$$d_c(x, y) = \inf\{T > 0 : \text{ there is a sub-unit curve } \gamma \text{ with } \gamma(0) = x, \gamma(T) = y\}.$$

The set of sub-unit curves joining x and y is not empty, by Chow's theorem. We shall denote $B_c(x, r)$ the open balls associated with d_c. The cc-distance is well behaved with respect to left translations and dilatations, that is

$$d_c(z \circ x, z \circ y) = d_c(x, y), \quad d_c(\delta_\lambda(x), \delta_\lambda(y)) = \lambda d_c(x, y)$$

for $x, y, z \in \mathbb{H}$ and $\lambda > 0$.

We also have

$$|B_c(x, r)| = r^4 |B_c(0, 1)| \quad \text{and} \quad |\partial B_c(x, r)| = r^3 |\partial B_c(0, 1)| \tag{3.19}$$

(recall that $4 =$ homogeneous dimension of $\mathbb{H}$).

We can define a group convolution in $\mathbb{H}$: if, for instance, $f \in \mathcal{D}(\mathbb{H})$ and $g \in L^1_{\text{loc}}(\mathbb{H})$, we set

$$f * g(x) := \int_{\mathbb{H}} f(y) g(y^{-1} \circ x)\, \mathrm{d}y \quad \text{for } x \in \mathbb{H} \tag{3.20}$$

(here y^{-1} denotes the inverse in $\mathbb{H}$). We remind that the convolution is well defined when $f, g \in \mathcal{D}'(\mathbb{H})$, provided at least one of them has compact support.

3.2.2 Fractional powers of sub-elliptic Laplacians

Here, we collect some results on fractional powers of sub-Laplacian in the Heisenberg group (see [40, 42]).

To begin with, let us characterize $(-\Delta_{\mathbb{H}})^s$ as the spectral resolution of $\Delta_{\mathbb{H}}$ in $L^2(\mathbb{H})$ (see [40, Theorem 3.10] and [42, Section 3]).

Theorem 3.11. *The operator $\Delta_{\mathbb{H}}$ is a positive self-adjoint operator with domain $W^{2,2}_{\mathbb{H}}(\mathbb{H})$. Denote now by $\{E(\lambda)\}$ the spectral resolution of $\Delta_{\mathbb{H}}$ in $L^2(\mathbb{H})$. If $\alpha > 0$ then*

$$(-\Delta_{\mathbb{H}})^{\alpha/2} = \int_0^{+\infty} \lambda^{\alpha/2}\, \mathrm{d}E(\lambda)$$

with domain

$$W^{\alpha,2}_{\mathbb{H}}(\mathbb{H}) := \{v \in L^2(\mathbb{H}) : \int_0^{+\infty} \lambda^{\alpha} \mathrm{d}\langle E(\lambda)v, v\rangle < \infty\},$$

endowed with the graph norm.

Before giving a more "explicit" expression of the fractional sub-Laplacian, we recall some definitions. Denote by $h = h(t,x)$ the fundamental solution of $\Delta_{\mathbb{H}} + \partial/\partial t$ (see [42, Proposition 3.3]). For all $0 < \beta < 4$ the integral

$$R_\beta(x) = \frac{1}{\Gamma(\beta/2)} \int_0^{+\infty} t^{\frac{\beta}{2}-1} h(t,x)\, \mathrm{d}t$$

converges absolutely for $x \neq 0$. Moreover

(i) R_2 is the fundamental solution of $\Delta_{\mathbb{H}}$;

(ii) if $v \in \mathcal{D}(\mathbb{H})$, then
$$(-\Delta_{\mathbb{H}})^s = \Delta_{\mathbb{H}} * R_{2-2s};$$

(iii) the kernels R_α admit the following convolution rule: if $\alpha, \beta > 0$, $\alpha + \beta < 4$, $y^{-1}x \neq 0$ and $x \neq 0$, then
$$R_{\alpha+\beta}(x) = R_\alpha(x) * R_\beta(x).$$

If $\beta < 0, \beta \notin \{0, -2. -4, ...\}$, then

$$\widetilde{R}_\beta(x) = \frac{\frac{\beta}{2}}{\Gamma(\beta/2)} \int_0^{+\infty} t^{\frac{\beta}{2}-1} h(t,x)\, \mathrm{d}t$$

defines a smooth function in $\mathbb{H} \setminus \{0\}$, since $t \mapsto h(t,x)$ vanishes of infinite order as $t \to 0$ if $x \neq 0$. In addition, $\widetilde{R}_\beta$ is positive and $\mathbb{H}$-homogeneous of degree $\beta - 4$.

We also set

$$\rho(x) = R_{2-\alpha}^{-\frac{1}{2+\alpha}}(x), \quad 0 < \alpha < 2. \tag{3.21}$$

ρ is an $\mathbb{H}$-homogeneous norm in $\mathbb{H}$, smooth outside of the origin. In addition, $d(x,y) := \rho(y^{-1} \circ x)$ is a quasi-distance in $\mathbb{H}$. In turn, d is equivalent to the Carnot-Carathéodoty distance on $\mathbb{H}$, as well as to any other $\mathbb{H}$-homogeneous left invariant distance on $\mathbb{H}$.

Recall that, as usual, $\mathcal{S}$ denotes the Schwartz space of rapidly decreasing C^∞ functions. We have the following representation formula:

Theorem 3.12 ([40], Theorem 3.11)**.** *For every $v \in \mathcal{S}(\mathbb{H})$, $(-\Delta_{\mathbb{H}})^s v \in L^2(\mathbb{H})$ and*

$$\begin{aligned}(-\Delta_{\mathbb{H}})^s v(x) &= \int_{\mathbb{H}} (v(x \circ y) - v(x) - \omega(y)\langle \nabla_{\mathbb{H}} v(x), y\rangle)\, \widetilde{R}_{-2s}(y)\, \mathrm{d}y \\ &= \mathrm{P.V.} \int_{\mathbb{H}} (v(y) - v(x)) \widetilde{R}_{-2s}(y^{-1} \circ x)\, \mathrm{d}y,\end{aligned}$$

where ω is the characteristic function of the unit ball $B_\rho(0,1)$.

