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Fully nonlinear elliptic equations and semilinear fractional elliptic equations

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Fully nonlinear elliptic equations and semilinear fractional elliptic equations

Abstract

This thesis is divided into six parts.

The first part is devoted to prove Hadamard properties and Liouville type theorems for viscosity solutions of fully nonlinear elliptic partial differential equations with gradient term

$$\mathcal{M}^-(|x|, D^2u) + \sigma(|x|)|Du| + f(x,u) \leq 0, \qquad x \in \Omega, \tag{1}$$

where $\Omega = \mathbb{R}^N$ or an exterior domain, the functions $\sigma : [0,\infty) \to \mathbb{R}$ and $f : \Omega \times (0,\infty) \to (0,\infty)$ are continuous which satisfy some extra conditions.

In the second part, we study existence of boundary blow up solutions for semilinear fractional elliptic equations

$$\begin{aligned} (-\Delta)^\alpha u(x) + |u|^{p-1}u(x) &= h(x), \quad & x \in \Omega, \\ u(x) &= 0, & x \in \bar{\Omega}^c, \\ \lim_{x\in\Omega, x\to\partial\Omega} u(x) &= +\infty, \end{aligned} \tag{2}$$

where $p > 1$, Ω is an open bounded C^2 domain of $\mathbb{R}^N (N \geq 2)$, the operator $(-\Delta)^\alpha$ with $\alpha \in (0,1)$ is the fractional Laplacian and $h : \Omega \to \mathbb{R}$ is a continuous function which satisfies some extra conditions. Moreover, we analyze the uniqueness and asymptotic behavior of solutions to problem (2).

The main goal of the third part is to investigate positive solutions for fractional elliptic equations

$$\begin{aligned} (-\Delta)^\alpha u(x) + |u|^{p-1}u(x) &= 0, \qquad & x \in \Omega \setminus \mathcal{C}, \\ u(x) &= 0, & x \in \Omega^c, \\ \lim_{x\in\Omega\setminus\mathcal{C},\ x\to\mathcal{C}} u(x) &= +\infty, \end{aligned} \tag{3}$$

where $p > 1$ and Ω is an open bounded C^2 domain of $\mathbb{R}^N(N \geq 2)$, $\mathcal{C} \subset \Omega$ is the boundary of domain G which is C^2 and satifies $\bar{G} \subset \Omega$. We consider the existence of positive solutions for problem (3). In the meantime, we further analyze uniqueness, asymptotic behaviour and nonexistence to problem (3).

In the forth part, we study the existence of weak solutions to (F) $(-\Delta)^\alpha u+g(u) = \nu$ in an open bounded C^2 domain Ω of $\mathbb{R}^N(N \geq 2)$ which vanish in Ω^c, where $\alpha \in (0,1)$, ν is a Radon measure and g is a nondecreasing function satisfying some extra hypotheses. When g satisfies a subcritical integrability condition, we prove the existence and uniqueness of a weak solution for problem (F) for any measure. In the case where ν is a Dirac mass, we characterize the asymptotic behavior of solutions to (F). In addition, when $g(r) = |r|^{k-1}r$ with k supercritical, we show that a condition of absolute continuity of the measure with respect to some Bessel capacity is a necessary and sufficient condition in order (F) to be solved.

The purpose of fifth part is to investigate weak and strong singular solutions of semilinear fractional elliptic equations. Let $p \in (0, \frac{N}{N-2\alpha})$, $\alpha \in (0,1)$, $k > 0$ and $\Omega \subset \mathbb{R}^N(N \geq 2)$ be an open bounded C^2 domain containing 0 and δ_0 be the Dirac mass at 0, we study that the weak solution of $(E)_k$ $(-\Delta)^\alpha u + u^p = k\delta_0$ in Ω which vanishes in Ω^c is a weakly singular solution of (E^*) $(-\Delta)^\alpha u + u^p = 0$ in $\Omega \setminus \{0\}$ with the same outer data. Moreover, we study the limit of weak solutions of $(E)_k$ when $k \to \infty$. For $p \in (0, 1 + \frac{2\alpha}{N}]$, the limit is infinity in Ω. For $p \in (1 + \frac{2\alpha}{N}, \frac{N}{N-2\alpha})$, the limit is a strongly singular solution of (E^*).

Finally, in sixth part we study semilinear fractional elliptic equation (E1) $(-\Delta)^\alpha u$ $+\epsilon g(|\nabla u|) = \nu$ in an open bounded C^2 domain Ω of $\mathbb{R}^N(N \geq 2)$, which vanish in Ω^c, where $\epsilon = \pm 1$, $\alpha \in (1/2, 1)$, ν is a Radon measure and $g : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is a continuous function. We prove the existence of weak solutions for problem (E1) when g is subcritical. Furthermore, the asymptotic behavior and uniqueness of solutions are described when $\epsilon = 1$, ν is a Dirac mass and $g(s) = s^p$ with $p \in (0, \frac{N}{N-2\alpha+1})$.

Key words: Hadamard property, Liouville type theorem, Viscosity solution, Fully nonlinear elliptic PDE, Fractional Laplacian, Existence, Uniqueness, Asymptotic behavior, Blow-up solution, Radon measure, Dirac mass, Green kernel, Bessel capacities, Isolated singularity, Weak solution, Weak singular solution, Strong singular solution.

Resumen

Esta tesis esta dividida en seis partes.

La primera parte está dedicada a probar propiedades de Hadamard y teoremas del tipo de Liouville para soluciones viscosas de ecuaciones diferenciales parciales elípticas completamente no lineales con término gradiente

$$\mathcal{M}^-(|x|, D^2u) + \sigma(|x|)|Du| + f(x,u) \le 0, \qquad x \in \Omega, \tag{4}$$

donde $\Omega = \mathbb{R}^N$ o un dominio exterior, las funciones $\sigma : [0,\infty) \to \mathbb{R}$ y $f : \Omega \times (0,\infty) \to (0,\infty)$ son continuas las cuales satisfacen algunas condiciones extras.

En la segunda parte se estudia la existencia de soluciones que explotan en la frontera para ecuaciones elípticas fraccionarias semilineales

$$\begin{aligned} (-\Delta)^\alpha u(x) + |u|^{p-1}u(x) = h(x), &\quad x \in \Omega, \\ u(x) = 0, &\quad x \in \bar{\Omega}^c, \\ \lim_{x\in\Omega, x\to\partial\Omega} u(x) = +\infty, & \end{aligned} \tag{5}$$

donde $p > 1$, Ω es un dominio abierto acotado C^2 de $\mathbb{R}^N (N \ge 2)$, el operador $(-\Delta)^\alpha$ con $\alpha \in (0,1)$ es el Laplaciano fraccionario y $h : \Omega \to \mathbb{R}$ es una función continua la cual satisface algunas condiciones extras. Por otra parte, analizamos la unicidad y el comportamiento asimptótico de soluciones al problema (5).

El objetivo principal de la tercera parte es investigar soluciones positivas para ecuaciones elípticas fraccionarias

$$\begin{aligned} (-\Delta)^\alpha u(x) + |u|^{p-1}u(x) = 0, &\quad x \in \Omega \setminus \mathcal{C}, \\ u(x) = 0, &\quad x \in \Omega^c, \\ \lim_{x\in\Omega\setminus\mathcal{C},\ x\to\mathcal{C}} u(x) = +\infty, & \end{aligned} \tag{6}$$

donde $p > 1$ y Ω es un dominio abierto acotado C^2 de $\mathbb{R}^N (N \ge 2)$, $\mathcal{C} \subset \Omega$ es el frontera de dominio G que es C^2 y satisface $\bar{G} \subset \Omega$. Consideramos la existencia de soluciones positivas para el problema (6). Mas aún, analizamos la unicidad, el comportamiento asimptótico y la no existencia al problema (6).

En la cuarta parte, estudiamos la existencia de soluciones débiles de (F) $(-\Delta)^\alpha u + g(u) = \nu$ en un dominio Ω abierto acotado C^2 de $\mathbb{R}^N (N \geq 2)$ el cual se desvanece en Ω^c, donde $\alpha \in (0,1)$, ν es una medida de Radon y g es una función no decreciente satisfaciendo algunas hipótesis extras. Cuando g satisface una condición de integrabilidad subcrítica, probamos la existencia y unicidad de una solución débil para el problema (F) para cualquier medida. En el caso donde ν es una masa de Dirac, caracterizamos el comportamiento asimptótico de soluciones a (F). Asimismo, cuando $g(r) = |r|^{k-1}r$ con k supercrítico, mostramos que una condición de absoluta continuidad de la medida con respecto a alguna capacidad de Bessel es una condición necesaria y suficiente para que (F) sea resuelta.

El propósito de la quinta parte es investigar soluciones singulares débiles y fuertes de ecuaciones elípticas fraccionarias semilineales. Sean $p \in (0, \frac{N}{N-2\alpha})$, $\alpha \in (0,1)$, $k > 0$ y $\Omega \subset \mathbb{R}^N (N \geq 2)$ un dominio abierto acotado C^2 conteniendo a 0 y δ_0 la masa de Dirac en 0, estudiamos que la solución débil de $(E)_k$ $(-\Delta)^\alpha u + u^p = k\delta_0$ en Ω la cual se desvanece en Ω^c es una solución débil singular de (E^*) $(-\Delta)^\alpha u + u^p = 0$ en $\Omega \setminus \{0\}$ con el mismo dato externo. Por otra parte, estudiamos el límite de soluciones débiles de $(E)_k$ cuando $k \to \infty$. Para $p \in (0, 1 + \frac{2\alpha}{N}]$, el límite es infinito en Ω. Para $p \in (1 + \frac{2\alpha}{N}, \frac{N}{N-2\alpha})$, el límite es una solución fuertemente singular de (E^*).

Finalmente, en la sexta parte estudiamos la ecuación elíptica fraccionaria semilineal (E1) $(-\Delta)^\alpha u + \epsilon g(|\nabla u|) = \nu$ en un dominio Ω abierto acotado C^2 de $\mathbb{R}^N (N \geq 2)$, el cual se desvanece en Ω^c, donde $\epsilon = \pm 1$, $\alpha \in (1/2, 1)$, ν es una medida de Radon y $g : \mathbb{R}_+ \mapsto \mathbb{R}_+$ es una función continua. Probamos la existencia de soluciones débiles para el problema (E1) cuando g es subcrítico. Además, el comportamiento asimptótico y la unicidad de soluciones son descritas cuando $\epsilon = 1$, ν es una masa de Dirac y $g(s) = s^p$ con $p \in (0, \frac{N}{N-2\alpha+1})$.

Palabras claves: Propiedad de Hadamard, Teorema del tipo de Liouville, soluciones Viscosas, EDP elípticas completamente no lineales, Laplaciano fraccionario, Existencia, Unicidad, Comportamiento asimptótico, Soluciones blow-up, Medida de Radon, Masa de Dirac, Núcleo de Green, Capacidades de Bessel, Singularidad aislada, Soluciones débiles, Soluciones singulares débiles, Soluciones singulares fuertes.

Résumé

Cette thèse est divisée en six parties.

La première parties est consacrée à l'étude de propriétés de Hadamard et à l'obtention de théorèmes de Liouville pour des solutions de viscosité d'équations aux dérivées partielles elliptiques complètement non-linéaires avec des termes de gradient,

$$\mathcal{M}^-(|x|, D^2u) + \sigma(|x|)|Du| + f(x,u) \leq 0, \qquad x \in \Omega, \tag{7}$$

où Ω est ou bien $\mathbb{R}^N$ ou bien un domaine extérieur, et les fonctions $\sigma : [0, \infty) \to \mathbb{R}$ et $f : \Omega \times (0, \infty) \to (0, \infty)$ sont continues et vérifient certaines conditions.

Dans la seconde partie nous étudions l'existence de grandes solutions, c'est à dire de solutions que explosent au bord, d'équations elliptiques fractionnaires semi linéaires

$$\begin{aligned} (-\Delta)^\alpha u(x) + |u|^{p-1}u(x) = h(x), &\quad x \in \Omega, \\ u(x) = 0, &\quad x \in \bar{\Omega}^c, \\ \text{lím}_{x\in\Omega, x\to\partial\Omega}\, u(x) = +\infty, & \end{aligned} \tag{8}$$

où $p > 1$, Ω est un ouvert borné de classe C^2 de $\mathbb{R}^N(N \geq 2)$, $(-\Delta)^\alpha$ avec $\alpha \in (0,1)$ est le Laplacien fractionnaire et $h : \Omega \to \mathbb{R}$ est continue et vérifie des conditions de croissance qui seront précisées. En outre nous étudions les questions d'unicité et de comportement asymptotique des solutions du problème (8).

Le but essentiel de la troisime partie est d'étudier les solutions positives de l'équation elliptique fractionnaire

$$\begin{aligned} (-\Delta)^\alpha u(x) + |u|^{p-1}u(x) = 0, &\quad x \in \Omega \setminus \mathcal{C}, \\ u(x) = 0, &\quad x \in \Omega^c, \\ \text{lím}_{x\in\Omega\setminus\mathcal{C},\ x\to\mathcal{C}}\, u(x) = +\infty, & \end{aligned} \tag{9}$$

où $p > 1$ et Ω est un domaine borné de classe C^2 de $\mathbb{R}^N(N \geq 2)$, $\mathcal{C} \subset \Omega$ est le

bord d'un domaine G de classe C^2 tel que $\bar{G} \subset \Omega$. Nous intéressons à l'existence de solutions positives au problème (9). Par la même occasion, nous analysons aussi les questions d'unicité, de comportement asymptotique et, le caséchéant, la non-existence de solutions au problème (9).

Dand la quatrième partie nous étudions l'existence de solutions faibles à l'équation $(-\Delta)^\alpha u + g(u) = \nu$ dans un domaine de classe C^2 borné $\Omega \subset \mathbb{R}^N (N \geq 2)$, qui s'annulent dans Ω^c, où $\alpha \in (0,1)$, ν est une mesure de Radon et g une fonction croissante vérifiant une condition de croissance. Quand g satisfait à une condition intégrale de sous-criticalité, nous montrons l'existence et l'unicité de solutions au problème (F) pour n'importe quelle mesure bornée. Dans le cas où ν est une mesure de Dirac, nous caractérisons le comportement asymptotique des solutions de (F). En outre, quand $g(r) = |r|^{k-1}r$ avec k sur-critique nous obtenons une condition nécessaire portant sur une mesure ν positive pour que le problème (F) admette une solution, sous forme d'une condition d'absolue continuité de la mesure par rapport à une certaine capacité de Bessel.

L'objectif de la cinquième partie est d'étudier les propriétés des solutions singulières de solutions d'équations elliptiques fractionnaires semi-linéaires. Soit $p \in (0, \frac{N}{N-2\alpha})$, $\alpha \in (0,1)$, $k > 0$, $\Omega \subset \mathbb{R}^N (N \geq 2)$ est un domaine borné de classe C^2 contenant 0 et δ_0 la mesure de Dirac en 0. Nous montrons que la solution faible u_k de (E_k) $(-\Delta)^\alpha u + u^p = k\delta_0$ qui s'annule dans Ω^c est une solution singulière faible de (E^*) $(-\Delta)^\alpha u + u^p = 0$ dans $\Omega \setminus \{0\}$ vérifiant la même condition dans Ω^c. En outre, nous montrons que lorsqur k tend vers l'infini et $0 < p \leq 1 + \frac{2\alpha}{N}$, la limite de u_k est infinie dans tout Ω, alors que cette limite est une solution de (E^*) fortement singulière quand $1 + \frac{2\alpha}{N} < p < \frac{N}{N-2\alpha}$.

Dand la sixième partie nous étudions les équations de la forme (E1) $(-\Delta)^\alpha u + \epsilon g(|\nabla u|) = \nu$ dans un domaine borné $\Omega \subset \mathbb{R}^N (N \geq 2)$ de classe C^2 qui s'annulent dans Ω^c, où $\epsilon = \pm 1$, $\alpha \in (1/2, 1)$, ν est une mesure de Radon et $g : \mathbb{R}_+ \mapsto \mathbb{R}_+$ une fonction continue. Nous montrons l'existence de solutions du problème (E1) quand g vérifie une condition intégrale de sous-criticalité. En outre, nous analysons aussi les questions d'unicité, de comportement asymptotique au problème (E1) quand $\epsilon = \pm 1$, ν est mesure de Dirac, $g(s) = s^p$ avec $p \in (0, \frac{N}{N-2\alpha+1})$.

Mots clefs: Propriété d'Hadamard, Théorème de typr Liouville, Solutions de ciscosité, Équations elliptiques complètement non-linéaires, Laplacien fractionnaire, Existence, Unicité, Comportement asymptotique, Grandes solutions, Mesures de Radon, Mesure de Dirac, Noyau de Green, Capacité de Bessel, Singularités isolées, Solutions faibles, Singularité faible, Singularité forte.

Índice general

Introduction

This thesis is to study Liouville theorems for fully nonlinear elliptic equations, to do research for large solutions of semiliear fractional elliptic equations and to investigate weak solutions of semiliear fractional elliptic equations involving measures.

0.1. Liouville type theorems for fully nonlinear elliptic equations with gradient term

In the study of nonlinear elliptic equations in bounded domains, non-existence results for entire solutions of related limiting equations appear as a crucial ingredient. In the search for positive solutions for semi-linear elliptic equations with nonlinearity behaving as a power at infinity, one is interested in the non-negative solutions of the equation

$$\Delta u + u^p = 0, \qquad \text{in} \quad \mathbb{R}^N. \tag{10}$$

The question is for which value of p, typically $p > 1$, this equation has or has no solution. This has been one of the motivations that has pushed forward the study of Liouville type theorems for general equations in $\mathbb{R}^N$ and in unbounded domains like cones or exterior domains. On the other hand, the understanding of structural characteristics of general linear or nonlinear operators has been another motivation for advancing the study of Liouville type theorems that have attracted many researchers. See the work in [1, 2, 19, 61, 86].

If we consider the Pucci's operators instead of the Laplacian, the question set above becomes very interesting, since most of the techniques used in the case of the Laplacian are not available. The Liouville type theorem for the equation analogous to equation (10) has not been proved in full generality, but only in the radial case. On the other hand, the Liouville type theorem for non-negative solutions of

$$\mathcal{M}^- u + u^p \le 0, \qquad \text{in} \quad \mathbb{R}^N, \tag{11}$$

has been studied in full extent by Cutrì and Leoni [45] and generalized in various directions by Felmer and Quaas [52, 54, 55] Capuzzo-Dolcetta and Cutrì [30] and Armstrong and Sirakov [3]. In all these cases the solutions of the inequality are considered in the viscosity sense.

In a recent paper, Armstrong and Sirakov in [4] made significant progress in the understanding of the structure of positive solutions of equations generalizing (11), shading light even for equations of the form

$$\Delta u + f(u) \leq 0, \qquad \text{in} \quad \mathbb{R}^N. \tag{12}$$

They propose a general approach to non-existence and existence of solutions of the general inequality

$$Q(u) + f(x,u) \leq 0, \qquad \text{in} \quad \mathbb{R}^N, \tag{13}$$

where the second order differential operator Q satisfies certain scaling property, it possesses fundamental solutions behaving as power asymptotically and it satisfies some other properties, common to elliptic operators, like a weak comparison principle, a quantitative strong comparison principle and a very weak Harnack inequality, see hypothesis (H1)-(H5) in [4]. Regarding the nonlinearity f, the results in [4] unravel a very interesting property, that is, that the behavior of the function f only matters near $u = 0$ and for x large. These results are new even for the case of (12). Moreover, the authors in [4] are able to apply their approach to equation (13) in exterior domains without any boundary condition, providing another truly new result.

It is the purpose of this chapter 1 to extend the results described above in order to include elliptic operators with first order term. The introduction of a first order term may brake the scaling property of the differential operator and it allows for the appearance of non-homogeneous fundamental solutions, not even asymptotically. Thus, the approach in [4] cannot be applied to this more general situation and we have to find different arguments. Interestingly, to prove our results we use the more elementary approach taken in the original work by Cutrì and Leoni, where the Hadamard property, obtained through the comparison principle, is combined with the appropriate choice of a function to test the equation. The underline principle is the asymptotic comparison between the solutions of the inequality and the fundamental solution. This can be interpreted as the interaction between the elliptic operator, including first order term, and the nonlinearity (the zero order term).

We start the precise description of our results by recalling the definition of the Pucci's operators. In chapter 1, we consider

$$\mathcal{M}^-(r, D^2u) = \lambda(r)\sum_{e_i \geq 0} e_i + \Lambda(r)\sum_{e_i<0} e_i, \tag{14}$$

where $e_1, \dots, e_N$ are the eigenvalues of D^2u, $\lambda, \Lambda : [0, \infty) \to \mathbb{R}$ are continuous, λ_0 and Λ_0 are positive constants and

$$0 < \lambda_0 \leq \lambda(r) \leq \Lambda(r) \leq \Lambda_0 < +\infty, \quad \forall r = |x|,\ x \in \mathbb{R}^N. \tag{15}$$

Our purpose is to study the non-negative solutions of

$$\mathcal{M}^-(r, D^2u) + \sigma(r)|Du| + f(x,u) \leq 0, \quad \text{ in } \Omega, \tag{16}$$

with $\Omega = \mathbb{R}^N$ or an exterior domain and $\sigma : [0,\infty) \to \mathbb{R}$ and $f : \Omega \times (0,\infty) \to (0,\infty)$ are continuous functions. By an exterior domain we mean a set $\Omega = \mathbb{R}^N \setminus K$ connected, where K is nonempty compact subset of $\mathbb{R}^N$.

We consider the fundamental solutions for the second order differential operator $\varphi, \psi : (0,\infty) \to \mathbb{R}$ in given (1.41) and (1.42), which are non-trivial radially symmetric solutions of

$$\mathcal{M}^-(r, D^2u) + \sigma(r)|Du| = 0, \quad \text{ in } \mathbb{R}^N \setminus \{0\}, \tag{17}$$

satisfying

(i) ψ is increasing and either $\lim_{r\to\infty} \psi(r) = \infty$ or $\lim_{r\to\infty} \psi(r) = 0$ and

(ii) φ is decreasing and either $\lim_{r\to\infty} \varphi(r) = -\infty$ or $\lim_{r\to\infty} \varphi(r) = 0$.

Now we are in a position to make precise assumptions about the interaction between the differential operator and the nonlinearity. We assume that

(f_1) $f : \Omega \times (0,\infty) \to (0,\infty)$, $\lambda, \Lambda, \sigma : [0,\infty) \to \mathbb{R}$ are continuous.

(f_2) We have

$$\lim_{r=|x|\to\infty} \frac{r^2}{1+\sigma_-(r)r} f(x,s) = \infty$$

uniformly on compact subsets of $(0,\infty)$. Here and in what follows $\sigma_- = \max\{-\sigma, 0\}$.

In order to state the next assumption we need a definition. Given $\mu > 0$, $a > 1$, $k > 0$ and $\tau > 0$ we define

$$\Psi_k(\tau) = \frac{\varphi(a\tau)}{\varphi(\tau)} \inf_{x\in B_{a\tau}\setminus B_\tau} \left\{ \frac{r^2}{\sigma_-(r)r+1} \inf_{k\varphi(ar)\leq s\leq\mu} \frac{f(x,s)}{s} \right\}. \tag{18}$$

We assume:

(f_3) If $\lim_{r\to\infty} \varphi(r) = 0$ then we assume the existence of constants $\mu > 0$ and $a > 1$ such that, defining

$$h(k) = \limsup_{\tau\to\infty} \Psi_k(\tau),$$

one of the following holds:

(i) for all $k > 0$ we have $h(k) = \infty$ or

(ii) for all $k > 0$ we have

$$0 < \liminf_{\tau \to \infty} \Psi_k(\tau) \quad \text{and} \quad \lim_{k \to \infty} h(k) = \infty \tag{19}$$

and there is a constant $C \in \mathbb{R}$ such that

$$r\sigma(r) > C, \qquad \text{for all } r > 0. \tag{20}$$

Now we state our first Liouville type theorem for inequality (16) in $\mathbb{R}^N$:

Theorem 0.1.1 *Assume that f satisfies (f_1), (f_2) and (f_3). Then inequality (16) in $\mathbb{R}^N$ does not have a non-trivial viscosity solution $u \geq 0$.*

We observe that hypothesis (f_3) does restrict f when $\lim_{r\to\infty} \varphi(r) = -\infty$.

Regarding hypotheses (f_2) and (f_3) we would like to notice that they are natural extensions of hypotheses $(f2)-(f3)$ in [4], when $\sigma \not\equiv 0$ and the fundamental solution φ is not necessarily power-like. Thus, we are generalizing the results in [4] in the case of a one-homogeneous differential operator in $\mathbb{R}^N$. It is also interesting to notice that hypotheses (f_2) and (f_3) appear explicitly and in a natural way in our proof of the theorem.

When the condition (i) is satisfied we say that inequality (16) is sub-critical and when condition (ii) holds, we say it is critical. In case of

$$\Delta u + u^p \leq 0,$$

we say the inequality is sub-critical when $p < N/(N-2)$ and when $p = N/(N-2)$ it is critical. When $p > N/(N-2)$ we say the inequality is super-critical and here the existence of positive solution holds. Accordingly, we would like to define a notion of super-criticality the cases (i) and (ii) do not hold. However, in Theorem 1.2.3 we provide an example where there is no solution in a super-critical sub-region, showing that further study is required to understand the critical boundary.

In the case of an exterior domain, we need to consider also the interaction between the differential operator and the nonlinearity at ∞. We need a definition in order to state our assumptions. Given $\mu > 0$, $a > 1$, $k > 0$ and $\tau > 0$ we define

$$\tilde{\Psi}_k(\tau) = \frac{\psi(\tau)}{\psi(a\tau)} \inf_{x \in B_{a\tau} \setminus B_\tau} \left\{ \frac{r^2}{\sigma_-(r)r + 1} \inf_{\mu \leq s \leq k\psi(ar)} \frac{f(x,s)}{s} \right\}.$$

Now we assume that

(f_4) If $\lim_{r\to\infty} \psi(r) = \infty$ then we assume the existence of constants $\mu > 0$ and $a > 1$ such that, defining

$$\tilde{h}(k) = \limsup_{\tau \to \infty} \tilde{\Psi}_k(\tau),$$

one of the following holds:

(i) for all $k > 0$ we have $\tilde{h}(k) = \infty$ or

(ii) for all $k > 0$ we have

$$0 < \liminf_{\tau\to\infty} \tilde{\Psi}_k(\tau) \quad \text{and} \quad \lim_{k\to 0^+} \tilde{h}(k) = \infty \tag{21}$$

and there is a constant $C \in \mathbb{R}$ such that (20) holds.

For an exterior domain we have the following non-existence result.

Theorem 0.1.2 *Assume that Ω is an exterior domain and f satisfies (f_1), (f_2), (f_3) and (f_4). Then inequality (16) in Ω does not have a non-trivial viscosity solution $u \geq 0$.*

We observe that hypothesis (f_4) does restrict f when $\lim_{r\to\infty} \psi(r) = 0$.

As for (f_3), hypothesis (f_4) is the natural extension of $(f4)$ in [4] to our case. Here we allow $\sigma \not\equiv 0$ and ψ not power-like, thus generalizing [4].

In case of (f_4) we may also define the notion of criticality for (16) in an analogous way as for (f_3). Since here the behavior of f is relevant at zero and infinity mixed cases appear, as for example, an inequality critical at 0 and sub-critical at ∞ or vice verse.

In the proofs of Theorem 0.1.1 and 0.1.2 we use some basic properties of the functions

$$m(r) = \inf_{x\in B_r} u(x), \quad m_0(r) = \inf_{x\in B_r\setminus B_{r_0}} u(x) \quad \text{and} \quad M(r) = \inf_{x\notin B_r} u(x) \tag{22}$$

in connection with the fundamental solutions, as given by the Hadamard property provided in Theorem 1.4.3. Then we test the equation with an adequate function and we use the asymptotic assumptions on f and the fundamental solutions to obtain a contradiction with the existence of non-trivial non-negative solutions. In the proofs of our theorems we only consider $a = 2$.

For the existence of positive solutions of (16), it is nature to consider the super-critical assumption, that is, the case when hypotheses (f_3) and (f_4) are not satisfied, which means

$$\liminf_{\tau\to\infty} \Psi_k(\tau) = 0 \quad \text{or} \quad \limsup_{k\to\infty} h(k) < \infty$$

and

$$\liminf_{\tau\to\infty} \tilde{\Psi}_k(\tau) = 0 \quad \text{or} \quad \limsup_{k\to\infty} \tilde{h}(k) < \infty,$$

where h, $\tilde{h}$, Ψ_k and $\tilde{\Psi}_k$ were defined in (f_3) and (f_4). We observe that super-criticality holds when $h(k) = 0$ or $\tilde{h}(k) = 0$ for any $k > 0$, but it is not true that under this notion of super-criticality a positive solution of (16) always exists.

Finally, we consider a Liouville type theorem in the case f is a linear function, that is, $f(x,s) = h(x)s$, that interestingly can be proved using the same techniques considered in the nonlinear case. This problem has been recently studied by Rossi [89] after some previous work by Berestycki, Hamel and Nadirashvili [20], Berestycki, Hamel and Roques [21] and Berestycki, Hamel and Rossi [22]. Rossi [89] proved a Liouville type theorem for generally unbounded domains, assuming that

$$\liminf_{x\in\Omega,|x|\to\infty} \frac{u(x)+1}{dist(x,\partial\Omega)} = 0. \tag{23}$$

It is clear that when Ω is an exterior domain then $dist(x,\partial\Omega) \le |x|$, so that (23) implies a linear growth constraint on u. Thus, it is interesting to investigate the existence or non-existence of positive solutions of the corresponding equation when (23) does no longer hold. Here is our result:

Theorem 0.1.3 *Let u be a viscosity nonnegative solution of*

$$\mathcal{M}^-(r, D^2u) + \sigma(r)|Du| + h(x)u \le 0, \quad \text{in} \quad \Omega, \tag{24}$$

where Ω is an exterior domain. Assume further that λ and Λ satisfy (15) and that

(h_1) *$h : \mathbb{R}^N \to \mathbb{R}$ and $\sigma : \mathbb{R}_+ \to \mathbb{R}$ are continuous, h is positive and σ is negative.*

(h_2) *There exists a function $\kappa : \mathbb{R}_+ \to \mathbb{R}_+$ of class C^1 such that*

$$\lim_{r\to\infty} \kappa'(r) = 0 \tag{25}$$

and there is a constant $\mu \ge 1$ such that

$$1 \le \kappa(r) \max_{r-\kappa(r)\le s\le r} |\sigma(s)| \le \mu, \quad \text{for all } r > 0. \tag{26}$$

(h_3) *There exists a sequence $r_n \to \infty$ such that*

$$\lim_{n\to\infty} \inf_{r\in(r_n-\kappa(r_n),r_n)} \left\{ h(r) - e^{\frac{\mu}{\lambda_0}}(2\Lambda_0+1)\sigma^2(r) \right\} > 0. \tag{27}$$

Then $u \equiv 0$.

0.2. Semilinear fractional elliptic equations

0.2.1. Large solutions to semilinear fractional elliptic equations

In 1957, a fundamental contribution due to Keller [66] and Osserman [84] is the study of the nonlinear reaction diffusion equation

$$\begin{aligned} -\Delta u + h(u) = 0, \quad \text{in } \Omega, \\ \text{lím}_{x\in\Omega, x\to\partial\Omega} u(x) = +\infty, \end{aligned} \tag{28}$$

where Ω is an open bounded C^2 domain of $\mathbb{R}^N (N \geq 2)$ and h is a nondecreasing positive function. They proved that this equation admits a solution if and only if h satisfies

$$\int_1^{+\infty} \frac{ds}{\sqrt{H(s)}} < +\infty, \tag{29}$$

where $H(s) = \int_0^s h(t)dt$, that in the case of $h(u) = u^p$ means $p > 1$. This integral condition on the non-linearity is known as the Keller-Osserman criteria. The solution of (28) found in [66] and [84] exists as a consequence of the interaction between the reaction and the diffusion term, without the influence of an external source that blows up at the boundary. Solutions exploding at the boundary are usually called boundary blow up solutions or large solutions. From then on, the result of Keller and Osserman has been extended by numerous mathematicians in various ways, weakening the assumptions on the domain, generalizing the differential operator and the nonlinear term for equations and systems. The case of $h(u) = u_+^p$ with $p = \frac{N+2}{N-2}$ is studied by Loewner and Nirenberg [72], where in particular uniqueness and asymptotic behavior were obtained. After that, Bandle and Marcus [6] obtained uniqueness and asymptotic for more general non-linearties h. Later, Le Gall in [70] established a uniqueness result of problem (28) in the domain whose boundary is non-smooth when $h(u) = u_+^2$. Marcus and Véron [74, 75] extended the uniqueness of blow-up solution for (28) in general domains whose boundary is locally represented as a graph of a continuous function when $h(u) = u_+^p$ for $p > 1$. For another interesting contributions to boundary blow-up solutions see [5, 7, 44, 47, 48, 59, 73, 87].

During the last years there has been a renewed and increasing interest in the study of linear and nonlinear integral operators, especially, the fractional Laplacian, motivated by great applications in physics and by important links on the theory of Lévy processes, refer to [23, 26, 27, 52, 54, 55, 57, 85, 90]. In a recent work, Felmer and Quaas [51] considered an analog of (28) where the Laplacian is replaced by the fractional Laplacian

$$\begin{aligned} (-\Delta)^\alpha u(x) + |u|^{p-1}u(x) &= f(x), \quad x \in \Omega, \\ u(x) &= g(x), \quad x \in \bar{\Omega}^c, \\ \text{lím}_{x\in\Omega, x\to\partial\Omega} u(x) &= +\infty, \end{aligned} \tag{30}$$

where Ω is an open bounded C^2 domain of $\mathbb{R}^N (N \geq 2)$, $p > 1$ and the fractional Laplacian operator is defined as

$$(-\Delta)^\alpha u(x) = -\frac{1}{2}\int_{\mathbb{R}^N} \frac{\delta(u,x,y)}{|y|^{N+2\alpha}}dy, \quad x \in \Omega, \tag{31}$$

with $\alpha \in (0,1)$ and $\delta(u,x,y) = u(x+y) + u(x-y) - 2u(x)$. The authors proved the existence of a solution to (30) provided that g explodes at the boundary and satisfies other technical conditions. In case the function g blows up with an explosion rate as $d(x)^\beta$, with $\beta \in (-\frac{2\alpha}{p-1}, 0)$ and $d(x) = dist(x, \partial\Omega)$, the solution satisfies

$$0 < \liminf_{x\in\Omega, x\to\partial\Omega} u(x)d(x)^{-\beta} \leq \limsup_{x\in\Omega, x\to\partial\Omega} u(x)d(x)^{\frac{2\alpha}{p-1}} < +\infty.$$

In [51] the explosion is driven by the function g. The external source f has a secondary role, not intervening in the explosive character of the solution. f may be bounded or unbounded, in later case the explosion rate has to be controlled by $d(x)^{-2\alpha p/(p-1)}$.

One interesting question not answered in [51] is the existence of a boundary blow up solution without external source, that is assuming $g = 0$ in $\bar{\Omega}^c$ and $f = 0$ in Ω, thus extending the original result by Keller and Osserman, where solutions exists due to the pure interaction between the reaction and the diffusion terms. It is the purpose of chapter 2 to answer positively this question and to better understand how the non-local character influences the large solutions of (30) and what is the structure of the large solutions of (30) with or without sources. Comparing with the Laplacian case, where well possedness holds for (30), a much richer structure for the solution set appears for the non-local case, depending on the parameters and the data f and g. In particular, Theorem 0.2.1 shows that existence, uniqueness, non-existence and infinite existence may occur at different values of p and α.

Our first result in chapter 2 is on the existence of blowing up solutions driven by the sole interaction between the diffusion and reaction term, assuming the external value g vanishes. Thus we will be considering the equation

$$\begin{aligned} (-\Delta)^\alpha u(x) + |u|^{p-1}u(x) &= f(x), \quad x \in \Omega, \\ u(x) &= 0, \qquad x \in \bar{\Omega}^c, \\ \text{lím}_{x\in\Omega, x\to\partial\Omega} u(x) &= +\infty, \end{aligned} \tag{32}$$

On the external source f we will assume the following hypotheses

(H1) The external source $f : \Omega \to \mathbb{R}$ is a $C^{\beta}_{loc}(\Omega)$, for some $\beta > 0$.

(H2) Defining $f_-(x) = \text{máx}\{-f(x), 0\}$ and $f_+(x) = \text{máx}\{f(x), 0\}$ we have

$$\limsup_{x\in\Omega, x\to\partial\Omega} f_+(x)d(x)^{\frac{2\alpha p}{p-1}} < +\infty \quad \text{and} \quad \lim_{x\in\Omega, x\to\partial\Omega} f_-(x)d(x)^{\frac{2\alpha p}{p-1}} = 0.$$

A related condition that we need for non-existence results

(H2*) The function f satisfies

$$\limsup_{x\in\Omega, x\to\partial\Omega} |f(x)|d(x)^{2\alpha} < +\infty.$$

Now we are in a position to state our first theorem in this part.

Theorem 0.2.1 *Assume that Ω is an open, bounded and connected domain of class C^2 and $\alpha \in (0, 1)$. Then we have:*
Existence: *Assume that f satisfies (H1) and (H2), then there exists $\tau_0(\alpha) \in (-1, 0)$ such that for every p satisfying*

$$1 + 2\alpha < p < 1 - \frac{2\alpha}{\tau_0(\alpha)}, \tag{33}$$

the equation (32) possesses at least one solution u satisfying

$$0 < \liminf_{x\in\Omega, x\to\partial\Omega} u(x)d(x)^{\frac{2\alpha}{p-1}} \le \limsup_{x\in\Omega, x\to\partial\Omega} u(x)d(x)^{\frac{2\alpha}{p-1}} < +\infty. \tag{34}$$

Uniqueness: *If f further satisfies $f \ge 0$ in Ω, then $u > 0$ in Ω and u is the unique solution of (32) satisfying (34).*
Nonexistence: *If f satisfies (H1), (H2*) and $f \ge 0$, then in the following three cases:*

i) For any $\tau \in (-1, 0) \setminus \{-\frac{2\alpha}{p-1},\ \tau_0(\alpha)\}$ and p satisfying (33) or

ii) For any $\tau \in (-1, 0)$ and

$$p \ge 1 - \frac{2\alpha}{\tau_0(\alpha)} \quad \text{or} \tag{35}$$

iii) For any $\tau \in (-1, 0) \setminus \{\tau_0(\alpha)\}$ and

$$1 < p \le 1 + 2\alpha, \tag{36}$$

equation (32) does not have a solution u satisfying

$$0 < \liminf_{x\in\Omega, x\to\partial\Omega} u(x)d(x)^{-\tau} \leq \limsup_{x\in\Omega, x\to\partial\Omega} u(x)d(x)^{-\tau} < +\infty. \tag{37}$$

Special existence for $\tau = \tau_0(\alpha)$. *Assume $f(x) \equiv 0,\ x \in \Omega$ and that*

$$\max\{1 - \frac{2\alpha}{\tau_0(\alpha)} + \frac{\tau_0(\alpha)+1}{\tau_0(\alpha)}, 1\} < p < 1 - \frac{2\alpha}{\tau_0(\alpha)}. \tag{38}$$

Then, there exist constants $C_1 \geq 0$ and $C_2 > 0$, such that for any $t > 0$ there is a positive solution u of equation (32) satisfying

$$C_1 d(x)^{\min\{\tau_0(\alpha)p+2\alpha,0\}} \leq td(x)^{\tau_0(\alpha)} - u(x) \leq C_2 d(x)^{\min\{\tau_0(\alpha)p+2\alpha,0\}}. \tag{39}$$

Remark 0.2.1 *We remark that hypothesis (H2) and (*H2**) are satisfied when $f \equiv 0$, so this theorem answer the question on existence rised in [51]. We also observe that a function f satisfying (H2) may also satisfy*

$$\lim_{x\in\Omega, x\in\partial\Omega} f(x) = -\infty,$$

what matters is that the rate of explosion is smaller than $\frac{2\alpha p}{p-1}$.

For proving the existence part of this theorem we will construct appropriate super and sub-solutions. This construction involves the one dimensional truncated Laplacian of power functions given by

$$C(\tau) = \int_0^{+\infty} \frac{\chi_{(0,1)}(t)|1-t|^\tau + (1+t)^\tau - 2}{t^{1+2\alpha}} dt, \tag{40}$$

for $\tau \in (-1, 0)$ and where $\chi_{(0,1)}$ is the characteristic function of the interval $(0, 1)$. The number $\tau_0(\alpha)$ appearing in the statement of our theorems is precisely the unique $\tau \in (-1, 0)$ satisfying $C(\tau) = 0$. See Proposition 2.3.1 for details.

Remark 0.2.2 *For the uniqueness, we would like to mention that, by using iteration technique, Kim in [67] has proved the uniqueness of solution to the problem*

$$\begin{cases} -\Delta u + u_+^p = 0, & in \quad \Omega, \\ u = +\infty, & on \quad \partial\Omega, \end{cases} \tag{41}$$

where $u_+ = \max\{u, 0\}$, under the hypotheses that $p > 1$ and Ω is bounded and satisfying $\partial\Omega = \partial\bar{\Omega}$. García-Melián in [59, 60] introduced some improved iteration technique to obtain the uniqueness for problem (41) with replacing nonlinear term by $a(x)u^p$. However, there is a big difficulty for us to extend the iteration technique *to our problem (32) involving fractional Laplacian, which is caused by the nonlocal character.*

Next we are also interested in considering the existence of blowing up solutions driven by external source f on which we assume the following hypothesis

(H3) There exists $\gamma \in (-1-2\alpha, 0)$ such that

$$0 < \liminf_{x\in\Omega, x\to\partial\Omega} f(x)d(x)^{-\gamma} \leq \limsup_{x\in\Omega, x\to\partial\Omega} f(x)d(x)^{-\gamma} < +\infty.$$

Depending on the size of γ we will say that the external source is weak or strong. In order to gain in clarity, in this case we will state separately the existence, uniqueness and non-existence theorem in this source-driven case.

Theorem 0.2.2 (Existence) *Assume that Ω is an open, bounded and connected domain of class C^2. Assume that f satisfies (H1) and let $\alpha \in (0,1)$, then we have:*

(i) (weak source) If f satisfies (H3) with

$$-2\alpha - \frac{2\alpha}{p-1} \leq \gamma < -2\alpha, \tag{42}$$

then, for every p such that (35) holds, equation (32) possesses at least one solution u, with asymptotic behavior near the boundary given by

$$0 < \liminf_{x\in\Omega, x\to\partial\Omega} u(x)d(x)^{-\gamma-2\alpha} \leq \limsup_{x\in\Omega, x\to\partial\Omega} u(x)d(x)^{-\gamma-2\alpha} < +\infty. \tag{43}$$

(ii) (strong source) If f satisfies (H3) with

$$-1-2\alpha < \gamma < -2\alpha - \frac{2\alpha}{p-1} \tag{44}$$

then, for every p such that

$$p > 1 + 2\alpha, \tag{45}$$

equation (32) possesses at least one solution u, with asymptotic behavior near the boundary given by

$$0 < \liminf_{x\in\Omega, x\to\partial\Omega} u(x)d(x)^{-\frac{\gamma}{p}} \leq \limsup_{x\in\Omega, x\to\partial\Omega} u(x)d(x)^{-\frac{\gamma}{p}} < +\infty. \tag{46}$$

As we already mentioned, in Theorem 0.2.1 the existence of blowing up solutions results from the interaction between the reaction u^p and the diffusion term $(-\Delta)^\alpha$, while the role of the external source f is secondary. In contrast, in Theorem 0.2.2 the existence of blowing up solutions results on the interaction between the external source, and the diffusion term in case of weak source and the interaction between the external source and the reaction term in case of strong source.