3.2.3 A Poisson Kernel

With a natural notion of group convolution, the Heisenberg group makes possible to recover, starting from the abstract representation in terms of spectral resolution, another explicit form of the fractional power in terms of the convolution with suitable Poisson kernel (see [40, Theorem 4.4]).

If $v \in L^2(\mathbb{H})$ and $y > 0$ (recall that $-1 < a < 1$), we set

$$u(\cdot, y) := \phi(\theta y^{1-a}(-\Delta_{\mathbb{H}})^{(1-a)/2})v := \int_0^{+\infty} \phi(\theta y^{1-a}\lambda^{(1-a)/2})\, \mathrm{d}E(\lambda)v,$$

where $\theta := (1-a)^{a-1}$ and $\phi : [0,\infty) \to \mathbb{R}$ solves the boundary value problem

$$\begin{cases} -t^\alpha \phi'' + \phi = 0, \\ \phi(0) = 1, \\ \lim_{t\to+\infty} \phi(t) = 0, \end{cases}$$

($\alpha = -\frac{2a}{1-a}$).

We denote by $h(t,\cdot)$ the heat kernel associated with $-\Delta_{\mathbb{H}}$ as in [42, Proposition 3.3], and by $P_{\mathbb{H}}(\cdot, y)$ the "Poisson kernel"

$$P_{\mathbb{H}}(\cdot, y) := C_a y^{1-a} \int_0^\infty t^{(a-3)/2} \mathrm{e}^{-\frac{y^2}{4t}} h(t,\cdot)\, \mathrm{d}t, \tag{3.22}$$

where

$$C_a = \frac{2^{a-1}}{\Gamma((1-a)/2)}.$$

Then

$$P_{\mathbb{H}}(\cdot, y) \geq 0$$

and

$$u(\cdot, y) = v * P_{\mathbb{H}}(\cdot, y). \tag{3.23}$$

Neumann condition: With this representation at hand, we can show a relation between the extension u and the fractional sub-Laplacian of v; this relation is an analogous version of that for the Euclidean setting in (3.2)–(3.3). Indeed,

$$\begin{aligned} y^a \frac{u(x,y) - u(x,0)}{y} &= y^a \frac{v * P_{\mathbb{H}}(\cdot, y) - v(x)}{y} \\ &= C_a \int_0^\infty t^{(a-3)/2} \mathrm{e}^{-\frac{y^2}{4t}} v * h(t,\cdot)\, \mathrm{d}t \\ &\quad - C_a v(x) \int_{\mathbb{H}} \int_0^\infty t^{(a-3)/2} \mathrm{e}^{-\frac{y^2}{4t}} h(t, \xi^{-1} \circ x)\, \mathrm{d}t \mathrm{d}\xi \\ &= C_a \int_{\mathbb{H}} \int_0^\infty t^{(a-3)/2} \mathrm{e}^{-\frac{y^2}{4t}} h(t, \xi^{-1} \circ x)(v(\xi) - v(x))\, \mathrm{d}t \mathrm{d}\xi. \end{aligned}$$

On the other hand

$$\lim_{y\to 0^+} C_a \int_0^\infty t^{(a-3)/2} \mathrm{e}^{-\frac{y^2}{4t}} h(t, \xi^{-1} \circ x)\, \mathrm{d}t = \widetilde{C}_a \widetilde{R}_{a-1}.$$

Thus

$$\lim_{y\to 0^+} y^a \frac{u(x,y)-u(x,0)}{y} = \widetilde{C}_a \int_{\mathbb{H}} (v(\xi)-v(x))\widetilde{R}_{a-1}(\xi)\,\mathrm{d}\xi \\ = \widetilde{C}_a(-\Delta_{\mathbb{H}})^{\frac{1-a}{2}} v(x). \tag{3.24}$$

Remark 3.13. We note that $v \in C^2(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ is a stable solution of (3.1) if and only if its lifting $u(\cdot,y) = v * P_{\mathbb{H}}(\cdot,y)$ is a stable solution of (3.4).

3.2.4 Regularity theory for (3.1) and (3.4)

In the following, we prove several regularity properties for solutions of (3.1) and (3.4). Some classical pointwise estimates: the Harnack inequality and the Hölder continuity of the weak solutions (De Giorgi-Nash-Moser theorem), can be extended to a class of strongly degenerate elliptic operators of the second order, like that in (3.4).

The following result, that is the counterpart in the sub-elliptic framework of the Euclidean setting, can be found in [54]; this idea goes back to [44, 45]. Basically, this is possible thanks to weighted Sobolev-Poincaré inequalities in Carnot Groups.

Before stating the regularity result, we define the notion of A_2-weight:

Definition 3.14. Let $\mathbb{G}$ be a Carnot group (see [8, Chapter 1] for the precise definition). A function $\omega \in L^1_{\mathrm{loc}}(\mathbb{G})$ is said to be an A_2-weight with respect to the cc-metric of $\mathbb{G}$ if

$$\sup_{x\in\mathbb{G}, r>0} \fint_{B_c(x,r)} \omega(y)\,\mathrm{d}y \cdot \fint_{B_c(x,r)} \omega(y)^{-1}\,\mathrm{d}y < \infty.$$

In particular, $\mathbb{H}\times\mathbb{R}$ can be naturally provided with a structure of Carnot group (see [40, Section 2]) in which the function $\omega(x,y) = y^a$ is an A_2-weight with respect to the cc-metric of $\mathbb{H}\times\mathbb{R}$ if and only if $-1 < a < 1$.