Regarding uniqueness result for solutions of (32), as in Theorem 0.2.1 we will assume that f is non-negative, hypothesis that we need for technical reasons. We have

Theorem 0.2.3 (Uniqueness) *Assume that Ω is an open, bounded and connected domain of class C^2, $\alpha \in (0,1)$ and f satisfies (H1) and $f \geq 0$. Then we have*

i) (weak source) the solution of (32) satisfying (43) is positive and unique, and

ii) (strong source) the solution of (32) satisfying (46) is positive and unique.

We complete our theorems with a non-existence result for solution with a previously defined asymptotic behavior, as we see in Theorem 0.2.1. We have

Theorem 0.2.4 (Non-existence) *Assume that Ω is an open, bounded and connected domain of class C^2, $\alpha \in (0,1)$ and f satisfies $(H1)$, $(H3)$ and $f \geq 0$. Then we have*

i) (weak source) Suppose that p satisfies (35), γ satisfies (42) and $\tau \in (-1,0) \setminus \{\gamma + 2\alpha\}$. Then equation (32) does not have a solution u satisfying (37).

ii) (strong source) Suppose that p satisfies (45), γ satisfies (44) and $\tau \in (-1,0) \setminus \{\frac{\gamma}{p}\}$. Then, equation (32) does not have a solution u satisfying (37).

All these results stated so far deal with equation (30) in the case $g \equiv 0$, but they may also be applied when $g \not\equiv 0$ and, in particular, these result improve those given in [51]. In what follows we describe how to obtain this. We start with some notation, we consider $L^1_\omega(\bar{\Omega}^c)$ the weighted L^1 space in $\bar{\Omega}^c$ with weight

$$\omega(y) = \frac{1}{1 + |y|^{N+2\alpha}}, \quad \text{for all } y \in \mathbb{R}^N.$$

Our hypothesis on the external values g is the following

$$(H4) \qquad \text{The function } g : \bar{\Omega}^c \to \mathbb{R} \text{ is measurable and } g \in L^1_\omega(\bar{\Omega}^c).$$

Given g satisfying $(H4)$, we define

$$G(x) = \frac{1}{2} \int_{\mathbb{R}^N} \frac{\tilde{g}(x+y)}{|y|^{N+2\alpha}} dy, \quad x \in \Omega, \tag{47}$$

where

$$\tilde{g}(x) = \begin{cases} 0, & x \in \bar{\Omega}, \\ g(x), & x \in \bar{\Omega}^c. \end{cases} \tag{48}$$

We observe that

$$G(x) = -(-\Delta)^\alpha \tilde{g}(x), \quad x \in \Omega.$$

Hypothesis $(H4)$ implies that G is continuous in Ω as seen in Lemma 2.2.1 and has an explosion of order $d(x)^{\beta-2\alpha}$ towards the boundary $\partial\Omega$, if g has an explosion of order $d(x)^\beta$ for some $\beta \in (-1,0)$, as we shall see in Proposition 2.3.3. We observe that under the hypothesis $(H4)$, if u is a solution of equation (30), then $u - \tilde{g}$ is the solution of

$$\begin{aligned} (-\Delta)^\alpha u(x) + |u|^{p-1}u(x) &= f(x) + G(x), \quad && x \in \Omega, \\ u(x) &= 0, && x \in \bar{\Omega}^c, \\ \text{lím}_{x\in\Omega, x\to\partial\Omega}\, u(x) &= +\infty, \end{aligned} \tag{49}$$

and vice versa, if v is a solution of (49), then $v + \tilde{g}$ is a solution of (30).

Thus, using Theorem 0.2.1-0.2.4, we can state the corresponding results of existence, uniqueness and non-existence for (30), combining f with g to define a new external source

$$F(x) = G(x) + f(x), \quad x \in \Omega. \tag{50}$$

With this we can state appropriate hypothesis for g and thus we can write theorems, one corresponding to each of Theorem 0.2.1-0.2.4.

Moreover, in chapter 3 we study self-generated interior blow-up solutions to fractional elliptic equations

$$\begin{aligned} (-\Delta)^\alpha u(x) + |u|^{p-1}u(x) &= 0, \qquad && x \in \Omega \setminus \mathcal{C}, \\ u(x) &= 0, && x \in \Omega^c, \\ \text{lím}_{x\in\Omega\setminus\mathcal{C},\ x\to\mathcal{C}}\, u(x) &= +\infty, \end{aligned} \tag{51}$$

where $p > 1$, Ω is an open bounded C^2 domain in $\mathbb{R}^N$, $\mathcal{C} \subset \Omega$ is a compact C^2 manifold with $N-1$ multiples dimensions and without boundary. The explosion of solutions to (51) near $\mathcal{C}$ is governed by a function $c : (-1, 0] \to \mathbb{R}$, defined as

$$c(\tau) = \int_0^{+\infty} \frac{|1-t|^\tau + (1+t)^\tau - 2}{t^{1+2\alpha}} dt. \tag{52}$$

This function plays the role of the function C defined by (40), but it has certain differences. In Section 3.2 we prove the existence of a number $\alpha_0 \in (0,1)$ such that $\alpha \in [\alpha_0, 1)$ the function c is always positive in $(-1,0)$, while if $\alpha \in (0,\alpha_0)$ then there exists exists a unique $\tau_1(\alpha) \in (-1,0)$ such that $c(\tau_1(\alpha)) = 0$ and $c(\tau) > 0$ in $(-1,\tau_1(\alpha))$ and $c(\tau) < 0$ in $(\tau_1(\alpha), 0)$, see Proposition 3.2.1. We notice here that $\tau_1(\alpha) > \tau_0(\alpha)$ if $\alpha \in (0,\alpha_0)$, where $\tau_0(\alpha)$ is from Theorem 0.2.1 (also see Proposition 2.3.1 for details).

Throughout this part we denote the distance function

$$D : \Omega \setminus \mathcal{C} \to \mathbb{R}_+, \quad D(x) = dist(x, \mathcal{C}). \tag{53}$$

Now we are ready to state our main theorems in chapter 3 on the existence uniqueness and asymptotic behavior of interior blow-up solutions to equation (51). These theorems deal separately the case $\alpha \in (0, \alpha_0)$ and $\alpha \in [\alpha_0, 1)$.

Theorem 0.2.5 *Assume that $\alpha \in (0, \alpha_0)$ and the assumptions on Ω and $\mathcal{C}$ as above. Then we have:*
(i) If

$$1 + 2\alpha < p < 1 - \frac{2\alpha}{\tau_1(\alpha)}, \tag{54}$$

then problem (51) admits a unique positive solution u satisfying

$$0 < \liminf_{x\in\Omega\setminus\mathcal{C}, x\to\mathcal{C}} u(x)D(x)^{\frac{2\alpha}{p-1}} \leq \limsup_{x\in\Omega\setminus\mathcal{C}, x\to\mathcal{C}} u(x)D(x)^{\frac{2\alpha}{p-1}} < +\infty. \tag{55}$$

(ii) If

$$\max\{1 - \frac{2\alpha}{\tau_1(\alpha)} + \frac{\tau_1(\alpha) + 1}{\tau_1(\alpha)}, 1\} < p < 1 - \frac{2\alpha}{\tau_1(\alpha)}. \tag{56}$$

Then, for any $t > 0$, there is a positive solution u of problem (51) satisfying

$$\lim_{x\in\Omega\setminus\mathcal{C}, x\to\mathcal{C}} u(x)D(x)^{-\tau_1(\alpha)} = t. \tag{57}$$

(iii) If one of the following three conditions holds

a) $1 < p \leq 1 + 2\alpha$ *and* $\tau \in (-1, 0) \setminus \{\tau_1(\alpha)\}$,

b) $1 + 2\alpha < p < 1 - \frac{2\alpha}{\tau_1(\alpha)}$ *and* $\tau \in (-1, 0) \setminus \{\tau_1(\alpha), -\frac{2\alpha}{p-1}\}$ *or*

c) $p \geq 1 - \frac{2\alpha}{\tau_1(\alpha)}$ *and* $\tau \in (-1, 0)$,

then problem (51) does not admit any solution u satisfying

$$0 < \liminf_{x\in\Omega\setminus\mathcal{C}, x\to\mathcal{C}} u(x)D(x)^{-\tau} \leq \limsup_{x\in\Omega\setminus\mathcal{C}, x\to\mathcal{C}} u(x)D(x)^{-\tau} < +\infty. \tag{58}$$

Theorem 0.2.6 *Assume that $\alpha \in [\alpha_0, 1)$ and the assumptions on Ω and $\mathcal{C}$ as above. Then we have:*
(i) If $p > 1 + 2\alpha$, then problem (51) admits a unique positive solution u satisfying (55).
(ii) If $p > 1$, then problem (51) does not admit any solution u satisfying (58) for any $\tau \in (-1, 0) \setminus \{-\frac{2\alpha}{p-1}\}$.

0.2.2. Semilinear fractional elliptic equations involving measures

In chapter 4, we are concerned with the existence of weak solutions to the semilinear fractional elliptic problem

$$\begin{aligned} (-\Delta)^\alpha u + g(u) = \nu, \quad &\text{in} \quad \Omega, \\ u = 0, \quad &\text{in} \quad \Omega^c, \end{aligned} \tag{59}$$

where $\Omega \subset \mathbb{R}^N$ is an open bounded C^2 domain, $g : \mathbb{R} \mapsto \mathbb{R}$ is a continuous function and ν is a Radon measure such that $\int_\Omega \rho^\beta d|\nu| < +\infty$ for some $\beta \in [0, \alpha]$ and $\rho(x) = dist(x, \Omega^c)$. The fractional Laplacian $(-\Delta)^\alpha$ with $\alpha \in (0,1)$ is defined by

$$(-\Delta)^\alpha u(x) = \lim_{\epsilon \to 0^+} (-\Delta)^\alpha_\epsilon u(x),$$

where for $\epsilon > 0$,

$$(-\Delta)^\alpha_\epsilon u(x) = -\int_{\mathbb{R}^N} \frac{u(z) - u(x)}{|z-x|^{N+2\alpha}} \chi_\epsilon(|x-z|) dz \tag{60}$$

and

$$\chi_\epsilon(t) = \begin{cases} 0, & \text{if} \quad \mathrm{t} \in [0, \epsilon], \\ 1, & \text{if} \quad \mathrm{t} > \epsilon. \end{cases}$$

We remark that (60) is equivalent to (31).

When $\alpha = 1$, the semilinear elliptic problem

$$\begin{aligned} -\Delta u + g(u) = \nu, \quad &\text{in} \quad \Omega, \\ u = 0, \quad &\text{on} \quad \partial\Omega, \end{aligned} \tag{61}$$

has been extensively studied by numerous authors in the last 30 years. A fundamental contribution is due to Brezis [17], Bénilan and Brezis [10], where ν is a bounded measure in Ω and the function $g : \mathbb{R} \to \mathbb{R}$ is nondecreasing, positive on $(0, +\infty)$ and satisfies the subcritical assumption:

$$\int_1^{+\infty} (g(s) - g(-s)) s^{-2\frac{N-1}{N-2}} ds < +\infty.$$

They proved the existence and uniqueness of the solution for problem (61). Baras and Pierre [9] studied (61) when $g(u) = |u|^{p-1}u$ for $p > 1$ and ν is absolutely continuous with respect to the Bessel capacity $C_{2,\frac{p}{p-1}}$, to obtain a solution. In [101] Véron extended Benilan and Brezis results in replacing the Laplacian by a general uniformly elliptic second order differential operator with Lipschitz continuous coefficients; he obtained existence and uniqueness results for solutions, when $\nu \in \mathfrak{M}(\Omega, \rho^\beta)$ with

$\beta \in [0,1]$, $\mathfrak{M}(\Omega, \rho^\beta)$ denotes the space of Radon measures in Ω satisfying

$$\int_\Omega \rho^\beta d|\nu| < +\infty, \tag{62}$$

in particular, $\mathfrak{M}(\Omega, \rho^0) = \mathfrak{M}^b(\Omega)$ is the set of bounded Radon measures, the function g is nondecreasing and satisfies the β-subcritical assumption:

$$\int_1^{+\infty} (g(s) - g(-s)) s^{-2\frac{N+\beta-1}{N+\beta-2}} ds < +\infty.$$

The study of general semilinear elliptic equations with measure data have been investigated, such as the equations involving measures boundary data which was initiated by Gmira and Véron [62] who adapted the method introduced by Bénilan and Brezis to obtain the existence and uniqueness of solution. This subject has been vastly expanded in recent years, see the papers of Marcus and Véron [74, 76, 77, 78, 79], Bidaut-Véron and Vivier [14], Bidaut-Véron, Hung and Véron [13].

In this chapter, our interesting is to study the existence and uniqueness of solutions of semilinear fractional elliptic problem (59) in a measure framework. Before stating our main theorem we make precise the notion of weak solution used in this thesis.

Definition 0.2.1 *We say that u is a weak solution of (59), if $u \in L^1(\Omega)$, $g(u) \in L^1(\Omega, \rho^\alpha dx)$ and*

$$\int_\Omega [u(-\Delta)^\alpha \xi + g(u)\xi] dx = \int_\Omega \xi d\nu, \quad \forall \xi \in \mathbb{X}_\alpha, \tag{63}$$

where $\mathbb{X}_\alpha \subset C(\mathbb{R}^N)$ is the space of functions ξ satisfying:

(i) $\mathrm{supp}(\xi) \subset \bar{\Omega}$,

(ii) $(-\Delta)^\alpha \xi(x)$ exists for all $x \in \Omega$ and $|(-\Delta)^\alpha \xi(x)| \le C$ for some $C > 0$,

(iii) there exist $\varphi \in L^1(\Omega, \rho^\alpha dx)$ and $\epsilon_0 > 0$ such that $|(-\Delta)^\alpha_\epsilon \xi| \le \varphi$ a.e. in Ω, for all $\epsilon \in (0, \epsilon_0]$.

We notice that for $\alpha = 1$, the test space $\mathbb{X}_\alpha$ is used as $C_0^{1,L}(\Omega)$, which has similar properties like (i) and (ii). The counter part for the Laplacian of assumption (iii) would be that the difference quotient $\nabla_{x_j,h}[u](.) := h^{-1}[\partial_{x_j} u(. + h\mathbf{e}_j) - \partial_{x_j} u(.)]$ is bounded by an L^1-function, which is true since

$$\nabla_{x_j,h}[u](x) = h^{-1} \int_0^h \partial^2_{x_j,x_j} u(x + s\mathbf{e}_j) ds.$$

We denote by G_α the Green kernel of $(-\Delta)^\alpha$ in Ω and by $\mathbb{G}_\alpha[.]$ the Green operator defined by

$$\mathbb{G}_\alpha[f](x) = \int_\Omega G_\alpha(x,y)f(y)dy, \qquad \forall f \in L^1(\Omega, \rho^\alpha dx). \tag{64}$$

For $N \geq 2$, $\alpha \in (0,1)$ and $\beta \in [0,\alpha]$, we define the critical exponent

$$k_{\alpha,\beta} = \begin{cases} \frac{N}{N-2\alpha}, & \text{if} \quad \beta \in [0, \frac{\mathrm{N}-2\alpha}{\mathrm{N}}\alpha], \\ \frac{N+\alpha}{N-2\alpha+\beta}, & \text{if} \quad \beta \in (\frac{\mathrm{N}-2\alpha}{\mathrm{N}}\alpha, \alpha]. \end{cases} \tag{65}$$

Our main result in this part is the following.

Theorem 0.2.7 *Assume that $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is an open bounded C^2 domain, $\alpha \in (0,1)$, $\beta \in [0,\alpha]$ and $k_{\alpha,\beta}$ is defined by (65). Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous, nondecreasing function, satisfying*

$$g(r)r \geq 0, \quad \forall r \in \mathbb{R} \quad \text{and} \quad \int_1^{+\infty} (g(s) - g(-s))s^{-1-k_{\alpha,\beta}}ds < +\infty. \tag{66}$$

Then for any $\nu \in \mathfrak{M}(\Omega, \rho^\beta)$ problem (59) admits a unique weak solution u. Furthermore, the mapping: $\nu \mapsto u$ is increasing and

$$-\mathbb{G}_\alpha[\nu_-] \leq u \leq \mathbb{G}_\alpha[\nu_+] \quad \text{a.e.} \;\; \text{in } \Omega, \tag{67}$$

where ν_+ and ν_- are respectively the positive and negative part in the Jordan decomposition of ν.

We note that for $\alpha = 1$ and $\beta \in [0,1)$, we have

$$k_{1,\beta} > \frac{N+\beta}{N-2+\beta}, \tag{68}$$

where $k_{1,\beta}$ is given in (65) and the number in right hand side of (68) is from Theorem 3.7 in [101]. Inspired by [62, 101], the existence of solution could be extended in assuming that $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies *the* (N,α,β)*-weak-singularity assumption*, that is, there exists $r_0 > 0$ such that

$$g(x,r)r \geq 0, \quad \forall (x,r) \in \Omega \times (\mathbb{R} \setminus (-r_0, r_0)),$$

and

$$|g(x,r)| \leq \tilde{g}(|r|), \quad \forall (x,r) \in \Omega \times \mathbb{R},$$

where $\tilde{g} : [0,\infty) \to [0,\infty)$ is continuous, nondecreasing and satisfies that

$$\int_1^{+\infty} \tilde{g}(s)s^{-1-k_{\alpha,\beta}}ds < +\infty.$$

We also give a stability result which shows that problem (59) is weakly closed in the space of measures $\mathfrak{M}(\Omega, \rho^\beta)$. Moreover, we characterize the behaviour of the solution u of (59) when $\nu = \delta_a$ for some $a \in \Omega$. We also study the case where $g(r) = |r|^{k-1}r$ when $k \geq k_{\alpha,\beta}$, which doesn't satisfy (66). We show that a necessary and sufficient condition in order a weak solution to problem

$$\begin{aligned} (-\Delta)^\alpha u + |u|^{k-1}u = \nu, \quad &\text{in} \quad \Omega, \\ u = 0, \quad &\text{in} \quad \Omega^c, \end{aligned} \tag{69}$$

to exist where ν is a positive bounded measure and vanishes on compact subsets K of Ω with zero $C_{2\alpha,k'}$ Bessel-capacity.

0.2.3. Weakly and strongly singular solutions of semilinear fractional elliptic equations

The aim of chapter 5 is to study the properties of the weak solution to problem

$$\begin{aligned} (-\Delta)^\alpha u + u^p = k\delta_0, \quad &\text{in} \quad \Omega, \\ u = 0, \qquad &\text{in} \quad \Omega^c, \end{aligned} \tag{70}$$

where Ω is an open bounded C^2 domain of $\mathbb{R}^N (N \geq 2)$ containing 0, $\alpha \in (0,1)$, $k > 0$, $p \in (0, \frac{N}{N-2\alpha})$ and δ_0 denotes the Dirac measure at 0.

In 1980, Brezis in[16] (also see [10]) obtained that the problem

$$\begin{aligned} -\Delta u + u^q = k\delta_0 \quad &\text{in} \quad \Omega, \\ u = 0 \qquad &\text{on} \quad \partial\Omega \end{aligned} \tag{71}$$

admits a unique solution u_k for $1 < q < N/(N-2)$, while no solution exists when $q \geq N/(N-2)$. Later on, Brezis and Véron in [18] proved that the problem

$$\begin{aligned} -\Delta u + u^q = 0 \quad &\text{in} \quad \Omega \setminus \{0\}, \\ u = 0 \quad &\text{on} \quad \partial\Omega \end{aligned} \tag{72}$$

admits only the zero solution when $q \geq N/(N-2)$. When $1 < q < N/(N-2)$, Véron in [100] described all the possible singular behaviour of positive solutions of (72). In particular he proved that this behaviour is always isotropic (when $(N+1)/(N-1) \leq q < N/(N-2)$ the assumption of positivity is unnecessary) and that two types of singular behaviour occur:

(i) either $u(x) \sim c_N k|x|^{2-N}$ as $x \to 0$ and k can take any positive value; u is said to have a *weak singularity* at 0, and actually $u = u_k$,

(ii) or $u(x) \sim c_{N,q}|x|^{-\frac{2}{q-1}}$ as $x \to 0$; u is said to have a *strong singularity* at 0, and

$u = u_\infty := \lim_{k\to\infty} u_k$.

In a recent work, Chen and Véron [39] derived that for $1 + \frac{2\alpha}{N} < p < \frac{N}{N-2\alpha}$, the problem

$$\begin{aligned} (-\Delta)^\alpha u + u^p &= 0 \quad \text{in} \quad \Omega \setminus \{0\}, \\ u &= 0 \quad \text{in} \quad \Omega^c \end{aligned} \tag{73}$$

admits a solution u_s satisfying

$$\lim_{x\to 0} u_s(x)|x|^{\frac{2\alpha}{p-1}} = c_p, \tag{74}$$

for some $c_p > 0$. Moreover u_s is the unique positive solution of (73) in the class set of

$$0 < \liminf_{x\to 0} u(x)|x|^{\frac{2\alpha}{p-1}} \le \limsup_{x\to 0} u(x)|x|^{\frac{2\alpha}{p-1}} < +\infty. \tag{75}$$

We say that u is a *weakly singular solution* of (73) if $\limsup_{x\to 0} |u(x)||x|^{N-2\alpha} < +\infty$, or *strongly singular solution* if $\lim_{x\to 0} |u(x)||x|^{N-2\alpha} = +\infty$.

We also in [40] obtained that there exists a unique weak solution to the problem

$$\begin{aligned} (-\Delta)^\alpha u + g(u) &= \nu \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{in} \quad \Omega^c, \end{aligned} \tag{76}$$

where g is a subcritical nonlinearity, ν is a Radon measure in Ω. In the fractional framework, the definition of weak solution is given as follows.

Definition 0.2.2 *A function $u \in L^1(\Omega)$ is a weak solution of (76) if $g(u) \in L^1(\Omega, \rho^\alpha dx)$ and*

$$\int_\Omega [u(-\Delta)^\alpha \xi + g(u)\xi] dx = \int_\Omega \xi d\nu, \quad \forall \xi \in \mathbb{X}_\alpha, \tag{77}$$

where $\rho(x) = dist(x, \Omega^c)$ and $\mathbb{X}_\alpha \subset C(\mathbb{R}^N)$ is the space of functions ξ satisfying:

(i) $\mathrm{supp}(\xi) \subset \bar{\Omega}$,

(ii) $(-\Delta)^\alpha \xi(x)$ exists for all $x \in \Omega$ and $|(-\Delta)^\alpha \xi(x)| \le C$ for some $C > 0$,

(iii) there exist $\varphi \in L^1(\Omega, \rho^\alpha dx)$ and $\epsilon_0 > 0$ such that $|(-\Delta)^\alpha_\epsilon \xi| \le \varphi$ a.e. in Ω, for all $\epsilon \in (0, \epsilon_0]$.

According to Theorem 0.2.7with $g(s) = |s|^{p-1}s$ and $\nu = k\delta_0$, we have following result for problem (70).

Proposition 0.2.1 *Assume that $p \in (0, \frac{N}{N-2\alpha})$. Then for any $k > 0$, problem (70) admits a unique weak solution u_k satisfying*

$$\mathbb{G}_\alpha[k\delta_0] - \mathbb{G}_\alpha[(\mathbb{G}_\alpha[k\delta_0])^p] \le u_k \le \mathbb{G}_\alpha[k\delta_0] \quad \text{in} \quad \Omega. \tag{78}$$

Moreover, (i) *u_k is positive in* Ω;
(ii) $\{u_k\}_k$ *is a sequence increasing functions, i.e.*

$$u_k(x) \le u_{k+1}(x), \quad \forall x \in \Omega. \tag{79}$$

Here $\mathbb{G}_\alpha[\cdot]$ is the Green operator defined by

$$\mathbb{G}_\alpha[\nu](x) = \int_\Omega G_\alpha(x,y)d\nu(y), \quad \forall \nu \in \mathfrak{M}(\Omega, \rho^\alpha), \tag{80}$$

where G_α is the Green kernel of $(-\Delta)^\alpha$ in $\Omega \times \Omega$. By monotonicity of $\{u_k\}_k$,

$$u_\infty(x) := \lim_{k\to\infty} u_k(x), \quad \forall x \in \mathbb{R}^N \setminus \{0\} \tag{81}$$

and then $u_\infty(x) \in \mathbb{R}_+ \cup \{+\infty\}$ for $x \in \mathbb{R}^N \setminus \{0\}$.

Our purpose in this chapter is to do further study on the properties of u_k, including the regularity and the limit of u_k, which is the unique weak solution of (70).

Theorem 0.2.8 *Assume that* $1+\frac{2\alpha}{N} \ge \frac{2\alpha}{N-2\alpha}$, $p \in (0, \frac{N}{N-2\alpha})$, *$u_k$ is the weak solution of (70) and u_∞ is given by (81).*

Then u_k is a classical solution of (73). Furthermore,
(i) *if* $p \in (0, 1+\frac{2\alpha}{N})$,

$$u_\infty(x) = \infty, \quad \forall x \in \Omega; \tag{82}$$

(ii) *if* $p \in (1+\frac{2\alpha}{N}, \frac{N}{N-2\alpha})$,

$$u_\infty = u_s,$$

where u_s is the solution of (73) satisfying (74).

The result of part (i) indicates that there is no strongly singular solution to problem (73) for $p \in (0, 1+\frac{2\alpha}{N})$, which is different from the result for Laplacian case. This phenomenon comes from the fact that the fractional Laplacian is a nonlocal operator, which requires the solution to belong to $L^1(\Omega)$, therefore no barrier can be constructed for $p < 1 + \frac{2\alpha}{N}$. On the contrary, part (ii) points out that u_∞ is the least strongly singular solution of (73).

Next we consider the case $1 + \frac{2\alpha}{N} < \frac{2\alpha}{N-2\alpha}$. It occurs only when

$$\frac{\sqrt{5}-1}{4} N < \alpha < 1, \quad N = 2, 3.$$

In this situation, it is obvious that $\frac{N}{2\alpha} < 1 + \frac{2\alpha}{N}$. Now we state our second theorem as following.

Theorem 0.2.9 *Assume that* $1+\frac{2\alpha}{N} < \frac{2\alpha}{N-2\alpha}$, $p \in (0, \frac{N}{N-2\alpha})$, u_k *is the weak solution of (70) and* u_∞ *is given by (81).*

Then u_k *is a classical solution of (73). Furthermore,*
(i) if $p \in (0, \frac{N}{2\alpha})$, *then*

$$u_\infty(x) = \infty, \qquad \forall x \in \Omega;$$

(ii) if $p \in (1+\frac{2\alpha}{N}, \frac{2\alpha}{N-2\alpha})$, *then* u_∞ *is a classical solution of (73) and there exist* $\rho_0 > 0$ *and* $c_0 > 0$ *such that*

$$c_0|x|^{-\frac{(N-2\alpha)p}{p-1}} \le u_\infty \le u_s, \qquad \forall x \in B_{\rho_0}(0) \setminus \{0\}; \tag{83}$$

(iii) if $p = \frac{2\alpha}{N-2\alpha}$, *then* u_∞ *is a classical solution of (73) and there exist* $\rho_0 > 0$ *and* $c_1 > 0$ *such that*

$$c_1 \frac{|x|^{-\frac{(N-2\alpha)p}{p-1}}}{(1+|\log(|x|)|)^{\frac{1}{p-1}}} \le u_\infty \le u_s, \qquad \forall x \in B_{\rho_0}(0) \setminus \{0\}; \tag{84}$$

(iv) if $p \in (\frac{2\alpha}{N-2\alpha}, \frac{N}{N-2\alpha})$, *then*

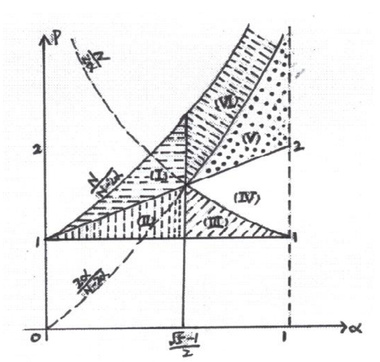
$$u_\infty = u_s,$$

where u_s *is the solution of (73) satisfying (74)*

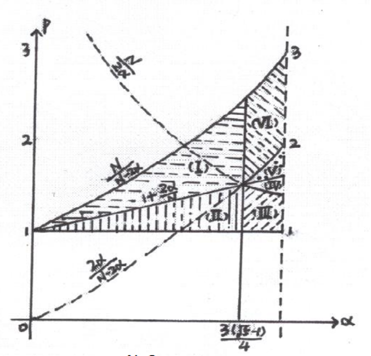
We note that Theorem 0.2.8 and Theorem 0.2.9 do not provide description of u_∞ in the region

$$\begin{aligned}\mathcal{U} := &\left\{(\alpha,p) \in (0,1)\times(1,\tfrac{N}{N-2}) : 1+\tfrac{2\alpha}{N} < \tfrac{2\alpha}{N-2\alpha},\ \tfrac{N}{2\alpha} \le p \le 1+\tfrac{2\alpha}{N}\right\} \\ &\bigcup \left\{(\alpha,p) \in (0,1)\times(1,\tfrac{N}{N-2}) : 1+\tfrac{2\alpha}{N} \ge \tfrac{2\alpha}{N-2\alpha},\ p = 1+\tfrac{2\alpha}{N}\right\},\end{aligned}$$

which is region (IV) and the segment $p = 1+\frac{2\alpha}{N}$, see the pictures $N=2$ and $N=3$.



N=2



N=3

0.2.4. Semilinear fractional elliptic equations with gradient nonlinearity involving measure

The purpose of chapter 6 is to study the existence of weak solutions to the semilinear fractional elliptic problem

$$\begin{aligned} (-\Delta)^\alpha u + \epsilon g(|\nabla u|) &= \nu, \quad \text{in} \quad \Omega, \\ u &= 0, \quad \text{in} \quad \Omega^c, \end{aligned} \tag{85}$$

where $\Omega \subset \mathbb{R}^N (N \geq 2)$ is an open bounded C^2 domain, $\alpha \in (1/2, 1)$, $g : \mathbb{R}_+ \mapsto \mathbb{R}_+$ be a continuous function, $\epsilon = 1$ or -1 and $\nu \in \mathfrak{M}(\Omega, \rho^\beta)$ with $\beta \in [0, 2\alpha - 1)$. In particular, we denote $\mathfrak{M}^b(\Omega) = \mathfrak{M}(\Omega, \rho^0)$. The associated positive cones are respectively $\mathfrak{M}_+(\Omega, \rho^\beta)$ and $\mathfrak{M}_+^b(\Omega)$. According to the value of ϵ, we speak of an absorbing nonlinearity the case $\epsilon = 1$ and a source nonlinearity the case $\epsilon = -1$. In a recent work, Nguyen-Phuoc and Véron [82] obtained the existence of solutions to the viscous Hamilton-Jacobi equation

$$\begin{aligned} -\Delta u + h(|\nabla u|) &= \nu, \quad \text{in} \quad \Omega, \\ u &= 0, \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{86}$$

when $\nu \in \mathfrak{M}^b(\Omega)$, h is a continuous nondecreasing function vanishing at 0 which satisfies

$$\int_1^{+\infty} h(s) s^{-\frac{2N-1}{N-1}} ds < +\infty.$$

More recently, Bidaut-Véron, García-Huidobro and Véron in [12] studied the existence of solutions to the Dirichlet problem

$$\begin{aligned} -\Delta_p u + \epsilon |\nabla u|^q &= \nu, \quad \text{in} \quad \Omega, \\ u &= 0, \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{87}$$

with $1 < p \leq N$, $\epsilon = 1$ or -1, $q > 0$ and $\nu \in \mathfrak{M}^b(\Omega)$.

Our interest in this part is to investigate the existence of weak solutions to fractional equations involving nonlinearity in the gradient term and with Radon measure. In order the fractional Laplacian be the dominant operator in terms of order of differentiation, it is natural to assume that $\alpha \in (1/2, 1)$.

Definition 0.2.3 *We say that u is a weak solution of (85), if $u \in L^1(\Omega)$, $|\nabla u| \in L^1_{loc}(\Omega)$, $g(|\nabla u|) \in L^1(\Omega, \rho^\alpha dx)$ and*

$$\int_\Omega [u(-\Delta)^\alpha \xi + \epsilon g(|\nabla u|)\xi] dx = \int_\Omega \xi d\nu, \qquad \forall\, \xi \in \mathbb{X}_\alpha, \tag{88}$$

where $\mathbb{X}_\alpha$ is defined in Definition 0.2.1.

Our main result in the case $\epsilon = 1$ is the following.

Theorem 0.2.10 *Assume that* $\epsilon = 1$ *and* $g : \mathbb{R}_+ \mapsto \mathbb{R}_+$ *is a continuous function verifying* $g(0) = 0$ *and*

$$\int_1^{+\infty} g(s)s^{-1-p^*_\alpha}ds < +\infty, \tag{89}$$

where

$$p^*_\alpha = \frac{N}{N-2\alpha+1}. \tag{90}$$

Then for any $\nu \in \mathfrak{M}_+(\Omega, \rho^\beta)$ *with* $\beta \in [0, 2\alpha-1)$, *problem (85) admits a nonnegative weak solution* u_ν *which satisfies*

$$u_\nu \le \mathbb{G}_\alpha[\nu]. \tag{91}$$

When $\epsilon = -1$, we have to consider the critical value $p^*_{\alpha,\beta}$ which depends truly on β and is expressed by

$$p^*_{\alpha,\beta} = \frac{N}{N-2\alpha+1+\beta}. \tag{92}$$

We observe that $p^*_{\alpha,0} = p^*_\alpha$ and $p^*_{\alpha,\beta} < p^*_\alpha$ when $\beta > 0$. In the source case, the assumptions on g are of a different nature from in the absorption case, namely

(G) $g : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is a continuous function which satisfies

$$g(s) \le c_1 s^p + \sigma_0, \quad \forall s \ge 0, \tag{93}$$

for some $p \in (0, p^*_{\alpha,\beta})$, where $c_1 > 0$ and $\sigma_0 > 0$.

Our main result concerning the source case is the following.

Theorem 0.2.11 *Assume that* $\epsilon = -1$, $\nu \in \mathfrak{M}(\Omega, \rho^\beta)$ *with* $\beta \in [0, 2\alpha - 1)$ *is nonnegative,* g *satisfies* (G) *and*

(i) $p \in (0, 1)$, *or*

(ii) $p = 1$ *and* c_1 *is small enough, or*

(iii) $p \in (1, p^*_{\alpha,\beta})$, σ_0 *and* $\|\nu\|_{\mathfrak{M}(\Omega,\rho^\beta)}$ *are small enough.*

Then problem (85) admits a weak nonnegative solution u_ν *which satisfies*

$$u_\nu \ge \mathbb{G}_\alpha[\nu]. \tag{94}$$

In the last section of this part, we assume that Ω contains 0 and give pointwise estimates of the positive solutions

$$\begin{aligned} (-\Delta)^\alpha u + |\nabla u|^p &= \delta_0 \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{in} \quad \Omega^c \end{aligned} \tag{95}$$

with $0 < p < p_\alpha^*$. Combining properties of the Riesz kernel with a bootstrap argument, we prove that any weak solution of (95) is regular outside 0 and is actually a classical solution of

$$\begin{aligned} (-\Delta)^\alpha u + |\nabla u|^p = 0 \quad &\text{in} \quad \Omega \setminus \{0\}, \\ u = 0 \quad &\text{in} \quad \Omega^c. \end{aligned} \tag{96}$$

These pointwise estimates are quite easy to establish in the case $\alpha = 1$, but much more delicate when the diffusion operator is non-local. We give sharp asymptotics of the behaviour of u near 0 and prove that the solution of (95) is unique in the class of positive solutions.

Capítulo 1

On Liouville type theorems for fully nonlinear elliptic equations with gradient term

Abstract: in this chapter[1], we prove a Hadamard property and Liouville type theorems for viscosity solutions of fully nonlinear elliptic partial differential equations with a gradient term, both in the whole space and in an exterior domain.

1.1. Introduction

In the study of nonlinear elliptic equations in bounded domains, non-existence results for entire solutions of related limiting equations appear as a crucial ingredient. In the search for positive solutions for semi-linear elliptic equations with nonlinearity behaving as a power at infinity, one is interested in the non-negative solutions of the equation

$$\Delta u + u^p = 0, \qquad \text{in} \quad \mathbb{R}^N. \tag{1.1}$$

The question is for which value of p, typically $p > 1$, this equation has or has no solution. This has been one of the motivations that has pushed forward the study of Liouville type theorems for general equations in $\mathbb{R}^N$ and in unbounded domains like cones or exterior domains. On the other hand, the understanding of structural characteristics of general linear or nonlinear operators has been another motivation for advancing the study of Liouville type theorems that have attracted many researchers. See the work in [1, 2, 19, 61, 86].

If we consider the Pucci's operators instead of the Laplacian, the question set above becomes very interesting, since most of the techniques used in the case of the

[1]This chapter is based on the paper: H. Chen and P. Felmer, *On Liouville type theorems for fully nonlinear elliptic equations with gradient term,* J. Differential Equations 255, 2167-2195 (2013).

Laplacian are not available. The Liouville type theorem for the equation analogous to equation (1.1) has not been proved in full generality, but only in the radial case. On the other hand, the Liouville type theorem for non-negative solutions of

$$\mathcal{M}^{-}u + u^{p} \leq 0, \qquad \text{in} \quad \mathbb{R}^{N}, \tag{1.2}$$

has been studied in full extent by Cutrì and Leoni [45] and generalized in various directions by Felmer and Quaas [52, 54, 55] Capuzzo-Dolcetta and Cutrì [30] and Armstrong and Sirakov in [3]. In all these cases the solutions of the inequality are considered in the viscosity sense.

In a recent paper, Armstrong and Sirakov in [4] made significant progress in the understanding of the structure of positive solutions of equations generalizing (1.2), shading light even for equations of the form

$$\Delta u + f(u) \leq 0, \qquad \text{in} \quad \mathbb{R}^{N}. \tag{1.3}$$

They propose a general approach to non-existence and existence of solutions of the general inequality

$$Q(u) + f(x,u) \leq 0, \qquad \text{in} \quad \mathbb{R}^{N}, \tag{1.4}$$

where the second order differential operator Q satisfies certain scaling property, it possesses fundamental solutions behaving as power asymptotically and it satisfies some other properties, common to elliptic operators, like a weak comparison principle, a quantitative strong comparison principle and a very weak Harnack inequality, see hypothesis (H1)-(H5) in [4]. Regarding the nonlinearity f, the results in [4] unravel a very interesting property, that is, that the behavior of the function f only matters near $u = 0$ and for x large. These results are new even for the case of (1.3). Moreover, the authors in [4] are able to apply their approach to equation (1.4) in exterior domains without any boundary condition, providing another truly new result.

It is the purpose of this chapter to extend the results described above in order to include elliptic operators with first order term. The introduction of a first order term may brake the scaling property of the differential operator and it allows for the appearance of non-homogeneous fundamental solutions, not even asymptotically. Thus, the approach in [4] cannot be applied to this more general situation and we have to find different arguments. Interestingly, to prove our results we use the more elementary approach taken in the original work by Cutrì and Leoni, where the Hadamard property, obtained through the comparison principle, is combined with the appropriate choice of a function to test the equation. The underline principle is the asymptotic comparison between the solutions of the inequality and the fundamental solution. This can be interpreted as the interaction between the elliptic operator, including first order term, and the nonlinearity (the zero order term).

We start the precise description of our results by recalling the definition of the

Pucci's operators. In this chapter, we consider

$$\mathcal{M}^+(r, D^2u) = \Lambda(r)\sum_{e_i\geq 0} e_i + \lambda(r)\sum_{e_i<0} e_i \tag{1.5}$$

and

$$\mathcal{M}^-(r, D^2u) = \lambda(r)\sum_{e_i\geq 0} e_i + \Lambda(r)\sum_{e_i<0} e_i, \tag{1.6}$$

where $e_1, \ldots, e_N$ are the eigenvalues of D^2u, $\lambda, \Lambda : [0,\infty) \to \mathbb{R}$ are continuous, λ_0 and Λ_0 are positive constants and

$$0 < \lambda_0 \leq \lambda(r) \leq \Lambda(r) \leq \Lambda_0 < +\infty, \quad \forall r = |x|, x \in \mathbb{R}^N. \tag{1.7}$$

Our purpose is to study the non-negative solutions of

$$\mathcal{M}^-(r, D^2u) + \sigma(r)|Du| + f(x,u) \leq 0 \ \text{ in } \ \Omega, \tag{1.8}$$

with $\Omega = \mathbb{R}^N$ or an exterior domain and $\sigma : [0,\infty) \to \mathbb{R}$ and $f : \Omega \times (0,\infty) \to (0,\infty)$ are continuous. In this chapter, by an exterior domain we mean a set $\Omega = \mathbb{R}^N \setminus K$ connected, where K is nonempty compact subset of $\mathbb{R}^N$.

We consider the fundamental solutions for the second order differential operator $\varphi, \psi : (0,\infty) \to \mathbb{R}$ in given (1.41) and (1.42), which are non-trivial radially symmetric solutions of

$$\mathcal{M}^-(r, D^2u) + \sigma(r)|Du| = 0, \qquad x \in \mathbb{R}^N \setminus \{0\}. \tag{1.9}$$

satisfying

(i) ψ is increasing and either $\text{lím}_{r\to\infty}\psi(r) = \infty$ or $\text{lím}_{r\to\infty}\psi(r) = 0$ and

(ii) φ is decreasing and either $\text{lím}_{r\to\infty}\varphi(r) = -\infty$ or $\text{lím}_{r\to\infty}\varphi(r) = 0$.

Now we are in a position to make precise assumptions about the interaction between the differential operator and the nonlinearity. We assume that

(f_1) $f : \Omega \times (0,\infty) \to (0,\infty)$, $\lambda, \Lambda, \sigma : [0,\infty) \to \mathbb{R}$ are continuous.

(f_2) We have

$$\text{lím}_{r=|x|\to\infty} \frac{r^2}{1+\sigma_-(r)r} f(x,s) = \infty$$

uniformly on compact subsets of $(0,\infty)$. Here and in what follows $\sigma_- = \text{máx}\{-\sigma, 0\}$.

In order to state the next assumption we need a definition. Given $\mu > 0$, $a > 1$, $k > 0$ and $\tau > 0$ we define

$$\Psi_k(\tau) = \frac{\varphi(a\tau)}{\varphi(\tau)} \inf_{x\in B_{a\tau}\setminus B_\tau} \left\{ \frac{r^2}{\sigma_-(r)r+1} \inf_{k\varphi(ar)\leq s\leq\mu} \frac{f(x,s)}{s} \right\}. \tag{1.10}$$

We assume:

(f_3) If $\lim_{r\to\infty}\varphi(r)=0$ then we assume the existence of constants $\mu>0$ and $a>1$ such that, defining

$$h(k)=\limsup_{\tau\to\infty}\Psi_k(\tau),$$

one of the following holds:

(i) for all $k>0$ we have $h(k)=\infty$ or

(ii) for all $k>0$ we have

$$0<\liminf_{\tau\to\infty}\Psi_k(\tau)\quad\text{and}\quad\lim_{k\to\infty}h(k)=\infty \tag{1.11}$$

and there is a constant $C\in\mathbb{R}$ such that

$$r\sigma(r)>C,\qquad\text{for all } r>0. \tag{1.12}$$

Now we state our first Liouville type theorem for inequality (1.8) in $\mathbb{R}^N$.

Theorem 1.1.1 *Assume that f satisfies (f_1), (f_2) and (f_3). Then inequality (1.8) in $\mathbb{R}^N$ does not have a non-trivial viscosity solution $u\geq 0$.*

We observe that hypothesis (f_3) does restrict f when $\lim_{r\to\infty}\varphi(r)=-\infty$.

Regarding hypotheses (f_2) and (f_3) we would like to notice that they are natural extensions of hypotheses $(f2)-(f3)$ in [4], when $\sigma\not\equiv 0$ and the fundamental solution φ is not necessarily power-like. Thus, we are generalizing the results in [4] in the case of a one-homogeneous differential operator in $\mathbb{R}^N$. It is also interesting to notice that hypotheses (f_2) and (f_3) appear explicitly and in a natural way in our proof of the theorem.