Theorem 3.15. *Let $\mathbb{G}$ be a Carnot group and $\Omega \subset \mathbb{G}$ be an open set. Let now $\omega \in L^1_{\mathrm{loc}}(\mathbb{G})$ be an A_2-weight with respect to the Carnot-Carathéodory metric d_c of $\mathbb{G}$. Then, if $u \in W^{1,2}_{\mathbb{G}}(\Omega, \omega \mathrm{d}x)$ is a weak solution of*

$$\mathrm{div}_{\mathbb{G}}(\omega \nabla_{\mathbb{G}} u) = 0, \tag{3.25}$$

u is locally Hölder continuous in Ω. If, in addition, $u \geq 0$, then there exist $C, b > 0$ (independent of u) such that the following invariant Harnack inequality holds:

$$\sup_{B_c(x,r)} u \leq C \inf_{B_c(x,r)} u$$

for any metric ball $B_c(x,r)$ such that $B_c(x,br) \subset \Omega$.

Suppose now Ω satisfies the following local condition: for any $x_0 \in \partial\Omega$ there exists $r_0 > 0$ and $\alpha > 0$ such that

$$|B_c(x_0,r) \cap \Omega^c| \geq \alpha |B_c(x_0,r)| \quad \text{for } r < r_0.$$

Then u is locally Hölder continuous in $\overline{\Omega}$.

To end the section, we state a result which let us control further derivatives in x. Basically, this is possible thanks to the fact that the operator is independent of the variable $x \in \mathbb{H}$.

Lemma 3.16. *Let u be a bounded weak solution of (3.4). Then,*

$$y^a|\nabla_{\widehat{\mathbb{H}}}Xu|^2, y^a|\nabla_{\widehat{\mathbb{H}}}Yu|^2 \in L^1(B_R^+)$$

for every $R > 0$.

Proof. Given $R > 0$, let us prove that $y^a|\nabla_{\widehat{\mathbb{H}}}Xu|^2 \in L^1(B_R^+)$.

We consider the incremental quotient

$$u_h(x,y) = \frac{u(x \circ (he_1), y) - u(x,y)}{h} \quad \text{for all } (x,y) \in \widehat{\mathbb{H}},$$

where $e_1 = (1,0,0)$. Recall that (see Proposition 1.2.11 in [8])

$$\lim_{h\to 0} u_h(x,y) = Xu(x,y) \quad \text{for all } (x,y) \in \widehat{\mathbb{H}}. \tag{3.26}$$

Thanks to (3.10) and the smoothness of f, we have

$$[f(u)]_h \le C \tag{3.27}$$

for some $C > 0$.

Let now ξ be as requested in (3.6). We have

$$\begin{aligned}\int_{\widehat{\mathbb{H}}} y^a\langle\nabla_{\widehat{\mathbb{H}}}u_h, \nabla_{\widehat{\mathbb{H}}}\xi\rangle_{\widehat{\mathbb{H}}} - \int_{\mathbb{H}}[f(u)]_h\xi &= -\int_{\widehat{\mathbb{H}}} y^a\langle\nabla_{\widehat{\mathbb{H}}}u, \nabla_{\widehat{\mathbb{H}}}\xi_{-h}\rangle_{\widehat{\mathbb{H}}} + \int_{\mathbb{H}} f(u)\xi_{-h} \\ &= 0.\end{aligned}$$

We now consider a smooth cut-off function τ such that $0 \le \tau \in C_0^\infty(B_{R+1})$, with $\tau = 1$ in B_R and $\nabla\tau \le 2$. Taking $\xi := u_h\tau^2$ in the above expression, we find that

$$2\int_{\widehat{\mathbb{H}}} y^a\tau u_h\langle\nabla_{\widehat{\mathbb{H}}}u_h, \nabla_{\widehat{\mathbb{H}}}\tau\rangle_{\widehat{\mathbb{H}}} + \int_{\widehat{\mathbb{H}}} y^a\tau^2|\nabla_{\widehat{\mathbb{H}}}u_h|^2 = \int_{\mathbb{H}}[f(u)]_h u_h\tau^2. \tag{3.28}$$

Note that ξ satisfies (3.7) thanks to (3.5) and $u \in L^\infty_{\mathrm{loc}}(\overline{\mathbb{R}^4_+})$.

Now, by Cauchy-Schwarz inequality, we have

$$\begin{aligned}\int_{\widehat{\mathbb{H}}} y^a\tau u_h\langle\nabla_{\widehat{\mathbb{H}}}u_h, \nabla_{\widehat{\mathbb{H}}}\tau\rangle_{\widehat{\mathbb{H}}} \ge &-\frac{\varepsilon}{2}\int_{\widehat{\mathbb{H}}} y^a\tau^2|\nabla_{\widehat{\mathbb{H}}}u_h|^2 \\ &-\frac{1}{2\varepsilon}\int_{\widehat{\mathbb{H}}} y^a u_h^2|\nabla_{\widehat{\mathbb{H}}}\tau|^2\end{aligned}$$

for any $\varepsilon > 0$. Choosing ε small, (3.28) reads

$$\int_{\widehat{\mathbb{H}}} y^a\tau^2|\nabla_{\widehat{\mathbb{H}}}u_h|^2 \le C\left(\int_{B^+_{R+1}} y^a u_h^2 + \int_{\{|x|\le R\}\times\{y=0\}} |[f(u)]_h u_h|\right) \tag{3.29}$$

for some $C > 0$. This inequality, together (3.10) and (3.27), allows to control

$$\int_{\widehat{\mathbb{H}}} y^a\tau^2|\nabla_{\widehat{\mathbb{H}}}u_h|^2$$

uniformly in h.

By sending $h \to 0$ and using Fatou lemma (recall also (3.26)), we obtain the desired claim. ∎

3.3 Analytic and geometric inequalities

In this section we develop the analytical and geometrical tools toward (3.11), we follow the ideas in [41, 68]. We summarize the main points of the argument and omit some technical computations.