When the condition (i) is satisfied we say that inequality (1.8) is sub-critical and when condition (ii) holds, we say it is critical. In case of

$$\Delta u+u^p\leq 0,$$

we say the inequality is sub-critical when $p<N/(N-2)$ and when $p=N/(N-2)$ it is critical. When $p>N/(N-2)$ we say the inequality is super-critical and here the existence of positive solution holds. Accordingly, we would like to define a notion of super-criticality the cases (i) and (ii) do not hold. However, in Theorem 1.2.3 we provide an example where there is no solution in a super-critical sub-region, showing that further study is required to understand the critical boundary.

In the case of an exterior domain, we need to consider also the interaction between the differential operator and the nonlinearity at ∞. We need a definition in order to state our assumptions. Given $\mu > 0$, $a > 1$, $k > 0$ and $\tau > 0$ we define

$$\tilde{\Psi}_k(\tau) = \frac{\psi(\tau)}{\psi(a\tau)} \inf_{x \in B_{a\tau} \setminus B_\tau} \left\{ \frac{r^2}{\sigma_-(r)r + 1} \inf_{\mu \le s \le k\psi(ar)} \frac{f(x,s)}{s} \right\}.$$

Now we assume that

(f_4) If $\lim_{r\to\infty} \psi(r) = \infty$ then we assume the existence of constants $\mu > 0$ and $a > 1$ such that, defining

$$\tilde{h}(k) = \limsup_{\tau\to\infty} \tilde{\Psi}_k(\tau),$$

one of the following holds:

(i) for all $k > 0$ we have $\tilde{h}(k) = \infty$ or

(ii) for all $k > 0$ we have

$$0 < \liminf_{\tau\to\infty} \tilde{\Psi}_k(\tau) \quad \text{and} \quad \lim_{k\to 0^+} \tilde{h}(k) = \infty \tag{1.13}$$

and there is a constant $C \in \mathbb{R}$ such that (1.12) holds.

For an exterior domain we have the following non-existence result.

Theorem 1.1.2 *Assume that Ω is an exterior domain and f satisfies (f_1), (f_2), (f_3) and (f_4). Then inequality (1.8) in Ω does not have a non-trivial viscosity solution $u \ge 0$.*

We observe that hypothesis (f_4) does restrict f when $\lim_{r\to\infty} \psi(r) = 0$.

As for (f_3), hypothesis (f_4) is the natural extension of $(f4)$ in [4] to our case. Here we allow $\sigma \not\equiv 0$ and ψ not power-like, thus generalizing [4].

In case of (f_4) we may also define the notion of criticality for (1.8) in an analogous way as for (f_3). Since here the behavior of f is relevant at zero and infinity mixed cases appear, as for example, an inequality critical at 0 and sub-critical at ∞ or vice verse.

In the proofs of Theorem 1.1.1 and 1.1.2 we use some basic properties of the functions

$$m(r) = \inf_{x \in B_r} u(x), \quad m_0(r) = \inf_{x \in B_r \setminus B_{r_0}} u(x) \quad \text{and} \quad M(r) = \inf_{x \notin B_r} u(x) \tag{1.14}$$

in connection with the fundamental solutions, as given by the Hadamard property provided in Theorem 1.4.3. Then we test the equation with an adequate function and we use the asymptotic assumptions on f and the fundamental solutions to obtain a contradiction with the existence of non-trivial non-negative solutions. In the proofs of our theorems we only consider $a = 2$.

The interaction between the elliptic operator and the nonlinearity, that is expressed in hypotheses (f_3) and (f_4), is not easy to understand in full generality. However, beyond the cases studied in [4], there are many interesting examples that well illustrate the relevance of our results to understand the general structure of solutions for these equations. In particular, in Section 1.2 we discuss some examples for the inequality

$$\Delta u + \sigma(r)|Du| + f(u) \geq 0, \quad \text{in } \mathbb{R}^{\mathrm{N}}, \tag{1.15}$$

which are not covered in the literature. In the first example we analyze the nonlinearity $f(u) = u^p$ with a function σ associated to a fundamental solution with oscillatory power, see (1.26). In this case, it is interesting to observe the way to introduce σ which affects the critical power of the nonlinearity. In the second example we analyze the case of $f(u) = u^p(1 + \log|u|)^\nu$ and a function σ providing a fundamental solution matching the non-homogeneous nonlinearity, see (1.31). In this case we analyze the range of p and ν for non-existence of solutions to (1.15).

For the existence of positive solutions of (1.8), it is nature to consider the supercritical assumption, that is, the case when hypotheses (f_3) and (f_4) are not satisfied, which means

$$\liminf_{\tau\to\infty} \Psi_k(\tau) = 0 \quad \text{or} \quad \limsup_{k\to\infty} h(k) < \infty$$

and

$$\liminf_{\tau\to\infty} \tilde{\Psi}_k(\tau) = 0 \quad \text{or} \quad \limsup_{k\to\infty} \tilde{h}(k) < \infty,$$

where h, $\tilde{h}$, Ψ_k and $\tilde{\Psi}_k$ were defined in (f_3) and (f_4). We observe that supercriticality holds when $h(k) = 0$ or $\tilde{h}(k) = 0$ for any $k > 0$, but it is not true that under this notion of super-criticality a positive solution of (1.8) always exists, as we see in Section 1.2 through an example.

In the last part of this chapter we consider a Liouville type theorem in the case f is a linear function, that is, $f(x, s) = h(x)s$, that interestingly can be proved using the same techniques considered in the nonlinear case. This problem has been recently studied by Rossi [89] after some previous work by Berestycki, Hamel and Nadirashvili [20], Berestycki, Hamel and Roques [21] and Berestycki, Hamel and Rossi [22]. Rossi [89] proved a Liouville type theorem for generally unbounded domains, assuming that

$$\liminf_{x\in\Omega, |x|\to\infty} \frac{u(x) + 1}{dist(x, \partial\Omega)} = 0. \tag{1.16}$$

It is clear that when Ω is an exterior domain then $dist(x, \partial\Omega) \leq |x|$, so that (1.16) implies a linear growth constraint on u. Thus, it is interesting to investigate the existence or non-existence of positive solutions of the corresponding equation when (1.16) does no longer hold. Here is our result:

Theorem 1.1.3 *Let u be a viscosity nonnegative solution of*

$$\mathcal{M}^{-}(r, D^2u) + \sigma(r)|Du| + h(x)u \leq 0, \quad \text{in} \quad \Omega, \tag{1.17}$$

where Ω is an exterior domain. Assume further that λ and Λ satisfy (1.7) and that

(h_1) $h : \mathbb{R}^N \to \mathbb{R}$ *and* $\sigma : \mathbb{R}_+ \to \mathbb{R}$ *are continuous, h is positive and σ is negative.*

(h_2) *There exists a function $\kappa : \mathbb{R}_+ \to \mathbb{R}_+$ of class C^1 such that*

$$\lim_{r\to\infty} \kappa'(r) = 0 \tag{1.18}$$

and there is a constant $\mu \geq 1$ such that

$$1 \leq \kappa(r) \max_{r-\kappa(r)\leq s\leq r} |\sigma(s)| \leq \mu, \quad \text{for all } r > 0. \tag{1.19}$$

(h_3) *There exists a sequence $r_n \to \infty$ such that*

$$\lim_{n\to\infty} \inf_{r\in(r_n-\kappa(r_n),r_n)} \left\{ h(r) - e^{\frac{\mu}{\lambda_0}}(2\Lambda_0 + 1)\sigma^2(r) \right\} > 0. \tag{1.20}$$

Then $u \equiv 0$.

1.2. Discussion and examples

We devote this section to present various examples that illustrate the relevance of our results. We start discussing the relation between σ and the fundamental solution, then we present two examples for Theorem 1.1.1 and we conclude the section with a theorem related with the concept of super-criticality. We concentrate our discussion on Theorem 1.1.1 regarding inequality (1.8) in $\mathbb{R}^N$ in the case of the elliptic operator

$$Q(r, u) = \Delta u + \sigma(r)|Du|. \tag{1.21}$$

We may certainly construct examples for Theorem 1.1.2 regarding the inequality in an exterior domain and for general Pucci's operators as in our theorems.

In Section 1.3 we study with details the fundamental solutions associated to the differential operator in equation (1.8). We see that in the case of Q, the decreasing fundamental solution is given by

$$\varphi(r) = -\int_1^r s^{1-N} e^{\int_1^s \sigma(\tau)d\tau} ds + L_\varphi,$$

where L_φ is a constant so that when $\lim_{r\to\infty} \varphi(r)$ exists, it becomes equal to 0, see Proposition 1.3.1 and its proof. With this formula we may construct many examples of fundamental solutions with a whole variety of asymptotic behavior. We start showing the effect of the first order term on the behavior of the fundamental solution. Our first example is for

$$\sigma(r) = \frac{d}{dr}(\sin r \log r), \quad r \geq 1,$$

properly extended to $[0,1)$. Then we have, for a constant L_φ,

$$\varphi(r) = -\int_1^r s^{1-N+\sin(s)} ds + L_\varphi, \quad r \geq 1. \tag{1.22}$$

We observe that this fundamental solution does not behave like a power at infinity. The second example is given by

$$\sigma(r) = \frac{d}{dr}(\cos(\log\log r)\log r), \quad r \geq e,$$

properly extended to $[0,e)$. The associated fundamental solution does not behave like a power, not even asymptotically. Its behavior is oscillatory, with slower rate than (1.22). For a third example we consider

$$\sigma(r) = -\frac{d}{dr}((\alpha + 2 - N)\log r + \log\log r), \quad r \geq e, \tag{1.23}$$

extended to $[0,e)$ as a continuous function with fundamental solution

$$\varphi(r) = -e^{\alpha+1}\int_e^r \frac{s^{-1-\alpha}}{\log s} ds + L_\varphi, \quad r \geq e. \tag{1.24}$$

This example is different from earlier ones since it is not oscillatory, but with an asymptotic behavior which is not power-like because of its logarithmic term.

It is interesting to see that we may prescribe explicit fundamental solutions by providing functions q like

$$\varphi(r) = e^{-q(r)}, \quad r \geq 0, \tag{1.25}$$

assuming that q is increasing and $\lim_{r\to\infty} q(r) = \infty$. It is easy to check that this fundamental solution is obtained when the function σ is given by

$$\sigma(r) = \frac{N-1}{r} - q'(r) + \frac{q''(r)}{q'(r)}, \quad r \geq 0. \tag{1.26}$$

In view of our examples later, we will require q to be such that $r\sigma(r)$ is bounded. This condition is not necessary to use Theorem 1.1.1, but under this condition (f_2)

and (f_3) greatly simplify. Assume that $N \geq 3$ and

$$q(r) = (N-2)\log r + \frac{1}{2}\sin(\log(\log r))\log r, \quad r > e. \tag{1.27}$$

After some direct calculation, we see that $q'(r) > 0$ and, if σ is defined as in (1.26), $r\sigma(r)$ is bounded. In this case, the fundamental solution is

$$\varphi(r) = r^{-(N-2+\frac{1}{2}\sin(\log\log r))}, \quad r \geq e,$$

which is a power exhibiting an oscillatory exponent. In this situation we have

Theorem 1.2.1 *Assume that $N \geq 3$ and*

$$1 < p < \frac{N-\frac{1}{2}}{N-\frac{5}{2}}, \tag{1.28}$$

then there is no positive solution to the nonlinear inequality

$$\Delta u + \sigma(|x|)|Du| + u^p \leq 0, \quad \text{in } \mathbb{R}^N. \tag{1.29}$$

This theorem shows the effect of the first order term on the critical exponent. It is interesting to notice that the critical exponent is enlarged because the dimension is decreased by 1/2, the amplitude of the oscillatory power.

Proof of Theorem 1.2.1. The application of Theorem 1.1.1 requires to analyze the function Ψ_k in (f_3), since all other hypotheses are satisfied. Using the definition of Ψ_k, that $p > 1$ and that $r\sigma(r)$ is bounded, we find that for r large

$$\Psi_k(r) = k^{p-1}e^{-q(2r)p+q(r)+2\log r}. \tag{1.30}$$

Computing the exponent, from (1.27) we see that

$$\begin{aligned} -q(2r)p + q(r) = & -(N-2)p\log 2 - \frac{p}{2}\sin(\log(\log(2r)))\log 2 \\ & + \left[-(N-2)(p-1) - \frac{p}{2}\sin(\log(\log(2r))) + \frac{1}{2}\sin(\log(\log(r)))\right]\log r. \end{aligned}$$

We claim that there exists a sequence $\{r_n\}$ such that $\lim_{n\to\infty} r_n = \infty$,

$$\lim_{n\to\infty} \sin(\log(\log(2r_n))) = -1 \quad \text{and} \quad \lim_{n\to\infty} \sin(\log(\log(r_n))) = -1.$$

Assume the claim is true now, then we get $\lim_{n\to\infty} \Psi_k(r_n) = \infty$ if we have $-(N-2)(p-1) + (p-1)/2 + 2 > 0$, which is exactly (1.28). To complete the proof we check the claim. We let r_n be the positive solution of the equation

$$\sin\left(\frac{\log(\log(2r_n)) + \log(\log(r_n))}{2}\right) = -1, \quad n \in \mathbb{N},$$

that satisfies $\text{lím}_{n\to\infty} r_n = \infty$. Then we have

$$\sin(\log(\log(2r_n))) + \sin(\log(\log(r_n))) = -2\cos\left(\frac{\log(\log(2r_n)/\log(r_n))}{2}\right),$$

from where the claim follows, since

$$\lim_{n\to\infty} [\sin(\log(\log(2r_n))) + \sin(\log(\log(r_n)))] = -2.$$

□

Now we consider another example for the function

$$q(r) = (N-2)\log r + \log(\log r), \quad r > e. \tag{1.31}$$

Its associated fundamental solution is a power with a logarithmic factor

$$\varphi(r) = \frac{1}{r^{N-2}\log r}, \quad r \geq 1,$$

and $r\sigma(r)$ is bounded, for σ as in (1.26). Next we apply Theorem 1.1.1 to the nonlinearity $f(u) = u^p(|\log u| + 1)^\nu$ with differential term Q with σ as above.

Theorem 1.2.2 *Assume that* $N \geq 3$ *and*

$$1 < p < \frac{N}{N-2} \quad \text{and} \quad \nu \in \mathbb{R}, \quad \text{or}$$

$$p = \frac{N}{N-2} \quad \text{and} \quad \nu \geq -\frac{2}{N-2},$$

then there is no positive solution to the nonlinear inequality

$$\Delta u + \sigma(|x|)|Du| + u^p(|\log u| + 1)^\nu \leq 0, \quad \text{in} \ \mathbb{R}^N. \tag{1.32}$$

This theorem provides an example of a non-existence result where the nonlinearity and the fundamental solution are not homogeneous and they match in such a way that the hypothesis (f_3) is satisfied.

Proof of Theorem 1.2.2. In this case, the function Ψ_k in (f_3) is given by

$$\Psi_k(r) = k^{p-1}e^{-q(2r)p+q(r)+2\log r}[|\log k - q(2r)| + 1]^\nu. \tag{1.33}$$

From here and (1.31) we have

$$-q(2r)p + q(r) = (N-2)((1-p)\log r - p\log 2) - p\log(\log(2r)) + \log(\log(r)).$$

From here, there exists a constant $C > 0$ so that, for r large, we have

$$\Psi_k(r) \geq Ck^{p-1} r^{(N-2)(p-1)} (\log r)^{p-1} (\log r - \log k)^{\nu}.$$

If $p < \frac{N}{N-2}$ with $\nu \in \mathbb{R}$ or $p = \frac{N}{N-2}$ with $\nu > -\frac{2}{N-2}$, then $\lim_{r\to\infty} \Psi_k(r) = \infty$. In the limit case, when $p = \frac{N}{N-2}$ with $\nu = -\frac{2}{N-2}$, we have $\lim_{r\to\infty} \Psi_k(r) \geq Ck^{p-1}$, from where we complete the proof using Theorem 1.1.1. □

In the examples discussed above the fact that $f(s)/s$ is decreasing allowed to get the inner most infimum easily. Then, the monotonicity of the remaining term in r allowed to get the second infimum and thus Ψ_k was obtained explicitly. In what follows we give simplified versions of hypothesis (f_3).

Remark 1.2.1 *In hypothesis (f_3), we may define the function h in a different way, namely we may consider*

$$h_1(k) = \liminf_{\tau\to\infty} \Psi_k(\tau) \quad \text{or}$$

$$h_2(k) = \liminf_{r=|x|\to\infty} \frac{\varphi(ar)}{\varphi(r)} \frac{r^2}{\sigma_-(r)r + 1} \inf_{k\varphi(ar)\leq s\leq \mu} \frac{f(x,s)}{s}.$$

These two definitions give rise to two stronger versions of hypothesis (f_3). We may use this condition to deal with the example given by (1.26).

Remark 1.2.2 *If we assume that there exists $C \in \mathbb{R}$ such that*

$$\int_R^{2R} \sigma(r)dr \geq C > -\infty \tag{1.34}$$

for each $R > 1$, for the function h_2 defined above, we have

$$h_2(k) = \liminf_{r=|x|\to\infty} \frac{r^2}{\sigma_-(r)r + 1} \inf_{k\varphi(ar)\leq s\leq \mu} \frac{f(x,s)}{s}.$$

In case $f(x,s) = s^p$ and assuming $\lim_{r\to\infty}\varphi(r) = 0$, the function h_2 becomes

$$h_2(r) = k^{p-1} \lim_{r\to\infty} \frac{r^2\varphi(r)^{p-1}}{\sigma_-(r)r + 1}.$$

We conclude this section discussing an example for the notion of super-criticality suggested by (f_3). For the power nonlinearity and the Laplacian

$$-\Delta u + u^p \leq 0, \quad x \in \mathbb{R}^N,$$

it is well known that a solution exists in the super-critical case, that is, when $p > \frac{N}{N-2}$. In our case, we defined super-critical inequality in the introduction, but our example below shows that this may not be fully appropriate.

We assume that α and ν are positive numbers and $p > 1$. We let $\sigma : [0, \infty) \to \mathbb{R}$ as in (1.23) and $f : (0, \infty) \to \mathbb{R}$ be as in

$$f(s) = s^p(|\log s| + 1)^\nu, \quad s \in (0, \infty). \tag{1.35}$$

Considering the corresponding fundamental solution given in (1.22), we get

$$\lim_{r \to \infty} \varphi(r) = 0 \quad \text{and} \quad \lim_{r \to \infty} \frac{\varphi(r)}{r^{-\alpha}(\log r)^{-1}} = e^{\alpha+2}\alpha^{-1}.$$

Next, given $k > 0$, we find positive constants C and $\bar{R}$ such that for $r > \bar{R}$

$$\frac{Ck^{p-1}(\log r - \log k)^\nu}{r^{\alpha(p-1)-2}(\log r)^{p-1}} \leq \Psi_k(r) \leq \frac{k^{p-1}(\log r - \log k)^\nu}{Cr^{\alpha(p-1)-2}(\log r)^{p-1}},$$

where Ψ_k was defined in (1.10). Then we obtain the following three cases:

$$(C_1) \text{ sub} - \text{critical} \quad p < \frac{2}{\alpha} + 1, \quad \text{or} \quad p = \frac{2}{\alpha} + 1 \quad \text{and} \quad \nu > \frac{2}{\alpha},$$

$$(C_2) \text{ critical} \quad p = \frac{2}{\alpha} + 1 \quad \text{and} \quad \nu = \frac{2}{\alpha},$$

$$(C_3) \text{ super} - \text{critical} \quad p > \frac{2}{\alpha} + 1, \quad \text{or} \quad p = \frac{2}{\alpha} + 1 \quad \text{and} \quad \nu < \frac{2}{\alpha}.$$

And we obtain some non-existence and existence results as following:

Theorem 1.2.3 *Suppose that σ and f are given as above and $\Omega = \mathbb{R}^N$, then:*
(i) If (C_1) or (C_2) holds, then (1.8) does not have a positive solution.
(ii) If

$$p = \frac{2}{\alpha} + 1 \quad \text{and} \quad 0 < \frac{2}{\alpha} - 1 < \nu < \frac{2}{\alpha}, \tag{1.36}$$

then (1.8) does not have a positive solution.
(iii) If $p > \frac{2}{\alpha} + 1$, then (1.8) has a positive solution.

We see that the sub-region for (p, α) given in (1.36) is super-critical, however we can prove non-existence of a positive solution there. This fact shows that more analysis in needed to understand the critical boundary in general.

1.3. Fundamental solutions and basic properties

In this section we construct the fundamental solutions of the nonlinear second order operator with first order term given in (1.9). These special radial solutions are important tools for understanding the behavior of general viscosity solutions of (1.9).

We start defining the dimension like numbers, which are relevant in our construction. We let $n, N : (0, \infty) \to \mathbb{R}$ be the functions given by

$$n(r) = \begin{cases} \frac{\Lambda(r)}{\lambda(r)}(N-1)+1 & \text{if} \quad r\sigma(r) \leq \Lambda(r)(N-1), \\ N & \text{if} \quad r\sigma(r) > \Lambda(r)(N-1), \end{cases} \tag{1.37}$$

and

$$N(r) = \begin{cases} \frac{\lambda(r)}{\Lambda(r)}(N-1)+1 & \text{if} \quad r\sigma(r) > -\lambda(r)(N-1), \\ N & \text{if} \quad r\sigma(r) \leq -\lambda(r)(N-1). \end{cases} \tag{1.38}$$

We also need to consider the following functions

$$m_\lambda(r) = \begin{cases} \lambda(r) & \text{if} \quad r\sigma(r) \leq \Lambda(r)(N-1), \\ \Lambda(r) & \text{if} \quad r\sigma(r) > \Lambda(r)(N-1) \end{cases} \tag{1.39}$$

and

$$M_\lambda(r) = \begin{cases} \lambda(r) & \text{if} \quad r\sigma(r) \leq -\lambda(r)(N-1), \\ \Lambda(r) & \text{if} \quad r\sigma(r) > -\lambda(r)(N-1). \end{cases} \tag{1.40}$$

Given $r_1 > 0$ and constants L_φ and L_ψ we define the functions $\varphi, \psi : (0, \infty) \to \mathbb{R}$ as follows:

$$\varphi(r) = -\int_{r_1}^{r} s e^{\int_{r_1}^{s}(\frac{\sigma(\tau)}{m_\lambda(\tau)} - \frac{n(\tau)}{\tau})d\tau} ds + L_\varphi \tag{1.41}$$

and

$$\psi(r) = \int_{r_1}^{r} s e^{-\int_{r_1}^{s}(\frac{\sigma(\tau)}{M_\lambda(\tau)} + \frac{N(\tau)}{\tau})d\tau} ds + L_\psi. \tag{1.42}$$

Proposition 1.3.1 *(i) The function φ defined in (1.41), is of class $C^{1,1}$ and it satisfies equation (1.9). Moreover, φ is a decreasing function and, by choosing the constant L_φ adequately, it satisfies*

$$\lim_{r\to\infty} \varphi(r) = -\infty \quad \text{or} \quad \lim_{r\to\infty} \varphi(r) = 0. \tag{1.43}$$

(ii) The function ψ defined in (1.42) is of class $C^{1,1}$ and it satisfies equation (1.9). Moreover, ψ is an increasing function and, by choosing the constant L_ψ ade-

quately, it satisfies

$$\lim_{r\to\infty} \psi(r) = \infty \quad \text{or} \quad \lim_{r\to\infty} \psi(r) = 0. \tag{1.44}$$

The functions φ and ψ satisfying (1.43) and (1.44), respectively, are called fundamental solutions of the operator (1.9).

Proof of Proposition 1.3.1. We recall that, given a C^2 radially symmetric function $u(x) = v(|x|)$, the eigenvalues of D^2u are $v''(r)$ with multiplicity 1 and $v'(r)/r$ with multiplicity $N-1$.

(i) By the definition (1.41), we have

$$\varphi'(r) = -re^{\int_{r_1}^{r}(\frac{\sigma(\tau)}{m_\lambda(\tau)} - \frac{n(\tau)}{\tau})d\tau} \quad \text{and} \quad \varphi''(r) = \left[\frac{1-n(r)}{r} + \frac{\sigma(r)}{m_\lambda(r)}\right]\varphi'(r).$$

Then we readily see that $\varphi'(r) < 0$, so that φ is a decreasing function, and using (1.37) and (1.39) we find that

$$\begin{array}{lll} \varphi''(r) \geq 0 & \text{if} \quad r\sigma(r) \leq \Lambda(r)(N-1) & \text{and} \\ \varphi''(r) < 0 & \text{if} \quad r\sigma(r) > \Lambda(r)(N-1). & \end{array}$$

Thus, whenever $r\sigma(r) \leq \Lambda(r)(N-1)$, we obtain

$$\begin{aligned} \mathcal{M}^-(r, D^2\varphi) + \sigma(r)|D\varphi| &= \lambda(r)\varphi''(r) + \Lambda(r)\frac{N-1}{r}\varphi'(r) - \sigma(r)\varphi'(r) \\ &= \lambda(r)\left[\varphi''(r) + \frac{n(r)-1}{r}\varphi'(r) - \frac{\sigma(r)}{\lambda(r)}\varphi'(r)\right] = 0 \end{aligned}$$

and, whenever $r\sigma(r) > \Lambda(r)(N-1)$, we obtain

$$\begin{aligned} \mathcal{M}^-(r, D^2\varphi) + \sigma(r)|D\varphi| &= \Lambda(r)\varphi''(r) + \Lambda(r)\frac{N-1}{r}\varphi'(r) - \sigma(r)\varphi'(r) \\ &= \Lambda(r)\left[\varphi''(r) + \frac{N-1}{r}\varphi'(r) - \frac{\sigma(r)}{\Lambda(r)}\varphi'(r)\right] = 0. \end{aligned}$$

We conclude then, that φ is a solution of equation (1.9), it is of class $C^{1,1}$ and, since φ is decreasing, the limit in (1.43) exists. If it is bounded, we may find L_φ so that φ has limit equal to zero.

(ii) can be proved in a completely analogous way. □

Remark 1.3.1 *We observe that the functions φ and ψ are not necessarily convex or concave and that they may change their concavity along r.*

In what follows we derive various properties of the fundamental solutions that we need in the sequel. We start with properties for the function φ.

Lemma 1.3.1 *If* $\lim_{r\to\infty}\varphi(r) = 0$, *then there exists a sequence* $\{r_n\}$ *diverging to infinity such that*

$$\lim_{n\to\infty} r_n\varphi'(r_n) = 0. \tag{1.45}$$

Proof. This is equivalent to $\limsup_{r\to\infty} r\varphi'(r) < 0$ implies $\lim_{r\to\infty}\varphi(r) = -\infty$, which is obviously true. $\square$

Proposition 1.3.2 *Suppose* $\lim_{r\to\infty}\varphi(r) = 0$ *and assume that (1.12) holds, then there is a constant* $C_0 > 0$ *such that*

$$-\frac{r\varphi'(r)}{\varphi(r)} \leq C_0, \qquad \textit{for all } r \geq 1.$$

Proof. We first see that, from definition of φ and (1.12), we have

$$\frac{(r\varphi'(r))'}{\varphi'(r)} = \frac{r\varphi''(r)+\varphi'(r)}{\varphi'(r)} = -n(r) + 2 + \frac{r\sigma(r)}{m_\lambda(r)} \geq C,$$

for a certain negative constant C and all $r \geq 1$. Then, since φ is decreasing,

$$(r\varphi'(r))' \leq C\varphi'(r), \quad \text{for all } r \geq 1.$$

Considering the sequence given in Lemma 1.3.1, we integrate to obtain

$$r_n\varphi'(r_n) - r\varphi'(r) \leq C(\varphi(r_n) - \varphi(r)), \quad \text{for all } n \in \mathbb{N}.$$

Then, taking limit as $n \to \infty$ and using the hypothesis, we find

$$-r\varphi'(r) \leq -C\varphi(r),$$

from where we conclude, taking $C_0 = -C$. $\square$

Proposition 1.3.3 *Assume that* $\lim_{r\to\infty}\varphi(r) = 0$ *and* σ *satisfies*

$$\int_r^{2r} \sigma(\tau)d\tau \geq C, \tag{1.46}$$

for some $C \in \mathbb{R}$ *and for all* $r \geq 1$. *Then, there exists* $C_0 > 0$ *such that*

$$\frac{\varphi(2r)}{\varphi(r)} \geq C_0, \quad \textit{for all } r \geq 1.$$

Proof. By definition of φ and hypothesis (1.46), we have

$$\frac{\varphi'(2r)}{\varphi'(r)} \quad = \quad 2e^{\int_r^{2r}(\frac{\sigma(\tau)}{m_\lambda(\tau)} - \frac{n(\tau)}{\tau})d\tau} \geq 2e^{c(\int_r^{2r}\sigma(\tau)d\tau) - C\log 2} \geq C_0,$$

for certain constants c, C and C_0. Then, since φ is decreasing, we have

$$\varphi'(2r) \leq C_0\varphi'(r), \qquad \text{for all } r \geq 1.$$

Thus, integrating in $[r, R]$, taking limit as $R \to \infty$ and using the hypothesis we get the result. $\square$

Next we obtain two other propositions, but now regarding the function ψ.

Proposition 1.3.4 *Assume* $\lim_{r\to\infty}\psi(r) = \infty$ *and* σ *satisfy (1.12), then there exist* $C_0 > 0$ *and* $r_1 > 0$ *such that*

$$\frac{r\psi'(r)}{\psi(r)} \leq C_0, \qquad \textit{for all } r \geq r_1.$$

Proof. From (1.12) and definition of ψ we have

$$\frac{(r\psi'(r))'}{\psi'(r)} = \frac{r\psi''(r) + \psi'(r)}{\psi'(r)} = -N(r) + 2 - \frac{r\sigma(r)}{M_\lambda(r)} \leq C,$$

for some $C > 0$. Let r_1 be such that $\psi(r_1) > 0$ and consider that

$$(r\psi'(r))' \leq C\psi'(r),$$

then we integrate in $[r_1, r]$ and get

$$\frac{r\psi'(r)}{\psi(r)} \leq C + \frac{r_1\psi'(r_1) - C\psi(r_1)}{\psi(r)} \leq C + \frac{r_1\psi'(r_1)}{\psi(r_1)} \equiv C_0,$$

for all $r \geq r_1$ completing the proof. $\square$

Proposition 1.3.5 *Assuming that* $\lim_{r\to\infty}\psi(r) = \infty$ *and* σ *satisfies (1.46), then there exists* $C_0 > 0$ *and* $r_1 > 0$ *such that*

$$\frac{\psi(r)}{\psi(2r)} \geq C_0, \qquad \textit{for all } r \geq r_1.$$

Proof. By definition of ψ and from (1.46) we have

$$\frac{\psi'(r)}{\psi'(2r)} \quad = \quad 2^{-1}e^{\int_r^{2r}\left(\frac{\sigma(\tau)}{M_\lambda(\tau)} + \frac{N(\tau)}{\tau}\right)d\tau} \geq 4C_0,$$

for a certain positive constant C_0, and then

$$\psi'(r) \geq 4C_0\psi'(2r), \qquad \text{for all } r \geq 1.$$

We let r_0 so that $\psi(2r_0) > 0$ and we integrate from r_0 to r to obtain

$$\frac{\psi(r)}{\psi(2r)} \geq 2C_0 + \frac{\psi(r_0) - C_0\psi(2r_0)}{\psi(2r)}.$$

From here we find r_1 such that the desired inequality holds for all $r \geq r_1$. $\square$

1.4. The Hadamard property

The Hadamard property and the Liouville type theorems are based on the Strong Maximum Principle and the Comparison Principle. Here we recall a version of these principles that are best suited for our purposes. We start with the Comparison Principle for viscosity solutions:

Theorem 1.4.1 (See Ishii [63].) *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set. Let λ, Λ and σ satisfy hypothesis (f_1) and the functions λ and Λ satisfy (1.7). If u and v are respectively super- and sub-solutions in the viscosity sense of*

$$\mathcal{M}^-(r, D^2u) + \sigma(|x|)|Du| = 0, \quad in \ \ \Omega,$$

respectively and $u \geq v$ on $\partial\Omega$, then $u \geq v$ in Ω.

Next we have the Strong Minimum Principle:

Theorem 1.4.2 (See Bardi and Da Lio [8].) *Let u be a super-solution in the viscosity solution of*

$$\mathcal{M}^-(r, D^2u) + \sigma(|x|)|Du| = 0, \quad in \ \ \Omega$$

If u attains its minimum at an interior point of Ω, then u is a constant.

Now we are in a position of proving the Hadamard property, a nonlinear Hadamard theorem. This theorem allows to obtain estimates for the behavior of super-solutions of (1.9) with regards to fundamental solutions. We have

Theorem 1.4.3 *Let $\Omega = \mathbb{R}^N$ or an exterior domain and suppose that $u \in C(\Omega)$ is a positive viscosity super-solution of (1.9) in Ω. We let $r_0 > 0$ be such that $B^c_{r_0} \subset \Omega$ and $r_0 < r_1 < r_2$. Then*

(*i*) *if $\Omega = \mathbb{R}^N$, for the function $m(r)$ defined in (1.14), we have*

$$m(r) \geq \frac{\varphi(r) - \varphi(r_1)}{\varphi(r_2) - \varphi(r_1)} m(r_2) + \frac{\varphi(r_2) - \varphi(r)}{\varphi(r_2) - \varphi(r_1)} m(r_1), \quad r_1 < r < r_2; \qquad (1.47)$$

(ii) *if Ω is an exterior domain, $m_0(r)$ is defined as in (1.14) and r_1 is large enough then for all $r_1 < r < r_2$ we have*

$$m_0(r) \geq \frac{\varphi(r) - \varphi(r_1)}{\varphi(r_2) - \varphi(r_1)} m_0(r_2) + \frac{\varphi(r_2) - \varphi(r)}{\varphi(r_2) - \varphi(r_1)} m_0(r_1); \tag{1.48}$$

(iii) *if Ω is an exterior domain and the function $M(r)$ is defined as in (1.14), for $r_0 < r < r_1$ we have*

$$\frac{M(r_1)}{\psi(r_1) - \psi(r_0)} \leq \frac{M(r)}{\psi(r) - \psi(r_0)}. \tag{1.49}$$

Proof. (i) It is clear that $m(r)$ is positive and non-increasing. By Proposition 1.3.1, we know that the function $\Phi(r) = C_1(\varphi(r) - \varphi(r_1)) + C_2$ with

$$C_1 = \frac{m(r_2) - m(r_1)}{\varphi(r_2) - \varphi(r_1)} > 0 \quad \text{and} \quad C_2 = m(r_1)$$

satisfies (1.9) for $0 < r_1 < r_2$ and $\Phi(r_1) = m(r_1)$ and $\Phi(r_2) = m(r_2)$. By the Comparison Principle (Theorem 1.4.1), we have

$$u(x) \geq \Phi(x), \ x \in B_{r_2} \setminus B_{r_1}. \tag{1.50}$$

But, also by the Comparison Principle (Theorem 1.4.1), we have that $m(r) = \text{mín}\{u(x) \,|\, x \in \mathbb{R}^N, |x| = r\}$, so the conclusion follows from (1.50).

(ii) In the case of m_0 we observe that by the Strong Maximum Principle either $m_0(r)$ is constant for all $r \geq r_0$ or $m(r) = \text{mín}\{u(x) \,|\, x \in \mathbb{R}^N, |x| = r\}$, for all $r \geq r_1$ and r_1 large enough. Then the result is obtained in the same way as for m.

(iii) Let $r_1 > r_0$ and

$$\Phi(r) := M(r_1) \frac{\psi(r) - \psi(r_0)}{\psi(r_1) - \psi(r_0)}, \quad r \in (r_0, r_1),$$

which satisfies (1.9) and we see that $\Phi(r_1) = M(r_1) \leq u(x)$, for all $|x| = r_1$ and $0 = \Phi(r_0) \leq u(x)$ for all $|x| = r_0$. Then, by the Comparison Principle, we have

$$M(r_1) \frac{\psi(r) - \psi(r_0)}{\psi(r_1) - \psi(r_0)} \leq u(x),$$

for all $r_0 \leq r = |x| \leq r_1$. On the other hand, by the Strong Maximum Principle we see that either $M(r)$ is equal to a constant for all $r \geq r_0$ or

$$M(r) = \text{mín}\{u(x) \,|\, x \in \mathbb{R}^N, |x| = r\}, \qquad \text{for all} \quad r \geq r_0.$$

This completes the proof. $\square$

From Theorem 1.4.3 we have

Corollary 1.4.1 *Assume that u is a nonnegative viscosity solution of (1.8) in Ω, the whole space or an exterior domain, then we have*

(i) If $\lim_{r\to\infty}\varphi(r)=0$, then

$$m(r)\geq \frac{m(r_1)}{\varphi(r_1)}\varphi(r) \quad \textit{and} \quad m_0(r)\geq \frac{m_0(r_1)}{\varphi(r_1)}\varphi(r), \quad \textit{for all} \quad r\geq r_1\geq r_0.$$

(ii) If $\lim_{r\to\infty}\varphi(r)=-\infty$, then

$$m(r)\geq m(r_1) \quad \textit{and} \quad m_0(r)\geq m_0(r_1), \quad \textit{for all} \quad r\geq r_1\geq r_0.$$

Proof. Since φ is decreasing, the result follows directly from Theorem 1.4.3 taking $r_2\to\infty$ in (1.47) and (1.48). $\square$

The next proposition provides additional properties of m, m_0 and M.

Proposition 1.4.1 *Suppose that u is a positive viscosity solution of (1.8). Let*

$$g(r):=\min_{|x|=r} u(x).$$

Then there exists $\bar{r}$ such that g is either strictly increasing or strictly decreasing for $r>\bar{r}$. Either $m_0(r)$ is constant and $M(r)=g(r)$ strictly increasing or $m_0(r)=g(r)$ is strictly decreasing and $M(r)$ is constant for $r>\bar{r}$.

Proof. Let $r_1<r_2<r_3$ and $g(r_1)\geq g(r_2)$ and $g(r_3)\geq g(r_2)$, then u has a minimum point $x\in B_{r_3}\setminus B_{r_1}$, which contradicts with Minimum Principle. Then $g(t)$ may change monotonicity just once. So g is decreasing strictly or increasing strictly or first increasing and then decreasing. In the third case, let $\bar{r}$ be such that g is decreasing for $r\geq\bar{r}$. From here the result follows if we define $m_0(r)=\min_{\bar{r}\leq|x|\leq r}u(x)$. $\square$

1.5. Proof of Theorems 1.1.1 and 1.1.2

In this section we prove Theorems 1.1.1 and 1.1.2. The idea of the proof is to assume (1.8) has a solution and use an appropriate test function in order to get the behavior of u at infinity, which in view of our hypothesis is incompatible with the Hadamard property proved in the previous section.

Proof of Theorem 1.1.1. If the fundamental solution satisfies $\varphi(r) \to -\infty$, then by Corollary 1.4.1 we have

$$m(r) \geq m(r_1), \qquad \text{for } r \geq r_1.$$

Since $m(r)$ is a non-increasing function, we conclude that u attains an interior minimum, but then by the Strong Minimum Principle u is constant. From here $u \equiv 0$ since $f(x,s) > 0$ if $s > 0$ from our assumption (f_1).

If $\varphi(r) \to 0$, then we consider two cases: the critical and subcritical equations.

Subcritical Case. We assume hypothesis (f_3) in case (i) holds. We may assume that $u > 0$ by the Strong Maximum Principle. From Corollary 1.4.1 we have

$$m(r) \geq \frac{m(r_1)}{\varphi(r_1)}\varphi(r). \tag{1.51}$$

We also see that $m(r)$ is strictly decreasing. Considering $0 < \tau < R$ as parameters, we define the test function

$$\zeta(x) = m(\tau)\left[1 - \left\{\frac{(|x| - \tau)_+}{(R-\tau)}\right\}^3\right].$$

We observe that $\zeta(x) \leq 0 < u(x)$ for $|x| \geq R$, $\zeta(x) \equiv m(\tau) < u(x)$ for $|x| < \tau$ and since m is strictly decreasing, $\zeta(\bar{x}) = u(\bar{x})$ at some $\bar{x}$ with $|\bar{x}| = \tau$. Therefore, $u - \zeta$ attains a non-positive global minimum at some point x_R^τ such that $\tau \leq |x_R^\tau| < R$. By definition of viscosity solution we have

$$f(x_R^\tau, u(x_R^\tau)) \leq -\mathcal{M}^-(r, D^2\zeta(x_R^\tau)) - \sigma(|x_R^\tau|)|D\zeta(x_R^\tau)|. \tag{1.52}$$

Since ζ is radial we directly compute the right hand side and get

$$\begin{aligned} & f(x_R^\tau, u(x_R^\tau)) \\ \leq \quad & \frac{3\Lambda(|x_R^\tau|)m(\tau)}{(R-\tau)^3}\left\{2 + \left(\frac{N-1}{|x_R^\tau|} - \frac{\sigma(|x_R^\tau|)}{\Lambda(|x_R^\tau|)}\right)(|x_R^\tau| - \tau)_+\right\}(|x_R^\tau| - \tau)_+. \end{aligned}$$

If $|x_R^\tau| = \tau$, then $f(x_R^\tau, u(x_R^\tau)) \leq 0$, contradicting (f_1). Thus, we may assume that $\tau < |x_R^\tau| < R$ and we have

$$f(x_R^\tau, u(x_R^\tau)) \leq Cm(\tau)\frac{1 + \sigma_-(|x_R^\tau|)(R-\tau)}{(R-\tau)^2}, \tag{1.53}$$

for certain constant $C > 0$. Now use the hypothesis (f_3)(i) to find a sequence $\{r_n\}$ diverging to infinity so that

$$\lim_{n\to\infty} \Psi_k(r_n) = h(k_1) = \infty, \tag{1.54}$$

with $k_1 = m(r_1)/\varphi(r_1)$. We let $\tau = r_n$, $R = 2r_n$ and $x_n = x_{2r_n}^{r_n}$, and recall that $r_n \leq |x_n| \leq 2r_n$. Next we see that $u(x_n) \to 0$ as $n \to \infty$, because (1.53) gives

$$\frac{|x_n|^2 f(x_n, u(x_n))}{4(1+\sigma_-(|x_n|)|x_n|)} \leq Cm(r_n),$$

that contradicts (f_2) if $u(x_n)$, or a subsequence, is bounded away from zero. Then we use the monotonicity of $m(r)/\varphi(r)$ given by (1.51) and the fact that $u(x_n) \geq m(r_n)$ to obtain

$$\frac{\varphi(2r_n)}{\varphi(r_n)} \frac{r_n^2}{1+\sigma_-(|x_n|)r_n} \frac{f(x_n, u(x_n))}{u(x_n)} \leq C. \tag{1.55}$$

But this contradicts (1.54), since by (1.51) $u(x_n) \geq m(2r_n) \geq k_1\varphi(2r_n)$, so that (1.55) gives that $\Psi_k(r_n)$ is bounded, completing the proof in this case.

Critical Case. If case (f_3) (ii) holds then there is no contradiction in case $h(k_1) < \infty$. In this case, arguing as above, we obtain $u(x_n) \to 0$ and, using hypothesis (1.12) and Proposition 1.3.3, then

$$\frac{r_n^2 f(x_n, u(x_n))}{u(x_n)} \leq C, \tag{1.56}$$

for any sequence $\{r_n\}$ diverging to ∞. At this point we claim that

$$\lim_{r\to\infty} \frac{m(r)}{\varphi(r)} = \infty. \tag{1.57}$$

Assuming for a moment that (1.57) holds, we find M_k for every k so that $u(x) \geq k\varphi(x)$, for all $|x| \geq M_k$, consequently, from (1.56) we obtain that

$$\Psi_k(r_n) \leq \frac{r_n^2 f(x_n, u(x_n))}{u(x_n)} \leq C,$$

for n large. Since the sequence $\{r_n\}$ is arbitrary, we conclude that $h(k) \leq C$ for all k, which is a contradiction that completes the proof of the theorem.

Now we prove the claim (1.57). Let $\Omega_\tau = \{x \in \mathbb{R}^N : |x| > \tau,\ u(x) < \mu\}$, which $\tau > r_1$ and $\mu > 0$ appears in (f_3). Ω_τ is open and nonempty. Next we consider the function

$$\Gamma(x) := -\varphi(|x|)\log\varphi(|x|)$$

and choose $\bar{r} \geq r_1$ such that $m(\tau) \leq \mu$ and $\Gamma(x) \leq \mu$, for all $|x| = \tau \geq \bar{r}$. Then we use (1.51) and the monotonicity of φ to find

$$\begin{aligned}
\frac{|x|^2}{\varphi(|x|)} f(x,u(x)) &\geq k_1 \frac{|x|^2}{1+\sigma_-(|x|)|x|} \frac{f(x,u(x))}{u(x)} \\
&\geq k_1 \frac{|x|^2}{1+\sigma_-(|x|)|x|} \inf_{k_1\varphi(|x|)\leq s\leq \mu} \frac{f(x,s)}{s} \\
&\geq k_1 \frac{\varphi(2\tau)}{\varphi(\tau)} \inf_{y\in B_{2\tau}\setminus B_\tau} \frac{|y|^2}{1+\sigma_-(|y|)|y|} \inf_{k_1\varphi(|y|)\leq s\leq \mu} \frac{f(y,s)}{s} \\
&\geq k_1 \Psi_{k_1}(\tau).
\end{aligned}$$

From here, taking $\tau = |x|$ and using (1.11) we obtain

$$f(x,u(x)) \geq C\frac{\varphi(|x|)}{|x|^2}, \tag{1.58}$$

for certain constant C, for all $x \in \Omega_{\bar{r}}$. On the other hand, computing directly and using Proposition 1.3.2 we find C_0 such that

$$\mathcal{M}^-(r, D^2\Gamma) + \sigma(r)|D\Gamma| \geq -C_0\frac{\varphi(|x|)}{|x|^2}, \qquad |x| \geq \bar{r}. \tag{1.59}$$

Then we let $\tilde{C} := \min\{\frac{\varepsilon}{C_0}, -k_1/\log\varphi(\bar{r}), 1\}$ and from (1.8), (1.58) and (1.59) we obtain

$$\mathcal{M}^-(r, D^2(u+\varepsilon)) + \sigma(r)|D(u+\varepsilon)| \leq \tilde{C}(\mathcal{M}^-(r, D^2\Gamma) + \sigma(r)|D\Gamma|),$$

for all $x \in \Omega_{\bar{r}}$ and $\varepsilon > 0$. By the choice of $\tilde{C}$ we have then

$$u(x) + \varepsilon \geq m(\bar{r}) \geq k_1\varphi(\bar{r}) = \frac{k_1}{-\log\varphi(\bar{r})}\Gamma(\bar{r}) \geq \tilde{C}\Gamma(\bar{r}), \qquad \text{for all } x \in \partial B_{\bar{r}}$$

and, since $\lim_{r\to\infty}\Gamma(r) = 0$, there is R such that

$$u(x) + \varepsilon \geq \varepsilon \geq \Gamma(R) \geq \tilde{C}\Gamma(R), \qquad \text{for all } x \in \partial B_R.$$

We also have and $u(x) = \mu \geq \tilde{C}\Gamma(|x|)$ for $x \in (B_R \setminus \bar{B}_{\bar{r}}) \cap \partial\Omega_{\bar{r}}$, thus

$$u(x) + \epsilon \geq \tilde{C}\Gamma(|x|), \quad x \in \partial(B_R \cap \Omega_{\bar{r}}).$$

Then we use the Comparison Principle and then take $R \to \infty$ and $\epsilon \to 0^+$ to get

$$u(x) \geq \tilde{C}\Gamma(|x|), \quad x \in \Omega_{\bar{r}},$$

which implies (1.57). □

Remark 1.5.1 *If we have a non-negative super-solution u of (1.9) and the fundamental solution φ satisfies $\varphi(r) \to -\infty$ then u has to be constant. This result is usually known as Liouville property.*

Now we prove Theorem 1.1.2 on the Liouville property in an exterior domain.
Proof of theorem 1.1.2. According to Proposition 1.4.1 for some $\bar{r} \geq r_0$:

Case 1: $m_0(r)$ is strictly decreasing and $M(r)$ is constant for $r > \bar{r}$ or
Case 2: $M(r)$ is strictly increasing and $m_0(r)$ is constant for $r > \bar{r}$.

We recall the new definition of m_0 given in the proof of Proposition 1.4.1, for notational convenience, we just write m instead of m_0, from now on.

Proof in Case 1: If $\varphi(r) \to 0$ as $r \to \infty$, the proof follows step by step that of Theorem 1.1.1. A small change is needed in the complementary case: Given $\bar{r} < r_1 < r_2$ we use inequality (1.48) and that $m(r_2) \geq 0$, to find

$$m(r) \geq m(r_1)\left(1 - \frac{\varphi(r)}{\varphi(r_2)}\right) \quad \text{for } r \in [r_1, r_2]. \tag{1.60}$$

Then, we let $r_2 \to \infty$ obtain $m(r) \geq m(r_1)$ for $r \geq r_1$, which is impossible since $m(r)$ is strictly decreasing.

Proof in Case 2 and sub-critical: We consider the test function

$$\zeta(|x|) = M(R)\left[1 - \left\{\frac{(R-|x|)_+}{(R-\tau)}\right\}^3\right],$$

where $R > \tau \geq \bar{r}$ are parameters. As in the proof of Theorem 1.1.1, we see that $u - \zeta$ attains a non-positive global minimum at some point x_R^τ such that $r < |x_R^\tau| \leq R$ and $u(x_R^\tau) \leq M(R)$. Then, by the definition of viscosity solution and computing the differential operator we obtain

$$f(x_R^\tau, u(x_R^\tau)) \leq CM(R)\frac{1 + \sigma_-(|x_R^\tau|)(R-\tau)}{(R-\tau)^2}. \tag{1.61}$$

Assuming that $\lim_{r\to\infty} \psi(r) = 0$ then, by Theorem 1.4.3 we have that $M(R)$ is bounded. Let us choose $\{r_n\}$ diverging to infinity and let $\tau = r_n, R = 2r_n$ and write $x_n = x_{2r_n}^{r_n}$. We notice that $r_n \leq |x_n| \leq 2r_n$ and $u(x_n) \leq M(2r_n)$, so that $u(x_n)$ is bounded. But then, from (1.61), we find that

$$\frac{r_n^2}{1 + \sigma(|x_n|)r_n} f(x_n, u(x_n)) \leq CM(2r_n), \tag{1.62}$$

contradiction (f_2).

Now we assume that $\lim_{r\to\infty}\psi(r)=\infty$ and we take, without loss of generality, that $\psi(r_0)=0$. From (1.49) and (1.61) we have

$$\frac{f(x_R^\tau,u(x_R^\tau))}{u(x_R^\tau)}\leq\frac{f(x_R^\tau,u(x_R^\tau))}{M(\tau)}\leq C\frac{\psi(R)}{\psi(\tau)}\frac{1+\sigma_-(|x_R^\tau|)(R-\tau)}{(R-\tau)^2}. \tag{1.63}$$

Next we use the hypothesis (f_4)(i) to find $\{r_n\}$ diverging to infinity so that

$$\lim_{n\to\infty}\tilde{\Psi}_{k_1}(r_n)=\tilde{h}(k_1)=\infty, \tag{1.64}$$

with $k_1=M(\bar{r})/\psi(\bar{r})$. We let $\tau=r_n$, $R=2r_n$ and we write $x_n=x_{2r_n}^{r_n}$. We notice that $r_n\leq|x_n|\leq 2r_n$ and $u(x_n)\leq M(2r_n)$, so that $u(x_n)\leq k_1\psi(2r_n)$, where this last inequality comes from (1.49). Again we have (1.62), but now we conclude that $M(2r_n)$ and consequently, $M(r_n)$ and $u(x_n)$ diverge to infinity. Now, from (1.63) we have the following inequality that contradicts (1.64)

$$\frac{\psi(r_n)}{\psi(2r_n)}\frac{r_n^2}{1+\sigma(|x_n|)r_n}\frac{f(x_n,u(x_n))}{u(x_n)}\leq C.$$

Proof in Case 2 and critical: Under hypothesis (f_4)(ii) then there is no contradiction in case $\tilde{h}(k_1)<\infty$. Arguing as above, using hypothesis (1.12) and Proposition 1.3.4 we obtain

$$\frac{r_n^2 f(x_n,u(x_n))}{u(x_n)}\leq C, \tag{1.65}$$

for any sequence $\{r_n\}$ diverging to ∞. At this point we claim that

$$\lim_{r\to\infty}\frac{M(r)}{\psi(r)}=0. \tag{1.66}$$

Assuming that the claim is true, for every k there is M_k so that

$$M(r)\leq k\psi(r),\qquad\text{for all } r\geq M_k,$$

consequently, from (1.65), we obtain that

$$\tilde{\Psi}_k(r_n)\leq\frac{r_n^2 f(x_n,u(x_n))}{u(x_n)}\leq C$$

for all n large and then

$$\limsup_{n\to\infty}\tilde{\Psi}_k(r_n)\leq C.$$

Since this inequality holds for all sequence $\{r_n\}$ diverging to infinity, we find that $\tilde{h}(k)\leq C$ for all k, contradicting (f_4)(ii).

Thus, we only need to prove (1.66) to complete the proof. We define the open set $\tilde{\Omega}_{\bar{r}} := \{x \in \Omega, |x| > \bar{r},\ u(x) < 3k_1\psi(|x|)\}$, which is nonempty since, given $r > \bar{r}$ we can find $\bar{x}$ with $|\bar{x}| = r$ and $u(\bar{x}) = M(r) \leq k_1\psi(r) < 3k_1\psi(r)$.

Assume our claim is not true, then there exists $\tilde{k} \in (0, k_1]$ such that

$$\lim_{r\to\infty} \frac{M(r)}{\psi(r)} = \tilde{k}. \tag{1.67}$$

Then we have

$$\tilde{k}\psi(|x|) \leq M(|x|) \leq k_1\psi(|x|), \qquad |x| > \bar{r}.$$

and $\tilde{k}\psi(|x|) \leq u(x)$ for all $x \in \Omega_{\bar{r}}$. From here and monotonicity of ψ we find

$$\begin{aligned} \frac{|x|^2}{\psi(|x|)} f(x, u(x)) &\geq \tilde{k}\frac{\psi(|x|)}{\psi(2|x|)} \inf_{y\in B_{2|x|}\setminus B_{|x|}} \frac{|y|^2}{1+\sigma_-(|y|)|y|} \inf_{\mu\leq s\leq \tilde{k}\psi(2|y|)} \frac{f(y,s)}{s} \\ &\geq \tilde{k}\tilde{\Psi}_{\tilde{k}}(|x|). \end{aligned}$$

Then, from (1.13), there exists $c > 0$ such that

$$f(x, u(x)) \geq c\frac{\psi(|x|)}{|x|^2}, \quad x \in \Omega_{\bar{r}}. \tag{1.68}$$

Next we define the auxiliary function

$$\tilde{\Gamma}(r) = \frac{\psi(r)}{\log \psi(r)}, \quad r = |x|.$$

Computing directly we obtain

$$\begin{aligned} \mathcal{M}^-(r, D^2\tilde{\Gamma}) + \sigma(r)|D\tilde{\Gamma}| &\geq \frac{\log\psi(r) - 1}{\log^2\psi(r)}(\mathcal{M}^-(r, D^2\psi) + \sigma(r)|D\psi|) \\ &\quad -\Lambda\frac{\log\psi(r) - 2}{\log^3\psi(r)}\frac{(\psi'(r))^2}{\psi(r)}. \end{aligned}$$

Since ψ is the fundamental solution, by Proposition 1.3.4 we get

$$\mathcal{M}^-(r, D^2\tilde{\Gamma}) + \sigma(r)|D\tilde{\Gamma}| \geq -C\frac{\psi(r)}{r^2\log^2\psi(r)}.$$

On the other hand we can find $r_1 < r_2 < r_3$ such that

$$\log(\psi(r_1)) = n^2, \quad \log(\psi(r_2)) = 2n^2 \quad \text{and} \quad \log(\psi(r_3)) = 3n^2,$$

with $n \in \mathbb{N}$ to be chosen later. We define

$$w(x) := \frac{M(r_3)}{\tilde{\Gamma}(r_3)}(\tilde{\Gamma}(r) - \tilde{\Gamma}(r_1)), \qquad x \in B_{r_3} \setminus B_{r_1}.$$

There exists $n_0 > 0$ such that, for $n \geq n_0$ and $x \in (B_{r_3} \setminus B_{r_1}) \cap \Omega_{\bar{r}}$, we have

$$\begin{aligned} \mathcal{M}^-(r, D^2 w) + \sigma(r)|Dw| &\geq -C\frac{\psi(r)}{r^2 \log^2(\psi(r))}\frac{M(r_3)\log\psi(r_3)}{\psi(r_3)} \\ &\geq -f(x,u) \geq \mathcal{M}^-(r, D^2 u) + \sigma(r)|Du|, \end{aligned}$$

where we used (1.68). Next we prove that

$$u(x) \geq w(|x|), \ x \in \partial((B_{r_3} \setminus B_{r_1}) \cap \Omega_{\bar{r}}).$$

This is obvious for $|x| = r_3$ or $|x| = r_1$. For $x \in (B_{r_3} \setminus \bar{B}_{r_1}) \cap \partial\Omega_{\bar{r}}$ we have

$$\begin{aligned} w(x) &= \frac{M(r_3)\log\psi(r_3)}{\psi(r_3)}\left(\frac{\psi(r)}{\log\psi(r)} - \frac{\psi(r_1)}{\log\psi(r_1)}\right) \\ &\leq k_1\psi(r)\frac{\log\psi(r_3)}{\log\psi(r)} \leq k_1\psi(r)\frac{\log\psi(r_3)}{\log\psi(r_1)} = 3k_1\psi(r) = u(x). \end{aligned}$$

Then we apply the Comparison Principle to obtain

$$u(x) \geq w(x) = \frac{M(r_3)\log\psi(r_3)}{\psi(r_3)}\left(\frac{\psi(r)}{\log\psi(r)} - \frac{\psi(r_1)}{\log\psi(r_1)}\right),$$

for $x \in (B_{r_3} \setminus \bar{B}_{r_1}) \cap \Omega_{\bar{r}}$. Then we take $x \in \partial B_{r_2} \cap \Omega_{\bar{r}}$, and we get

$$M(r_2) \geq \frac{M(r_3)\log\psi(r_3)}{\psi(r_3)}\left(\frac{\psi(r_2)}{\log\psi(r_2)} - \frac{\psi(r_1)}{\log\psi(r_1)}\right)$$

and then

$$\frac{M(r_2)}{\psi(r_2)} \geq \frac{M(r_3)}{\psi(r_3)}\left(\frac{3}{2} - \frac{3}{e^{n^2}}\right),$$

which is impossible if n is large enough, in view of (1.67). $\square$

1.6. Proof of Theorem 1.2.3

In this section we prove Theorem 1.2.3. We observe that part (i) is a consequence of Theorem 1.1.1. In order to prove part (ii) we need a preliminary lemma. Given

$\delta > 0$ we define

$$U_\delta(r) = \varphi(r)(-\log\varphi(r))^\delta, \tag{1.69}$$

where $r > \bar{r} \geq e$ and $\bar{r}$ is such that $\varphi(\bar{r}) < 1$.

Lemma 1.6.1 *Assume the hypothesis of Theorem 1.2.3 and let $u > 0$ be a solution of (1.8). Then, for any $\delta > 0$, there exists $C_\delta \in (0, 1)$ such that*

$$u(x) \geq C_\delta U_\delta(|x|), \quad |x| \geq \bar{r}.$$

Proof. By direct computation we find a constant $c > 0$ such that

$$\mathcal{M}^-(r, D^2U_\delta) + \sigma(r)|DU_\delta| \geq -c\frac{(\log|x|)^{-2+\delta}}{|x|^{2+\alpha}}. \tag{1.70}$$

On the other hand, by Hadamard theorem, there exists $c > 0$ such that $u(x) \geq c\varphi(|x|)$, for $r \geq \bar{r}$ and then there exists $\tilde{C} > 0$ such that

$$f(x, u) \geq f(x, c\varphi(|x|)) \geq \tilde{C}\frac{(\log|x|)^{\nu-p}}{|x|^{\alpha p}}, \quad \text{for all } \ |x| \geq \bar{r}. \tag{1.71}$$

If $0 < \delta \leq \delta_0 = 1 + \nu - \frac{2}{\alpha}$ and $\varepsilon > 0$, using (1.70) and (1.71) we get

$$\mathcal{M}^-(r, D^2U_\delta) + \sigma(r)|DU_\delta| \geq \mathcal{M}^-(r, D^2(u + \varepsilon)) + \sigma(r)|D(u + \varepsilon)|, \quad |x| \geq \bar{r}.$$

By appropriately choosing C and R we find that

$$u(x) + \varepsilon \geq CU_\delta(|x|), \quad x \in \partial(B_R \setminus B_{\bar{r}}),$$

thus, by the Comparison Principle and letting $R \to \infty$ and $\varepsilon \to 0$, we obtain

$$u(x) \geq CU_\delta(|x|), \qquad x \in B^c_{\bar{r}}. \tag{1.72}$$

For $\delta \in (\delta_o, (2 + \frac{2}{\alpha})\delta_0]$, we use (1.72) with $\delta = \delta_0$ to get, as in (1.71), that

$$f(x, u) \geq f(x, CU_\delta(|x|)) \geq \tilde{C}\frac{(\log|x|)^{\nu-p+\delta_0 p}}{|x|^{\alpha p}}. \tag{1.73}$$

Then, by making $\tilde{C}$ smaller if necessary, we obtain

$$\mathcal{M}^-(r, D^2U_\delta) + \sigma(r)|DU_\delta| \geq -f(x, u), \tag{1.74}$$

for all $\delta \in (\delta_o, (2 + \frac{2}{\alpha})\delta_0]$. Then we use the Comparison Principle as before to prove that, for certain constant C, we have $u(x) \geq CU_\delta(|x|)$, for $x \in B^c_{\bar{r}}$. Repeating the argument we can prove similar results for every $\delta > 0$. $\square$

Proof of Theorem 1.2.3(ii). We assume that there exists a positive solution u of (1.8). By arguments as in the proof of Theorem 1.1.1, we find x_R^r such that $r < |x_R^r| < R$ and

$$u(x_R^r)^p(|\log u(x_R^r)| + 1)^\nu \leq 3m(r)\frac{\Lambda(x_R^r)(N+1) + \sigma_-(|x_R^r|)(R-r)}{(R-r)^2}.$$

From here and the monotonicity of $r \to \frac{m(r)}{\varphi(r)}$, we obtain

$$u(x_R^r)^{p-1}(|\log u(x_R^r)| + 1)^\nu \leq C\frac{\varphi(r)}{\varphi(R)}\frac{1 + \sigma_-(|x_R^r|)(R-r)}{(R-r)^2}.$$

At this point we choose $R = 2r$, we write $x_r = x_{2r}^r$ and we obtain

$$|x_r|^2 u(x_r)^{p-1}(|\log u(x_r)| + 1)^\nu \leq C, \tag{1.75}$$

for certain positive constant C. From here we easily conclude that $u(x_r) \to 0$ as $r \to \infty$. Now we choose $\delta > 0$ such that

$$\nu - \frac{2}{\alpha} + \frac{2\delta}{\alpha} > 0$$

and we use Lemma 1.6.1 to obtain

$$|x_r|^2 u(x_r)^{p-1}(|\log u(x_r)| + 1)^\nu \geq |x_r|^2(CU_\delta(x_r))^{p-1}(|\log(CU_\delta(x_r))| + 1)^\nu.$$

From the choice of δ and the definition of U_δ we see that the right hand side diverges to infinity, while from (1.75) the left hand side is bounded. This is a contradiction that completes the proof. $\square$

We continue by proving the existence of a positive solutions.

Proof of Theorem 1.2.3(iii). We consider the function $U(x) = \varphi(|x|)^\theta$, where $\theta \in (0,1)$ will be chosen later. By direct computation we find a constant $C > 0$ and $R > 0$ so that

$$\mathcal{M}^-(r, D^2U) + \sigma(r)|DU| \leq -C\frac{(\log|x|)^{-\theta}}{|x|^{2+\alpha\theta}}$$

for $|x| > R$. On the other hand, we have

$$U^p(x)(|\log U(x)| + 1)^\nu \leq C|x|^{-\alpha\theta p}(\log|x|)^{\nu - \theta p}.$$

Now we choose

$$\theta = \frac{1}{2}\left(1 + \frac{2}{\alpha}\frac{1}{p-1}\right) < 1$$

and we use our assumption $p > \frac{2}{\alpha} + 1$ to obtain $\theta < 1$ and $(p-1)\theta > \frac{2}{\alpha}$. From here we find $\bar{R} > R$ such that for all $x \in B^c_{\bar{R}}$

$$\mathcal{M}^-(r, D^2U(x)) + \sigma(r)|DU(x)| + U^p(x)(|\log U(x)| + 1)^\nu \leq 0. \tag{1.76}$$

We notice that $U_\varepsilon(x) = \varepsilon U(x)$ also satisfies (1.76) if ε is small, since

$$\mathcal{M}^-(r, D^2U_\varepsilon(x)) + \sigma(r)|DU_\varepsilon(x)| \leq -C\varepsilon, \quad x \in B_{\bar{R}} \setminus B_{\frac{1}{2}}$$

and

$$U^p_\varepsilon(x)(|\log U_\varepsilon(x)| + 1)^\nu = o(\varepsilon), \quad x \in B_{\bar{R}} \setminus B_{\frac{1}{2}}$$

for $\varepsilon > 0$ small enough and $C > 0$. Thus (1.76) can be extended to $B^c_{\frac{1}{2}}$. Finally we let w be the unique radial solution of the problem

$$\begin{aligned} \lambda\Delta w + \sigma(|x|)|Dw| &= -1 \quad \text{in} \quad \mathrm{B}_1, \\ w &= 0 \qquad \text{on} \quad \partial\mathrm{B}_1, \end{aligned} \tag{1.77}$$

and let $w_\varepsilon = \varepsilon w$, with $\varepsilon > 0$. It is easy to see that there exists ε_0 small so that w_{ε_0} satisfies (1.8) in B_1. Since w is positive in B_1, there exists $\varepsilon_1 > 0$ such that $w_{\varepsilon_o}(x) > U_{\varepsilon_1}(x)$ for $|x| = 1/2$. On the other hand $U_{\varepsilon_1} \to \infty$ as $r \to 0$, so there exists $r \in (0, 1/2)$ such that $w(x) = U_{\varepsilon_1}(x)$ for all $|x| = r$. Now we define $V(x) = U_{\varepsilon_1(x)}$ if $x \in B^c_r$ and $V(x) = w_{\varepsilon_0}(x)$ if $x \in B_r$, which is a solution of (1.8) in $\mathbb{R}^N$, completing the proof. □

1.7. Liouville property for $f(x,u) = h(x)u$

In this section, we study the Liouville type theorem for equation (1.17) in exterior domains, when the functions h and σ satisfy $(h_1), (h_2)$ and (h_3). Before continuing we give two examples of functions satisfying (h_2):

Example 1. σ is a negative function such that $\text{lím inf}_{r\to\infty} \sigma(r) = c_0$, for some $c_0 < 0$. Then there is R_0 such that $c_0/2 \geq \sigma(r) \geq 2c_0$ for all $r \geq R_0$ and we can choose $\kappa(r) \equiv -\frac{1}{c_0}$.

We observe that if $\text{lím}_{r\to\infty} \sigma(r) = 0$, we may change σ by $\sigma - \varepsilon$, with $\varepsilon > 0$ and small enough so that inequality (1.17) and (h_3) are still satisfied.

Example 2. If σ is of class C^1 and satisfies

$$\text{lím}_{r\to\infty} \sigma(r) = -\infty \quad \text{and} \quad \text{lím}_{r\to\infty} \sigma'(r)/\sigma^2(r) = 0,$$

then we just let $\kappa = 1/\sigma$. If σ is not C^1, but the first limit still holds and $1/\sigma$ is convex, or if it does not differ too much from a convex function, then taking κ as an appropriate approximation of $1/\sigma$ will work.

Lemma 1.7.1 *Assume that σ and κ satisfy hypotheses (h_1) and (h_2). Then $\varphi(r) \to 0$ and $\psi(r) \to \infty$ as $r \to \infty$, and for $\varepsilon > 0$, there exists $\bar{R} > R_0$ such that*

$$\frac{\varphi(r-\kappa(r))}{\varphi(r)} \leq (1+\varepsilon)e^{\frac{\mu}{\lambda_0}} \quad \text{and} \quad \frac{\psi(r)}{\psi(r-\kappa(r))} \leq (1+\varepsilon)e^{\frac{\mu}{\lambda_0}}, \quad \forall r \in \bar{R}.$$

Proof. As we have observed above, we may always assume that $|\sigma(r)| \geq \sigma_0 > 0$ for all r. We also see from (1.18) that for $r \geq \bar{R}$ we have $\kappa(r) \leq r/2$. Next we see that

$$\int_{r-\kappa(r)}^{r} \frac{n(\tau)}{\tau} d\tau \leq C \int_{r-\kappa(r)}^{r} \frac{1}{\tau} d\tau \leq C \frac{\kappa(r)}{r-\kappa(r)} \leq 2C \frac{\kappa(r)}{r}. \tag{1.78}$$

By definition of φ and for $\varepsilon > 0$, we find $\bar{R} > 0$ large such that

$$\begin{aligned}
\frac{(\varphi(r-\kappa(r)))'}{(\varphi(r))'} &= \frac{\varphi'(r-\kappa(r))(1-\kappa'(r))}{\varphi'(r)} \\
&= (1-\kappa'(r)) \exp\left(\int_{r}^{r-\kappa(r)} \left(\frac{\sigma(\tau)}{m_\lambda(\tau)} - \frac{n(\tau)}{\tau} \right) d\tau \right) \\
&\leq (1-\kappa'(r)) \exp\left(\frac{\kappa(r)}{\lambda_0} \left(\max_{r-\kappa(r)\leq s \leq r} |\sigma(s)| + \frac{2C\lambda_0}{R} \right) \right) \\
&\leq (1+\varepsilon)e^{\frac{\mu}{\lambda_0}},
\end{aligned}$$

where we have used (1.78) and (1.19). Then we have

$$(\varphi(r-\kappa(r)))' \geq (1+\varepsilon)e^{\frac{\mu}{\lambda_0}}(\varphi(r))', \quad r > \bar{R}.$$

Integrating in $[r, R]$, letting R go to infinity and using the fact that $\varphi(r) \to 0$ we get the result. Proceeding as above, for $\varepsilon > 0$ there exists $\bar{R}$ so that

$$(\psi(r))' \leq (1+\frac{\varepsilon}{2})e^{\frac{\mu}{\lambda_0}}(\psi(r-\kappa(r)))', \quad r > \bar{R}.$$

Then we integrate in $[\bar{R}, r]$ and we divide by $\psi(r-\kappa(r))$ to get

$$\frac{\psi(r)}{\psi(r-\kappa(r))} - \frac{\psi(\bar{R})}{\psi(r-\kappa(r))} \leq (1+\frac{\varepsilon}{2})e^{\frac{\mu}{\lambda_0}} \left(1 - \frac{\psi(r-\kappa(r))}{\psi(\bar{R}-\kappa(\bar{R}))}\right).$$

Using that $\psi(r) \to \infty$ as $r \to \infty$, we get the result. $\square$

Proof of Theorem 1.1.3. If $u \geq 0$ is a non-trivial solution of (1.17), then

$$\mathcal{M}^-(r, D^2u) + \sigma(r)|Du| \leq 0, \qquad x \in \Omega$$

and $u > 0$ in Ω. Then we use Proposition 1.4.1 to consider two cases in the proof, depending on the behavior of $m_0(r)$ and $M(r)$, as defined in (1.14).

Case 1. $m_0(r)$ is strictly decreasing and $M(r)$ is constant for $r > \bar{r}$. We consider the test function

$$\zeta(x) = m_0(r) \left[1 - \left\{ \frac{(|x| - r)_+}{R - r} \right\}^3 \right],$$

where r, R are parameters such that $R > r > \max\{\bar{r}, r_0\}$. Proceeding as in the proof of Theorem 1.1.1, we obtain x_R^r such that $r < |x_R^r| < R$ and

$$h(|x_R^r|)u(x_R^r) \leq \frac{3\Lambda(x_R^r)m_0(r)}{(R - r)^3} \left\{ 2 + \left(\frac{N - 1}{|x_R^r|} - \frac{\sigma(|x_R^r|)}{\Lambda(x_R^r)} \right) (|x_R^r| - r) \right\} (|x_R^r| - r).$$

From here we obtain

$$h(|x_R^r|)u(x_R^r) \leq 3m_0(r) \left\{ \frac{2\Lambda_0 + \sigma(|x_R^r|)(R - r) + (N - 1)(R - r)r^{-1}}{(R - r)^2} \right\}$$

and then, by the monotonicity of $r \to \frac{m_0(r)}{\varphi(r)}$,

$$h(|x_R^r|) \leq 3\frac{\varphi(r)}{\varphi(R)} \left\{ \frac{2\Lambda_0 + \sigma(|x_R^r|)(R - r) + (N - 1)(R - r)r^{-1}}{(R - r)^2} \right\}. \tag{1.79}$$

Next we choose $r = R - \kappa(R)$ with $R \geq \bar{R}$, and we use Lemma 1.7.1 to find

$$h(|x_R^r|) \leq (1 + \varepsilon)e^{\frac{\mu}{\lambda_0}} \left\{ \frac{2\Lambda_0 + \sigma(|x_R^r|)\kappa(R) + \frac{(N-1)\kappa(R)}{R-\kappa(R)}}{(\kappa(R))^2} \right\}. \tag{1.80}$$

From here, taking $R = r_n$ as in the hypothesis, we obtain

$$\lim_{n\to\infty} \inf_{r\in(r_n-\kappa(r_n),r_n)} [h(r) - (1 + \varepsilon)e^{\frac{\mu}{\lambda_0}}(2\Lambda_0 + 1)\sigma^2(r)] \leq 0.$$

If $\varepsilon > 0$ is chosen properly, we obtain a contradiction with (1.20).

Case 2. $M(r)$ is strictly increasing and $m_0(r)$ is constant for $r > \bar{r}$. In this case we replace m_0 by $M(r)$ in the definition of the test function and we repeat step by step the proof, using Theorem 1.4.3 and the properties of ψ given in Lemma 1.7.1. □

Capítulo 2

Large solutions to elliptic equations involving fractional Laplacian

Abstract: in this chapter[1], we study existence of boundary blow up solutions for some fractional elliptic equations

$$\begin{cases} (-\Delta)^{\alpha} u(x) + |u|^{p-1}u(x) = f(x), & x \in \Omega, \\ u(x) = 0, & x \in \bar{\Omega}^c, \\ \lim_{x\in\Omega,\ x\to\partial\Omega} u(x) = +\infty, \end{cases} \tag{2.1}$$

where Ω is an open bounded domain of class C^2, the operator $(-\Delta)^{\alpha}$ with $\alpha \in (0,1)$ is the fractional Laplacian and $f : \Omega \to \mathbb{R}$ is a continuous function which satisfies some extra conditions. Moreover, we analyze the uniqueness and asymptotic behavior of solutions to problem (2.1).

2.1. Introduction

In their pioneering work, Keller [66] and Osserman [84] studied the existence of solutions to the nonlinear reaction diffusion equation

$$\begin{cases} -\Delta u + h(u) = 0, & \text{in} \quad \Omega, \\ u = +\infty, & \text{on} \quad \partial\Omega, \end{cases} \tag{2.2}$$

where Ω is an open bounded domain of $\mathbb{R}^N (N \geq 2)$ and h is a nondecreasing positive function. They independently proved that this equation admits a solution if and only

[1]This chapter is based on the paper: H. Chen, P. Felmer and A. Quaas, *Large solution to elliptic equations involving fractional Laplacian,* submitted.

if h satisfies

$$\int_1^{+\infty} \frac{ds}{\sqrt{H(s)}} < +\infty, \tag{2.3}$$

where $H(s) = \int_0^s h(t)dt$, that in the case of $h(u) = u^p$ means $p > 1$. This integral condition on the non-linearity is known as the Keller-Osserman criteria. The solution of (2.2) found in [66] and [84] exists as a consequence of the interaction between the reaction and the diffusion term, without the influence of an external source that blows up at the boundary. Solutions exploding at the boundary are usually called boundary blow up solutions or large solutions. From then on, more general boundary blow-up problem

$$\begin{cases} -\Delta u(x) + h(x,u) = f(x), & x \in \Omega, \\ \text{lím}_{x\in\Omega,\ x\to\partial\Omega} u(x) = +\infty \end{cases} \tag{2.4}$$

has been extensively studied, see [5, 6, 7, 44, 47, 48, 50, 59, 72, 73, 74, 87]. It has being extended in various ways, weakened the assumptions on the domain and the nonlinear terms, extended to more general class of equations and obtained more information on the uniqueness and the asymptotic behavior of solution at the boundary.

During the last years there has been a renewed and increasing interest in the study of linear and nonlinear integral operators, especially, the fractional Laplacian, motivated by great applications and by important advances on the theory of nonlinear partial differential equations, see [23, 26, 27, 32, 51, 52, 54, 55, 85, 90] for details.

In a recent work, Felmer and Quaas [51] considered an analog of (2.2) where the Laplacian is replaced by the fractional Laplacian

$$\begin{cases} (-\Delta)^\alpha u(x) + |u|^{p-1}u(x) = f(x), & x \in \Omega, \\ u(x) = g(x), & x \in \bar{\Omega}^c, \\ \text{lím}_{x\in\Omega,\ x\to\partial\Omega} u(x) = +\infty, \end{cases} \tag{2.5}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 2$, with boundary $\partial\Omega$ of class C^2, $p > 1$ and the fractional Laplacian operator is defined as

$$(-\Delta)^\alpha u(x) = -\frac{1}{2}\int_{\mathbb{R}^N} \frac{\delta(u,x,y)}{|y|^{N+2\alpha}} dy, \quad x \in \Omega,$$

with $\alpha \in (0,1)$ and $\delta(u,x,y) = u(x+y) + u(x-y) - 2u(x)$. The authors proved the existence of a solution to (2.5) provided that g explodes at the boundary and satisfies other technical conditions. In case the function g blows up with an explosion rate as $d(x)^\beta$, with $\beta \in (-\frac{2\alpha}{p-1}, 0)$ and $d(x) = dist(x, \partial\Omega)$, the solution satisfies

$$0 < \liminf_{x\in\Omega, x\to\partial\Omega} u(x)d(x)^{-\beta} \leq \limsup_{x\in\Omega, x\to\partial\Omega} u(x)d(x)^{\frac{2\alpha}{p-1}} < +\infty.$$

In [51] the explosion is driven by the function g. The external source f has a secondary role, not intervening in the explosive character of the solution. f may be bounded or unbounded, in latter case the explosion rate has to be controlled by $d(x)^{-2\alpha p/(p-1)}$.

One interesting question not answered in [51] is the existence of a boundary blow up solution without external source, that is assuming $g = 0$ in $\bar{\Omega}^c$ and $f = 0$ in Ω, thus extending the original result by Keller and Osserman, where solutions exists due to the pure interaction between the reaction and the diffusion terms. It is the purpose of this chapter to answer positively this question and to better understand how the non-local character influences the large solutions of (2.5) and what is the structure of the large solutions of (2.5) with or without sources. Comparing with the Laplacian case, where well possedness holds for (2.5), a much richer structure for the solution set appears for the non-local case, depending on the parameters and the data f and g. In particular, Theorem 2.1.1 shows that existence, uniqueness, non-existence and infinite existence may occur at different values of p and α.

Our first result in this chapter is on the existence of blowing up solutions driven by the sole interaction between the diffusion and reaction term, assuming the external value g vanishes. Thus we will be considering the equation

$$\begin{cases} (-\Delta)^\alpha u(x) + |u|^{p-1}u(x) = f(x), & x \in \Omega, \\ u(x) = 0, & x \in \bar{\Omega}^c, \\ \lim_{x\in\Omega,\ x\to\partial\Omega} u(x) = +\infty. \end{cases} \tag{2.6}$$

On the external source f we will assume the following hypotheses

(H1) The external source $f : \Omega \to \mathbb{R}$ is a $C^\beta_{loc}(\Omega)$, for some $\beta > 0$.

(H2) Defining $f_-(x) = \max\{-f(x), 0\}$ and $f_+(x) = \max\{f(x), 0\}$ we have

$$\limsup_{x\in\Omega, x\to\partial\Omega} f_+(x)d(x)^{\frac{2\alpha p}{p-1}} < +\infty \quad \text{and} \quad \lim_{x\in\Omega, x\to\partial\Omega} f_-(x)d(x)^{\frac{2\alpha p}{p-1}} = 0.$$

A related condition that we need for non-existence results

(H2*) The function f satisfies

$$\limsup_{x\in\Omega, x\to\partial\Omega} |f(x)|d(x)^{2\alpha} < +\infty.$$

Now we are in a position to state our first theorem in this chapter.

Theorem 2.1.1 *Assume that Ω is an open, bounded and connected domain of class C^2 and $\alpha \in (0,1)$. Then we have:*

Existence*: Assume that f satisfies (H1) and (H2), then there exists $\tau_0(\alpha) \in (-1,0)$ such that for every p satisfying*

$$1 + 2\alpha < p < 1 - \frac{2\alpha}{\tau_0(\alpha)}, \tag{2.7}$$

the equation (2.6) possesses at least one solution u satisfying

$$0 < \liminf_{x\in\Omega, x\to\partial\Omega} u(x)d(x)^{\frac{2\alpha}{p-1}} \leq \limsup_{x\in\Omega, x\to\partial\Omega} u(x)d(x)^{\frac{2\alpha}{p-1}} < +\infty. \tag{2.8}$$

Uniqueness*: If f further satisfies $f \geq 0$ in Ω, then $u > 0$ in Ω and u is the unique solution of (2.6) satisfying (2.8).*

Nonexistence*: If f satisfies (H1), (H2*) and $f \geq 0$, then in the following three cases:*

i) For any $\tau \in (-1,0) \setminus \{-\frac{2\alpha}{p-1},\ \tau_0(\alpha)\}$ and p satisfying (2.7) or

ii) For any $\tau \in (-1,0)$ and

$$p \geq 1 - \frac{2\alpha}{\tau_0(\alpha)} \quad or \tag{2.9}$$

iii) For any $\tau \in (-1,0) \setminus \{\tau_0(\alpha)\}$ and

$$1 < p \leq 1 + 2\alpha, \tag{2.10}$$

equation (2.6) does not have a solution u satisfying

$$0 < \liminf_{x\in\Omega, x\to\partial\Omega} u(x)d(x)^{-\tau} \leq \limsup_{x\in\Omega, x\to\partial\Omega} u(x)d(x)^{-\tau} < +\infty. \tag{2.11}$$

Special existence for $\tau = \tau_0(\alpha)$. *Assume $f(x) \equiv 0,\ x \in \Omega$ and that*

$$\text{máx}\{1 - \frac{2\alpha}{\tau_0(\alpha)} + \frac{\tau_0(\alpha)+1}{\tau_0(\alpha)}, 1\} < p < 1 - \frac{2\alpha}{\tau_0(\alpha)}. \tag{2.12}$$

Then, there exist constants $C_1 \geq 0$ and $C_2 > 0$, such that for any $t > 0$ there is a positive solution u of equation (2.6) satisfying

$$C_1 d(x)^{\text{mín}\{\tau_0(\alpha)p+2\alpha,0\}} \leq td(x)^{\tau_0(\alpha)} - u(x) \leq C_2 d(x)^{\text{mín}\{\tau_0(\alpha)p+2\alpha,0\}}. \tag{2.13}$$

Remark 2.1.1 *We remark that hypothesis (H2) and (H2*) are satisfied when $f \equiv 0$, so this theorem answer the question on existence rised in [51]. We also observe that* *a function f satisfying (H2) may also satisfy*

$$\lim_{x\in\Omega, x\in\partial\Omega} f(x) = -\infty,$$

what matters is that the rate of explosion is smaller than $\frac{2\alpha p}{p-1}$.

For proving the existence part of this theorem we will construct appropriate super and sub-solutions. This construction involves the one dimensional truncated laplacian of power functions given by

$$C(\tau) = \int_0^{+\infty} \frac{\chi_{(0,1)}(t)|1-t|^\tau + (1+t)^\tau - 2}{t^{1+2\alpha}} dt, \tag{2.14}$$

for $\tau \in (-1, 0)$ and where $\chi_{(0,1)}$ is the characteristic function of the interval $(0, 1)$. The number $\tau_0(\alpha)$ appearing in the statement of our theorems is precisely the unique $\tau \in (-1, 0)$ satisfying $C(\tau) = 0$. See Proposition 2.3.1 for details.

Remark 2.1.2 *For the uniqueness, we would like to mention that, by using iteration technique, Kim in [67] has proved the uniqueness of solution to the problem*

$$\begin{cases} -\Delta u + u_+^p = 0, & in \quad \Omega, \\ u = +\infty, & on \quad \partial\Omega, \end{cases} \tag{2.15}$$

where $u_+ = \text{máx}\{u, 0\}$, under the hypotheses that $p > 1$ and Ω is bounded and satisfying $\partial\Omega = \partial\bar{\Omega}$. García-Melián in [59, 60] introduced some improved iteration technique to obtain the uniqueness for problem (2.15) with replacing nonlinear term by $a(x)u^p$. However, there is a big difficulty for us to extend the iteration technique to our problem (2.6) involving fractional Laplacian, which is caused by the nonlocal character.

In the second part of this chapter, we are also interested in considering the existence of blowing up solutions driven by external source f on which we assume the following hypothesis

(H3) There exists $\gamma \in (-1-2\alpha, 0)$ such that

$$0 < \liminf_{x\in\Omega, x\to\partial\Omega} f(x)d(x)^{-\gamma} \le \limsup_{x\in\Omega, x\to\partial\Omega} f(x)d(x)^{-\gamma} < +\infty.$$

Depending on the size of γ we will say that the external source is weak or strong. In order to gain in clarity, in this case we will state separately the existence, uniqueness and non-existence theorem in this source-driven case.

Theorem 2.1.2 (Existence) *Assume that Ω is an open, bounded and connected domain of class C^2. Assume that f satisfies (H1) and let $\alpha \in (0,1)$, then we have:*

(i) (weak source) If f satisfies (H3) with

$$-2\alpha - \frac{2\alpha}{p-1} \leq \gamma < -2\alpha, \tag{2.16}$$

then, for every p such that (2.9) holds, equation (2.6) possesses at least one solution u, with asymptotic behavior near the boundary given by

$$0 < \liminf_{x\in\Omega, x\to\partial\Omega} u(x)d(x)^{-\gamma-2\alpha} \leq \limsup_{x\in\Omega, x\to\partial\Omega} u(x)d(x)^{-\gamma-2\alpha} < +\infty. \tag{2.17}$$

(ii) (strong source) If f satisfies (H3) with

$$-1-2\alpha < \gamma < -2\alpha - \frac{2\alpha}{p-1} \tag{2.18}$$

then, for every p such that

$$p > 1 + 2\alpha, \tag{2.19}$$

equation (2.6) possesses at least one solution u, with asymptotic behavior near the boundary given by

$$0 < \liminf_{x\in\Omega, x\to\partial\Omega} u(x)d(x)^{-\frac{\gamma}{p}} \leq \limsup_{x\in\Omega, x\to\partial\Omega} u(x)d(x)^{-\frac{\gamma}{p}} < +\infty. \tag{2.20}$$

As we already mentioned, in Theorem 2.1.1 the existence of blowing up solutions results from the interaction between the reaction u^p and the diffusion term $(-\Delta)^\alpha$, while the role of the external source f is secondary. In contrast, in Theorem 2.1.2 the existence of blowing up solutions results on the interaction between the external source, and the diffusion term in case of weak source and the interaction between the external source and the reaction term in case of strong source.