3.3.1 Analytical computations

We start with two lemmas, the fist one is a version in the Heisenberg group of a classical result (see [41]):

Lemma 3.17. *Let $c \in \mathbb{R}$. Suppose that Ω is an open domain of $\widehat{\mathbb{H}}$ and that $w : \Omega \to \mathbb{R}$ is Lipschitz with respect to the metric structure of $\widehat{\mathbb{H}}$. Then, $\nabla_{\widehat{\mathbb{H}}} w = 0$ for almost any $x \in \{w = c\}$.*

And the second one, an elementary observation.

Lemma 3.18. *Let u be as in Theorem 3.2. Assume that $\xi \in C^\infty(\mathbb{R}^4_+, \mathbb{R})$ and vanishes outside a ball. Then*

$$\int_{\widehat{\mathbb{H}}} y^a \langle \nabla_{\widehat{\mathbb{H}}} u, \nabla_{\widehat{\mathbb{H}}} X\xi \rangle_{\widehat{\mathbb{H}}} = \int_{\widehat{\mathbb{H}}} y^a (-\langle \nabla_{\widehat{\mathbb{H}}} Xu, \nabla_{\widehat{\mathbb{H}}} \xi \rangle_{\widehat{\mathbb{H}}} + 2TYu\xi) \tag{3.30}$$

and

$$\int_{\widehat{\mathbb{H}}} y^a \langle \nabla_{\widehat{\mathbb{H}}} u, \nabla_{\widehat{\mathbb{H}}} Y\xi \rangle_{\widehat{\mathbb{H}}} = \int_{\widehat{\mathbb{H}}} y^a (-\langle \nabla_{\widehat{\mathbb{H}}} Yu, \nabla_{\widehat{\mathbb{H}}} \xi \rangle_{\widehat{\mathbb{H}}} - 2TXu\xi). \tag{3.31}$$

Proof. Using integration by parts we deduce that

$$\begin{aligned}
\int_{\widehat{\mathbb{H}}} y^a \langle \nabla_{\widehat{\mathbb{H}}} u, \nabla_{\widehat{\mathbb{H}}} X\xi \rangle_{\widehat{\mathbb{H}}} &= \int_{\widehat{\mathbb{H}}} y^a (XuXX\xi + YuYX\xi + \partial_y u \partial_y (X\xi)) \\
&= \int_{\widehat{\mathbb{H}}} y^a (-XXuX\xi + YuYX\xi - \partial_y Xu \partial_y \xi) \\
&= \int_{\widehat{\mathbb{H}}} y^a (-\langle \nabla_{\widehat{\mathbb{H}}} Xu, \nabla_{\widehat{\mathbb{H}}} \xi \rangle_{\widehat{\mathbb{H}}} + YXuY\xi + YuYX\xi) \\
&= \int_{\widehat{\mathbb{H}}} y^a (-\langle \nabla_{\widehat{\mathbb{H}}} Xu, \nabla_{\widehat{\mathbb{H}}} \xi \rangle_{\widehat{\mathbb{H}}} + YXuY\xi - XYuY\xi + YuYX\xi - YuXY\xi) \\
&= \int_{\widehat{\mathbb{H}}} y^a (-\langle \nabla_{\widehat{\mathbb{H}}} Xu, \nabla_{\widehat{\mathbb{H}}} \xi \rangle_{\widehat{\mathbb{H}}} - TuY\xi - YuT\xi) \\
&= \int_{\widehat{\mathbb{H}}} y^a (-\langle \nabla_{\widehat{\mathbb{H}}} Xu, \nabla_{\widehat{\mathbb{H}}} \xi \rangle_{\widehat{\mathbb{H}}} + 2TYu\xi)
\end{aligned}$$

(recall that $TX = XT$ and $TY = YT$). The proof of (3.31) is similar. □

Next result gives the first part of the inequality (3.11). The proof is inspired by some computations in [38, 39, 68, 69, 70].

Theorem 3.19. *Under the hypothesis of Theorem 3.2, we have*

$$\int_{\widehat{\mathbb{H}}} y^a |\nabla_{\mathbb{H}} u|^2 |\nabla_{\widehat{\mathbb{H}}} \phi|^2 \geq \int_{\mathcal{R}^4_+} y^a \left(|Hu|^2 - \langle (Hu)^2 \nu_{x,y}, \nu_{x,y} \rangle_{\mathbb{H}} - 2(TYuXu - TXuYu) \right) \phi^2. \tag{3.32}$$

Proof. Let us consider $\xi = |\nabla_{\mathbb{H}} u|\phi$ as a test function in (3.8). Thanks to (3.10) and Lemma 3.16 (see also [68, Lemma 7]), it is possible to use here such a test function. We deduce that

$$\int_{\widehat{\mathbb{H}}} y^a(|\nabla_{\mathbb{H}}(|\nabla_{\mathbb{H}} u|\phi)|^2 + |\partial_y(|\nabla_{\mathbb{H}} u|\phi)|^2) - \int_{\mathbb{H}} f'(u)|\nabla_{\mathbb{H}} u|^2\phi^2 \geq 0. \tag{3.33}$$

The first term can be computed in the same way as in [41, Theorem 1.3]. We find that, in $\mathcal{R}^4_+$,

$$|\nabla_{\mathbb{H}}(|\nabla_{\mathbb{H}} u|\phi)|^2 = |\nabla_{\mathbb{H}} u|^2|\nabla_{\mathbb{H}}\phi|^2 + \phi^2\langle (Hu)^2\nu_{x,y}, \nu_{x,y}\rangle_{\mathbb{H}} + 2\langle Hu\nabla_{\mathbb{H}}\phi, \nabla_{\mathbb{H}} u\rangle_{\mathbb{H}}\phi. \tag{3.34}$$