Regarding uniqueness result for solutions of (2.6), as in Theorem 2.1.1 we will assume that f is non-negative, hypothesis that we need for technical reasons. We have

Theorem 2.1.3 (Uniqueness) *Assume that Ω is an open, bounded and connected domain of class C^2, $\alpha \in (0,1)$ and f satisfies (H1) and $f \geq 0$. Then we have*

i) (weak source) the solution of (2.6) satisfying (2.17) is positive and unique, and

ii) (strong source) the solution of (2.6) satisfying (2.20) is positive and unique.

We complete our theorems with a non-existence result for solution with a previously defined asymptotic behavior, as we saw in Theorem 2.1.1. We have

Theorem 2.1.4 (Non-existence) *Assume that Ω is an open, bounded and connected domain of class C^2, $\alpha \in (0,1)$ and f satisfies $(H1)$, $(H3)$ and $f \geq 0$. Then we have*

i) (weak source) Suppose that p satisfies (2.9), γ satisfies (2.16) and $\tau \in (-1,0) \setminus \{\gamma + 2\alpha\}$. Then equation (2.6) does not have a solution u satisfying (2.11).

ii) (strong source) Suppose that p satisfies (2.19), γ satisfies (2.18) and $\tau \in (-1,0) \setminus \{\frac{\gamma}{p}\}$. Then equation (2.6) does not have a solution u satisfying (2.11).

All theorems stated so far deal with equation (2.5) in the case $g \equiv 0$, but they may also be applied when $g \not\equiv 0$ and, in particular, these result improve those given in [51]. In what follows we describe how to obtain this. We start with some notation, we consider $L^1_\omega(\bar{\Omega}^c)$ the weighted L^1 space in $\bar{\Omega}^c$ with weight

$$\omega(y) = \frac{1}{1 + |y|^{N+2\alpha}}, \quad \text{for all } y \in \mathbb{R}^N.$$

Our hypothesis on the external values g is the following

$(H4)$ The function $g : \bar{\Omega}^c \to \mathbb{R}$ is measurable and $g \in L^1_\omega(\bar{\Omega}^c)$.

Given g satisfying $(H4)$, we define

$$G(x) = \frac{1}{2} \int_{\mathbb{R}^N} \frac{\tilde{g}(x+y)}{|y|^{N+2\alpha}} dy, \quad x \in \Omega, \tag{2.21}$$

where

$$\tilde{g}(x) = \begin{cases} 0, & x \in \bar{\Omega}, \\ g(x), & x \in \bar{\Omega}^c. \end{cases} \tag{2.22}$$

We observe that

$$G(x) = -(-\Delta)^\alpha \tilde{g}(x), \quad x \in \Omega.$$

Hypothesis $(H4)$ implies that G is continuous in Ω as seen in Lemma 2.2.1 and has an explosion of order $d(x)^{\beta - 2\alpha}$ towards the boundary $\partial\Omega$, if g has an explosion of order $d(x)^\beta$ for some $\beta \in (-1, 0)$, as we shall see in Proposition 2.3.3. We observe that under the hypothesis $(H4)$, if u is a solution of equation (2.5), then $u - \tilde{g}$ is the solution of

$$\begin{cases} (-\Delta)^\alpha u(x) + |u|^{p-1} u(x) = f(x) + G(x), & x \in \Omega, \\ u(x) = 0, & x \in \bar{\Omega}^c, \\ \lim_{x \in \Omega,\ x \to \partial\Omega} u(x) = +\infty \end{cases} \tag{2.23}$$

and vice versa, if v is a solution of (2.23), then $v + \tilde{g}$ is a solution of (2.5).

Thus, using Theorem 2.1.1-2.1.4, we can state the corresponding results of existence, uniqueness and non-existence for (2.5), combining f with g to define a new external source

$$F(x) = G(x) + f(x), \quad x \in \Omega. \tag{2.24}$$

With this we can state appropriate hypothesis for g and thus we can write theorems, one corresponding to each of Theorem 2.1.1, 2.1.2, 2.1.3 and 2.1.4. Even though, at first sight we need that $G(x)$ is $C^{\beta}_{loc}(\Omega)$, actually continuity of g is sufficient, as we discuss Remark 2.4.1. Moreover, in Remark 2.4.2 we explain how our results in this paper allows to give a different proof of those obtained by Felmer and Quaas in [51], generalizing them.

2.2. Preliminaries and existence theorem

The purpose of this section is to introduce some preliminaries and prove an existence theorem for blow-up solutions assuming the existence of ordered super-solution and sub-solution which blow up at the boundary. In order to prove this theorem we adapt the theory of viscosity to allow for boundary blow up.

We start this section by defining the notion of viscosity solution for non-local equation, allowing blow up at the boundary, see for example [27]. We consider the equation of the form:

$$(-\Delta)^{\alpha} u = h(x, u) \quad \text{in} \quad \Omega, \quad u = g \quad \text{in} \quad \Omega^c. \tag{2.25}$$

Definition 2.2.1 *We say that a function $u : (\partial\Omega)^c \to \mathbb{R}$, continuous in Ω and in $L^1_{\omega}(\mathbb{R}^N)$ is a viscosity super-solution (sub-solution) of (2.25) if*

$$u \geq g \ (\textit{resp. } u \leq g) \ \textit{in } \bar{\Omega}^c$$

and for every point $x_0 \in \Omega$ and some neighborhood V of x_0 with $\bar{V} \subset \Omega$ and for any $\phi \in C^2(\bar{V})$ such that $u(x_0) = \phi(x_0)$ and

$$u(x) \geq \phi(x) \ (\textit{resp. } u(x) \leq \phi(x)) \ \textit{for all } x \in V,$$

defining

$$\tilde{u} = \begin{cases} \phi & \textit{in} \quad V, \\ u & \textit{in} \quad V^c, \end{cases}$$

we have

$$(-\Delta)^\alpha \tilde{u}(x_0) \geq h(x_0, u(x_0)) \ (resp. (-\Delta)^\alpha \tilde{u}(x_0) \leq h(x_0, u(x_0)).$$

We say that u is a viscosity solution of (2.25) if it is a viscosity super-solution and also a viscosity sub-solution of (2.25).

It will be convenient for us to have also a notion of classical solution.

Definition 2.2.2 *We say that a function $u : (\partial\Omega)^c \to \mathbb{R}$, continuous in Ω and in $L^1_\omega(\mathbb{R}^N)$ is a classical solution of (2.25) if $(-\Delta)^\alpha u(x)$ is well defined for all $x \in \Omega$,*

$$(-\Delta)^\alpha u(x) = h(x, u(x)), \quad \textit{for all } x \in \Omega$$

and $u(x) = g(x)$ a.e. in $\overline{\Omega}^c$. Classical super and sub-solutions are defined similarly.

Next we have our first regularity theorem.

Theorem 2.2.1 *Let $g \in L^1_\omega(\mathbb{R}^N)$ and $f \in C^\beta_{loc}(\Omega)$, with $\beta \in (0,1)$, and u be a viscosity solution of*

$$(-\Delta)^\alpha u = f \quad in \quad \Omega, \quad u = g \quad in \quad \Omega^c,$$

then there exists $\gamma > 0$ such that $u \in C^{2\alpha+\gamma}_{loc}(\Omega)$

Proof. Suppose without loss of generality that $B_1 \subset \Omega$ and $f \in C^\beta(B_1)$. Let η be a non-negative, smooth function with support in B_1, such that $\eta = 1$ in $B_{1/2}$. Now we look at the equation

$$-\Delta w = \eta f \quad \text{in} \quad \mathbb{R}^N.$$

By Hölder regularity theory for the Laplacian we find $w \in C^{2,\beta}$, so that $(-\Delta)^{1-\alpha} w \in C^{2\alpha+\beta}$, see [94] or Theorem 3.1 in [53]. Then, since

$$(-\Delta)^\alpha (u - (-\Delta)^{1-\alpha} w) = 0 \quad \text{in} \quad B_{1/2},$$

we can use Theorem 1.1 and Remark 9.4 of [29] (see also Theorem 4.1 there), to obtain that there exist $\tilde{\beta}$ such that $u - (-\Delta)^{1-\alpha} w \in C^{2\alpha+\tilde{\beta}}(B_{1/2})$, from where we conclude. □

The Maximum and the Comparison Principles are key tools in the analysis, we present them here for completitude.

Theorem 2.2.2 *(Maximum principle) Let $\mathcal{O}$ be an open and bounded domain of $\mathbb{R}^N$ and u be a classical solution of*

$$(-\Delta)^\alpha u \leq 0 \quad in \quad \mathcal{O}, \tag{2.26}$$

continuous in $\bar{\mathcal{O}}$ and bounded from above in $\mathbb{R}^N$. Then $u(x) \leq M$, for all $x \in \mathcal{O}$, where $M = \sup_{x \in \mathcal{O}^c} u(x) < +\infty$.

Proof. If the conclusion is false, then there exists $x' \in \mathcal{O}$ such that $u(x') > M$. By continuity of u, there exists $x_0 \in \mathcal{O}$ such that

$$u(x_0) = \max_{x \in \mathcal{O}} u(x) = \max_{x \in \mathbb{R}^N} u(x)$$

and then $(-\Delta)^\alpha u(x_0) > 0$, which contradicts (2.26). □

Theorem 2.2.3 *(Comparison Principle) Let u and v be classical super-solution and sub-solution of*

$$(-\Delta)^\alpha u + h(u) = f \quad in \quad \mathcal{O},$$

respectively, where $\mathcal{O}$ is an open, bounded domain, the functions $f : \mathcal{O} \to \mathbb{R}$ is continuous and $h : \mathbb{R} \to \mathbb{R}$ is increasing. Suppose further that u and v are continuous in $\bar{\mathcal{O}}$ and $v(x) \leq u(x)$ for all $x \in \mathcal{O}^c$. Then

$$u(x) \geq v(x), \ x \in \mathcal{O}.$$

Proof. Suppose by contradiction that $w = u - v$ has a negative minimum in $x_0 \in \mathcal{O}$, then $(-\Delta)^\alpha w(x_0) < 0$ and so, by assumptions on u and v, $h(u(x_0)) > h(v(x_0))$, which contradicts the monotonicity of h. □

We devote the rest of the section to the proof of the existence theorem through super and sub-solutions. We prove the theorem by an approximation procedure for which we need some preliminary steps. We need to deal with a Dirichlet problem involving fractional Laplacian operator and with exterior data which blows up away from the boundary. Precisely, on the exterior data g, we assume the following hypothesis, given an open, bounded set $\mathcal{O}$ in $\mathbb{R}^N$ with C^2 boundary:

(G) $g : \mathcal{O}^c \to \mathbb{R}$ is in $L^1_\omega(\mathcal{O}^c)$ and it is of class C^2 in $\{z \in \mathcal{O}^c, dist(z, \partial\mathcal{O}) \leq \delta\}$, where $\delta > 0$.

In studying the nonlocal problem (2.5) with explosive exterior source, we have to adapt the stability theorem and the existence theorem for the linear Dirichlet problem. The following lemma is important in this direction.

Lemma 2.2.1 *Assume that $\mathcal{O}$ is an open, bounded domain in $\mathbb{R}^N$ with C^2 boundary. Let $w : \mathbb{R}^N \to \mathbb{R}$:*
(i) If $w \in L^1_\omega(\mathbb{R}^N)$ and w is of class C^2 in $\{z \in \mathbb{R}^N, d(z, \mathcal{O}) \leq \delta\}$ for some $\delta > 0$, then $(-\Delta)^\alpha w$ is continuous in $\bar{\mathcal{O}}$.
(ii) If $w \in L^1_\omega(\mathbb{R}^N)$ and w is of class C^2 in $\mathcal{O}$, then $(-\Delta)^\alpha w$ is continuous in $\mathcal{O}$.
(iii) If $w \in L^1_\omega(\mathbb{R}^N)$ and $w \equiv 0$ in $\mathcal{O}$, then $(-\Delta)^\alpha w$ is continuous in $\mathcal{O}$.

Proof. We first prove (ii). Let $x \in \Omega$ and $\eta > 0$ such that $B(x, 2\eta) \subset \Omega$. Then we consider

$$(-\Delta)^\alpha u(x) = L_1(x) + L_2(x),$$

where

$$L_1(x) = \int_{B(0,\eta)} \frac{\delta(u,x,y)}{|y|^{N+2\alpha}} dy \quad \text{and} \quad L_2(x) = \int_{B(0,\eta)^c} \frac{\delta(u,x,y)}{|y|^{N+2\alpha}} dy.$$

Since w is of class C^2 in $\mathcal{O}$, we may write L_1 as

$$L_1(x) = \int_0^\eta \left\{ \int_{S^{N-1}} \int_{-1}^1 \int_1^1 t\omega^t D^2 w(x + str\omega)\omega dt ds d\omega \right\} r^{1-\alpha} dr,$$

where the term inside the brackets is uniformly continuous in (x, r), so the resulting function L_1 is continuous. On the other hand we may write L_2 as

$$L_2(x) = -2w(x) \int_{B(0,\eta)^c} \frac{dy}{|y|^{N+2\alpha}} - 2 \int_{B(x,\eta)^c} \frac{w(z)dz}{|z-x|^{N+2\alpha}},$$

from where L_2 is also continuous. The proof of (i) and (iii) are similar. □

The next theorem gives the stability property for viscosity solutions in our setting.

Theorem 2.2.4 *Suppose that $\mathcal{O}$ is an open, bounded and C^2 domain and $h : \mathbb{R} \to \mathbb{R}$ is continuous. Assume that (u_n), $n \in \mathbb{N}$ is a sequence of functions, bounded in $L^1_\omega(\mathcal{O}^c)$ and f_n and f are continuous in $\mathcal{O}$ such that:*

$(-\Delta)^\alpha u_n + h(u_n) \geq f_n$ (resp. $(-\Delta)^\alpha u_n + h(u_n) \leq f_n$) in $\mathcal{O}$ in viscosity sense,

$u_n \to u$ locally uniformly in $\mathcal{O}$,

$u_n \to u$ in $L^1_\omega(\mathbb{R}^N)$, and

$f_n \to f$ locally uniformly in $\mathcal{O}$.

Then, $(-\Delta)^\alpha u + h(u) \geq f$ (resp. $(-\Delta)^\alpha u + h(u) \leq f$) in $\mathcal{O}$ in viscosity sense.

Proof. If $|u_n| \leq C$ in $\mathcal{O}$ then we use Lemma 4.3 of [27]. If u_n is unbounded in $\mathcal{O}$, then u_n is bounded in $\mathcal{O}_k = \{x \in \mathcal{O}, dist(x, \partial\mathcal{O}) \geq \frac{1}{k}\}$, since u_n is continuous in $\mathcal{O}$, and then by Lemma 4.3 of [27], u is a viscosity solution of $(-\Delta)^\alpha u + h(u) \geq f$ in $\mathcal{O}_k$ for any k. Thus u is a viscosity solution of $(-\Delta)^\alpha u + h(u) \geq f$ in $\mathcal{O}$ and the proof is completed. □

An existence result for the Dirichlet problem is given as follows:

Theorem 2.2.5 *Suppose that $\mathcal{O}$ is an open, bounded and C^2 domain, $g : \mathcal{O}^c \to \mathbb{R}$ satisfies (G), $f : \bar{\mathcal{O}} \to \mathbb{R}$ is continuous, $f \in C^\beta_{loc}(\mathcal{O})$, with $\beta \in (0,1)$, and $p > 0$.*

Then there exists a classical solution u of

$$\begin{cases} (-\Delta)^\alpha u(x) + |u|^{p-1}u(x) = f(x), & x \in \mathcal{O}, \\ u(x) = g(x), & x \in \mathcal{O}^c, \end{cases} \tag{2.27}$$

which is continuous in $\bar{\mathcal{O}}$.

In proving Theorem 2.2.5, we will use the following lemma:

Lemma 2.2.2 *Suppose that $\mathcal{O}$ is an open, bounded and C^2 domain, $f : \bar{\mathcal{O}} \to \mathbb{R}$ is continuous and $C > 0$. Then there exist a classical solution of*

$$\begin{cases} (-\Delta)^\alpha u(x) + Cu(x) = f(x), & x \in \mathcal{O}, \\ u(x) = 0, & x \in \mathcal{O}^c, \end{cases} \tag{2.28}$$

which is continuous in $\bar{\mathcal{O}}$.

Proof. For the existence of a viscosity solution u of (2.28), that is continuous in $\bar{\mathcal{O}}$, we refers to Theorem 3.1 in [51]. Now we apply Theorem 2.6 of [27] to conclude that u is $C^\theta_{loc}(\mathcal{O})$, with $\theta > 0$, and then we use Theorem 2.2.1 to conclude that the solution is classical (see also Proposition 1.1 and 1.4 in [88]). □

Using Lemma 2.2.2, we find $\bar{V}$, a classical solution of

$$\begin{cases} (-\Delta)^\alpha \bar{V}(x) = -1, & x \in \mathcal{O}, \\ \bar{V}(x) = 0, & x \in \mathcal{O}^c, \end{cases} \tag{2.29}$$

which is continuous in $\bar{\mathcal{O}}$ and negative in $\mathcal{O}$. it is classical since we apply Theorem 2.6 of [27] to conclude that u is $C^\theta_{loc}(\mathcal{O})$, with $\theta > 0$, and then we use Theorem 2.2.1 to conclude that the solution is classical (see also Proposition 1.1 and 1.4 in [88]).

Now we prove Theorem 2.2.5.

Proof of Theorem 2.2.5. Under assumption (G) and in view of the hypothesis on $\mathcal{O}$, we may extend g to $\bar{g}$ in $\mathbb{R}^N$ as a C^2 function in $\{z \in \mathbb{R}^N, d(z, \mathcal{O}) \leq \delta\}$. We certainly have $\bar{g} \in L^1_\omega(\mathbb{R}^N)$ and, by Lemma 2.2.1 $(-\Delta)^\alpha \bar{g}$ is continuous in $\bar{\mathcal{O}}$. Next we use Lemma 2.2.2 to find a solution v of equation (2.28) with $f(x)$ replaced by $f(x) - (-\Delta)^\alpha \bar{g}(x) - C\bar{g}(x)$, where $C > 0$. Then we define $u = v + \bar{g}$ and we see that u is continuous in $\bar{\mathcal{O}}$ and it satisfies in the viscosity sense

$$\begin{cases} (-\Delta)^\alpha u(x) + Cu(x) = f(x), & x \in \mathcal{O}, \\ u(x) = g(x), & x \in \mathcal{O}^c. \end{cases}$$

Now we use Theorem 2.6 in [27] and then Theorem 2.2.1 to conclude that u is a classical solution. Continuing the proof, we find super and sub-solutions for (2.27).

We define

$$u_\lambda(x) = \lambda \bar{V}(x) + \bar{g}(x), \;\; x \in \mathbb{R}^N,$$

where $\lambda \in \mathbb{R}$ and $\bar{V}$ is given in (2.29). We see that $u_\lambda(x) = g(x)$ in $\mathcal{O}^c$ for any λ and for λ large (negative), u_λ satisfies

$$(-\Delta)^\alpha u_\lambda(x) + |u_\lambda(x)|^{p-1}u_\lambda(x) - f(x) \geq (-\Delta)^\alpha \bar{g}(x) - \lambda - f(x) - |\bar{g}(x)|^p,$$

for $x \in \mathcal{O}$. Since $(-\Delta)^\alpha \bar{g}$, $\bar{g}$ and f are bounded in $\bar{\mathcal{O}}$, choosing $\lambda_1 < 0$ large enough we find that $u_{\lambda_1} \geq 0$ is a super-solution of (2.27) with $u_{\lambda_1} = g$ in $\mathcal{O}^c$.

On the other hand, for $\lambda > 0$ we have

$$(-\Delta)^\alpha u_\lambda(x) + |u_\lambda|^{p-1}u_\lambda(x) - f(x) \leq (-\Delta)^\alpha \bar{g}(x) - \lambda + |\bar{g}|^{p-1}\bar{g}(x) - f(x).$$

As before, there is $\lambda_2 > 0$ large enough, so that u_{λ_2} is a sub-solution of (2.27) with $u_{\lambda_2} = g$ in $\mathcal{O}^c$. Moreover, we have that $u_{\lambda_2} < u_{\lambda_1}$ in $\mathcal{O}$ and $u_{\lambda_2} = u_{\lambda_1} = g$ in $\mathcal{O}^c$.

Let $u_0 = u_{\lambda_2}$ and define iteratively, using the above argument, the sequence of functions u_n $(n \geq 1)$ as the classical solutions of

$$\begin{aligned} (-\Delta)^\alpha u_n(x) + Cu_n(x) &= f(x) + Cu_{n-1}(x) - |u_{n-1}|^{p-1}u_{n-1}(x), \;\; x \in \mathcal{O}, \\ u_n(x) &= g(x), \quad x \in \mathcal{O}^c, \end{aligned}$$

where $C > 0$ is so that the function $r(t) = Ct - |t|^{p-1}t$ is increasing in the interval $[\text{mín}_{x\in\bar{\mathcal{O}}} u_{\lambda_2}(x), \text{máx}_{x\in\bar{\mathcal{O}}} u_{\lambda_1}(x)]$. Next, using Theorem 2.2.3 we get

$$u_{\lambda_2} \leq u_n \leq u_{n+1} \leq u_{\lambda_1} \;\text{ in } \mathcal{O}, \quad \text{for all } n \in \mathbb{N}.$$

Then we define $u(x) = \text{lím}_{n\to+\infty} u_n(x)$, for $x \in \mathcal{O}$ and $u(x) = g(x)$, for $x \in \mathcal{O}^c$ and we have

$$u_{\lambda_2} \leq u \leq u_{\lambda_1} \;\text{ in }\; \mathcal{O}. \tag{2.30}$$

Moreover, $u_{\lambda_1}, u_{\lambda_2} \in L^1_\omega(\mathbb{R}^N)$ so that $u_n \to u$ in $L^1_\omega(\mathbb{R}^N)$, as $n \to \infty$.

By interior estimates as given in [26], for any compact set K of $\mathcal{O}$, we have that u_n has uniformly bounded $C^\theta(K)$ norm. So, by Ascoli-Arzela Theorem we have that u is continuous in K and $u_n \to u$ uniformly in K. Taking a sequence of compact sets $K_n = \{z \in \mathcal{O}, d(z, \partial\mathcal{O}) \geq \frac{1}{n}\}$, and $\mathcal{O} = \cup_{n=1}^{+\infty} K_n$, we find that u is continuous in $\mathcal{O}$ and, by Theorem 2.2.4, u is a viscosity solution of (2.27). Now we apply Theorem 2.6 of [27] to find that u is $C^\theta_{loc}(\mathcal{O})$, and then we use Theorem 2.2.1 con conclude that u is a classical solution. In addition, u is continuous up to the boundary by (2.30).

□

Now we are in a position to prove the main theorem of this section. We prove the existence of a blow-up solution of (2.6) assuming the existence of suitable super and sub-solutions.

Theorem 2.2.6 *Assume that Ω is an open, bounded domain of class C^2, $p > 1$ and f satisfy $(H1)$. Suppose there exists a super-solution $\bar{U}$ and a sub-solution $\underline{U}$ of (2.6) such that $\bar{U}$ and $\underline{U}$ are of class C^2 in Ω, $\underline{U}$, $\bar{U} \in L^1_\omega(\mathbb{R}^N)$,*

$$\bar{U} \geq \underline{U} \quad in\ \Omega, \quad \liminf_{x\in\Omega, x\to\partial\Omega} \underline{U}(x) = +\infty \quad and \quad \bar{U} = \underline{U} = 0 \quad in\ \bar{\Omega}^c.$$

Then there exists at least one solution u of (2.6) in the viscosity sense and

$$\underline{U} \leq u \leq \bar{U} \quad in \quad \Omega.$$

Additionally, if $f \geq 0$ in Ω, then $u > 0$ in Ω.

Proof. Let us consider $\Omega_n = \{x \in \Omega : d(x) > 1/n\}$ and use Theorem 2.2.5 to find a solution u_n of

$$\begin{cases} (-\Delta)^\alpha u(x) + |u|^{p-1}u(x) = f(x), & x \in \Omega_n, \\ u(x) = \underline{U}(x), & x \in \Omega_n^c, \end{cases} \tag{2.31}$$

We just replace $\mathcal{O}$ by Ω_n and define $\delta = \frac{1}{4n}$, so that $\underline{U}(x)$ satisfies assumption (G). We notice that Ω_n is of class C^2 for $n \geq N_0$, for certain N_0 large. Next we show that u_n is a sub-solution of (2.31) in Ω_{n+1}. In fact, since u_n is the solution of (2.31) in Ω_n and $\underline{U}$ is a sub-solution of (2.31) in Ω_n, by Theorem 2.2.3,

$$u_n \geq \underline{U} \quad \text{in} \quad \Omega_n.$$

Additionally, $u_n = \underline{U}$ in Ω_n^c. Then, for $x \in \Omega_{n+1} \setminus \Omega_n$, we have

$$(-\Delta)^\alpha u_n(x) = -\frac{1}{2}\int_{\mathbb{R}^N} \frac{\delta(u_n, x, y)}{|y|^{N+2\alpha}} dy \leq (-\Delta)^\alpha \underline{U}(x),$$

so that u_n is a sub-solution of (2.31) in Ω_{n+1}. From here and since u_{n+1} is the solution of (2.31) in Ω_{n+1} and $\bar{U}$ is a super-solution of (2.31) in Ω_{n+1}, by Theorem 2.2.3, we have $u_n \leq u_{n+1} \leq \bar{U}$ in Ω_{n+1}. Therefore, for any $n \geq N_0$,

$$\underline{U} \leq u_n \leq u_{n+1} \leq \bar{U} \quad \text{in} \quad \Omega.$$

Then we can define the function u as

$$u(x) = \lim_{n\to+\infty} u_n(x), \ x \in \Omega \quad \text{and} \quad u(x) = 0, \ x \in \bar{\Omega}^c$$

and we have

$$\underline{U}(x) \leq u(x) \leq \bar{U}(x), \ x \in \Omega.$$

Since $\underline{U}$ and $\bar{U}$ belong to $L^1_\omega(\mathbb{R}^N)$, we see that $u_n \to u$ in $L^1_\omega(\mathbb{R}^N)$, as $n \to \infty$. Now we repeat the arguments of the proof of Theorem 2.2.5 to find that u is a classical solution of (2.6). Finally, if f is positive we easily find that u is positive, again by a contradiction argument. $\square$

2.3. Some estimates

In order to prove our existence threorems we will use Theorem 2.2.6, so that it is crucial to have available super and sub-solutions to (2.5). In this section we provide the basic estimates that will allow to obtain in the next section the necessary super and sub-solutions.

To this end, we use appropriate powers of the distance function d and the main result in this section are the estimates given in Proposition 2.3.2, that provides the asymptotic behavior of the fractional operator applied to d.

But before going to this estimates, we describe the behavior of the function C defined in (2.14), which is a C^2 defined in $(-1, 2\alpha)$. We have:

Proposition 2.3.1 *For every $\alpha \in (0,1)$ there exists a unique $\tau_0(\alpha) \in (-1,0)$ such that $C(\tau_0(\alpha)) = 0$ and*

$$C(\tau)(\tau - \tau_0(\alpha)) < 0, \quad \textit{for all } \tau \in (-1,0) \setminus \{\tau_0(\alpha)\}. \tag{2.32}$$

Moreover, the function τ_0 satisfies

$$\lim_{\alpha \to 1^-} \tau_0(\alpha) = 0 \quad \textit{and} \quad \lim_{\alpha \to 0^+} \tau_0(\alpha) = -1. \tag{2.33}$$

Proof. We first observe that $C(0) < 0$ since the integrand in (2.14) is zero in $(0,1)$ and negative in $(1, +\infty)$. Next easily see that

$$\lim_{\tau \to -1^+} C(\tau) = +\infty, \tag{2.34}$$

since, as τ approaches -1, the integrand loses integrability at 0. Next we see that $C(\cdot)$ is strictly convex in $(-1,0)$, since

$$C'(\tau) = \int_0^{+\infty} \frac{|1-t|^\tau \chi_{(0,1)}(t) \log|1-t| + (1+t)^\tau \log(1+t)}{t^{1+2\alpha}} dt$$

and

$$C''(\tau) = \int_0^{+\infty} \frac{|1-t|^\tau [\chi_{(0,1)}(t) \log|1-t|]^2 + (1+t)^\tau [\log(1+t)]^2}{t^{1+2\alpha}} dt > 0.$$

The convexity $C(\cdot)$, $C(0) < 0$ and (2.34) allow to conclude the existence and uniqueness of $\tau_0(\alpha) \in (-1,0)$ such that (2.32) holds. To prove the first limit in (2.33), we proceed by contradiction, assuming that for $\{\alpha_n\}$ converging to 1 and $\tau_0 \in (-1,0)$

such that

$$\tau_0(\alpha_n) \le \tau_0 < 0.$$

Then, for a constant $c_1 > 0$ we have

$$\lim_{\alpha_n \to 1^-} \int_0^{\frac{1}{2}} \frac{(1-t)^{\tau_0(\alpha_n)} + (1+t)^{\tau_0(\alpha_n)} - 2}{t^{1+2\alpha_n}} dt \ge c_1 \lim_{\alpha_n \to 1^-} \int_0^{\frac{1}{2}} t^{1-2\alpha_n} dt = +\infty$$

and, for a constant c_2 independent of n, we have

$$\int_{\frac{1}{2}}^{+\infty} \left| \frac{\chi_{(0,1)}(t)(1-t)^{\tau_0(\alpha_n)} + (1+t)^{\tau_0(\alpha_n)} - 2}{t^{1+2\alpha_n}} \right| dt \quad \le \quad c_2,$$

contradicting the fact that $C(\tau_0(\alpha_n)) = 0$. For the second limit in (2.33), we proceed similarly, assuming that for $\{\alpha_n\}$ converging to 0 and $\bar{\tau}_0 \in (-1, 0)$ such that

$$\tau_0(\alpha_n) \ge \bar{\tau}_0 > -1.$$

There are positive constants c_1 and c_2 we have such that

$$\int_0^2 \left| \frac{\chi_{0,1}(t)(1-t)^{\tau_0(\alpha_n)} + (1+t)^{\tau_0(\alpha_n)} - 2}{t^{1+2\alpha_n}} \right| dt \le c_1$$

and

$$\lim_{n \to \infty} \int_2^{+\infty} \frac{(1+t)^{\tau_0(\alpha_n)} - 2}{t^{1+2\alpha_n}} dt \quad \le \quad -c_2 \lim_{n \to \infty} \int_2^{+\infty} \frac{1}{t^{1+2\alpha_n}} dt = -\infty,$$

contradicting again that $C(\tau_0(\alpha_n)) = 0$. □

Next we prove our main result in this section. We assume that $\delta > 0$ is such that the distance function $d(\cdot)$ is of class C^2 in $A_\delta = \{x \in \Omega, d(x) < \delta\}$ and we define

$$V_\tau(x) = \begin{cases} l(x), & x \in \Omega \setminus A_\delta, \\ d(x)^\tau, & x \in A_\delta, \\ 0, & x \in \Omega^c, \end{cases} \tag{2.35}$$

where τ is a parameter in $(-1, 0)$ and the function l is positive such that V_τ is C^2 in Ω. We have the following

Proposition 2.3.2 *Assume that Ω is a bounded, open subset of $\mathbb{R}^N$ with a C^2 boundary and let $\alpha \in (0, 1)$. Then there exists $\delta_1 \in (0, \delta)$ and a constant $C > 1$ such that:*

(i) If $\tau \in (-1, \tau_0(\alpha))$*, then*

$$\frac{1}{C} d(x)^{\tau-2\alpha} \leq -(-\Delta)^\alpha V_\tau(x) \leq C d(x)^{\tau-2\alpha}, \quad \textit{for all } x \in A_{\delta_1}.$$

(ii) If $\tau \in (\tau_0(\alpha), 0)$*, then*

$$\frac{1}{C} d(x)^{\tau-2\alpha} \leq (-\Delta)^\alpha V_\tau(x) \leq C d(x)^{\tau-2\alpha}, \quad \textit{for all } x \in A_{\delta_1}.$$

(iii) If $\tau = \tau_0(\alpha)$*, then*

$$|(-\Delta)^\alpha V_\tau(x)| \leq C d(x)^{\text{mín}\{\tau_0(\alpha), 2\tau_0(\alpha)-2\alpha+1\}}, \quad \textit{for all } x \in A_{\delta_1}.$$

Proof. By compactness we prove that the corresponding inequality holds in a neighborhood of any point $\bar{x} \in \partial\Omega$ and without loss of generality we may assume that $\bar{x} = 0$. For a given $0 < \eta \leq \delta$, we define

$$Q_\eta = \{z = (z_1, z') \in \mathbb{R} \times \mathbb{R}^{N-1}, |z_1| < \eta, |z'| < \eta\}$$

and $Q_\eta^+ = \{z \in Q_\eta, z_1 > 0\}$. Let $\varphi : \mathbb{R}^{N-1} \to \mathbb{R}$ be a C^2 function such that $(z_1, z') \in \Omega \cap Q_\eta$ if and only if $z_1 \in (\varphi(z'), \eta)$ and moreover, $(\varphi(z'), z') \in \partial\Omega$ for all $|z'| < \eta$. We further assume that $(-1, 0, \cdots, 0)$ is the outer normal vector of Ω at $\bar{x}$.

In the proof of our inequalities, we let $x = (x_1, 0)$, with $x_1 \in (0, \eta/4)$, be then a generic point in $A_{\eta/4}$. We observe that $|x - \bar{x}| = d(x) = x_1$. By definition we have

$$-(-\Delta)^\alpha V_\tau(x) = \frac{1}{2} \int_{Q_\eta} \frac{\delta(V_\tau, x, y)}{|y|^{N+2\alpha}} dy + \frac{1}{2} \int_{\mathbb{R}^N \setminus Q_\eta} \frac{\delta(V_\tau, x, y)}{|y|^{N+2\alpha}} dy \tag{2.36}$$

and we see that

$$|\int_{\mathbb{R}^N \setminus Q_\eta} \frac{\delta(V_\tau, x, y)}{|y|^{N+2\alpha}} dy| \leq c|x|^\tau, \tag{2.37}$$

where the constant c is independent of x. Thus we only need to study the asymptotic behavior of the first integral, that from now on we denote by $\frac{1}{2}E(x_1)$.

Our first goal is to get a lower bound for $E(x_1)$. For that purpose we first notice that, since $\tau \in (-1, 0)$, we have that

$$d(z)^\tau \geq |z_1 - \varphi(z')|^\tau, \quad \text{for all} \quad z \in Q_\delta \cap \Omega. \tag{2.38}$$

Now we assume that $0 < \eta \leq \delta/2$, then for all $y \in Q_\eta$ we have $x \pm y \in Q_\delta$. Thus $x \pm y \in \Omega \cap Q_\delta$ if and only if $\varphi(\pm y') < x_1 \pm y_1 < \delta$ and $|y'| < \delta$. Then, by (2.38), we have that

$$V_\tau(x + y) = d(x + y)^\tau \geq [x_1 + y_1 - \varphi(y')]^\tau, \quad x + y \in Q_\delta \cap \Omega \tag{2.39}$$

and

$$V_\tau(x-y) = d(x-y)^\tau \geq [x_1 - y_1 - \varphi(-y')]^\tau, \quad x-y \in Q_\delta \cap \Omega. \tag{2.40}$$

On the other side, for $y \in Q_\eta$ we have that if $x \pm y \in Q_\delta \cap \Omega^c$ then, by definition of V_τ, we have $V_\tau(x \pm y) = 0$. Now, for $y \in Q_\eta$ we define the intervals

$$I_+ = (\varphi(y') - x_1, \eta - x_1) \quad \text{and} \quad I_- = (x_1 - \eta, x_1 - \varphi(-y')) \tag{2.41}$$

and the functions

$$\begin{aligned}
I(y) &= \chi_{I_+}(y_1)|x_1 + y_1 - \varphi(y')|^\tau + \chi_{I_-}(y_1)|x_1 - y_1 - \varphi(-y')|^\tau - 2x_1^\tau, \\
J(y_1) &= \chi_{(x_1-\eta,x_1)}(y_1)|x_1 - y_1|^\tau + \chi_{(-x_1,\eta-x_1)}(y_1)|x_1 + y_1|^\tau - 2x_1^\tau, \\
I_1(y) &= \{\chi_{I_+}(y_1) - \chi_{(-x_1,\eta-x_1)}(y_1)\}|x_1 + y_1|^\tau, \\
I_2(y) &= \chi_{I_+}(y_1)(|x_1 + y_1 - \varphi(y')|^\tau - |x_1 + y_1|^\tau),
\end{aligned}$$

where χ_A denotes the characteristic function of the set A. Then, using these definitions and inequalities (2.39) and (2.40), we have that

$$E(x_1) \geq \int_{Q_\eta} \frac{I(y)}{|y|^{N+2\alpha}} dy = \int_{Q_\eta} \frac{J(y_1)}{|y|^{N+2\alpha}} dy + E_1(x_1) + E_2(x_1), \tag{2.42}$$

where

$$E_i(x_1) = \int_{Q_\eta} \frac{I_i(y) + I_{-i}(y)}{|y|^{N+2\alpha}} dy, \quad i = 1, 2. \tag{2.43}$$

Here we have considered that

$$I_{-1}(y) = \{\chi_{I_-}(y_1) - \chi_{(x_1-\eta,x_1)}(y_1)\}|x_1 - y_1|^\tau$$

and

$$I_{-2}(y) = \chi_{I_-}(y_1)(|x_1 - y_1 - \varphi(-y')|^\tau - |x_1 - y_1|^\tau),$$

for $y = (y_1, y') \in \mathbb{R}^N$. We start studying the first integral in the right hand side in (2.42). Changing variables we see that

$$\int_{Q_\eta} \frac{J(y_1)}{|y|^{N+2\alpha}} dy = x_1^{\tau-2\alpha} \int_{Q_{\frac{\eta}{x_1}}} \frac{J(x_1 z_1) x_1^{-\tau}}{|z|^{N+2\alpha}} dz = 2x_1^{\tau-2\alpha}(R_1 - R_2),$$

where

$$R_1 = \int_{Q^+_{\frac{\eta}{x_1}}} \frac{\chi_{(0,1)}(z_1)|1 - z_1|^\tau + (1 + z_1)^\tau - 2}{|z|^{N+2\alpha}} dz$$

and

$$R_2 = \int_{Q^+_{\frac{\eta}{x_1}}} \frac{\chi_{(\frac{\eta}{x_1}-1, \frac{\eta}{x_1})}(z_1)(1+z_1)^\tau}{|z|^{N+2\alpha}} dz.$$

Next we estimate these last two integrals. For R_1 we see that, for appropriate positive constants c_1 and c_2

$$\begin{aligned}
& \int_{\mathbb{R}^N_+} \frac{\chi_{(0,1)}(z_1)|1-z_1|^\tau + (1+z_1)^\tau - 2}{|z|^{N+2\alpha}} dz \\
= & \int_0^{+\infty} \frac{\chi_{(0,1)}(z_1)|1-z_1|^\tau + (1+z_1)^\tau - 2}{z_1^{1+2\alpha}} dz_1 \int_{\mathbb{R}^{N-1}} \frac{1}{(|z'|^2+1)^{\frac{N+2\alpha}{2}}} dz' \\
= & \; c_1\, C(\tau)
\end{aligned}$$

and

$$\int_{(Q^+_{\frac{\eta}{x_1}})^c} \frac{\chi_{(0,1)}(z_1)|1-z_1|^\tau + (1+z_1)^\tau - 2}{|z|^{N+2\alpha}} dz = -c_2\, x_1^{2\alpha}(1+o(1)).$$

Consequently we have, for some constant c that

$$R_1 = c_1(C(\tau) + c x_1^{2\alpha} + o(x_1^{2\alpha})). \tag{2.44}$$

For R_2 we have that

$$R_2 \;=\; \int_{\frac{\eta}{x_1}-1}^{\frac{\eta}{x_1}} \frac{(1+z_1)^\tau}{z_1^{1+2\alpha}} \int_{B_{\frac{\eta}{x_1}}} \frac{1}{(1+|z'|^2)^{\frac{N+2\alpha}{2}}} dz' dz_1 \le c_3 x_1^{2\alpha-\tau+1}, \tag{2.45}$$

where $c_3 > 0$. Here and in what follows we denote by B_σ the ball of radius σ in $\mathbb{R}^{N-1}$. From (2.44) and (2.45) we then conclude that

$$\int_{Q_\eta} \frac{J(y_1)}{|y|^{N+2\alpha}} dy = c_1 x_1^{\tau-2\alpha}(C(\tau) + c x_1^{2\alpha} + o(x_1^{2\alpha})). \tag{2.46}$$

Continuing with our analysis we estimate $E_1(x_1)$. We only consider the term $I_1(y)$, since the estimate for $I_{-1}(y)$ is similar. We have

$$\int_{Q_\eta} \frac{I_1(y)}{|y|^{N+2\alpha}} dy = -\int_{B_\eta} \int_{-x_1}^{\varphi(y')-x_1} \frac{|x_1+y_1|^\tau}{|y|^{N+2\alpha}} dy_1 dy' = -x_1^{\tau-2\alpha} F_1(x_1),$$

where

$$F_1(x_1) = \int_{B_{\frac{\eta}{x_1}}} \int_0^{\frac{\varphi(x_1 z')}{x_1}} \frac{|z_1|^\tau}{((z_1-1)^2+|z'|^2)^{(N+2\alpha)/2}} dz_1 dz'. \tag{2.47}$$

In what follows we write $\varphi_-(y') = \text{mín}\{\varphi(y'), 0\}$ and $\varphi_+(y') = \varphi(y') - \varphi_-(y')$. Next we see that assuming that $0 \le \varphi_+(y') \le C|y'|^2$ for $|y'| \le \eta$, for given (z_1, z') satisfying $0 \le z_1 \le \frac{\varphi_+(x_1 z')}{x_1}$ and $|z'| \le \frac{\eta}{x_1}$ then

$$(1-z_1)^2 + |z'|^2 \ge \frac{1}{4}(1+|z'|^2), \tag{2.48}$$

if we assume η small enough. Thus

$$\begin{aligned}
F_1(x_1) &\le C \int_{B_{\frac{\eta}{x_1}}} \int_0^{\frac{\varphi_+(x_1 z')}{x_1}} \frac{|z_1|^\tau}{(1+|z'|^2)^{(N+2\alpha)/2}} dz_1 dz' \\
&\le C x_1^{\tau+1} \int_{B_{\frac{\eta}{x_1}}} \frac{|z'|^{2(\tau+1)}}{(1+|z'|^2)^{(N+2\alpha)/2}} dz' \\
&\le C x_1^{\tau+1}(x_1^{-2\tau+2\alpha-1} + 1) \le C x_1^{\text{mín}\{\tau+1, 2\alpha-\tau\}}.
\end{aligned}$$

Thus we have obtained

$$E_1(x_1) \ge -C x_1^{\tau-2\alpha} x_1^{\text{mín}\{\tau+1, 2\alpha-\tau\}}. \tag{2.49}$$

We continue with the estimate of $E_2(x_1)$. As before we only consider the term $I_2(y)$,

$$\begin{aligned}
\int_{Q_\eta} \frac{I_2(y)}{|y|^{N+2\alpha}} dy &= \int_{B_\eta} \int_{\varphi(y')-x_1}^{\eta-x_1} \frac{|x_1+y_1-\varphi(y')|^\tau - |x_1+y_1|^\tau}{(y_1^2+|y'|^2)^{\frac{N+2\alpha}{2}}} dy_1 dy' \\
&\ge \int_{B_\eta} \int_{\varphi_-(y')-x_1}^{\eta-x_1} \frac{|x_1+y_1-\varphi_-(y')|^\tau - |x_1+y_1|^\tau}{(y_1^2+|y'|^2)^{\frac{N+2\alpha}{2}}} dy_1 dy' \\
&= \int_{B_\eta} \int_{\varphi_-(y')}^{\eta} \frac{|z_1-\varphi_-(y')|^\tau - |z_1|^\tau}{((z_1-x_1)^2+|y'|^2)^{\frac{N+2\alpha}{2}}} dz_1 dy' \\
&\ge \int_{B_\eta} \int_0^{\eta} \frac{|z_1-\varphi_-(y')|^\tau - |z_1|^\tau}{((z_1-x_1)^2+|y'|^2)^{\frac{N+2\alpha}{2}}} dz_1 dy' \\
&\quad + \int_{B_\eta} \int_{\varphi_-(y')}^{0} \frac{-|z_1|^\tau}{((z_1-x_1)^2+|y'|^2)^{\frac{N+2\alpha}{2}}} dz_1 dy' \\
&= E_{21}(x_1) + E_{22}(x_1).
\end{aligned} \tag{2.50}$$

We observe that $E_{22}(x_1)$ is similar to $F_1(x_1)$. In order to estimate $E_{21}(x_1)$ we use integration by parts

$$
\begin{aligned}
E_{21}(x_1) &= \frac{1}{\tau+1}\int_{B_\eta}\left\{\frac{(\eta-\varphi_-(y'))^{\tau+1}-\eta^{\tau+1}}{((\eta-x_1)^2+|y'|^2)^{\frac{N+2\alpha}{2}}}-\frac{(-\varphi_-(y'))^{\tau+1}}{(x_1^2+|y'|^2)^{\frac{N+2\alpha}{2}}}\right\}dy' \\
&+ \frac{N+2\alpha}{\tau+1}\int_{B_\eta}\int_0^\eta \frac{(z_1-\varphi_-(y'))^{\tau+1}-z_1^{\tau+1}}{((z_1-x_1)^2+|y'|^2)^{\frac{N+2\alpha}{2}+1}}(z_1-x_1)dz_1dy' \\
&= A_1+A_2.
\end{aligned}
$$

For the first integral we have

$$
\begin{aligned}
A_1 &\geq \frac{1}{\tau+1}\int_{B_\eta}\left\{\frac{-\eta^{\tau+1}}{((\eta-x_1)^2+|y'|^2)^{\frac{N+2\alpha}{2}}}-\frac{(-\varphi_-(y'))^{\tau+1}}{(x_1^2+|y'|^2)^{\frac{N+2\alpha}{2}}}\right\}dy' \\
&\geq -C(\eta)-C\int_{B_\eta}\frac{|y'|^{2\tau+2}}{(x_1^2+|y'|^2)^{\frac{N+2\alpha}{2}}}dy' \geq -Cx_1^{\tau-2\alpha+\tau+1}-C.
\end{aligned}
$$

For the second integral, since $\tau\in(-1,0)$ and $(z_1-\varphi_-(y'))^{\tau+1}-|z_1|^{\tau+1}>0$, we have that

$$
\begin{aligned}
A_2 &\geq \frac{N+2\alpha}{\tau+1}\int_{B_\eta}\int_0^{x_1}\frac{(z_1-\varphi_-(y'))^{\tau+1}-|z_1|^{\tau+1}}{((z_1-x_1)^2+|y'|^2)^{\frac{N+2\alpha}{2}+1}}(z_1-x_1)dz_1dy' \\
&\geq \frac{N+2\alpha}{(\tau+1)^2}\int_{B_\eta}\int_0^{x_1}\frac{-\varphi_-(y')z_1^\tau}{((z_1-x_1)^2+|y'|^2)^{\frac{N+2\alpha}{2}+1}}(z_1-x_1)dz_1dy' \\
&\geq C_3x_1^{2\tau-2\alpha+1}\int_{B_{\eta/x_1}}\int_0^1\frac{|z'|^2z_1^\tau}{((z_1-1)^2+|z'|^2)^{\frac{N+2\alpha}{2}+1}}(z_1-1)dz_1dz' \\
&\geq -C_4x_1^{2\tau-2\alpha+1}, \qquad (2.51)
\end{aligned}
$$

where $C_3, C_4>0$ independent of x_1 and the second inequality used $a=z_1$ and $b=-\varphi_-(y')$ in the fact that $(a+b)^{\tau+1}-a^{\tau+1}\leq\frac{a^\tau b}{\tau+1}$ for $a>0, b\geq 0$.

Thus, we have obtained

$$
E_2(x_1)\geq -Cx_1^{\tau-2\alpha}x_1^{\mathrm{mín}\{\tau+1,2\alpha-\tau\}}. \qquad (2.52)
$$

The next step is to obtain the other inequality for $E(x_1)$. By choosing δ smaller if necessary, we can prove that

Lemma 2.3.1 *Under the regularity conditions on the boundary and with the arrangements given at the beginning of the proof, there is $\eta>0$ and $C>0$ such that*

$$
d(z)\geq(z_1-\varphi(z'))(1-C|z'|^2) \quad \textit{for all } (z_1,z')\in\Omega\cap Q_\eta.
$$

Proof. Since φ is C^2 and $\nabla\varphi(0) = 0$, there exist $\eta_1 \in (0, 1/8)$ small and $C_1 > 0$ such that $C_1\eta_1 < 1/4$ and

$$|\varphi(y')| < C_1|y'|^2, \quad |\nabla\varphi(y')| \leq C_1|y'|, \quad \forall\, y' \in B_{\eta_1}. \tag{2.53}$$

Choosing $\eta_2 \in (0, \eta_1)$ such that for any $z = (z_1, z') \in Q_{\eta_2} \cap \Omega$, there exists y' satisfying $(\varphi(y'), y') \in \partial\Omega \cap Q_{\eta_1}$ and $d(z) = |z - (\varphi(y'), y')|$.

We observe that y' mentioned above, is the minimizer of

$$H(z') = (z_1 - \varphi(z'))^2 + |z' - y'|^2, \quad |z'| < \eta_1,$$

then

$$-(z_1 - \varphi(y'))\nabla\varphi(y') + (z' - y') = 0,$$

which, together with (2.53) implies that

$$\begin{aligned} |y'| - |z'| &\leq |z' - y'| = |(z_1 - \varphi(y'))\nabla\varphi(y')| \leq (|z_1| + C_1|y'|^2)|\nabla\varphi(y')| \\ &\leq C_1(\eta_2 + C_1\eta_1^2)|y'| \leq 2C_1\eta_1|y'| < \frac{1}{2}|y'|. \end{aligned}$$

Then

$$|y'| \leq 2|z'|. \tag{2.54}$$

Denote the points $z, (\varphi(y'), y'), (\varphi(z'), z')$ by A, B, C, respectively, and let θ be the angle between the segment BC and the hyper plane with normal vector $e_1 = (1, 0, ..., 0)$ and containing C. Then the angle $\angle C = \frac{\pi}{2} - \theta$. Denotes the arc from B to C in the plane ABC by $\mathrm{arc}(BC)$. By the geometry, there exists some point $x = (\varphi(x'), x') \in \mathrm{arc}(BC)$ such that line BC parallels the tangent line of $\mathrm{arc}(BC)$ at point x. Then, from (2.54) we have $|x'| \leq \text{máx}\{|z'|, |y'|\} \leq 2|z'|$ and so, from (2.53) we obtain

$$\tan(\theta) = |\frac{y' - z'}{|y' - z'|} \cdot \nabla\varphi(x')| \leq |\nabla\varphi(x')| \leq C_1|x'| \leq 2C_1|z'|,$$

which implies that for some $C > 0$,

$$\cos(\theta) \geq 1 - C|z'|^2. \tag{2.55}$$

Then we complete the proof using Sine Theorem and (2.55)

$$\begin{aligned} d(z) &= \frac{\sin(\angle C)}{\sin(\angle B)}(z_1 - \varphi(z')) \geq (z_1 - \varphi(z'))\sin(\frac{\pi}{2} - \theta) \\ &= (z_1 - \varphi(z'))\cos(\theta) \geq (z_1 - \varphi(z'))(1 - C|z'|^2). \qquad \square \end{aligned}$$

From this lemma, by making C and η smaller if necessary we obtain that

$$d^\tau(z) \leq (z_1 - \varphi(z'))^\tau (1 + C|z'|^2) \quad \text{for all } z \in \Omega \cap Q_\eta. \tag{2.56}$$

With $x = (x_1, 0)$ satisfying $x_1 \in (0, \eta/4)$ as at the beginning of the proof, we have that $d(x) = x_1$ and for any $y \in Q_\eta$ we see that $x \pm y \in Q_\delta$. We also see that $x \pm y \in \Omega \cap Q_\delta$ if and only if $\varphi(\pm y') < x_1 \pm y_1 < \delta$ and $|y'| < \delta$. Then, for $x \pm y \in \Omega \cap Q_\delta$, by (2.56) we have,

$$V_\tau(x \pm y) = d(x \pm y)^\tau \leq (x_1 \pm y_1 - \varphi(\pm y'))^\tau (1 + C|y'|^2). \tag{2.57}$$

For $y \in Q_\eta$, we define

$$I_3(y) = C|y'|^2 \chi_{I_+}(y_1) |x_1 + y_1 - \varphi(y')|^\tau$$

and

$$I_3(-y) = C|y'|^2 \chi_{I_-}(y_1) |x_1 - y_1 - \varphi(-y')|^\tau,$$

where I_+ and I_- were defined in (2.41). Using (2.57) as in (2.42) we find

$$\begin{aligned} E(x_1) &= \int_{Q_\eta} \frac{\delta(V_\tau, x, y)}{|y|^{N+2\alpha}} dy \leq \int_{Q_\eta} \frac{I(y)}{|y|^{N+2\alpha}} dy + E_3(x_1) \\ &= \int_{Q_\eta} \frac{J(y)}{|y|^{N+2\alpha}} dy + E_1(x_1) + E_2(x_1) + E_3(x_1), \end{aligned} \tag{2.58}$$

where E_1 and E_2 were defined in (2.43) and

$$E_3(x_1) = \int_{Q_\eta} \frac{I_3(y) + I_3(-y)}{|y|^{N+2\alpha}} dy. \tag{2.59}$$

We estimate $E_3(x_1)$ and for that we observe that it is enough to estimate the integral with one of the terms in (2.59) (the other is similar), say

$$\begin{aligned} &\int_{Q_\eta} \frac{I_3(y)}{|y|^{N+2\alpha}} dy = \int_{B_\eta} \int_{\varphi(y')-x_1}^{\eta - x_1} \frac{C|y'|^2 |x_1 + y_1 - \varphi(y')|^\tau}{|y|^{N+2\alpha}} dy_1 dy' \\ &= C x_1^{\tau - 2\alpha + 2} \int_{B_{\frac{\eta}{x_1}}} \int_{\frac{\varphi(x_1 z')}{x_1}}^{\frac{\eta}{x_1}} \frac{|z'|^2 |z_1 - \frac{\varphi(x_1 z')}{x_1}|^\tau}{((z_1 - 1)^2 + |z'|^2)^{(N+2\alpha)/2}} dz_1 dz' \\ &= C x_1^{\tau - 2\alpha + 2} (A_1 + A_2), \end{aligned} \tag{2.60}$$

where A_1 and A_2 are integrals over properly chosen subdomains, estimated separately.

$$
\begin{aligned}
A_1 &= \int_{B_{\frac{\eta}{x_1}}} \int_{\frac{\varphi(x_1 z')}{x_1}}^{\frac{\varphi(x_1 z')}{x_1}+\frac{1}{2}} \frac{|z'|^2 |z_1 - \frac{\varphi(x_1 z')}{x_1}|^\tau}{((z_1-1)^2+|z'|^2)^{(N+2\alpha)/2}} dz_1 dz' \\
&\leq \frac{c}{(\tau+1)2^{\tau+1}} \int_{B_{\frac{\eta}{x_1}}} \frac{|z'|^2}{(1+|z'|^2)^{(N+2\alpha)/2}} dz' \qquad (2.61) \\
&\leq c' \left(\frac{\eta}{x_1}\right)^{-2\alpha+1}. \qquad (2.62)
\end{aligned}
$$

The inequality in (2.61) is obtained noticing that the ball $B((1,0),1/2)$ in R^N does not touch the band

$$
\{(z_1, z') \,/\, |z'| \leq \eta, \frac{\varphi(x_1 z')}{x_1} \leq z_1 \leq \frac{\varphi(x_1 z')}{x_1} + 1/2\}
$$

if x_1 is small enough, and so $(z_1-1)^2 + |z'|^2 \geq \frac{1}{8} + \frac{1}{2}|z'|^2$. Then simple integration gives the next term. Next we estimate A_2

$$
\begin{aligned}
A_2 &= \int_{B_{\frac{\eta}{x_1}}} \int_{\frac{\varphi(x_1 z')}{x_1}+\frac{1}{2}}^{\frac{\eta}{x_1}} \frac{|z'|^2 |z_1 - \frac{\varphi(x_1 z')}{x_1}|^\tau}{((z_1-1)^2+|z'|^2)^{(N+2\alpha)/2}} dz_1 dz' \\
&\leq \frac{1}{2^\tau} \int_{B_{\frac{\eta}{x_1}}} \int_{\frac{\varphi(x_1 z')}{x_1}+\frac{1}{2}}^{\frac{\eta}{x_1}} \frac{|z'|^2}{((z_1-1)^2+|z'|^2)^{(N+2\alpha)/2}} dz_1 dz' \\
&\leq c' \left(\frac{\eta}{x_1}\right)^{-2\alpha+2}. \qquad (2.63)
\end{aligned}
$$

Putting together (2.60), (2.62), (2.63) and (2.59) we obtain

$$
E_3(x_1) = \int_{Q_\eta} \frac{(I_3(y) + I_3(-y))}{|y|^{N+2\alpha}} dy \leq c x_1^\tau. \qquad (2.64)
$$

From (2.47), but using the other inequality for F_1, that is,

$$
F_1(x_1) \geq C \int_{B_{\frac{\eta}{x_1}}} \int_0^{\frac{\varphi_-(x_1 z')}{x_1}} \frac{|z_1|^\tau}{(1+|z'|^2)^{(N+2\alpha)/2}} dz_1 dz'
$$

and arguing similarly we obtain as in (2.49)

$$
E_1(x_1) \leq C x_1^{\tau-2\alpha} x_1^{\text{mín}\{\tau+1, 2\alpha\}}. \qquad (2.65)
$$

Then we look at $E_2(x_1)$ and, as in (2.50), we only consider the term $I_2(y)$:

$$\int_{Q_\eta} \frac{I_2(y)}{|y|^{N+2\alpha}} dy \;\; \leq \;\; \int_{B_\eta} \int_{\varphi_+(y')}^{\eta} \frac{|z_1 - \varphi_+(y')|^\tau - |z_1|^\tau}{((z_1 - x_1)^2 + |y'|^2)^{\frac{N+2\alpha}{2}}} dz_1 dy' = \tilde{E}_{21}(x_1).$$

In order to estimate $\tilde{E}_{21}(x_1)$ we use integration by parts

$$\begin{aligned}
&\tilde{E}_{21}(x_1) = \\
&\frac{1}{\tau+1} \int_{B_\eta} \left\{ \frac{(\eta - \varphi_+(y'))^{\tau+1} - \eta^{\tau+1}}{((\eta - x_1)^2 + |y'|^2)^{\frac{N+2\alpha}{2}}} - \frac{(\varphi_+(y'))^{\tau+1}}{((\varphi_+(y') - x_1)^2 + |y'|^2)^{\frac{N+2\alpha}{2}}} \right\} dy' \\
&+ \frac{N+2\alpha}{\tau+1} \int_{B_\eta} \int_{\varphi_+(y')}^{\eta} \frac{(z_1 - \varphi_+(y'))^{\tau+1} - z_1^{\tau+1}}{((z_1 - x_1)^2 + |y'|^2)^{\frac{N+2\alpha}{2}+1}} (z_1 - x_1) dz_1 dy' \\
&\leq \frac{N+2\alpha}{\tau+1} \int_{B_\eta} \int_{\text{mín}\{\varphi_+(y'), x_1\}}^{x_1} \frac{(z_1 - \varphi_+(y'))^{\tau+1} - z_1^{\tau+1}}{((z_1 - x_1)^2 + |y'|^2)^{\frac{N+2\alpha}{2}+1}} (z_1 - x_1) dz_1 dy'.
\end{aligned}$$

This integral can be estimated in a similar way as E_{21}, see (2.51) and the estimates given before. We then obtain

$$E_2(x_1) \leq C x_1^{2\tau - 2\alpha + 1}. \tag{2.66}$$

Then we conclude from (2.36), (2.42), (2.46), (2.49), (2.52), (2.58), (2.64), (2.65) and (2.66) that

$$-(-\Delta)^\alpha V_\tau(x) = C x_1^{\tau - 2\alpha} (C(\tau) + O(x_1^{\text{mín}\{\tau+1, 2\alpha\}})), \tag{2.67}$$

where there exists a constant $c > 0$ so that

$$|O(x_1^{\text{mín}\{\tau+1,2\alpha\}})| \leq c x_1^{\text{mín}\{\tau+1,2\alpha\}}, \qquad \text{for all small } x_1 > 0.$$

From here, depending on the value of $\tau \in (-1, 0)$, conditions (i), (ii) and (iii) follows and the proof of the proposition is complete. $\square$

We end this section with an estimate we need when dealing with equation (2.5) when the external value g is not zero. We have the following proposition

Proposition 2.3.3 *Assume that Ω is a bounded, open and C^2 domain in $\mathbb{R}^N$. Assume that $g \in L^1_\omega(\Omega^c)$. Assume further that there are numbers $\beta \in (-1, 0)$, $\eta > 0$ and $c > 1$ such that*

$$\frac{1}{c} \leq g(x) d(x)^{-\beta} \leq c, \quad x \in \bar{\Omega}^c \text{ and } d(x) \leq \eta.$$

Then there exist $\eta_1 > 0$ and $C > 1$ such that G, defined in (2.21), satisfies

$$\frac{1}{C}d(x)^{\beta-2\alpha} \leq G(x) \leq Cd(x)^{\beta-2\alpha}, \quad x \in A_{\eta_1}. \tag{2.68}$$

Proof. The proof of this proposition requires estimates similar to those in the proof of Proposition 2.3.2 so we omit it. However, the function C used there and defined in (2.14), needs to be replaced here by $\tilde{C} : (-1, 0) \to \mathbb{R}$ given by

$$\tilde{C}(\beta) = \int_1^\infty \frac{|t-1|^\beta}{t^{1+2\alpha}} dt.$$

We observe that this function is always positive. □

2.4. Proof of existence results

In this section, we will give the proof of existence of large solution to (2.6). By Theorem 2.2.6 we only need to find ordered super and sub-solution, denoted by U and W, for (2.6) under our various assumptions. We begin with a simple lemma that reduce the problem to find them only in A_δ.

Lemma 2.4.1 *Let U and W be classical ordered super and sub-solution of (2.6) in the sub-domain A_δ. Then there exists λ large such that $U_\lambda = U - \lambda\bar{V}$ and $W_\lambda = W + \lambda\bar{V}$, where $\bar{V}$ is the solution of (2.29), with $\mathcal{O} = \Omega$, are ordered super and sub-solution of (2.6).*

Proof. Notice that by negativity $\bar{V}$ in Ω, we have that $U_\lambda \geq U$ and $W_\lambda \leq W$, so they are still ordered in A_δ. In addition U_λ satisfies

$$(-\Delta)^\alpha U_\lambda + |U_\lambda|^{p-1}U_\lambda - f(x) \geq (-\Delta)^\alpha U + |U|^{p-1}U - f(x) + \lambda > 0, \quad \text{in} \quad \Omega.$$

This inequality holds because of our assumption in A_δ, the fact that $(-\Delta)^\alpha U + |U|^{p-1}U - f(x)$ is continuous in $\Omega \setminus A_\delta$ and by taking λ large enough.

By the same type of arguments we find the W_λ is a sub-solution of the first equation in (2.6) and we complete the proof. □

Now we are in position to prove our existence results that we already reduced to find ordered super and sub-solution of (2.6) with the first equation in A_δ with the desired asymptotic behavior.

Proof of Theorem 2.1.1 (Existence). Define

$$U_\mu(x) = \mu V_\tau(x) \quad \text{and} \quad W_\mu(x) = \mu V_\tau(x), \tag{2.69}$$

with $\tau = -\frac{2\alpha}{p-1}$. We observe that $\tau = -\frac{2\alpha}{p-1} \in (-1, \tau_0(\alpha))$ and $\tau p = \tau - 2\alpha$, Then by Proposition 2.3.2 and $(H2)$ we find that for $x \in A_\delta$ and $\delta > 0$ small

$$(-\Delta)^\alpha U_\mu(x) + U_\mu^p(x) - f(x) \geq -C\mu d(x)^{\tau-2\alpha} + \mu^p d(x)^{\tau p} - Cd(x)^{\tau p},$$

for some $C > 0$. Then there exists a large $\mu > 0$ such that U_μ is a super-solution of (2.6) with the first equation in A_δ with the desired asymptotic behavior. Now by Proposition 2.3.2 we have that for $x \in A_\delta$ and $\delta > 0$ small

$$(-\Delta)^\alpha W_\mu(x) + W_\mu^p(x) - f(x) \leq -\frac{\mu}{C} d(x)^{\tau-2\alpha} + \mu^p d(x)^{\tau p} - f(x) \leq 0,$$

in the last inequality we have used $(H2)$ and $\mu > 0$ small. Then, by Theorem 2.2.6 there exists a solution, with the desired asymptotic behavior. □

Proof of Theorem 2.1.1 (Special case $\tau = \tau_0(\alpha)$). We define for $t > 0$,

$$U_\mu(x) = tV_{\tau_0(\alpha)}(x) - \mu V_{\tau_1}(x) \quad \text{and} \quad W_\mu(x) = tV_{\tau_0(\alpha)}(x) - \mu V_{\tau_1}(x), \tag{2.70}$$

where $\tau_1 = \text{mín}\{\tau_0(\alpha)p + 2\alpha, 0\}$. If $\tau_1 = 0$, we write $V_0 = \chi_\Omega$ and we have

$$(-\Delta)^\alpha V_0(x) = \int_{\mathbb{R}^N\setminus\Omega} \frac{1}{|z-x|^{N+2\alpha}} dz, \quad x \in \Omega.$$

By direct computation, there exists $C > 1$ such that

$$\frac{1}{C} d(x)^{-2\alpha} \leq (-\Delta)^\alpha V_0(x) \leq Cd(x)^{-2\alpha}, \quad x \in \Omega. \tag{2.71}$$

We see that $\tau_1 \in (\tau_0(\alpha), 0]$ and, if $\tau_1 < 0$, we have $\tau_1 - 2\alpha = \tau_0(\alpha)p$ and

$$\tau_1 - 2\alpha < \text{mín}\{\tau_0(\alpha), \tau_0(\alpha) - 2\alpha + \tau_0(\alpha) + 1\}.$$

Then, by Proposition 2.3.2 and (2.71), for $x \in A_\delta$, it follows that

$$\begin{aligned}(-\Delta)^\alpha U_\mu(x) + |U_\mu(x)|^{p-1}U_\mu(x) \quad &\geq \quad -Ctd(x)^{\text{mín}\{\tau_0(\alpha),\tau_0(\alpha)-2\alpha+\tau_0(\alpha)+1\}} \\ &\quad\quad -C\mu d(x)^{\tau_1-2\alpha} + t^p d(x)^{\tau_0(\alpha)p}.\end{aligned}$$

Thus, letting $\mu = t^p/(2C)$ if $\tau_1 < 0$ and $\mu = 0$ if $\tau_1 = 0$, for a possible smaller $\delta > 0$, we obtain

$$(-\Delta)^\alpha U_\mu(x) + |U_\mu(x)|^{p-1}U_\mu(x) \geq 0, \quad x \in A_\delta.$$

For the sub-solution, by Proposition 2.3.2 and (2.71), for $x \in A_\delta$, we have

$$\begin{aligned}(-\Delta)^\alpha W_\mu(x) + |W_\mu|^{p-1}W_\mu(x) \quad &\leq \quad Ctd(x)^{\text{mín}\{\tau_0(\alpha),\tau_0(\alpha)-2\alpha+\tau_0(\alpha)+1\}} \\ &\quad\quad -\frac{\mu}{C} d(x)^{\tau_1-2\alpha} + t^p d(x)^{\tau_0(\alpha)p},\end{aligned}$$

where $C > 1$. Then, for $\mu \geq 2Ct^p$ and a possibly smaller $\delta > 0$

$$(-\Delta)^\alpha W_\mu(x) + |W_\mu|^{p-1} W_\mu(x) \leq 0, \ x \in A_\delta,$$

completing the proof. $\square$

Proof of Theorem 2.1.2. We define U_μ and W_μ as in (2.69). In the case of a weak source, we take $\tau = \gamma + 2\alpha$ and we observe that $\gamma + 2\alpha \geq -\frac{2\alpha}{p-1} \geq \tau_0(\alpha)$ and $p(\gamma + 2\alpha) \geq \gamma$. Using Proposition 2.3.2 and $(H3)$ we find that U_μ is a super-solution for $\mu > 0$ large (resp. W_μ is a sub-solution for $\mu > 0$ small) of (2.6) with the first equation in A_δ for $\delta > 0$ small. In the case of a strong source, we take $\tau = \frac{\gamma}{p}$ and observe that $\gamma < \frac{\gamma}{p} - 2\alpha$. Using Proposition 2.3.2 we find

$$|(-\Delta)^\alpha U_\mu|, |(-\Delta)^\alpha W_\mu| \leq C d(x)^{\frac{\gamma}{p} - 2\alpha}.$$

By $(H3)$ we find that U_μ is a super-solution for μ large (resp. W_μ is a sub-solution for μ small) of (2.6) with the first equation in A_δ for δ small. $\square$

Remark 2.4.1 *In order to obtain the above existence results for classical solution to (2.5), that is when g is not necessarily zero, we only need use them with F as a right hand side as given in (2.24). Here we only need to assume that g satisfies $(H4)$. In fact, as above we find super and sub-solutions for (2.6), with f replaced by F. Then, as in the proof of Theorem 2.2.6, we find a viscosity solution of (2.6) and then $v = u + \tilde{g}$ is a viscosity solution of (2.5). Next we use Theorem 2.6 in [27] and then we use Theorem 2.2.1 to obtain that v is a classical solution of (2.5).*

Remark 2.4.2 *Now we compare Theorem 2.1.1 with the result in [51]. Let us assume that f and g satisfies hypothesis (F0)-(F2) and (G0)-(G3), respectively, given in [51]. We first observe that the function F, as defined above, satisfies $(H1)$ thanks to (G0), (G3) and (F0). Next we see that F satisfies $(H2)$, since (G2), (F1) and (F2) holds. Here we have to use Proposition 2.3.3. In the range of p given by (2.7), we then may apply Theorem 2.1.1 to obtain existence of a blow-up solution as given in Theorem 1.1 in [51]. We see that the existence is proved here, without assuming hypothesis $(G1)$, thus we generalized this earlier result. Moreover, here we obtain a uniqueness and non existence of blow-up solution, if we further assume hypotheses on f and g, guaranteeing hypothesis $(H2^*)$ in Theorem 2.1.1. The complementary range of p is obtained using Theorem 1.2 for the existence of solutions as given in Theorem 1.1 in [51] and uniqueness and non-existence as in Theorem 1.3 and 1.4 are truly new results. The hypotheses needed on g to obtain $(H3)$ for the function F are a bit stronger, since we are requiring in $(H3)$ that the explosion rate is the same from above and from below, while in (G2) and (G4) they may be different.*

2.5. Proof of uniqueness results

In this section we prove our uniqueness results, which are given in Theorem 2.1.1 and Theorem 2.1.3. These results are for positive solutions, so we assume that the external source f is non-negative. We assume that there are two positive solutions u and v of (2.6) and then define the set

$$\mathcal{A} = \{x \in \Omega,\ u(x) > v(x)\}. \tag{2.72}$$

This set is open, $\mathcal{A} \subset \Omega$ and we only need to prove that $\mathcal{A} = \emptyset$, to obtain that $u = v$, by interchanging the roles of u and v.

We will distinguish three cases, depending on the conditions satisfying u and v: Case a) u and v satisfy (2.7) and (2.8) (uniqueness part of Theorem 2.1.1), Case b) u and v (2.16) and (2.17) (weak source in Theorem 2.1.3) and Case c) u and v with (2.18) -(2.20) (strong source in Theorem 2.1.3).

We start our proof considering an auxiliary function

$$V(x) = \begin{cases} c(1-|x|^2)^3, & x \in B_1(0), \\ 0, & x \in B_1^c(0), \end{cases} \tag{2.73}$$

where the constant c may be chosen so that V satisfies

$$(-\Delta)^\alpha V(x) \le 1 \quad \text{and} \quad 0 < V(0) = \max_{x\in\mathbb{R}^N} V(x). \tag{2.74}$$

In order to prove the uniqueness result in the three cases, we need first some preliminary lemmas.

Lemma 2.5.1 *If $\mathcal{A}_k = \{x \in \Omega, u(x) - kv(x) > 0\} \neq \emptyset$, for $k > 1$. Then,*

$$\partial\mathcal{A}_k \cap \partial\Omega \neq \emptyset. \tag{2.75}$$

Proof. If (2.75) is not true, there exists $\bar{x} \in \Omega$ such that

$$u(\bar{x}) - kv(\bar{x}) = \max_{x\in\mathbb{R}^N}(u-kv)(x) > 0,$$

Then, we have

$$(-\Delta)^\alpha(u-kv)(\bar{x}) \ge 0,$$

which contradicts

$$\begin{aligned} (-\Delta)^\alpha(u-kv)(\bar{x}) &= -u^p(\bar{x}) + kv^p(\bar{x}) - (k-1)f(\bar{x}) \\ &\le -(k^p-k)v^p(\bar{x}) < 0. \qquad \square \end{aligned}$$

Lemma 2.5.2 *If $\mathcal{A}_k \neq \emptyset$, for $k > 1$, then*

$$\sup_{x\in\Omega}(u - kv)(x) = +\infty. \tag{2.76}$$

Proof. Assume that $\bar{M} = \sup_{x\in\Omega}(u - kv)(x) < +\infty$. We see that $\bar{M} > 0$ and there is no point $\bar{x} \in \Omega$ achieving the supreme of $u - kv$, by the same argument given above. Let us consider $x_0 \in \mathcal{A}_k$, $r = d(x_0)/2$ and define

$$w_k = u - kv \quad \text{in} \quad \mathbb{R}^N. \tag{2.77}$$

Under the conditions of Case a) and b) (resp. Case c)), for all $x \in B_r(x_0) \cap \mathcal{A}_k$ we have

$$(-\Delta)^\alpha w_k(x) = -u^p(x) + kv^p(x) + (1-k)f(x) \leq -K_1 r^{\tau-2\alpha}, \tag{2.78}$$

(resp. $\leq -K_1 r^\gamma$). Here we have used that $\tau = -2\alpha/(p-1)$ and, in Case a) (2.8) for v, in Case b) $(H3)$ and (2.16) and in Case c) $(H3)$. Moreover, in Case a) we have considered $K_1 = C(k^p - k)$ and in Cases b) and c) $K_1 = C(k-1)$ for some constant C. Now we define

$$w(x) = \frac{2\bar{M}}{V(0)} V\left(\frac{x - x_0}{r}\right)$$

for $x \in \mathbb{R}^N$, where V is given in (2.73), and we see that

$$w(x_0) = 2\bar{M} \tag{2.79}$$

and

$$(-\Delta)^\alpha w \leq \frac{2\bar{M}}{V(0)} r^{-2\alpha}, \qquad \text{in} \quad B_r(x_0). \tag{2.80}$$

Since $\tau < 0$ ($\gamma < -2\alpha$ in the Case c)), by Lemma 2.5.1 we can take $x_0 \in \mathcal{A}_k$ close to $\partial\Omega$, so that

$$\frac{2\bar{M}}{V(0)} \leq K_1 r^\tau \quad \Big(\frac{2\bar{M}}{V(0)} \leq K_1 r^{\gamma+2\alpha}, \qquad \text{in Case c)}\Big).$$

From here, combining (2.78) with (2.80), we have that

$$(-\Delta)^\alpha (w_k + w)(x) \leq 0, \quad x \in B_r(x_0) \cap \mathcal{A}_k.$$

Then, by the Maximum Principle, we obtain

$$w_k(x_0) + w(x_0) \leq \text{máx}\{\bar{M}, \sup_{x\in B_r(x_0)\cap\mathcal{A}_k^c} (w_k + w)\}. \tag{2.81}$$

In case we have

$$\bar{M} < \sup_{x \in B_r(x_0) \cap \mathcal{A}_k^c} (w_k + w), \tag{2.82}$$

then

$$\begin{aligned} w(x_0) < (w_k + w)(x_0) &\le \sup_{x \in B_r(x_0) \cap \mathcal{A}_k^c} (w_k + w)(x) \\ &\le \sup_{x \in B_r(x_0) \cap \mathcal{A}_k^c} w(x) \le l w(x_0), \end{aligned} \tag{2.83}$$

which is impossible. So that (2.82) is false and then, from (2.81) we get

$$w(x_0) < w_k(x_0) + w(x_0) \le \bar{M},$$

which is impossible in view of (2.79), completing the proof. $\square$

Lemma 2.5.3 *There exists a sequence $\{C_n\}$, with $C_n > 0$, satisfying*

$$\lim_{n \to +\infty} C_n = 0 \tag{2.84}$$

and such that for all $x_0 \in \mathcal{A}_k$ and $k > 1$ we have

$$0 < \int_{Q_n} \frac{w_k(z) - M_n}{|z - x|^{N+2\alpha}} dz \le C_n r^{\tau - 2\alpha}, \quad \forall x \in B_r(x_0),$$

where we consider $r = d(x_0)/2$, $Q_n = \{z \in A_{r/n} \,/\, w_k(z) > M_n\}$ and $M_n = \max_{x \in \Omega \setminus A_{r/n}} w_k(x)$.

Proof. In Case a): we see that $Q_n \subset A_{r/n}$ and $\lim_{n \to +\infty} |Q_n| = 0$, so that using (2.11) we directly obtain

$$\begin{aligned} \int_{Q_n} \frac{w_k(z) - M_n}{|z - x|^{N+2\alpha}} dz &\le C_0 r^{-N-2\alpha} \int_{A_{r/n}} d(z)^\tau dz \\ &\le C r^{-N-2\alpha} \int_0^{r/n} t^\tau t^{N-1} dt \le \frac{C}{n^{N+\tau}} r^{\tau - 2\alpha}, \end{aligned}$$

where C depends on C_0 and $\partial\Omega$. We complete the proof defining $C_n = \frac{C}{n^{N+\tau}}$.

In Case b) we argue similarly using (2.17) and define C_n as before, while in Case c) we argue similarly using (2.20), but defining $C_n = \frac{C}{n^{N+\gamma/p}}$. $\square$

Now we are in a position to prove our non-existence results.

Proof of uniqueness results in Cases a), b) and c). We assume that $\mathcal{A} \neq \emptyset$, then there exists $k > 1$ such that $\mathcal{A}_k \neq \emptyset$. By Lemma 2.5.2 there exists $x_0 \in \mathcal{A}_k$ such that

$$w_k(x_0) = \max\{w_k(x) \,/\, x \in \Omega \setminus A_{d(x_0)}\}.$$

Proceeding as in Lemma 2.5.2 with the function

$$w(x) = \frac{K_1}{2} r^{\tau} V(\frac{x - x_0}{r})$$

$$\text{and} \quad w(x) = \frac{K_1}{2} r^{\gamma+2\alpha} V(\frac{x - x_0}{r}), \text{ in Case c)},$$

we see that

$$(-\Delta)^{\alpha}(w_k + w)(x) \leq -\frac{K_1}{2} r^{\tau-2\alpha}, \quad x \in B_r(x_0) \cap \mathcal{A}_k. \tag{2.85}$$

$$\text{and} \quad (-\Delta)^{\alpha}(w_k + w)(x) \leq -\frac{K_1}{2} r^{\gamma}, \text{ in Case c)}. \tag{2.86}$$

With M_n, as given in Lemma 2.5.3, we define

$$\bar{w}_n(x) = \begin{cases} (w_k + w)(x), & \text{if } w_k(x) \leq M_n, \\ M_n, & \text{if } w_k(x) > M_n, \end{cases} \tag{2.87}$$

for $n > 1$. By Lemma 2.5.3 we find n_0 such that

$$\begin{aligned} (-\Delta)^{\alpha} \bar{w}_{n_0}(x) &= (-\Delta)^{\alpha}(w_k + w)(x) + 2 \int_{Q_{n_0}} \frac{w_k(z) - M_{n_0}}{|z - x|^{N+2\alpha}} dz \\ &\leq 0, \quad \text{in} \quad B_r(x_0) \cap \mathcal{A}_k. \end{aligned}$$

In Case b) we have use (2.16) and in Case c) we have use (2.18), to get similar conclusion. Then, by the Maximum Principle, we get

$$\bar{w}_{n_0}(x_0) \leq \text{máx}\{M_{n_0}, \sup_{x \in B_r(x_0) \cap \mathcal{A}_k^c} (w_{k_0} + w)\}.$$

Using the same argument as in (2.83), we conclude that

$$\sup_{x \in B_r(x_0) \cap \mathcal{A}_k^c} (w_{k_0} + w) > M_{n_0}$$

does not hold and therefore

$$\bar{w}_{n_0}(x_0) = w_k(x_0) + w(x_0) \leq M_{n_0}. \tag{2.88}$$

Next, by the definition of M_n, we choose $x_1 \in \Omega \setminus A_{r/n_0}$ such that $w_k(x_1) = M_{n_0}$. But then we have

$$w_k(x_0) + w(x_0) \geq w(x_0) = \frac{K_1}{2} V(0) r^{\tau} \quad \text{in Case a) and b)}$$

$$\text{and} \quad w_k(x_0) + w(x_0) \geq w(x_0) = \frac{K_1}{2} V(0) r^{\gamma+2\alpha} \quad \text{in Case c).}$$

Thus, by the asymptotic behavior of v, (2.7) in Case a), (2.16) in Case b) and (2.18) in Case c), we have

$$r^\tau \geq n_0^\tau C v(x_1) \quad \text{and} \quad r^{\gamma+2\alpha} \geq r^{\gamma/p} \geq n_0^{\gamma/p} C v(x_1) \quad \text{in Case c).}$$

We recall that in Case a) $K_1 = C(k^p - k)$, so from (2.88)

$$u(x_1) > (1 + c_0) k v(x_1), \tag{2.89}$$

where $c_0 > 0$ is a constant, not depending on x_0 and increasing in k. Now we repeat this process above initiating by x_1 and $k_1 = k(1 + c_0)$. Proceeding inductively, we can find a sequence $\{x_m\} \subset \mathcal{A}$ such that

$$u(x_m) > (1 + c_0)^m k v(x_m),$$

which contradicts the common asymptotic behavior of u and v.

In the Case b) and c) recall that $K_1 = C(k-1)$ and, as before, we can proceed inductively to find a sequence $\{x_m\} \subset \mathcal{A}$ such that

$$u(x_m) > (k + m c_0) v(x_m),$$

which again contradicts the common asymptotic behavior of u and v. □

2.6. Proof of our non-existence results

In this section we prove our non-existence results. Our arguments are based on the construction of some special super and sub-solutions and some ideas used in Section 2.5. The main portion of our proof is based on the following proposition that we state and prove next.

Proposition 2.6.1 *Assume that Ω is an open, bounded and connected domain of class C^2, $\alpha \in (0,1)$, $p > 1$ and f is nonnegative. Suppose that U is a sub or super-solution of (2.6) satisfying $U = 0$ in Ω^c and (2.11) for some $\tau \in (-1, 0)$. Moreover, if $\tau > -\frac{2\alpha}{p-1}$, assume there are numbers $\epsilon > 0$ and $\delta > 0$ such that, in case U is a sub-solution of (2.6),*

$$(-\Delta)^\alpha U(x) \leq -\epsilon d(x)^{\tau - 2\alpha} \quad \textit{or} \quad f(x) \geq \epsilon d(x)^{\tau-2\alpha}, \quad \textit{for } x \in A_\delta, \tag{2.90}$$

and in case U is a super-solution of (2.6),

$$(-\Delta)^\alpha U(x) \geq \epsilon d(x)^{\tau-2\alpha} \quad \textit{and} \quad f(x) \leq \frac{\epsilon}{2} d(x)^{\tau-2\alpha}, \quad \textit{for } x \in A_\delta. \tag{2.91}$$

Then there is no solution u of (2.6) such that, in case U is a sub-solution,

$$\begin{aligned} 0 < \liminf_{x\in\Omega,\ x\to\partial\Omega} u(x)d(x)^{-\tau} \quad &\leq \quad \limsup_{x\in\Omega,\ x\to\partial\Omega} u(x)d(x)^{-\tau} \\ &< \quad \liminf_{x\in\Omega,\ x\to\partial\Omega} U(x)d(x)^{-\tau} \end{aligned} \tag{2.92}$$

or in case U is a super-solution,

$$\begin{aligned} 0 < \limsup_{x\in\Omega,\ x\to\partial\Omega} U(x)d(x)^{-\tau} \quad &< \quad \liminf_{x\in\Omega,\ x\to\partial\Omega} u(x)d(x)^{-\tau} \\ &\leq \quad \limsup_{x\in\Omega,\ x\to\partial\Omega} u(x)d(x)^{-\tau} < \infty. \end{aligned} \tag{2.93}$$

We prove this proposition by a contradiction argument, so we assume that u is a solution of (2.6) satisfying (2.92) or (2.93), depending on the fact that U is a sub-solution or a super-solution. Since f is non-negative we have that $u > 0$ in Ω and by our assumptions on U, there is a constant $C_0 \geq 1$ so that, in case U is a sub-solution

$$C_0^{-1} \leq u(x)d(x)^{-\tau} < U(x)d(x)^{-\tau} \leq C_0, \quad x \in A_\delta \tag{2.94}$$

and, in case U is a super-solution

$$C_0^{-1} \leq U(x)d(x)^{-\tau} < u(x)d(x)^{-\tau} \leq C_0, \quad x \in A_\delta. \tag{2.95}$$

Here δ is decreased if necessary so that (2.90), (2.91), (2.94) and (2.95) hold. We define

$$\pi_k(x) = \begin{cases} U(x) - ku(x), & \text{in case } U \text{ is a sub-solution,} \\ u(x) - kU(x), & \text{in case } U \text{ is a super-solution,} \end{cases} \tag{2.96}$$

where $k \geq 0$. In order to prove Proposition 2.6.1, we need the following two preliminary lemmas.

Lemma 2.6.1 *Under the hypotheses of Proposition 2.6.1. If $\mathcal{A}_k = \{x \in \Omega \,/\, \pi_k(x) > 0\} \neq \emptyset$, for $k > 1$. Then,*

$$\partial\mathcal{A}_k \cap \partial\Omega \neq \emptyset. \tag{2.97}$$

The proof of this lemma follows the same arguments as the proof of Lemma 2.5.1 so we omit it.

Lemma 2.6.2 *Under the hypotheses of Proposition 2.6.1. If $\mathcal{A}_k \neq \emptyset$, for $k > 1$, then*

$$\sup_{x\in\Omega} \pi_k(x) = +\infty. \tag{2.98}$$

Proof. If (2.98) fails, then we have $M = \sup_{x\in\Omega} \pi_k(x) < +\infty$. We see that $M > 0$ and, as in Lemma 2.5.2, there is no point $\bar{x} \in \Omega$ achieving M. By Lemma 2.6.1 we may choose $x_0 \in \mathcal{A}_k$ and $r = d(x_0)/4$ such that $B_r(x_0) \subset A_\delta$, where r could be chosen as small as we want. Here δ is as in (2.90) and (2.91).