By exploiting Lemma 3.17 with $w = |\nabla_{\mathbb{H}} u|$, we obtain that $\nabla_{\widehat{\mathbb{H}}}(|\nabla_{\mathbb{H}} u|\phi) = 0$ almost everywhere outside $\mathcal{R}^4_+$. Analogously, using Lemma 3.17 with $w = Xu$ or $w = Yu$, we conclude that $\nabla_{\mathbb{H}} Xu = \nabla_{\mathbb{H}} Yu = 0$ almost everywhere outside $\mathcal{R}^4_+$. Thus, (3.33) is equivalent to

$$\begin{aligned} 0 \leq &\int_{\mathcal{R}^4_+} y^a(|\nabla_{\mathbb{H}} u|^2|\nabla_{\mathbb{H}}\phi|^2 + \phi^2\langle (Hu)^2\nu_{x,y}, \nu_{x,y}\rangle_{\mathbb{H}} + 2\langle Hu\nabla_{\mathbb{H}}\phi, \nabla_{\mathbb{H}} u\rangle_{\mathbb{H}}\phi) \\ &+ \int_{\mathcal{R}^4_+} y^a|\partial_y(|\nabla_{\mathbb{H}} u|\phi)|^2 - \int_{\mathbb{H}} f'(u)|\nabla_{\mathbb{H}} u|^2\phi^2. \end{aligned} \tag{3.35}$$

Let us now compute the last term. First, note that

$$\begin{aligned} \int_{\mathbb{H}} f'(u)(Xu)^2\phi^2 &= \int_{\mathbb{H}} X(f(u))(Xu\phi^2) \\ &= -\int_{\mathbb{H}} f(u)X(Xu\phi^2). \end{aligned} \tag{3.36}$$

Let ξ be as in the previous lemma. By the weak solution notion (3.6) and the previous lemma, we deduce that

$$-\int_{\mathbb{H}} f(u)X\xi = \int_{\widehat{\mathbb{H}}} y^a\left[\langle \nabla_{\widehat{\mathbb{H}}} Xu, \nabla_{\widehat{\mathbb{H}}}\xi\rangle_{\widehat{\mathbb{H}}} - 2TYu\xi\right]. \tag{3.37}$$

A density argument (see Lemma 3.4 and Theorem 2.4 in [26]), implies that (3.37) holds for $\xi = -Xu\phi^2$, where ϕ is as in statement of Theorem 3.2. Therefore

$$-\int_{\mathbb{H}} f(u)X(Xu\phi^2) = \int_{\widehat{\mathbb{H}}} y^a\left[\langle \nabla_{\widehat{\mathbb{H}}} Xu, \nabla_{\widehat{\mathbb{H}}}(Xu\phi^2)\rangle_{\widehat{\mathbb{H}}} - 2TYuXu\phi^2\right]. \tag{3.38}$$

Similarly, we have

$$-\int_{\mathbb{H}} f(u)Y(Yu\phi^2) = \int_{\widehat{\mathbb{H}}} y^a\left[\langle \nabla_{\widehat{\mathbb{H}}} Yu, \nabla_{\widehat{\mathbb{H}}}(Yu\phi^2)\rangle_{\widehat{\mathbb{H}}} + 2TXuYu\phi^2)\right]. \tag{3.39}$$

Then, by (3.36) and then summing term by term in (3.38) and (3.39), we see that

$$\begin{aligned} \int_{\mathbb{H}} f'(u)|\nabla_{\mathbb{H}} u|^2\phi^2 &= \int_{\mathbb{H}} f'(u)\left[(Xu)^2 + (Yu)^2\right]\phi^2 \\ &= \int_{\widehat{\mathbb{H}}} y^a(|\nabla_{\mathbb{H}} Xu|^2 + |\nabla_{\mathbb{H}} Yu|^2)\phi^2 \\ &\quad + \int_{\widehat{\mathbb{H}}} y^a(\langle \nabla_{\mathbb{H}} Xu, \nabla_{\mathbb{H}}(\phi^2)\rangle_{\mathbb{H}} Xu + \langle \nabla_{\mathbb{H}} Yu, \nabla_{\mathbb{H}}(\phi^2)\rangle_{\mathbb{H}} Yu) \\ &\quad + 2\int_{\widehat{\mathbb{H}}} y^a(TXuYu - TYuXu)\phi^2 \\ &\quad + \int_{\widehat{\mathbb{H}}} y^a\left[\partial_y Xu\partial_y(Xu\phi^2) + \partial_y Yu\partial_y(Yu\phi^2)\right]. \end{aligned}$$

Putting this in (3.35) we deduce, after a rearrangement, that

$$0 \le \int_{\mathcal{R}^4_+} y^a |\nabla_{\mathbb{H}} u|^2 |\nabla_{\mathbb{H}} \phi|^2 \tag{3.40}$$
$$+ \int_{\mathcal{R}^4_+} y^a \Big[-|\nabla_{\mathbb{H}} X u|^2 - |\nabla_{\mathbb{H}} Y u|^2 + \langle (Hu)^2 \nu_{x,y}, \nu_{x,y} \rangle_{\mathbb{H}}$$
$$+2(TYuXu - TXuYu) \big] \phi^2$$
$$+ \int_{\mathcal{R}^4_+} y^a |\partial_y (|\nabla_{\mathbb{H}} u| \phi)|^2 - \int_{\widehat{\mathbb{H}}} y^a \Big[\partial_y X u \partial_y (X u \phi^2) + \partial_y Y u \partial_y (Y u \phi^2) \Big];$$

here we used the fact that:

$$2\langle Hu \nabla_{\mathbb{H}} \phi, \nabla_{\mathbb{H}} u \rangle_{\mathbb{H}} \phi - \langle \nabla_{\mathbb{H}} X u, \nabla_{\mathbb{H}} (\phi^2) \rangle_{\mathbb{H}} X u - \langle \nabla_{\mathbb{H}} Y u, \nabla_{\mathbb{H}} (\phi^2) \rangle_{\mathbb{H}} Y u = 0.$$