In what follows we consider $x \in B_r(x_0) \cap \mathcal{A}_k$ and we notice that $3r < d(x) < 5r$. We first analyze the case U is a sub-solution and $\tau \le -\frac{2\alpha}{p-1}$. We have

$$\begin{aligned}(-\Delta)^\alpha \pi_k(x) &\le -U^p(x) + ku^p(x) - (k-1)f(x)\\ &\le -(k^{p-1}-1)ku^p(x)\\ &\le -C_0^{-p}(k^{p-1}-1)kd(x)^{\tau p} \le -K_1 r^{\tau-2\alpha},\end{aligned}$$

where we have used $f \ge 0$, $k > 1$, (2.94), $K_1 = 5^{\tau-2\alpha}C_0^{-p}(k^{p-1}-1)k > 0$ and C_0 is taken from (2.94). Next we consider the case U is a sub-solution and $\tau > -\frac{2\alpha}{p-1}$. By the first inequality in (2.90), we have

$$\begin{aligned}(-\Delta)^\alpha \pi_k(x) &\le -\epsilon d(x)^{\tau-2\alpha} + ku^p(x) - kf(x)\\ &\le -(\epsilon - kC_0^p r^{2\alpha-\tau+\tau p})d(x)^{\tau-2\alpha} \le -K_1 r^{\tau-2\alpha},\end{aligned}$$

where the last inequality is achieved by choosing r small enough so that $(\epsilon - kC_0^p r^{2\alpha-\tau+\tau p}) \ge \frac{\epsilon}{2}$ and $K_1 = 5^{\tau-2\alpha}\frac{\epsilon}{2}$. On the other hand, if the second inequality in (2.90) holds, we have

$$\begin{aligned}(-\Delta)^\alpha \pi_k(x) &\le ku^p(x) - (k-1)\epsilon d(x)^{\tau-2\alpha}\\ &\le -((k-1)\epsilon - kC_0^p r^{2\alpha-\tau+\tau p})d(x)^{\tau-2\alpha} \le -K_1 r^{\tau-2\alpha},\end{aligned}$$

where r satisfies $(k-1)\epsilon - kC_0^p r^{2\alpha-\tau+\tau p} \ge \frac{k-1}{2}\epsilon$ and $K_1 = 5^{\tau-2\alpha}\frac{k-1}{2}\epsilon$.

In case U is a super-solution and $\tau \le -\frac{2\alpha}{p-1}$, we argue similarly to obtain

$$(-\Delta)^\alpha \pi_k(x) \le -u^p(x) + kU^p(x) - (k-1)f(x) \le -K_1 r^{\tau-2\alpha},$$

where $K_1 = 5^{\tau-2\alpha}C_0^{-p}(k^{p-1}-1)k > 0$. Finally, in case U is a super-solution and $\tau > -\frac{2\alpha}{p-1}$, using (2.91) we find

$$(-\Delta)^\alpha \pi_k(x) \le -u^p(x) - k\epsilon d(x)^{\tau-2\alpha} + f(x) \le -K_1 r^{\tau-2\alpha},$$

with $K_1 = 5^{\tau-2\alpha}\frac{k}{2}\epsilon > 0$. Thus, in all cases we have obtained

$$(-\Delta)^\alpha \pi_k(x) \le -K_1 r^{\tau-2\alpha}, \quad x \in B_r(x_0) \cap \mathcal{A}_k, \tag{2.99}$$

for some $K_1 = K_1(k) > 0$ non-decreasing with k. From here we can argue as in Lemma 2.5.2 to get a contradiction. $\square$

Now proof of Proposition 2.6.1 is easy.

Proof of Proposition 2.6.1. From (2.99), recalling that K_1 non-decreasing with k, we can argue as in the proof of uniqueness result in Case b) to get a sequence (x_m) in A_δ such that, for some $k_0 > 1$ and $\bar{k} > 0$, in case U is a sub-solution we have

$$U(x_m) > (k_0 + m\bar{k})u(x_m)$$

and, in case U is a super-solution we have

$$u(x_m) > (k_0 + m\bar{k})U(x_m).$$

From here we obtain a contradiction with (2.94) or (2.95), for m large. □

Proof of non-existence part of Theorem 2.1.1. For any $t > 0$ we construct a sub-solution or super-solution U of (2.6) such that

$$\lim_{x\in\Omega, x\to\partial\Omega} U(x)d(x)^{-\tau} = t, \tag{2.100}$$

and U satisfies the assumption of Proposition 2.6.1, for different combinations of the parameters p and τ. For $t > 0$ and $\mu \in \mathbb{R}$ we define

$$U_{\mu,t} = tV_\tau + \mu V_0 \quad \text{in} \quad \mathbb{R}^N, \tag{2.101}$$

where $V_0 = \chi_\Omega$ is the characteristic function of Ω and V_τ is defined in (2.35). It is obvious that (2.100) holds for $U_{\mu,t}$ for any $\mu \in \mathbb{R}$. To complete proof we show that for any $t > 0$, there is $\mu(t)$ such that $U_{\mu(t),t}$ is a sub-solution or super-solution of (2.6), depending on the zone to which (p, τ) belongs.

Zone 1: We consider $p > 1$ and $\tau \in (\tau_0(\alpha), 0)$. By Proposition 2.3.2 (ii), there exist $\delta_1 > 0$ and $C_1 > 0$ such that

$$(-\Delta)^\alpha V_\tau(x) > C_1 d(x)^{\tau-2\alpha}, \quad x \in A_{\delta_1}. \tag{2.102}$$

Combining with $(H2^*)$, for any $\mu > 0$, there exists $\delta_1 > 0$ depending on t such that

$$(-\Delta)^\alpha U_{\mu,t}(x) + U_{\mu,t}^p(x) - f(x) > C_1 t d(x)^{\tau-2\alpha} - Cd(x)^{-2\alpha} \geq 0, \quad x \in A_{\delta_1}.$$

On the other hand, since V_τ is of class C^2, f is continuous in Ω and $\Omega \setminus A_{\delta_1}$ is compact, there exists $C_2 > 0$ such that

$$|f|,\ |(-\Delta)^\alpha V_\tau(x)| \leq C_2, \quad x \in \Omega \setminus A_{\delta_1}. \tag{2.103}$$

Then, using (2.71), there exists $\mu > 0$ such that

$$(-\Delta)^\alpha U_{\mu,t}(x) + U_{\mu,t}^p(x) - f(x) > -2C_2 + C_0\mu \geq 0, \quad x \in \Omega \setminus A_{\delta_1}. \tag{2.104}$$

We conclude that for any $t > 0$, there exists $\mu(t) > 0$ such that $U_{\mu(t),t}$ is a super-solution of (2.6) and, by $(H2^*)$ and (2.102), it satisfies (2.91).

Zone 2: We consider $p > 1 + 2\alpha$ and $\tau \in (-1, -\frac{2\alpha}{p-1})$. By Proposition 2.3.2 (i) and (ii), there exists $\delta_1 > 0$ depending on t such that

$$(-\Delta)^\alpha U_{\mu,t}(x) + U_{\mu,t}^p(x) - f(x) \geq -C_1 t d(x)^{\tau-2\alpha} + t^p d(x)^{\tau p} - C d(x)^{-2\alpha} \geq 0, \quad (2.105)$$

for $x \in A_{\delta_1}$ and for any $\mu > 0$, where we used that $0 > \tau - 2\alpha > \tau p$. On the other hand, for $x \in \Omega \setminus A_{\delta_1}$, (2.104) holds for some $\mu > 0$ and so we have constructed a super-solution of (2.6).

Zone 3: We consider $1+2\alpha < p \leq 1-\frac{2\alpha}{\tau_0(\alpha)}$ and $\tau \in (-\frac{2\alpha}{p-1}, \tau_0(\alpha))$, which implies that $\tau p > \tau - 2\alpha$. By Proposition 2.3.2 (i) and $f \geq 0$ in Ω, there exists $\delta_1 > 0$ so that for all $\mu \leq 0$

$$(-\Delta)^\alpha U_{\mu,t}(x) + U_{\mu,t}^p(x) - f(x) \leq -C_1 t d(x)^{\tau-2\alpha} + t^p d(x)^{\tau p} \leq 0, \qquad (2.106)$$

for $x \in A_{\delta_1}$. Then, using (2.71) and (2.103), there exists $\mu = \mu(t) < 0$ such that

$$(-\Delta)^\alpha U_{\mu,t}(x) + U_{\mu,t}^p(x) - f(x) < 2C_2 + C_0\mu \leq 0, \quad x \in \Omega \setminus A_{\delta_1}. \qquad (2.107)$$

We conclude that for any $t > 0$, there exists $\mu(t) < 0$ such that $U_{\mu(t),t}$ is a sub-solution of (2.6) and it satisfies (2.90).

We see that Zone 1, 2 and 3 cover the range of parameters in part (i) of Theorem 2.1.1, completing the proof in the case.

Zone 4: To cover part (ii) of Theorem 2.1.1 we only need to consider $p = 1-\frac{2\alpha}{\tau_0(\alpha)}$ with $\tau = \tau_0(\alpha) = -\frac{2\alpha}{p-1}$, which implies that $\tau p = \tau - 2\alpha < \text{mín}\{\tau - 2\alpha + \tau + 1, \tau\}$. By Proposition 2.3.2 (iii), there exists $\delta_1 > 0$ depending on t such that

$$\begin{aligned}(-\Delta)^\alpha U_{\mu,t}(x) + U_{\mu,t}^p(x) - f(x) \;\; \geq \;\; & -C_1 t d(x)^{\text{mín}\{\tau-2\alpha+\tau+1,\tau\}} + t^p d(x)^{\tau p} \\ & -C d(x)^{-2\alpha} \geq 0, \quad x \in A_{\delta_1}\end{aligned}$$

for any $\mu > 0$. For $x \in \Omega \setminus A_{\delta_1}$, (2.104) holds for some $\mu > 0$, so we have constructed a super-solution of (2.6).

We see that Zones 1, 2 and 4 cover the parameters in part (ii) of Theorem 2.1.1, so the proof is complete in this case too.

Zone 5: We consider $1 < p \leq 1 + 2\alpha$ and $\tau \in (-1, \tau_0(\alpha))$, which implies that $\tau p > \tau - 2\alpha$. By Proposition 2.3.2 (i) and $f \geq 0$ in Ω, there exists $\delta_1 > 0$ such that for all $\mu \leq 0$ and $x \in A_{\delta_1}$, inequality (2.106) holds. Then, using (2.71) and (2.103), there exists $\mu = \mu(t) < 0$ such that (2.107) holds and we conclude that for any $t > 0$, there exists $\mu(t) < 0$ such that $U_{\mu(t),t}$ satisfies the first inequality of (2.90) and it is a sub-solution of (2.6).

We see that Zones 1 and 5 cover the parameters in part (iii) of Theorem 2.1.1. This completes the proof. $\square$

Proof of Theorem 2.1.4. Here again we construct sub or super-solutions satisfying Proposition 2.6.1 to prove the theorem. In the case of a weak source, that is, part (i) of Theorem 2.1.4, we have $p \geq 1 - \frac{2\alpha}{\tau_0(\alpha)}$ and $-2\alpha - \frac{2\alpha}{p-1} \leq \gamma < -2\alpha$, which implies that $-1 < \tau_0(\alpha) \leq -\frac{2\alpha}{p-1} \leq \gamma + 2\alpha < 0$. We consider two zones depending on τ.

Zone 1: we consider $\tau \in (\gamma + 2\alpha, 0)$, so we have $\gamma < \tau p$ and $\gamma < \tau - 2\alpha$. By Proposition 2.3.2 (ii) and $(H3)$, we have that, for any $t > 0$ there exist $\delta_1 > 0$, $C_1 > 0$ and $C_2 > 0$ such that

$$(-\Delta)^\alpha U_{\mu,t}(x) + U_{\mu,t}^p(x) - f(x) \leq C_1 t d(x)^{\tau-2\alpha} + t^p d(x)^{\tau p} - C_2 d(x)^\gamma \leq 0, \quad (2.108)$$

for $x \in A_{\delta_1}$ and any $\mu \leq 0$. On the other hand, using (2.71) and (2.103) we find $\mu = \mu(t) < 0$ such that (2.107) holds for $x \in \Omega \setminus A_{\delta_1}$. We conclude that for any $t > 0$, there exists $\mu(t) < 0$ such that $U_{\mu(t),t}$ is is a sub-solution of (2.6) and by $(H3)$, it satisfies (2.90).

Zone 2: we consider $\tau \in (-1, \gamma + 2\alpha)$. For $\tau \in (\tau_0(\alpha), \gamma + 2\alpha)$ in case $\tau_0(\alpha) < \gamma + 2\alpha$, by Proposition 2.3.2 (i) there exists $\delta_1 > 0$, depending on t, such that

$$(-\Delta)^\alpha U_{\mu,t}(x) + U_{\mu,t}^p(x) - f(x) \geq C_1 t d(x)^{\tau-2\alpha} - C_2 d(x)^\gamma \geq 0, \quad (2.109)$$

for $x \in A_{\delta_1}$ and any $\mu \geq 0$. For $\tau \in (-1, \tau_0(\alpha)] \cap (-1, \gamma + 2\alpha)$, we have $\tau p < \gamma$ and $\tau p < \tau - 2\alpha$, so by Proposition 2.3.2 (i) and (iii), there exists $\delta_1 > 0$ dependent of t such that (2.105) holds for any $\mu \geq 0$, while for $x \in \Omega \setminus A_{\delta_1}$, (2.104) holds for some $\mu > 0$. We conclude that for any $t > 0$, there exists $\mu(t) > 0$ such that $U_{\mu(t),t}$ is a super-solution of (2.6) and by $(H3)$ it satisfies (2.91), completing the proof in the weak source case.

Next we consider the case of strong source, that is part (ii) of Theorem 2.1.4. Here we have that

$$-1 < \frac{\gamma}{p} < -\frac{2\alpha}{p-1} < 0.$$

Here again we have two zones, depending on the parameter τ.

Zone 1: we consider $\tau \in (\frac{\gamma}{p}, 0)$, in which case we have $\tau - 2\alpha > \gamma$ and $\tau p > \gamma$. Then there exist $\delta_1 > 0$, $C_1 > 0$ and $C_2 > 0$ such that (2.108) holds for any $\mu \leq 0$ and using (2.71) and (2.103), there exists $\mu = \mu(t) < 0$ such that (2.107) holds for $x \in \Omega \setminus A_{\delta_1}$. Thus, for any $t > 0$ there exists $\mu(t) < 0$ such that $U_{\mu(t),t}$ is a sub-solution of (2.6) and $(H3)$ implies the first inequality of (2.90).

Zone 2: we consider $\tau \in (-1, \frac{\gamma}{p})$, in which case we have $\tau p < \tau - 2\alpha$ and $\tau p < \gamma$. Then there exist $\delta_1 > 0$, $C_1 > 0$ and $C_2 > 0$ such that (2.109) holds for $x \in A_{\delta_1}$ and $\mu \geq 0$. We see also that for $x \in \Omega \setminus A_{\delta_1}$, inequality (2.104) holds for some $\mu > 0$and so for any $t > 0$, there exists $\mu(t) > 0$ such that $U_{\mu(t),t}$ is a super-solution of (2.6).

This completes the proof of the theorem. $\square$

Capítulo 3

Self-generated interior blow-up solutions in fractional elliptic equation with absorption

Abstract: in this chapter[1], we study positive solutions to problem involving the fractional Laplacian

$$\begin{cases} (-\Delta)^\alpha u(x) + |u|^{p-1}u(x) = 0, & x \in \Omega \setminus \mathcal{C}, \\ u(x) = 0, & x \in \Omega^c, \\ \lim_{x\in\Omega\setminus\mathcal{C},\ x\to\mathcal{C}} u(x) = +\infty, \end{cases} \tag{3.1}$$

where $p > 1$ and Ω is an open bounded C^2 domain in $\mathbb{R}^N$, $\mathcal{C} \subset \Omega$ is a compact C^2 manifold with $N-1$ multiples dimensions and without boundary, the operator $(-\Delta)^\alpha$ with $\alpha \in (0,1)$ is the fractional Laplacian. We consider the existence of positive solutions for problem (3.1). Moreover, we further analyze uniqueness, asymptotic behaviour and nonexistence to (3.1).

3.1. Introduction

In 1957, a fundamental contribution due to Keller in [66] and Osserman in [84] is the study of boundary blow-up solutions for the non-linear elliptic equation

$$\begin{cases} -\Delta u + h(u) = 0 \ \text{ in } \ \Omega, \\ \lim_{x\in\Omega, x\to\partial\Omega} u(x) = +\infty. \end{cases} \tag{3.2}$$

[1]This chapter is based on the paper: H. Chen, P. Felmer and A. Quaas, *Self-generated interior blow-up solutions in fractional elliptic equation with absorption,* submitted.

They proved the existence of solutions to (3.2) when $h : \mathbb{R} \to [0, +\infty)$ is a locally Lipschitz continuous function which is nondecreasing and satisfies the so called Keller-Osserman condition. From then on, the result of Keller and Osserman has been extended by numerous mathematicians in various ways, weakening the assumptions on the domain, generalizing the differential operator and the nonlinear term for equations and systems. The case of $h(u) = u_+^p$ with $p = \frac{N+2}{N-2}$ is studied by Loewner and Nirenberg [72], where in particular uniqueness and asymptotic behavior were obtained. After that, Bandle and Marcus [6] obtained uniqueness and asymptotic for more general non-linearties h. Later, Le Gall in [70] established a uniqueness result of problem (3.2) in the domain whose boundary is non-smooth when $h(u) = u_+^2$. Marcus and Véron [75, 74] extended the uniqueness of blow-up solution for (3.2) in general domains whose boundary is locally represented as a graph of a continuous function when $h(u) = u_+^p$ for $p > 1$. Under this special assumption on h, Kim [67] studied the existence and uniqueness of boundary blow-up solution to (3.2) in bounded domains Ω satisfying $\partial\Omega = \partial\bar{\Omega}$. For another interesting contributions to boundary blow-up solutions see for example Kondratev, Nikishkin [68], Lazer, McKenna [69], Arrieta and Rodríguez-Bernal [5], Chuaqui, Cortázar, Elgueta and J. García-Melián [44], del Pino and Letelier [47], Díaz and Letelier [48], Du and Huang [50], García-Melián [59], Véron [99], and the reference therein.

In a recent work, Felmer and Quaas [51] considered a version of Keller and Osserman problem for a class of non-local operator. Being more precise, they considered as a particular case the fractional elliptic problem

$$\begin{cases} (-\Delta)^\alpha u(x) + |u|^{p-1}u(x) = f(x), & x \in \Omega, \\ u(x) = g(x), & x \in \bar{\Omega}^c, \\ \text{lím}_{x\in\Omega,\ x\to\partial\Omega}\, u(x) = +\infty, \end{cases} \tag{3.3}$$

where $p > 1$, f and g are appropriate functions and Ω is a bounded domain with C^2 boundary. The operator $(-\Delta)^\alpha$ is the fractional Laplacian which is defined as

$$(-\Delta)^\alpha u(x) = -\frac{1}{2}\int_{\mathbb{R}^N} \frac{\delta(u,x,y)}{|y|^{N+2\alpha}}dy, \quad x \in \Omega, \tag{3.4}$$

with $\alpha \in (0,1)$ and $\delta(u,x,y) = u(x+y) + u(x-y) - 2u(x)$.

In [51] the authors proved the existence of a solution to (3.3) provided that g explodes at the boundary and satisfies other technical conditions. In case the function g blows up with an explosion rate as $d(x)^\beta$, with $\beta \in [-\frac{2\alpha}{p-1}, 0)$ and $d(x) = dist(x, \partial\Omega)$, it is shown that the solution satisfies

$$0 < \liminf_{x\in\Omega, x\to\partial\Omega} u(x)d(x)^{-\beta} \leq \limsup_{x\in\Omega, x\to\partial\Omega} u(x)d(x)^{\frac{2\alpha}{p-1}} < +\infty.$$

Here the explosion is driven by the external value g and the external source f has a secondary role, not intervening in the explosive character of the solution.

More recently, Chen, Felmer and Quaas [33] extended the results in [51] studying existence, uniqueness and non-existence of boundary blow-up solutions when the function g vanishes and the explosion on the boundary is driven by the external source f, with weak or strong explosion rate. Moreover, the results are extended even to the case where the boundary blow-up solutions in driven internally, when the external source and value, f and g, vanish. Existence, uniqueness, asymptotic behavior and non-existence results for blow-up solutions of (3.3) are considered in [33]. In the analysis developed in [33], a key role is played by the function $C : (-1,0] \to \mathbb{R}$, that governs the behavior of the solution near the boundary. The function C is defined as

$$C(\tau) = \int_0^{+\infty} \frac{\chi_{(0,1)}(t)|1-t|^\tau + (1+t)^\tau - 2}{t^{1+2\alpha}} dt \tag{3.5}$$

and it possess exactly one zero in $(-1,0)$ and we call it $\tau_0(\alpha)$. In what follows we explain with more details the results in the case of vanishing external source and values, that is $f = 0$ in Ω and $g = 0$ in $\bar{\Omega}^c$, which is the case we will consider in this paper. In Theorem 1.1 in [33], we proved that whenever

$$1 + 2\alpha < p < 1 - \frac{2\alpha}{\tau_0(\alpha)},$$

then problem (3.3) admits a unique positive solution u such that

$$0 < \liminf_{x\in\Omega, x\to\partial\Omega} u(x)d(x)^{\frac{2\alpha}{p-1}} \leq \limsup_{x\in\Omega, x\to\partial\Omega} u(x)d(x)^{\frac{2\alpha}{p-1}} < +\infty.$$

On the other hand, we proved that when $p \geq 1$, then problem (3.3) does not admit any solution u such that

$$0 < \liminf_{x\in\Omega, x\to\partial\Omega} u(x)d(x)^{-\tau} \leq \limsup_{x\in\Omega, x\to\partial\Omega} u(x)d(x)^{-\tau} < +\infty, \tag{3.6}$$

for any $\tau \in (-1,0) \setminus \{\tau_0(\alpha), -\frac{2\alpha}{p-1}\}$. We observe that the non-existence result does not include the case when u has an asymptotic behavior of the form $d(x)^{\tau_0(\alpha)}$, where $\tau_0(\alpha)$ is precisely where C vanishes. We have a a special existence result in this case, precisely if

$$\max\{1 - \frac{2\alpha}{\tau_0(\alpha)} + \frac{\tau_0(\alpha)+1}{\tau_0(\alpha)}, 1\} < p < 1 - \frac{2\alpha}{\tau_0(\alpha)},$$

then, for any $t > 0$, problem (3.3) admits a positive solution u such that

$$\lim_{x\in\Omega, x\to\partial\Omega} u(x)d(x)^{-\tau_0(\alpha)} = t.$$

Motivated by these results and in view of the non-local character of the fractional Laplacian we are interested in another class of blow-up solutions. We want to study solutions that vanish at the boundary of the domain Ω but that explodes at the interior of the domain, near a prescribed embedded manifold. From now on, we assume that Ω is an open bounded domain in $\mathbb{R}^N$ with C^2 boundary, and that there is a C^2, $(N-1)$-dimensional manifold $\mathcal{C}$ without boundary, embedded in Ω, such that, it separates $\Omega \setminus \mathcal{C}$ in exactly two connected components. We denote by Ω_1 the inner component and by Ω_2 the external component, that is $\bar{\Omega}_1 \cap \partial\Omega = \emptyset$ and $\bar{\Omega}_2 \cap \partial\Omega = \partial\Omega$. Throughout the paper we will consider the distance function

$$D : \Omega \setminus \mathcal{C} \to \mathbb{R}_+, \quad D(x) = \mathrm{dist}(x, \mathcal{C}). \tag{3.7}$$

Let us consider the equations, for $i = 1, 2$,

$$\begin{cases} (-\Delta)^\alpha u(x) + |u|^{p-1}u(x) = 0, & x \in \Omega_i, \\ u(x) = 0, & x \in \bar{\Omega}_i^c, \\ \text{lím}_{x\in\Omega_i,\ x\to\partial\Omega_i} u(x) = +\infty, & \end{cases} \tag{3.8}$$

which have solutions u_1 and u_2, for $i = 1, 2$ respectively, in the appropriate range of the parameters. In the local case, that is, $\alpha = 1$, these two solutions certainly do not interact among each other, but when $\alpha \in (0, 1)$, due to the non-local character of the fractional Laplacian and the non-linear character of the equation the solutions on each side of Ω interact and it is precisely the purpose of this paper to study their existence, uniqueness and non-existence.

In precise terms we consider the equation

$$\begin{cases} (-\Delta)^\alpha u(x) + |u|^{p-1}u(x) = 0, & x \in \Omega \setminus \mathcal{C}, \\ u(x) = 0, & x \in \Omega^c, \\ \text{lím}_{x\in\Omega\setminus\mathcal{C},\ x\to\mathcal{C}} u(x) = +\infty, & \end{cases} \tag{3.9}$$

where $p > 1$, Ω and $\mathcal{C} \subset \Omega$ are as described above. The explosion of the solution near $\mathcal{C}$ is governed by a function $c : (-1, 0] \to \mathbb{R}$, defined as

$$c(\tau) = \int_0^{+\infty} \frac{|1-t|^\tau + (1+t)^\tau - 2}{t^{1+2\alpha}} dt. \tag{3.10}$$

This function plays the role of the function C used in [33], but it has certain differences. In Section 3.2 we prove the existence of a number $\alpha_0 \in (0, 1)$ such that $\alpha \in [\alpha_0, 1)$ the function c is always positive in $(-1, 0)$, while if $\alpha \in (0, \alpha_0)$ then there exists exists a unique $\tau_1(\alpha) \in (-1, 0)$ such that $c(\tau_1(\alpha)) = 0$ and $c(\tau) > 0$ in $(-1, \tau_1(\alpha))$ and $c(\tau) < 0$ in $(\tau_1(\alpha), 0)$, see Proposition 3.2.1. We notice here that $\tau_1(\alpha) > \tau_0(\alpha)$ if $\alpha \in (0, \alpha_0)$.

Now we are ready to state our main theorems on the existence uniqueness and asymptotic behavior of interior blow-up solutions to equation (3.9). These theorems deal separately the case $\alpha \in (0, \alpha_0)$ and $\alpha \in [\alpha_0, 1)$.

Theorem 3.1.1 *Assume that $\alpha \in (0, \alpha_0)$ and the assumptions on Ω and $\mathcal{C}$. Then we have:*
(i) If

$$1 + 2\alpha < p < 1 - \frac{2\alpha}{\tau_1(\alpha)}, \tag{3.11}$$

then problem (3.9) admits a unique positive solution u satisfying

$$0 < \liminf_{x \in \Omega \setminus \mathcal{C}, x \to \mathcal{C}} u(x) D(x)^{\frac{2\alpha}{p-1}} \leq \limsup_{x \in \Omega \setminus \mathcal{C}, x \to \mathcal{C}} u(x) D(x)^{\frac{2\alpha}{p-1}} < +\infty. \tag{3.12}$$

(ii) If

$$\max\{1 - \frac{2\alpha}{\tau_1(\alpha)} + \frac{\tau_1(\alpha)+1}{\tau_1(\alpha)}, 1\} < p < 1 - \frac{2\alpha}{\tau_1(\alpha)}. \tag{3.13}$$

Then, for any $t > 0$, there is a positive solution u of problem (3.9) satisfying

$$\lim_{x \in \Omega \setminus \mathcal{C}, x \to \mathcal{C}} u(x) D(x)^{-\tau_1(\alpha)} = t. \tag{3.14}$$

(iii) If one of the following three conditions holds

a) $1 < p \leq 1 + 2\alpha$ *and* $\tau \in (-1, 0) \setminus \{\tau_1(\alpha)\}$,

b) $1 + 2\alpha < p < 1 - \frac{2\alpha}{\tau_1(\alpha)}$ *and* $\tau \in (-1, 0) \setminus \{\tau_1(\alpha), -\frac{2\alpha}{p-1}\}$ *or*

c) $p \geq 1 - \frac{2\alpha}{\tau_1(\alpha)}$ *and* $\tau \in (-1, 0)$,

then problem (3.9) does not admit any solution u satisfying

$$0 < \liminf_{x \in \Omega \setminus \mathcal{C}, x \to \mathcal{C}} u(x) D(x)^{-\tau} \leq \limsup_{x \in \Omega \setminus \mathcal{C}, x \to \mathcal{C}} u(x) D(x)^{-\tau} < +\infty. \tag{3.15}$$

We observe that this theorem is similar to Theorem 1.1 in [33], where the role of $\tau_0(\alpha)$ is played here by $\tau_1(\alpha)$. A quite different situation occurs when $\alpha \in [\alpha_0, 1)$ and the function c never vanishes in $(-1, 0)$. Precisely, we have

Theorem 3.1.2 *Assume that $\alpha \in [\alpha_0, 1)$ and the assumptions on Ω and $\mathcal{C}$. Then we have:*
(i) If $p > 1 + 2\alpha$, then problem (3.9) admits a unique positive solution u satisfying (3.12).
(ii) If $p > 1$, then problem (3.9) does not admit any solution u satisfying (3.15) for any $\tau \in (-1, 0) \setminus \{-\frac{2\alpha}{p-1}\}$.

Comparing Theorem 3.1.1 with Theorem 3.1.2 we see that the range of existence for the absorption term is quite larger for the second one, no constraint from above. The main difference with Theorem 2.1.1, with vanishing f and g occurs when α is large and the function c does not vanish, allowing thus for existence for all p large. This difference comes from the fact that the fractional Laplacian is a non-local operator so that in the interior blow-up, in each of the domains Ω_1 and Ω_2 there is a non-zero external value, the solutions itself acting on the other side of $\mathcal{C}$.

The proof of our theorems is obtained through the use of super and sub-solutions as in [33]. The main difficulty here is to find the appropriate super and sub-solutions to apply the iteration technique to fractional elliptic problem (3.9). Here we make use of some precise estimates based on the function c and the distance function D near $\mathcal{C}$.

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3.2. Preliminaries

In this section, we recall some basic results from [33] and obtain some useful estimate, which will be used in constructing super and sub-solutions of problem (3.9). The first result states as:

Theorem 3.2.1 *Assume that $p > 1$ and there are super-solution $\bar{U}$ and sub-solution $\underline{U}$ of problem (3.9) such that*

$$\bar{U} \geq \underline{U} \quad \text{in } \Omega \setminus \mathcal{C}, \qquad \liminf_{x\in\Omega\setminus\mathcal{C}, x\to\mathcal{C}} \underline{U}(x) = +\infty, \qquad \bar{U} = \underline{U} = 0 \quad \text{in } \Omega^c.$$

Then problem (3.9) admits at least one positive solution u such that

$$\underline{U} \leq u \leq \bar{U} \quad \text{in } \Omega \setminus \mathcal{C}.$$

Proof. The procedure is similar to the proof of Theorem 2.6 in [33], here we give the main differences.

Let us define $\Omega_n := \{x \in \Omega \,|\, D(x) > 1/n\}$ then we solve

$$\begin{cases} (-\Delta)^\alpha u_n(x) + |u_n|^{p-1} u_n(x) = 0, & x \in \Omega_n, \\ u_n(x) = \underline{U}, & x \in \Omega_n^c. \end{cases} \tag{3.16}$$

To find these solutions of (3.16) we observe that for fix n the method of section 3 of [51] applies even if the domain is not connected since the estimate of Lemma 3.2 holds with $\delta < 1/2n$ (see also Proposition 3.2 part ii) in [33]), form here sub and super-solution can be construct for the Dirichlet problem and then existence holds for (3.16) by an iteration technique (see also section 2 of [33] for that procedure).

Then as in Theorem 2.6 in [33] we have

$$\underline{U} \leq u_n \leq u_{n+1} \leq \bar{U} \ \text{ in } \ \Omega.$$

By monotonicity of u_n, we can define

$$u(x) := \lim_{n\to+\infty} u_n(x), \ x \in \Omega \ \text{ and } \ u(x) := 0, \ x \in \Omega^c.$$

Which, by a stability property, is a solution of problem (3.9) with the desired properties. □

In order to prove our existence result, it is crucial to have available super and sub-solutions to problem (3.9). To this end, we start describing the properties of $c(\tau)$ defined in (3.10), which is a C^2 function in $(-1,0)$.

Proposition 3.2.1 *There exists a unique $\alpha_0 \in (0,1)$ such that*
(i) For $\alpha \in [\alpha_0, 1)$, we have $c(\tau) > 0$, for all $\tau \in (-1,0)$;
(ii) For any $\alpha \in (0,\alpha_0)$, there exists unique $\tau_1(\alpha) \in (-1,0)$ satisfying

$$c(\tau) \begin{cases} > 0, & \text{if} \quad \tau \in (-1, \tau_1(\alpha)), \\ = 0, & \text{if} \quad \tau = \tau_1(\alpha), \\ < 0, & \text{if} \quad \tau \in (\tau_1(\alpha), 0) \end{cases} \tag{3.17}$$

and

$$\lim_{\alpha\to\alpha_0^-} \tau_1(\alpha) = 0 \quad \text{and} \quad \lim_{\alpha\to 0^+} \tau_1(\alpha) = -1. \tag{3.18}$$

Moreover, $\tau_1(\alpha) > \tau_0(\alpha)$, for all $\alpha \in (0,\alpha_0)$, where $\tau_0(\alpha) \in (-1,0)$ is the unique zero of $C(\tau)$, defined in (3.5).

Proof. From (3.10), differentiating twice we find that

$$c''(\tau) = \int_0^{+\infty} \frac{|1-t|^\tau (\log|1-t|)^2 + (1+t)^\tau (\log(1+t))^2}{t^{1+2\alpha}} dt > 0, \tag{3.19}$$

so that c is strictly convex in $(-1,0)$. We also see easily that

$$c(0) = 0 \quad \text{and} \quad \lim_{\tau\to-1^+} c(\tau) = \infty. \tag{3.20}$$

Thus, if $c'(0) \leq 0$ then $c(\tau) > 0$ for $\tau \in (-1,0)$ and if $c'(0) > 0$, then there exists $\tau_1(\alpha) \in (-1,0)$ such that $c(\tau) > 0$ for $\tau \in (-1,\tau_1(\alpha))$, $c(\tau) < 0$ for $\tau \in (\tau_1(\alpha), 0)$ and $c(\tau_1(\alpha)) = 0$. In order to complete our proof, we have to analyze the sign of $c'(0)$, which depends on α and to make this dependence explicit, we write $c'(0) = T(\alpha)$.

We compute $T(\alpha)$ from (3.10), differentiating and evaluating in $\tau = 0$

$$T(\alpha) = \int_0^{+\infty} \frac{\log|1-t^2|}{t^{1+2\alpha}} dt. \tag{3.21}$$

We have to prove that T possesses a unique zero in the interval $(0,1)$. For this purpose we start proving that

$$\lim_{\alpha\to 1^-} T(\alpha) = -\infty \quad \text{and} \quad \lim_{\alpha\to 0^+} T(\alpha) = +\infty. \tag{3.22}$$

The first limit follows from the fact that $\log(1-s) \leq -s$, for all $s \in [0, 1/4]$, and so

$$\lim_{\alpha\to 1^-} \int_0^{\frac{1}{2}} \frac{\log(1-t^2)}{t^{1+2\alpha}} dt \leq -\lim_{\alpha\to 1^-} \int_0^{\frac{1}{2}} t^{1-2\alpha} dt = -\infty$$

and the fact that exists a constant t_0 such that

$$\int_{\frac{1}{2}}^{+\infty} \frac{\log|1-t^2|}{t^{1+2\alpha}} dt \leq t_0, \qquad \text{for all } \alpha \in (1/2, 1).$$

The second limit in (3.22) follows from

$$\lim_{\alpha\to 0^+} \int_2^{+\infty} \frac{\log|1-t^2|}{t^{1+2\alpha}} dt \geq \log 3 \lim_{\alpha\to 0^+} \int_2^{+\infty} t^{-1-2\alpha} dt = +\infty$$

and the fact that there exists a constant t_1 such that

$$\int_0^2 \frac{\log|1-t^2|}{t^{1+2\alpha}} dt \leq t_1, \quad \text{for all } \alpha \in (0, 1/2).$$

On the other hand we claim that

$$T'(\alpha) = -2\int_0^{+\infty} \frac{\log|1-t^2|}{t^{1+2\alpha}} \log t dt < 0, \quad \alpha \in (0,1). \tag{3.23}$$

In fact, since $\log|1-t^2|\log t$ is negative only for $t \in (1, \sqrt{2})$, we have

$$\begin{aligned}
\int_0^{+\infty} \frac{\log|1-t^2|}{t^{1+2\alpha}} \log t dt &> \int_0^{\sqrt{2}-1} \frac{\log(1-t^2)}{t^{1+2\alpha}} \log t dt + \int_1^{\sqrt{2}} \log(t^2-1) \log t dt \\
&\geq \int_0^{\sqrt{2}-1} \frac{-t^2}{t^{1+2\alpha}} \log t dt + \int_1^{\sqrt{2}} \log(t-1) \log t dt \\
&= -\int_0^{\sqrt{2}-1} t^{1-2\alpha} \log t dt + \int_0^{\sqrt{2}-1} \log(1+t) \log t dt \\
&\geq -\int_0^{\sqrt{2}-1} t^{1-2\alpha} \log t dt + \int_0^{\sqrt{2}-1} t \log t dt > 0.
\end{aligned}$$

Then, (3.22) and (3.23) the existence of the desired $\alpha_0 \in (0,1)$ with the required properties follows, completing (i) and (3.17) in (ii).

To continue with the proof of our proposition, we study the first limit in (3.18). We assume that there exist a sequence $\alpha_n \in (0, \alpha_0)$ and $\tilde{\tau} \in (-1, 0)$ such that

$$\lim_{n\to+\infty} \alpha_n = \alpha_0 \quad \text{and} \quad \lim_{n\to+\infty} \tau_1(\alpha_n) = \tilde{\tau}$$

and so $c(\tilde{\tau}) = 0$. Moreover $c(0) = 0$ and $c'(0) = T(\alpha_0) = 0$, contradicting the strict convexity of c given by (3.19). Next we prove the second limit in (3.18). We proceed by contradiction, assuming that there exist a sequence $\{\alpha_n\} \subset (0,1)$ and $\bar{\tau} \in (-1, 0)$ such that

$$\lim_{n\to+\infty} \alpha_n = 0 \quad \text{and} \quad \tau_1(\alpha_n) \geq \bar{\tau} > -1, \quad \text{for all } n \in \mathbb{N}.$$

Then there exist $C_1, C_2 > 0$, depending on $\bar{\tau}$, such that

$$\int_0^2 \left|\frac{|1-t|^{\tau_1(\alpha_n)} + (1+t)^{\tau_1(\alpha_n)} - 2}{t^{1+2\alpha_n}}\right| dt \leq C_1$$

and

$$\lim_{n\to\infty} \int_2^{+\infty} \frac{|1-t|^{\tau_1(\alpha_n)} + (1+t)^{\tau_1(\alpha_n)} - 2}{t^{1+2\alpha_n}} dt \leq -C_2 \lim_{n\to\infty} \int_2^{+\infty} \frac{1}{t^{1+2\alpha_n}} dt = -\infty.$$

Then $c(\tau_1(\alpha_n)) \to -\infty$ as $n \to +\infty$, which is impossible since $c(\tau_1(\alpha_n)) = 0$.

We finally prove the last statement of the proposition. Since $\tau_0(\alpha) \in (-1, 0)$ is such that $C(\tau_0(\alpha)) = 0$ and we have, by the definition, that

$$c(\tau) = C(\tau) + \int_1^{+\infty} \frac{(t-1)^\tau}{t^{1+2\alpha}} dt,$$

we find that $c(\tau_0(\alpha)) > 0$, together with (3.17), implies that $\tau_0(\alpha) \in (-1, \tau_1(\alpha))$. □

Next we prove the main proposition in this section, which is on the basis of the construction of super and sub-solutions. By hypothesis on the domain Ω and the manifold $\mathcal{C}$, there exists $\delta > 0$ such that the distance functions $d(\cdot)$, to $\partial\Omega$, and $D(\cdot)$, to $\mathcal{C}$, are of class C^2 in B_δ and A_δ, respectively, and $dist(A_\delta, B_\delta) > 0$, where $A_\delta = \{x \in \Omega \mid D(x) < \delta\}$ and $B_\delta = \{x \in \Omega \mid d(x) < \delta\}$. Now we define the basic function V_τ as follows

$$V_\tau(x) := \begin{cases} D(x)^\tau, & x \in A_\delta \setminus \mathcal{C}, \\ d(x)^2, & x \in B_\delta, \\ l(x), & x \in \Omega \setminus (A_\delta \cup B_\delta), \\ 0, & x \in \Omega^c, \end{cases} \tag{3.24}$$

where τ is a parameter in $(-1, 0)$ and the function l is positive such that V_τ is of class C^2 in $\mathbb{R}^N \setminus \mathcal{C}$.

Proposition 3.2.2 *Let α_0 and $\tau_1(\alpha)$ be as in Proposition 3.2.1.*
(i) If $(\alpha, \tau) \in [\alpha_0, 1) \times (-1, 0)$ or $(\alpha, \tau) \in (0, \alpha_0) \times (-1, \tau_1(\alpha))$, then there exist $\delta_1 \in (0, \delta]$ and $C > 1$ such that

$$\frac{1}{C} D(x)^{\tau - 2\alpha} \le -(-\Delta)^\alpha V_\tau(x) \le C D(x)^{\tau - 2\alpha}, \quad x \in A_{\delta_1} \setminus \mathcal{C}.$$

(ii) If $(\alpha, \tau) \in (0, \alpha_0) \times (\tau_1(\alpha), 0)$, then there exist $\delta_1 \in (0, \delta]$ and $C > 1$ such that

$$\frac{1}{C} D(x)^{\tau - 2\alpha} \le (-\Delta)^\alpha V_\tau(x) \le C D(x)^{\tau - 2\alpha}, \quad x \in A_{\delta_1} \setminus \mathcal{C}.$$

(iii) If $(\alpha, \tau) \in (0, \alpha_0) \times \{\tau_1(\alpha)\}$, then there exist $\delta_1 \in (0, \delta]$ and $C > 1$ such that

$$|(-\Delta)^\alpha V_\tau(x)| \le C D(x)^{\text{mín}\{\tau, 2\tau - 2\alpha + 1\}}, \quad x \in A_{\delta_1} \setminus \mathcal{C}.$$

This proposition and its proof has many similarities with Proposition 3.2 in [33], but it has also important differences so we give a complete proof of it.

Proof. By compactness of $\mathcal{C}$, we just need to prove that the corresponding inequality holds in a neighborhood of any point $\bar{x} \in \mathcal{C}$ and, without loss of generality, we may assume $\bar{x} = 0$. For a given $0 < \eta \le \delta$, we define

$$Q_\eta = (-\eta, \eta) \times B_\eta \subset \mathbb{R} \times \mathbb{R}^{N-1},$$

where B_η denotes the ball centered at the origin and with radius η in $\mathbb{R}^{N-1}$. We observe that $Q_\eta \subset \Omega$. Let $\varphi : \mathbb{R}^{N-1} \to \mathbb{R}$ be a C^2 function such that $(z_1, z') \in \mathcal{C} \cap Q_\delta$ if and only if $z_1 = \varphi(z')$. We further assume that e_1 is normal to $\mathcal{C}$ at $\bar{x}$ and then there exists $C > 0$ such that $|\varphi(z')| \le C|z'|^2$ for $|z'| \le \delta$. Thus, choosing $\eta > 0$

smaller if necessary we may assume that $|\varphi(z')| < \frac{\eta}{2}$ for $|z'| \leq \eta$. In the proof of our inequalities, we will consider a generic point along the normal $x = (x_1, 0) \in A_{\eta/4}$, with $0 < |x_1| < \eta/4$. We observe that $|x - \bar{x}| = D(x) = |x_1|$. By definition we have

$$-(-\Delta)^\alpha V_\tau(x) = \frac{1}{2}\int_{Q_\eta} \frac{\delta(V_\tau, x, y)}{|y|^{N+2\alpha}} dy + \frac{1}{2}\int_{\mathbb{R}^N \setminus Q_\eta} \frac{\delta(V_\tau, x, y)}{|y|^{N+2\alpha}} dy. \tag{3.25}$$

It is not difficult to see that the second integral is bounded by Cx_1^τ, for an appropriate constant $C > 0$, so that we only need to study the first integral, that from now on we denote by $\frac{1}{2}E(x_1)$.