Finally, developing some calculations for the last terms in (3.40), we conclude that

$$\int_{\mathcal{R}^4_+} y^a |\partial_y (|\nabla_{\mathbb{H}} u| \phi)|^2 - \int_{\widehat{\mathbb{H}}} y^a \Big[\partial_y X u \partial_y (X u \phi^2) + \partial_y Y u \partial_y (Y u \phi^2) \Big]$$
$$= \int_{\widehat{\mathbb{H}}} y^a |\nabla_{\mathbb{H}} u|^2 (\partial_y \phi)^2 + \int_{\mathcal{R}^4_+} y^a [(\partial_y |\nabla_{\mathbb{H}} u| \phi)^2 + 2 |\nabla_{\mathbb{H}} u| \partial_y |\nabla_{\mathbb{H}} u| \phi \partial_y \phi$$
$$- |\partial_y \nabla_{\mathbb{H}} u|^2 \phi^2 - \frac{1}{2} \partial_y |\nabla_{\mathbb{H}} u|^2 \partial_y (\phi^2)]$$
$$= \int_{\widehat{\mathbb{H}}} y^a |\nabla_{\mathbb{H}} u|^2 (\partial_y \phi)^2 + \int_{\mathcal{R}^4_+} y^a [(\partial_y |\nabla_{\mathbb{H}} u|)^2 - |\partial_y \nabla_{\mathbb{H}} u|^2] \phi^2$$
$$\le \int_{\widehat{\mathbb{H}}} y^a |\nabla_{\mathbb{H}} u|^2 (\partial_y \phi)^2.$$

For the last inequality, note that, on $\mathcal{R}^4_+$,

$$(\partial_y |\nabla_{\mathbb{H}} u|)^2 = \left| \frac{\nabla_{\mathbb{H}} u \cdot \nabla_{\mathbb{H}} \partial_y u}{|\nabla_{\mathbb{H}} u|} \right|^2 \le |\partial_y \nabla_{\mathbb{H}} u|^2.$$

This and (3.40) complete the proof. □

3.3.2 Geometrical computations

To obtain the second part of (3.11), it is necessary a geometric analysis of the level sets of u at non-degenerate points P where $\{\nabla_{\mathbb{H}} u \neq 0\}$ (recall the smooth manifold L, defined in (3.9)). We omit the details and instead refer the reader to [2, 3] and [41, Section 2].

Lemma 3.20. *On the smooth manifold L we have*

$$|Hu|^2 - \langle (Hu)^2 \nu_{x,y}, \nu_{x,y} \rangle_{\mathbb{H}} = |\nabla_{\mathbb{H}} u|^2 \left[h^2_{x,y} + \left(p_{x,y} + \frac{\langle Hu v_{x,y}, \nu_{x,y} \rangle_{\mathbb{H}}}{|\nabla_{\mathbb{H}} u|} \right)^2 \right] \tag{3.41}$$

and

$$TYuXu - TXuYu = -|\nabla_{\mathbb{H}} u|^2 \langle T \nu_{x,y}, v_{x,y} \rangle_{\mathbb{H}}. \tag{3.42}$$

Proof of Theorem 3.2. Finally, the form of the geometric inequality given in (3.11) is a consequence of Theorem 3.19 and the previous lemma. □

3.4 Applications to entire stable solutions

3.4.1 Proof of Theorem 3.4

The strategy for proving Theorem 3.4 is to test the geometric formula of Theorem 3.2 against an appropriate capacity-type function to make the left-hand side vanish. This would give that the curvature of the level sets for fixed $y > 0$ vanishes.

For this, given $x = (x_1, x_2, x_3) \in \mathbb{H}$ we define its gauge norm as

$$|x|_{\mathbb{H}} = \Big((x_1^2 + x_2^2)^2 + x_3^2\Big)^{1/4}. \tag{3.43}$$

We also use the notation $Z := (x, y)$ for points in $\widehat{\mathbb{H}}$ and define the norm

$$|Z|_{\widehat{\mathbb{H}}} := \Big(|x|_{\mathbb{H}}^2 + y^2\Big)^{1/2} \tag{3.44}$$

(recall that $\widehat{\mathbb{H}} = \mathbb{H} \times \mathbb{R}_+$). Analogously, we denote the ball centered at 0 of radius R by

$$B(0, R) = \{Z \in \widehat{\mathbb{H}} \text{ s.t. } |Z|_{\widehat{\mathbb{H}}} < R\}.$$

and, given $r_1 \le r_2$, the semi-annulus by

$$\mathcal{A}_{r_1,r_2} := \{Z \in \mathbb{R}^4_+ \text{ s.t. } |Z|_{\widehat{\mathbb{H}}} \in [r_1, r_2]\}.$$

Lemma 3.21. *Let $g \in L^\infty_{\mathrm{loc}}(\mathbb{R}^4_+, [0, +\infty))$ and let $q > 0$. Let also, for any $\tau > 0$,*

$$\eta(\tau) = \int_{B(0,\tau)} g(Z)\,\mathrm{d}Z. \tag{3.45}$$

Then, for every $0 < r < R$,

$$\int_{\mathcal{A}_{r,R}} \frac{g(Z)}{|Z|^q_{\widehat{\mathbb{H}}}}\,\mathrm{d}Z \le q\int_r^R \frac{\eta(\tau)}{\tau^{q+1}}\,\mathrm{d}\tau + \frac{\eta(R)}{R^q}.$$

Proof. By changing order of integration,

$$\begin{aligned}
&\int_{\mathcal{A}_{r,R}} \frac{g(Z)}{|Z|^q_{\widehat{\mathbb{H}}}}\,\mathrm{d}Z \\
&\qquad = q\int_{\mathcal{A}_{r,R}} \left(\int_{|Z|_{\widehat{\mathbb{H}}}}^R \frac{g(Z)}{\tau^{q+1}}\,\mathrm{d}\tau\right)\mathrm{d}Z + \frac{1}{R^q}\int_{\mathcal{A}_{r,R}} g(Z)\,\mathrm{d}Z \\
&\qquad \le q\int_r^R \left(\int_{B(0,\tau)} \frac{g(Z)}{\tau^{q+1}}\,\mathrm{d}Z\right)\mathrm{d}\tau + \frac{\eta(R)}{R^q} \\
&\qquad \le q\int_r^R \frac{\eta(\tau)}{\tau^{q+1}}\,\mathrm{d}\tau + \frac{\eta(R)}{R^q}.
\end{aligned}$$

□

Proof of Theorem 3.4. Given $Z = (x, y) \in \widehat{\mathbb{H}}$, let us consider the function

$$g(Z) = 4y^a |\nabla_{\mathbb{H}} u(Z)|^2 (x_1^2 + x_2^2 + y^2)$$

(recall that $x = (x_1, x_2, x_3)$). Then, the function η defined in (3.12) is consistent with the notation in (3.45). Moreover, by (3.14) and the previous lemma,

$$\liminf_{R\to+\infty} \frac{1}{(\log R)^2} \int_{\mathcal{A}_{\sqrt{R},R}} \frac{g(Z)}{|Z|_{\widehat{\mathbb{H}}}^4} \, dZ = 0. \tag{3.46}$$

Now, we define for all $R > 1$ the test function

$$\phi_R(Z) = \begin{cases} 1 & \text{if } |Z|_{\widehat{\mathbb{H}}} \le \sqrt{R}, \\ \dfrac{2\log\left(\frac{R}{|Z|_{\widehat{\mathbb{H}}}}\right)}{\log R} & \text{if } \sqrt{R} < |Z|_{\widehat{\mathbb{H}}} < R, \\ 0 & \text{if } |Z|_{\widehat{\mathbb{H}}} \ge R, \end{cases}$$

and we observe that

$$\partial \phi_R = -\frac{2}{\log R} |Z|_{\widehat{\mathbb{H}}}^{-1} \partial(|Z|_{\widehat{\mathbb{H}}}),$$

where ∂ can be any of the operators X, Y or ∂_y. It is straightforward to verify that

$$\begin{aligned} X(|Z|_{\widehat{\mathbb{H}}}) &= |Z|_{\widehat{\mathbb{H}}}^{-1} |x|_{\mathbb{H}}^{-2} [x_1(x_1^2 + x_2^2) + x_2 x_3], \\ Y(|Z|_{\widehat{\mathbb{H}}}) &= |Z|_{\widehat{\mathbb{H}}}^{-1} |x|_{\mathbb{H}}^{-2} [x_2(x_1^2 + x_2^2) - x_1 x_3], \\ \partial_y(|Z|_{\widehat{\mathbb{H}}}) &= |Z|_{\widehat{\mathbb{H}}}^{-1} y. \end{aligned}$$

Therefore, for $Z \in \mathcal{A}_{\sqrt{R},R}$,

$$\begin{aligned} |\nabla_{\widehat{\mathbb{H}}} \phi_R(Z)|^2 &= (X\phi_R)^2 + (Y\phi_R)^2 + (\partial_y \phi_R)^2 \\ &= \frac{4}{(\log R)^2} |Z|_{\widehat{\mathbb{H}}}^{-2} [X(|Z|_{\widehat{\mathbb{H}}})^2 + Y(|Z|_{\widehat{\mathbb{H}}})^2 + \partial_y(|Z|_{\widehat{\mathbb{H}}})^2] \\ &= \frac{4}{(\log R)^2} |Z|_{\widehat{\mathbb{H}}}^{-4} (x_1^2 + x_2^2 + y^2). \end{aligned}$$

Thus, plugging ϕ_R inside the geometric inequality of Theorem 3.2, we deduce that

$$\begin{aligned} \int_{B(0,\sqrt{R})^+ \cap \mathcal{R}_+^4} y^a |\nabla_{\mathbb{H}} u|^2 \left[h_{x,y}^2 + \left(p_{x,y} + \frac{\langle Huv_{x,y}, \nu_{x,y}\rangle_{\mathbb{H}}}{|\nabla_{\mathbb{H}} u|} \right)^2 + 2\langle T\nu_{x,y}, v_{x,y}\rangle_{\mathbb{H}} \right] \\ = \frac{1}{(\log R)^2} \int_{\mathcal{A}_{\sqrt{R},R}} \frac{g(Z)}{|Z|_{\widehat{\mathbb{H}}}^4} \, dZ, \end{aligned}$$

for all $R > 1$.

Theorem 3.4 follows from the previous identity together (3.13) and (3.46). □

3.4.2 Proof of Theorem 3.7

Before proving Theorem 3.7, let us state the following regularity result for the extension with the Poisson kernel (see (3.23)). Recall that $C^{\alpha}(\mathbb{H})$ denotes the set of Hölder continuous functions with respect to the norm ρ defined in (3.21).

Lemma 3.22. *Let $v \in C^{2,\sigma}(\mathbb{H}) \cap L^{\infty}(\mathbb{H})$, $\sigma \in (0, 2s)$. Then the function $u(\cdot, y) = v * P_{\mathbb{H}}(\cdot, y)$, defined in (3.23), satisfies*

$$u \in C^{0,\sigma}(\overline{\widehat{\mathbb{H}}}).$$

Proof. Fix $\sigma \in (0, 1]$ and let $v \in C^{2,\sigma}(\mathbb{H}) \cap L^{\infty}(\mathbb{H})$. Recall the following homogeneity property of $h = h(t, x)$ (the fundamental solution of $\Delta_{\mathbb{H}} + \partial/\partial t$):

$$h(r^2 t, \delta_r(x)) = r^{-4} h(t, x) \quad \text{for all } (t, x) \in (0, +\infty) \times \mathbb{H} \tag{3.47}$$

(see (3.2) in [42]), where δ_r is the family of dilatations defined in (3.18).

We have, for all $\xi \in \mathbb{H}$ and $y > 0$,

$$\begin{aligned} P_{\mathbb{H}}(\delta_y(\xi), y) &= C_a y^{1-a} \int_0^{\infty} t^{(a-3)/2} \mathrm{e}^{-\frac{y^2}{4t}} h(t, \delta_y(\xi)) \, \mathrm{d}t \\ &= C_a \int_0^{\infty} t^{(a-3)/2} \mathrm{e}^{-\frac{1}{4t}} h(y^2 t, \delta_y(\xi)) \, \mathrm{d}t \\ &= C_a y^{-4} \int_0^{\infty} t^{(a-3)/2} \mathrm{e}^{-\frac{1}{4t}} h(t, \xi) \, \mathrm{d}t \\ &= C_a y^{-4} P_{\mathbb{H}}(\xi, 1). \end{aligned}$$

Then, given $(x, y) \in \widehat{\mathbb{H}}$, we have

$$\begin{aligned} u(x, y) &= \int_{\mathbb{H}} v(x \circ \xi^{-1}) P_{\mathbb{H}}(\xi, y) \, \mathrm{d}\xi \\ &= y^4 \int_{\mathbb{H}} v(x \circ \delta_y(\xi)^{-1}) P_{\mathbb{H}}(\delta_y(\xi), y) \, \mathrm{d}\xi \\ &= C_a \int_{\mathbb{H}} v(x \circ \delta_y(\xi)^{-1}) P_{\mathbb{H}}(\xi, 1) \, \mathrm{d}\xi. \end{aligned}$$

Therefore, for $(x^{(1)}, y_1), (x^{(2)}, y_2) \in \widehat{\mathbb{H}}$

$$\begin{aligned} &|u(x^{(1)}, y_1) - u(x^{(2)}, y_2)| \\ &\qquad \le C_a \int_{\mathbb{H}} |v(x^{(1)} \circ \delta_{y_1}(\xi)^{-1}) - v(x^{(2)} \circ \delta_{y_2}(\xi)^{-1})| P_{\mathbb{H}}(\xi, 1) \, \mathrm{d}\xi \\ &\qquad \le C \int_{\mathbb{H}} d(x^{(1)} \circ \delta_{y_1}(\xi)^{-1}, x^{(2)} \circ \delta_{y_2}(\xi)^{-1})^{\sigma} P_{\mathbb{H}}(\xi, 1) \, \mathrm{d}\xi, \end{aligned}$$

where d is the homogeneous distance associated to the homogeneous norm in (3.21).

Using the properties of homogeneous norms in Carnot groups (see [8, Section 5.1]), we deduce that

$$d(x^{(1)} \circ \delta_{y_1}(\xi)^{-1}, x^{(2)} \circ \delta_{y_2}(\xi)^{-1}) \le C[d(x^{(1)}, x^{(2)}) + |y_1 - y_2| \rho(\xi)].$$

Putting this in the previous inequality, we find that

$$|u(x^{(1)},y_1)-u(x^{(2)},y_2)| \le C\left[d(x^{(1)},x^{(2)})^\sigma + |y_1-y_2|^\sigma \int_{\mathbb{H}} \rho(\xi)^\sigma P_{\mathbb{H}}(\xi,1)\,\mathrm{d}\xi\right]. \tag{3.48}$$

The integral in this expression is finite because the function $P_{\mathbb{H}}(\xi,1):\mathbb{H}\to\mathbb{R}$ is bounded around the origin, $\sigma\in(0,2s)$ and

$$|P_{\mathbb{H}}(\xi,1)| \le C\rho(y)^{-2s-4}$$

for large ρ (see Remark 4.5 in [40] and (1.73) in [43]). We conclude that $u\in C^{0,\sigma}(\overline{\widehat{\mathbb{H}}})$. □

Proof of Theorem 3.7. Let v be a bounded stable solution of (3.1). We select the extension $u(\cdot,y)=v(\cdot)*P_{\mathbb{H}}(\cdot,y)$ by the Poisson kernel in (3.22), that is,

$$u(x,y)=\int_{\mathbb{H}} v(z)P_{\mathbb{H}}(z^{-1}\circ x,y)\,\mathrm{d}z. \tag{3.49}$$

Let us check that u satisfies the hypothesis of Theorem 3.2. Indeed, since $P_{\mathbb{H}}(\cdot,y)\in L^1(\mathbb{H})$ and $v\in L^\infty(\mathbb{H})$, u is well defined and bounded. Moreover, by a regularity property (see Proposition 4.3 in [40]),

$$u\in W^{1,2}_{\widehat{\mathbb{H}}}(B_R^+;y^a\mathrm{d}x\mathrm{d}y) \quad \text{for all } R>0, \tag{3.50}$$

which implies (3.5). Therefore, u is a stable weak solution (see Remark 3.13) of (3.4). On the other hand, gradient bound condition (3.10) follows by the following regularity argument: Given a multi-index I, the derivatives of $P_{\mathbb{H}}$ in $\mathbb{H}$ have the decay (see Remark 4.5 in [40])

$$|\partial^I P_{\mathbb{H}}(x,y)| \le C\rho^{-2s-4-|I|}$$

for large $\rho=\rho(x)$. Thus, if we take "$\mathbb{H}$-derivatives" in (3.49) and then use a similar argument to that in the proof of the previous lemma, we see that u and its second order derivatives in $\mathbb{H}$ are continuous up to the boundary in $\widehat{\mathbb{H}}$.

Therefore, u satisfies the hypothesis of Theorem 3.4, and the level sets of u intersected with L (recall (3.9)) are minimal surfaces in the Heisenberg group. Theorem 3.7 follows by taking $y\to 0^+$.

□

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