Our first goal is to obtain positive constants c_1, c_2 so that lower bound for $E(x_1)$

$$E(x_1) \geq c_1 c(\tau)|x_1|^{\tau - 2\alpha} - c_2|x_1|^{\text{mín}\{\tau, 2\tau - 2\alpha + 1\}} \tag{3.26}$$

holds, for all $|x_1| \leq \eta/4$. For this purpose we assume that $0 < \eta \leq \delta/2$, then for all $y = (y_1, y') \in Q_\eta$ we have that $x \pm y \in Q_\delta$, so that

$$D(x \pm y) \leq |x_1 \pm y_1 - \varphi(\pm y')|, \quad \text{for all} \ \ y \in Q_\eta.$$

From here and the fact that $\tau \in (-1, 0)$, we have that

$$E(x_1) = \int_{Q_\eta} \frac{\delta(V_\tau, x, y)}{|y|^{N+2\alpha}} dy \geq \int_{Q_\eta} \frac{I(y)}{|y|^{N+2\alpha}} dy + \int_{Q_\eta} \frac{J(y) + J(-y)}{|y|^{N+2\alpha}} dy, \tag{3.27}$$

where the functions I and J are defined, for $y \in Q_\eta$, as

$$I(y) = |x_1 - y_1|^\tau + |x_1 + y_1|^\tau - 2x_1^\tau \tag{3.28}$$

and

$$J(y) = |x_1 + y_1 - \varphi(y')|^\tau - |x_1 + y_1|^\tau. \tag{3.29}$$

In what follows we assume $x_1 > 0$ (the case $x_1 < 0$ is similar). For the first term of the right hand side in (3.27), we have

$$\int_{Q_\eta} \frac{I(y)}{|y|^{N+2\alpha}} dy = x_1^{\tau - 2\alpha} \int_{Q_{\frac{\eta}{x_1}}} \frac{|1 - z_1|^\tau + |1 + z_1|^\tau - 2}{|z|^{N+2\alpha}} dz.$$

On one hand we have that, for a constant c_1, we have

$$\int_{\mathbb{R}^N} \frac{|1 - z_1|^\tau + |1 + z_1|^\tau - 2}{|z|^{N+2\alpha}} dz = 2c(\tau) \int_{\mathbb{R}^{N-1}} \frac{1}{(|z'|^2 + 1)^{\frac{N+2\alpha}{2}}} dz' = c_1 c(\tau),$$

and, on the other hand, for constants C_2 and C_3 we have

$$
\begin{aligned}
& |\int_{-\frac{\eta}{x_1}}^{\frac{\eta}{x_1}} \int_{|z'|\geq \frac{\eta}{x_1}} \frac{|1-z_1|^\tau + |1+z_1|^\tau - 2}{|z|^{N+2\alpha}} dz| \\
\leq & \int_{-\frac{\eta}{x_1}}^{\frac{\eta}{x_1}} (|1-z_1|^\tau + |1+z_1|^\tau + 2) dz_1 \int_{|z'|\geq \frac{\eta}{x_1}} \frac{dz'}{|z'|^{N+2\alpha}} \leq C_2 x_1^{2\alpha}
\end{aligned}
$$

and

$$
\begin{aligned}
& |\int_{|z_1|\geq \frac{\eta}{x_1}} \int_{\mathbb{R}^{N-1}} \frac{|1-z_1|^\tau + |1+z_1|^\tau - 2}{|z|^{N+2\alpha}} dz| \\
\leq & \; 2\int_{\frac{\eta}{x_1}}^{+\infty} \frac{|1-z_1|^\tau + |1+z_1|^\tau + 2}{z_1^{1+2\alpha}} dz_1 \int_{\mathbb{R}^{N-1}} \frac{1}{(1+|z'|^2)^{\frac{N+2\alpha}{2}}} dz' \leq C_3 x_1^{2\alpha}.
\end{aligned}
$$

Consequently, for an appropriate constant c_2

$$
|\int_{Q_\eta} \frac{I(y)}{|y|^{N+2\alpha}} dy - c_1 c(\tau) x_1^{\tau-2\alpha}| \leq c_2 x_1^\tau. \tag{3.30}
$$

Next we estimate the second term of the right hand side in (3.27). Since

$$
\int_{Q_\eta} \frac{J(-y)}{|y|^{N+2\alpha}} dy = \int_{Q_\eta} \frac{J(y)}{|y|^{N+2\alpha}} dy,
$$

we only need to estimate

$$
\int_{Q_\eta} \frac{J(y)}{|y|^{N+2\alpha}} dy = \int_{B_\eta} \int_{-\eta}^{\eta} \frac{|x_1+y_1-\varphi(y')|^\tau - |x_1+y_1|^\tau}{(y_1^2+|y'|^2)^{\frac{N+2\alpha}{2}}} dy_1 dy'. \tag{3.31}
$$

We notice that $|x_1+y_1-\varphi(y')| \geq |x_1+y_1|$ if and only if

$$
\varphi(y')(x_1+y_1-\frac{\varphi(y')}{2}) \leq 0.
$$

From here and (3.31), we have

$$
\begin{aligned}
\int_{Q_\eta} \frac{J(y)}{|y|^{N+2\alpha}} dy \;\; \geq & \;\; \int_{B_\eta} \int_{-\eta}^{-x_1+\frac{\varphi_+(y')}{2}} \frac{|x_1+y_1-\varphi_+(y')|^\tau - |x_1+y_1|^\tau}{(y_1^2+|y'|^2)^{\frac{N+2\alpha}{2}}} dy_1 dy' \\
& + \int_{B_\eta} \int_{-x_1+\frac{\varphi_-(y')}{2}}^{\eta} \frac{|x_1+y_1-\varphi_-(y')|^\tau - |x_1+y_1|^\tau}{(y_1^2+|y'|^2)^{\frac{N+2\alpha}{2}}} dy_1 dy' \\
= & \;\; E_1(x_1) + E_2(x_1),
\end{aligned}
$$

where $\varphi_+(y') = \text{máx}\{\varphi(y'), 0\}$ and $\varphi_-(y') = \text{mín}\{\varphi(y'), 0\}$. We only estimate $E_1(x_1)$ ($E_2(x_1)$ is similar). Using integration by parts, we obtain

$$\begin{aligned}
& E_1(x_1) = \int_{B_\eta}\int_{x_1-\eta}^{\frac{\varphi_+(y')}{2}} \frac{|y_1-\varphi_+(y')|^\tau - |y_1|^\tau}{((y_1-x_1)^2+|y'|^2)^{\frac{N+2\alpha}{2}}} dy_1 dy' \\
= & \int_{B_\eta}\int_{x_1-\eta}^{0} \frac{(\varphi_+(y')-y_1)^\tau - (-y_1)^\tau}{((y_1-x_1)^2+|y'|^2)^{\frac{N+2\alpha}{2}}} dy_1 dy' \\
& + \int_{B_\eta}\int_{0}^{\frac{\varphi_+(y')}{2}} \frac{(\varphi_+(y')-y_1)^\tau - y_1^\tau}{((y_1-x_1)^2+|y'|^2)^{\frac{N+2\alpha}{2}}} dy_1 dy' \\
= & \frac{1}{\tau+1}\int_{B_\eta} [\frac{-\varphi_+(y')^{\tau+1}}{(x_1^2+|y'|^2)^{\frac{N+2\alpha}{2}}} + \frac{(\eta-x_1+\varphi_+(y'))^{\tau+1} - (\eta-x_1)^{\tau+1}}{(\eta^2+|y'|^2)^{\frac{N+2\alpha}{2}}}] dy' \\
& - \frac{N+2\alpha}{\tau+1}\int_{B_\eta}\int_{x_1-\eta}^{0} \frac{(\varphi_+(y')-y_1)^{\tau+1} - (-y_1)^{\tau+1}}{((y_1-x_1)^2+|y'|^2)^{\frac{N+2\alpha}{2}+1}} (y_1-x_1) dy_1 dy' \\
& + \frac{1}{\tau+1}\int_{B_\eta} [\frac{-2^{-\tau}\varphi_+(y')^{\tau+1}}{((\frac{\varphi_+(y')}{2}-x_1)^2+|y'|^2)^{\frac{N+2\alpha}{2}}} + \frac{\varphi_+(y')^{\tau+1}}{(x_1^2+|y'|^2)^{\frac{N+2\alpha}{2}}}] dy' \\
& + \frac{N+2\alpha}{\tau+1}\int_{B_\eta}\int_{0}^{\frac{\varphi_+(y')}{2}} \frac{(\varphi_+(y')-y_1)^{\tau+1} + y_1^{\tau+1}}{((y_1-x_1)^2+|y'|^2)^{\frac{N+2\alpha}{2}+1}} (y_1-x_1) dy_1 dy' \\
\geq & \frac{-2^{-\tau}}{\tau+1}\int_{B_\eta} \frac{\varphi_+(y')^{\tau+1}}{((\frac{\varphi_+(y')}{2}-x_1)^2+|y'|^2)^{\frac{N+2\alpha}{2}}} dy' \\
& + \frac{N+2\alpha}{\tau+1}\int_{B_\eta}\int_{0}^{\text{mín}\{\frac{\varphi_+(y')}{2}, x_1\}} \frac{(\varphi_+(y')-y_1)^{\tau+1} + y_1^{\tau+1}}{((y_1-x_1)^2+|y'|^2)^{\frac{N+2\alpha}{2}+1}} (y_1-x_1) dy_1 dy' \\
= & A_1(x_1) + A_2(x_1). \qquad (3.32)
\end{aligned}$$

In order to estimate $A(x_1)$, we split B_η in $O = \{y' \in B_\eta : |\frac{\varphi_+(y')}{2} - x_1| \geq \frac{x_1}{2}\}$ and $B_\eta \setminus O$. On one hand we have

$$\begin{aligned}
\int_O \frac{|y'|^{2\tau+2}}{((\frac{\varphi_+(y')}{2}-x_1)^2+|y'|^2)^{\frac{N+2\alpha}{2}}} dy' & \leq x_1^{2\tau-2\alpha+1}\int_{B_{\eta/x_1}} \frac{|z'|^{2\tau+2}}{(1/4+|z'|^2)^{\frac{N+2\alpha}{2}}} dz' \\
& \leq C(x_1^{2\tau-2\alpha+1} + x_1^\tau).
\end{aligned}$$

On the other hand, for $y' \in B_\eta \setminus O$ we have that $|y'| \geq c_1\sqrt{x_1}$, for some constant c_1, and then

$$\begin{aligned}
\int_{B_\eta\setminus O} \frac{|y'|^{2\tau+2}}{((\frac{\varphi_+(y')}{2}-x_1)^2+|y'|^2)^{\frac{N+2\alpha}{2}}} dy' & \leq \int_{B_\eta\setminus B_{c_1\sqrt{x_1}}} |y'|^{2\tau+2-N-2\alpha} dy' \\
& \leq C(x_1^{\tau-\alpha+\frac{1}{2}} + 1).
\end{aligned}$$

Thus, for some $C > 0$,

$$A_1(x_1) \geq -Cx_1^{\text{mín}\{\tau, 2\tau-2\alpha+1\}}. \tag{3.33}$$

Next we estimate $A_2(x_1)$:

$$\begin{aligned} A_2(x_1) &\geq \frac{2(N+2\alpha)}{\tau+1} \int_{B_\eta} \int_0^{x_1} \frac{\varphi_+(y')^{\tau+1}(y_1-x_1)}{((y_1-x_1)^2+|y'|^2)^{\frac{N+2\alpha}{2}+1}} dy_1 dy' \\ &\geq C \int_{B_\eta} \int_0^{x_1} \frac{|y'|^{2\tau+2}(y_1-x_1)}{((y_1-x_1)^2+|y'|^2)^{\frac{N+2\alpha}{2}+1}} dy_1 dy' \\ &\geq Cx_1^{2\tau-2\alpha+1} \int_{B_{\eta/x_1}} \int_0^1 \frac{|z'|^{2\tau+2}(z_1-1)}{((z_1-1)^2+|z'|^2)^{\frac{N+2\alpha}{2}+1}} dz_1 dz' \\ &\geq -C_1 x_1^{\text{mín}\{\tau, 2\tau-2\alpha+1\}}, \end{aligned}$$

for some $C, C_1 > 0$. From here, (3.32) and (3.33) we obtain, for some $C > 0$

$$E_1(x_1) \geq -Cx_1^{\text{mín}\{\tau, 2\tau-2\alpha+1\}}.$$

Using the similar estimate for $E_2(x_1)$, we obtain

$$\int_{Q_\eta} \frac{J(y)+J(-y)}{|y|^{N+2\alpha}} dy \geq -Cx_1^{\text{mín}\{\tau, 2\tau-2\alpha+1\}}. \tag{3.34}$$

Thus, from (3.27), (3.30), (3.34) and noticing that these inequalities also hold with $x_1 < 0$ with the obvious changes, we conclude the lower bound for $E(x_1)$ we gave in (3.26). Our second goal is to get an upper bound for $E(x_1)$ and for this, we first recall Lemma 3.1 in [33] to obtain

$$D(x \pm y)^\tau \leq (x_1 \pm y_1 - \varphi(y'))^\tau (1 + C|y'|^2), \quad \text{for all } \; |x_1| \leq \eta/4, y = (y_1, y') \in Q_\eta.$$

From here we see that

$$E(x_1) \leq \int_{Q_\eta} \frac{I(y)}{|y|^{N+2\alpha}} dy + \int_{Q_\eta} \frac{J(y)+J(-y)}{|y|^{N+2\alpha}} dy + C \int_{Q_\eta} \frac{I(y)+J(y)+J(-y)}{|y|^{N+2\alpha}} |y'|^2 dy.$$

We denote by $E_3(x_1)$ the third integral above. The first integral was studied in (3.30), so we study the second integral and that we only need to consider the term $J(y)$, since the other is completely analogous. We see that $|x_1+y_1-\varphi(y')| \leq |x_1+y_1|$ if and only if

$$\varphi(y')(x_1 + y_1 - \frac{\varphi(y')}{2}) \geq 0.$$

As before, we will consider only the case $x_1 > 0$, since the other one is analogous. From (3.31) we have

$$\begin{aligned}
\int_{Q_\eta} \frac{J(y)}{|y|^{N+2\alpha}} dy &\leq \int_{B_\eta} \int_{-\eta}^{-x_1+\frac{\varphi_-(y')}{2}} \frac{|x_1+y_1-\varphi_-(y')|^\tau - |x_1+y_1|^\tau}{(y_1^2+|y'|^2)^{\frac{N+2\alpha}{2}}} dy_1 dy' \\
&\quad + \int_{B_\eta} \int_{-x_1+\frac{\varphi_+(y')}{2}}^{\eta} \frac{|x_1+y_1-\varphi_+(y')|^\tau - |x_1+y_1|^\tau}{(y_1^2+|y'|^2)^{\frac{N+2\alpha}{2}}} dy_1 dy' \\
&= F_1(x_1) + F_2(x_1).
\end{aligned}$$

Next we estimate $F_1(x_1)$ ($F_2(x_1)$ is similar), using integration by parts

$$\begin{aligned}
F_1(x_1) &= \int_{B_\eta} \int_{x_1-\eta}^{\frac{\varphi_-(y')}{2}} \frac{|y_1-\varphi_-(y')|^\tau - |y_1|^\tau}{((y_1-x_1)^2+|y'|^2)^{\frac{N+2\alpha}{2}}} dy_1 dy' \\
&= \int_{B_\eta} \left[\int_{x_1-\eta}^{\varphi_-(y')} \frac{(\varphi_-(y')-y_1)^\tau - (-y_1)^\tau}{((y_1-x_1)^2+|y'|^2)^{\frac{N+2\alpha}{2}}} dy_1 + \int_{\varphi_-(y')}^{\frac{\varphi_-(y')}{2}} \frac{(y_1-\varphi_-(y'))^\tau - (-y_1)^\tau}{((y_1-x_1)^2+|y'|^2)^{\frac{N+2\alpha}{2}}} dy_1 \right] dy' \\
&= \frac{1}{\tau+1} \int_{B_\eta} [\frac{(-\varphi_-(y'))^{\tau+1}}{((x_1-\varphi_-(y'))^2+|y'|^2)^{\frac{N+2\alpha}{2}}} + \frac{(\eta-x_1+\varphi_-(y'))^{\tau+1} - (\eta-x_1)^{\tau+1}}{(\eta^2+|y'|^2)^{\frac{N+2\alpha}{2}}}] dy' \\
&\quad - \frac{N+2\alpha}{\tau+1} \int_{B_\eta} \int_{x_1-\eta}^{\varphi_-(y')} \frac{(\varphi_-(y')-y_1)^{\tau+1} - (-y_1)^{\tau+1}}{((y_1-x_1)^2+|y'|^2)^{\frac{N+2\alpha}{2}+1}} (y_1-x_1) dy_1 dy' \\
&\quad + \frac{1}{\tau+1} \int_{B_\eta} [\frac{2^{-\tau}(-\varphi_-(y'))^{\tau+1}}{((\frac{\varphi_-(y')}{2}-x_1)^2+|y'|^2)^{\frac{N+2\alpha}{2}}} + \frac{-(-\varphi_-(y'))^{\tau+1}}{((x_1-\varphi_-(y'))^2+|y'|^2)^{\frac{N+2\alpha}{2}}}] dy' \\
&\quad + \frac{N+2\alpha}{\tau+1} \int_{B_\eta} \int_{\varphi_-(y')}^{\frac{\varphi_-(y')}{2}} \frac{(y_1-\varphi_-(y'))^{\tau+1} + (-y_1)^{\tau+1}}{((y_1-x_1)^2+|y'|^2)^{\frac{N+2\alpha}{2}+1}} (y_1-x_1) dy_1 dy' \\
&\leq \frac{1}{\tau+1} \int_{B_\eta} \frac{2^{-\tau}(-\varphi_-(y'))^{\tau+1}}{((\frac{\varphi_-(y')}{2}-x_1)^2+|y'|^2)^{\frac{N+2\alpha}{2}}} dy' = B(x_1).
\end{aligned}$$

Since $(\frac{\varphi_-(y')}{2}-x_1)^2 \geq x_1^2$, we have

$$\begin{aligned}
B(x_1) &\leq \frac{2^{-\tau}}{\tau+1} \int_{B_\eta} \frac{(-\varphi_-(y'))^{\tau+1}}{(x_1^2+|y'|^2)^{\frac{N+2\alpha}{2}}} dy' \\
&\leq C \int_{B_\eta} \frac{|y'|^{2\tau+2}}{(x_1^2+|y'|^2)^{\frac{N+2\alpha}{2}}} dy' \leq C x_1^{\text{mín}\{\tau, 2\tau-2\alpha+1\}},
\end{aligned}$$

for some $C > 0$ independent of x_1. Thus we have obtained that

$$F_1(x_1) \leq C x_1^{\text{mín}\{\tau, 2\tau-2\alpha+1\}}. \tag{3.35}$$

Similarly, we can get an analogous estimate for $F_2(x_1)$ and these two estimates imply

$$\int_{Q_\eta} \frac{J(y)+J(-y)}{|y|^{N+2\alpha}}dy \leq Cx_1^{\text{mín}\{\tau,2\tau-2\alpha+1\}}. \tag{3.36}$$

Finally we obtain

$$\begin{aligned}\int_{Q_\eta} \frac{I(y)}{|y|^{N+2\alpha}}|y'|^2 dy &= x_1^{\tau-2\alpha+2}\int_{Q_{\frac{\eta}{x_1}}} \frac{|1-z_1|^\tau+|1+z_1|^\tau-2}{|z|^{N+2\alpha}}|z'|^2 dz \\ &\leq Cx_1^{\text{mín}\{\tau,\tau-2\alpha+2\}}\end{aligned}$$

and, in a similar way,

$$\int_{Q_\eta} \frac{J(y)|y'|^2}{|y|^{N+2\alpha}}dy \leq Cx_1^{\text{mín}\{\tau,2\tau-2\alpha+1\}}.$$

From the last two inequalities we obtain

$$E_3(x_1) \leq Cx_1^{\text{mín}\{\tau,2\tau-2\alpha+1\}}. \tag{3.37}$$

Then, taking into account (3.35), (3.30), (3.36), (3.37) and considering also the case $x_1<0$, we obtain

$$E(x_1) \leq c_1 c(\tau)|x_1|^{\tau-2\alpha} + c_2|x_1|^{\text{mín}\{\tau,2\tau-2\alpha+1\}}. \tag{3.38}$$

From inequalities (3.26), (3.38) and Proposition 3.2.1 the result follows. □

3.3. Existence of large solution

This section is devoted to use Proposition 3.2.2 to prove the existence of solution of problem (3.9). To this purpose, our main goal is to construct appropriate sub-solution and super-solution of problem (3.9) under the hypotheses of Theorem 3.1.1 (i), (ii) and Theorem 3.1.2 (i).

We begin with a simple lemma that reduces the problem to find them only in $A_\delta \setminus \mathcal{C}$.

Lemma 3.3.1 *Let U and W be classical ordered super and sub-solution of (3.9) in the sub-domain $A_\delta \setminus \mathcal{C}$. Then there exists λ large such that $U_\lambda = U + \lambda\bar{V}$ and $W_\lambda = W - \lambda\bar{V}$, are ordered super and sub-solution of (3.9), where $\bar{V}$ is the solution of*

$$\begin{cases}(-\Delta)^\alpha \bar{V}(x) = 1, & x\in\Omega,\\ \bar{V}(x)=0, & x\in\Omega^c.\end{cases} \tag{3.39}$$

Remark 3.3.1 *Here $U, W : \mathbb{R}^N \to \mathbb{R}$ are classical ordered of super and sub-solution of (3.9) in the sub-domain $A_\delta \setminus \mathcal{C}$ if U satisfies*

$$(-\Delta)^\alpha U + |U|^{p-1}U \geq 0 \quad in \quad A_\delta \setminus \mathcal{C}$$

and W satisfies the reverse inequality. Moreover, they satisfy

$$U \geq W \quad \text{in } \Omega \setminus \mathcal{C}, \qquad \liminf_{x \in \Omega\setminus\mathcal{C}, x \to \mathcal{C}} W(x) = +\infty, \qquad U = W = 0 \quad \text{in } \Omega^c.$$

Proof of Lemma 3.3.1. Notice that by the maximum principle $\bar{V}$ is nonnegative in Ω, therefore $U_\lambda \geq U$ and $W_\lambda \leq W$, so they are still ordered. In addition U_λ satisfies

$$(-\Delta)^\alpha U_\lambda + |U_\lambda|^{p-1}U_\lambda \geq (-\Delta)^\alpha U + |U|^{p-1}U + \lambda > 0, \quad \text{in} \quad \Omega \setminus \mathcal{C}.$$

This inequality holds because of our assumption in $A_\delta \setminus \mathcal{C}$ and the fact that $(-\Delta)^\alpha U + |U|^{p-1}U$ is continuous in $\Omega \setminus A_\delta$ and by taking λ large enough.

By the same type of arguments we find that W_λ is a sub-solution. □

Proof of existence results in Theorem 3.1.1 (i) **and Theorem 3.1.2** (i)**.** We define

$$U_\lambda(x) = \lambda V_\tau(x) \text{ and } W_\lambda(x) = \lambda V_\tau(x), \; x \in \mathbb{R}^N \setminus \mathcal{C}, \tag{3.40}$$

where V_τ is defined in (3.24) with $\tau = -\frac{2\alpha}{q-1}$

1. U_λ **is Super-solution.** By hypotheses of Theorem 3.1.1 (i) and Theorem 3.1.2 (i), we notice that

$$\tau \in (-1, 0), \quad \text{for } \alpha \in [\alpha_0, 1),$$
$$\tau \in (-1, \tau_1(\alpha)), \quad \text{for } \alpha \in (0, \alpha_0)$$

and $\tau p = \tau - 2\alpha$, then we use Proposition 3.2.2 part (i) to obtain that there exist $\delta_1 \in (0, \delta]$ and $C > 1$ such that

$$(-\Delta)^\alpha U_\lambda(x) + U_\lambda^p(x) \geq -C\lambda D(x)^{\tau-2\alpha} + \lambda^p D(x)^{\tau p}, \quad x \in A_{\delta_1} \setminus \mathcal{C}.$$

Then there exist $\lambda_1 > 1$ such that for $\lambda \geq \lambda_1$, we have

$$(-\Delta)^\alpha U_\lambda(x) + U_\lambda^p(x) \geq 0, \; x \in A_{\delta_1} \setminus \mathcal{C}.$$

2. W_λ **is Sub-solution.** We use Proposition 3.2.2 part (i) to obtain that there exist $\delta_1 \in (0, \delta]$ and $C > 1$ such that for $x \in A_{\delta_1} \setminus \mathcal{C}$, we have

$$\begin{aligned}(-\Delta)^\alpha W_\lambda(x) + |W_\lambda|^{p-1}W_\lambda(x) &\leq -\frac{\lambda}{C}D(x)^{\tau-2\alpha} + \lambda^p D(x)^{\tau p} \\ &\leq (-\frac{\lambda}{C} + \lambda^p)D(x)^{\tau-2\alpha}.\end{aligned}$$

Then there exists $\lambda_3 \in (0,1)$ such that for all $\lambda \in (0, \lambda_3)$, it has

$$(-\Delta)^\alpha W_\lambda(x) + |W_\lambda|^{p-1} W_\lambda(x) \le 0, \;\; x \in A_{\delta_1} \setminus \mathcal{C}.$$

To conclude the proof we use Lemma 3.3.1 and Proposition 3.2.2. $\square$

Proof of Theorem 3.1.1 *(ii)***.** For any given $t > 0$, we denote

$$U(x) = tV_{\tau_1(\alpha)}(x), \quad x \in \mathbb{R}^N \setminus \mathcal{C},$$

$$W_\mu(x) = tV_{\tau_1(\alpha)}(x) - \mu V_{\bar{\tau}}(x), \quad x \in \mathbb{R}^N \setminus \mathcal{C}$$

where $\bar{\tau} = \text{mín}\{\tau_1(\alpha)p + 2\alpha, \frac{1}{2}\tau_1(\alpha)\} < 0$. By (3.13), we have

$$\bar{\tau} \in (\tau_1(\alpha), 0), \;\; \bar{\tau} - 2\alpha < \text{mín}\{\tau_1(\alpha), 2\tau_1(\alpha) - 2\alpha + 1\} \text{ and } \bar{\tau} - 2\alpha < \tau_1(\alpha)\text{p}. \quad (3.41)$$

1. U **is Super-solution.** We use Proposition 3.2.2 *(iii)* to obtain that for any $x \in A_{\delta_1} \setminus \mathcal{C}$,

$$(-\Delta)^\alpha U(x) + U^p(x) \ge -CtD(x)^{\text{mín}\{\tau_1(\alpha), 2\tau_1(\alpha)-2\alpha+1\}} + t^p D(x)^{\tau_1(\alpha)p},$$

together with $\tau_1(\alpha)p < \text{mín}\{\tau_1(\alpha), 2\tau_1(\alpha) - 2\alpha + 1\}$, then there exists $\delta_2 \in (0, \delta_1]$ such that

$$(-\Delta)^\alpha U(x) + U^p(x) \ge 0, \quad x \in A_{\delta_2} \setminus \mathcal{C}.$$

2. W_μ **is Sub-solution.** We use Proposition 3.2.2 *(ii)* and *(iii)* to obtain that for $x \in A_{\delta_1} \setminus \mathcal{C}$,

$$\begin{aligned}(-\Delta)^\alpha W_\mu(x) + |W_\mu|^{p-1} W_\mu(x) \;\; &\le \;\; CtD(x)^{\text{mín}\{\tau_1(\alpha), 2\tau_1(\alpha)-2\alpha+1\}} \\ &\quad -\frac{\mu}{C} D(x)^{\bar{\tau}-2\alpha} + t^p D(x)^{\tau_1(\alpha)p}.\end{aligned}$$

Then there exists $\delta_2 \in (0, \delta_1]$ such that for any $\mu \ge 1$, we have

$$(-\Delta)^\alpha W_\mu(x) + |W_\mu|^{p-1} W_\mu(x) \le 0, \;\; x \in A_{\delta_2} \setminus \mathcal{C}.$$

To conclude the proof we use Lemma 3.3.1 and Proposition 3.2.2. $\square$

3.4. Uniqueness and nonexistence

We prove the uniqueness statement by contradiction. Assume that u and v are solutions of problem (3.9) satisfying (3.12). Then there exist $C_0 \ge 1$ and $\bar{\delta} \in (0, \delta)$ such that

$$\frac{1}{C_0} \le v(x)D(x)^{-\tau}, \;\; u(x)D(x)^{-\tau} \le C_0, \quad \forall x \in A_{\bar{\delta}} \setminus \mathcal{C}, \tag{3.42}$$

where $\tau = -\frac{2\alpha}{p-1}$. We denote

$$\mathcal{A} = \{x \in \Omega \setminus \mathcal{C} \mid u(x) > v(x)\}. \tag{3.43}$$

Then $\mathcal{A}$ is open and $\mathcal{A} \subset \Omega$. Then the uniqueness in Theorem 3.1.2 (i) and Theorem 3.1.1 (i) is a consequence of the following result:

Proposition 3.4.1 *Under the hypotheses of Theorem 3.1.2* (i) *and Theorem 3.1.1* (i)*, we have*

$$\mathcal{A} = \emptyset.$$

Proof. The procedure of proof is similar as Section 5 in [33], noting that we need to replace $d(x)$ by $D(x)$ and $\partial\Omega$ by $\mathcal{C}$. □

From Proposition 3.4.1, we can prove uniqueness part in Theorem 3.1.1 (i) and Theorem 3.1.2 (i) .

The final goal in this note is to consider the nonexistence of solutions of problem (3.9) under the hypotheses of Theorem 3.1.1 (iii) and Theorem 3.1.2 (ii).

Proposition 3.4.2 *Under the hypotheses of Theorem 3.1.1* (iii) *and Theorem 3.1.2* (ii)*, we assume that* U_1 *and* U_2 *are both sub-solutions (or both super-solutions) of (3.9) satisfying that* $U_1 = U_2 = 0$ *in* Ω^c *and*

$$\begin{aligned} 0 &< \liminf_{x\in\Omega\setminus\mathcal{C},\ x\to\mathcal{C}} U_1(x)D(x)^{-\tau} \le \limsup_{x\in\Omega\setminus\mathcal{C},\ x\to\mathcal{C}} U_1(x)D(x)^{-\tau} \\ &< \liminf_{x\in\Omega\setminus\mathcal{C},\ x\to\mathcal{C}} U_2(x)D(x)^{-\tau} \le \limsup_{x\in\Omega\setminus\mathcal{C},\ x\to\mathcal{C}} U_2(x)D(x)^{-\tau} < +\infty, \end{aligned}$$

for $\tau \in (-1, 0)$. *For the case* $\tau p > \tau - 2\alpha$, *we further assume that*
(i) *if* U_1, U_2 *are sub-solutions, there exist* $C > 0$ *and* $\tilde{\delta} > 0$,

$$(-\Delta)^\alpha U_2(x) \le -CD(x)^{\tau-2\alpha}, \quad x \in A_{\tilde{\delta}} \setminus \mathcal{C}; \tag{3.44}$$

or
(ii) *if* U_1, U_2 *are super-solutions, there exist* $C > 0$ *and* $\tilde{\delta} > 0$,

$$(-\Delta)^\alpha U_1(x) \ge CD(x)^{\tau-2\alpha}, \quad x \in A_{\tilde{\delta}} \setminus \mathcal{C}. \tag{3.45}$$

Then there doesn't exist any solution u *of (3.9) such that*

$$\limsup_{x\in\Omega\setminus\mathcal{C},\ x\to\mathcal{C}} \frac{U_1(x)}{u(x)} < 1 < \liminf_{x\in\Omega\setminus\mathcal{C},\ x\to\mathcal{C}} \frac{U_2(x)}{u(x)}. \tag{3.46}$$

Proof. The proof is similar as Proposition 6.1 in [33], noting again that we need to replace $d(x)$ by $D(x)$ and $\partial\Omega$ by $\mathcal{C}$. □

With the help of Proposition 3.2.2, for given $t_1 > t_2 > 0$, we construct two sub-solutions (or both super-solutions) U_1 and U_2 of (3.9) such that

$$\lim_{x\in\Omega\setminus\mathcal{C},x\to\mathcal{C}} U_1(x)D(x)^{-\tau} = t_1, \quad \lim_{x\in\Omega\setminus\mathcal{C},x\to\mathcal{C}} U_2(x)D(x)^{-\tau} = t_2.$$

So what we have to do is to prove that for any $t > 0$, we can construct super-solution (sub-solution) of problem (3.9).

Proof of Theorem 3.1.1 *(iii)* **and Theorem 3.1.2** *(ii)***.** We divide our proof of the nonexistence results into several cases under the assumption $p > 1$.

Zone 1: We consider $\tau \in (\tau_1(\alpha), 0)$ and $\alpha \in (0, \alpha_0)$. By Proposition 3.2.2 *(ii)*, there exists $\delta_1 > 0$ such that

$$(-\Delta)^\alpha V_\tau(x) \geq \frac{1}{C} D(x)^{\tau-2\alpha}, \quad x \in A_{\delta_1} \setminus \mathcal{C}. \tag{3.47}$$

Since V_τ is C^2 in $\Omega \setminus \mathcal{C}$, then there exists $C > 0$ such that

$$|(-\Delta)^\alpha V_\tau(x)| \leq C, \quad x \in \Omega \setminus A_{\delta_1}. \tag{3.48}$$

Let $\bar{U} := V_\tau + C\bar{V} \quad \text{in} \ \mathbb{R}^{\mathrm{N}} \setminus \mathcal{C}$, then we have $\bar{U} > 0$ in $\Omega \setminus \mathcal{C}$,

$$(-\Delta)^\alpha \bar{U} \geq 0 \ \text{ in } \ \Omega \setminus \mathcal{C} \quad \text{and} \quad (-\Delta)^\alpha \bar{U}(x) \geq \frac{1}{C} D(x)^{\tau-2\alpha}, \quad x \in A_{\delta_1} \setminus \mathcal{C}.$$

Then, we have that $t\bar{U}$ is super-solution of (3.9) for any $t > 0$. Using Proposition 3.4.2, we see that there is no solution of (3.9) satisfying (3.15).

Zone 2: We consider $\tau - 2\alpha < \tau p$ and

$$\tau \in \begin{cases} (-1, 0), & \alpha \in [\alpha_0, 1), \\ (-1, \tau_1(\alpha)), & \alpha \in (0, \alpha_0). \end{cases}$$

Let us define

$$W_{\mu,t} = tV_\tau - \mu\bar{V} \quad \text{in} \ \mathbb{R}^N \setminus \mathcal{C},$$

where $t, \mu > 0$. By Proposition 3.2.2 *(i)*, for $x \in A_{\delta_1} \setminus \mathcal{C}$,

$$(-\Delta)^\alpha W_{\mu,t}(x) + |W_{\mu,t}|^{p-1} W_{\mu,t}(x) \leq -\frac{t}{C} D(x)^{\tau-2\alpha} + t^p D(x)^{\tau p}.$$

For any fixed $t > 0$, there exists $\delta_2 \in (0, \delta_1]$, for all $\mu \geq 0$,

$$(-\Delta)^\alpha W_{\mu,t}(x) + |W_{\mu,t}|^{p-1} W_{\mu,t}(x) \leq 0, \quad A_{\delta_2} \setminus \mathcal{C}. \tag{3.49}$$

To consider $x \in \Omega \setminus A_{\delta_2}$, in fact, there exists $C_1 > 0$ such that

$$t|(-\Delta)^\alpha V_\tau(x)| + t^p V_\tau^p(x) \le C_1, \quad x \in \Omega \setminus A_{\delta_2}$$

and

$$(-\Delta)^\alpha W_{\mu,t}(x) + |W_{\mu,t}|^{p-1} W_{\mu,t}(x) \le C_1 t - \mu, \quad x \in \Omega \setminus A_{\delta_2}$$

For given $t > 0$, there exists $\mu(t) > 0$ such that

$$(-\Delta)^\alpha W_{\mu(t),t}(x) + |W_{\mu,t}|^{p-1} W_{\mu(t),t}(x) \le 0, \quad x \in \Omega \setminus A_{\delta_2}. \tag{3.50}$$

Therefore, together with (3.49) and (3.50), for any given $t > 0$, there sub-solutions $W_{\mu(t),t}$ of problem (3.9) and by Proposition 3.4.2, we see that there is no solution u of (3.9) satisfying (3.15).

Zone 3: We consider $\tau - 2\alpha > \tau p$ and

$$\tau \in \begin{cases} (-1, 0), & \alpha \in [\alpha_0, 1), \\ (-1, \tau_1(\alpha)), & \alpha \in (0, \alpha_0). \end{cases}$$

We denote that

$$U_{\mu,t} = tV_\tau + \mu \bar{V} \quad \text{in } \mathbb{R}^N \setminus \mathcal{C},$$

where $t, \mu > 0$. Here $U_{\mu,t} > 0$ in $\Omega \setminus \mathcal{C}$. By Proposition 3.2.2 (i),

$$(-\Delta)^\alpha U_{\mu,t}(x) + U_{\mu,t}^p(x) \ge -CtD(x)^{\tau - 2\alpha} + t^p D(x)^{\tau p}, \quad x \in A_{\delta_1} \setminus \mathcal{C}.$$

For any fixed $t > 0$, there exists $\delta_2 \in (0, \delta_1]$, for all $\mu \ge 0$,

$$(-\Delta)^\alpha U_{\mu,t}(x) + U_{\mu,t}^p(x) \ge 0, \quad x \in A_{\delta_2} \setminus \mathcal{C}. \tag{3.51}$$

For $x \in \Omega \setminus A_{\delta_2}$, we see that $(-\Delta)^\alpha V_\tau$ is bounded and

$$(-\Delta)^\alpha U_{\mu,t}(x) + U_{\mu,t}^p(x) \ge -Ct + \mu.$$

For given $t > 0$, there exists $\mu(t) > 0$ such that

$$(-\Delta)^\alpha U_{\mu(t),t}(x) + U_{\mu(t),t}^p(x) \ge 0, \quad x \in \Omega \setminus A_{\delta_2}. \tag{3.52}$$

Combining with (3.51) and (3.52), we have that for any $t > 0$, there exists $\mu(t) > 0$ such that

$$(-\Delta)^\alpha U_{\mu(t),t}(x) + U_{\mu(t),t}^p(x) \ge 0, \quad x \in \Omega \setminus \mathcal{C}.$$

Therefore, for any given $t > 0$, there is a super-solution $U_{\mu(t),t}$ of problem (3.9) and by Proposition 3.4.2, we see that there is no solution of (3.9) satisfying (3.15).

We see that Zones 1 and 2 cover Theorem 3.1.1 part (iii) a) since $\tau > -2\alpha/(p-1)$. From Zones 1, 2 and 3 we cover Theorem 3.1.1 part (iii) b) since $\tau_1(\alpha) > 2\alpha/(p-1)$. Moreover, from Zone 1 to Zone 3, we cover the parameters in part (iii) c) of Theorem 3.1.1, since $\tau_1(\alpha) < 2\alpha/(p-1)$. Finally Theorem 3.1.2 part ii) can be obtained from Zone 2 and Zone 3. This complete the proof. $\square$

Capítulo 4

Semilinear fractional elliptic equations involving measures

Abstract: in this chapter [1], we study the existence of weak solutions to (E) $(-\Delta)^\alpha u + g(u) = \nu$ in an open bounded regular domain Ω in $\mathbb{R}^N (N \geq 2)$ which vanish in $\mathbb{R}^N \setminus \Omega$, where $(-\Delta)^\alpha$ denotes the fractional Laplacian with $\alpha \in (0,1)$, ν is a Radon measure and g is a nondecreasing function satisfying some extra hypotheses. When g satisfies a subcritical integrability condition, we prove the existence and uniqueness of a weak solution for problem (E) for any measure. In the case where ν is Dirac measure, we characterize the asymptotic behavior of the solution. When $g(r) = |r|^{k-1}r$ with k supercritical, we show that a condition of absolute continuity of the measure with respect to some Bessel capacity is a necessary and sufficient condition in order (E) to be solved.

4.1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be an open bounded C^2 domain and $g : \mathbb{R} \mapsto \mathbb{R}$ be a continuous function. We are concerned with the existence of weak solutions to the semilinear fractional elliptic problem

$$\begin{aligned} (-\Delta)^\alpha u + g(u) = \nu \quad &\text{in} \quad \Omega, \\ u = 0 \quad &\text{in} \quad \Omega^c, \end{aligned} \tag{4.1}$$

where $\alpha \in (0,1)$, ν is a Radon measure such that $\int_\Omega \rho^\beta d|\nu| < \infty$ for some $\beta \in [0, \alpha]$ and $\rho(x) = dist(x, \Omega^c)$. The fractional Laplacian $(-\Delta)^\alpha$ is defined by

$$(-\Delta)^\alpha u(x) = \lim_{\epsilon \to 0^+} (-\Delta)^\alpha_\epsilon u(x),$$

[1]This chapter is based on the paper: H. Chen and L. Véron, *Semilinear fractional elliptic equations involving measures,* submitted.

where for $\epsilon > 0$,

$$(-\Delta)^\alpha_\epsilon u(x) = -\int_{\mathbb{R}^N} \frac{u(z) - u(x)}{|z - x|^{N+2\alpha}} \chi_\epsilon(|x - z|) dz \tag{4.2}$$

and

$$\chi_\epsilon(t) = \begin{cases} 0, & \text{if} \quad \mathrm{t} \in [0, \epsilon], \\ 1, & \text{if} \quad \mathrm{t} > \epsilon. \end{cases}$$

When $\alpha = 1$, the semilinear elliptic problem

$$\begin{aligned} -\Delta u + g(u) = \nu \quad & \text{in} \quad \Omega, \\ u = 0 \quad & \text{on} \quad \partial\Omega, \end{aligned} \tag{4.3}$$

has been extensively studied by numerous authors in the last 30 years. A fundamental contribution is due to Brezis [16], Bénilan and Brezis [10], where ν is a bounded measure in Ω and the function $g : \mathbb{R} \to \mathbb{R}$ is nondecreasing, positive on $(0, +\infty)$ and satisfies the subcritical assumption:

$$\int_1^{+\infty} (g(s) - g(-s)) s^{-2\frac{N-1}{N-2}} ds < +\infty.$$

They proved the existence and uniqueness of the solution for problem (4.3). Baras and Pierre [9] studied (4.3) when $g(u) = |u|^{p-1}u$ for $p > 1$ and ν is absolutely continuous with respect to the Bessel capacity $C_{2,\frac{p}{p-1}}$, to obtain a solution. In [101] Véron extended Benilan and Brezis results in replacing the Laplacian by a general uniformly elliptic second order differential operator with Lipschitz continuous coefficients; he obtained existence and uniqueness results for solutions, when $\nu \in \mathfrak{M}(\Omega, \rho^\beta)$ with $\beta \in [0, 1]$ where $\mathfrak{M}(\Omega, \rho^\beta)$ denotes the space of Radon measures in Ω satisfying

$$\int_\Omega \rho^\beta d|\nu| < +\infty, \tag{4.4}$$

$\mathfrak{M}(\Omega, \rho^0) = \mathfrak{M}^b(\Omega)$ is the set of bounded Radon measures and g is nondecreasing and satisfies the β-subcritical assumption:

$$\int_1^{+\infty} (g(s) - g(-s)) s^{-2\frac{N+\beta-1}{N+\beta-2}} ds < +\infty.$$

The study of general semilinear elliptic equations with measure data have been investigated, such as the equations involving measures boundary data which was initiated by Gmira and Véron [62] who adapted the method introduced by Benilan and Brezis to obtain the existence and uniqueness of solution. This subject has been vastly expanded in recent years, see the papers of Marcus and Véron [76, 77, 78, 79], Bidaut-Véron and Vivier [14], Bidaut-Véron, Hung and Véron [13].

Recently, great attention has been devoted to non-linear equations involving fractional Laplacian or more general integro-differential operators and we mention the reference [24, 28, 26, 33, 35, 71, 88, 92]. In particular, Karisen, Petitta and Ulusoy in [65] used the duality approach to study the equations of

$$(-\Delta)^\alpha v = \mu \quad \text{in} \quad \mathbb{R}^N,$$

where μ is a Radon measure with compact support. Chen, Felmer and Quaas in [33] the authors obtained the existence of large solutions to equation

$$(-\Delta)^\alpha u + g(u) = f \quad \text{in} \quad \Omega, \tag{4.5}$$

where Ω is a bounded regular domain. In [39] we considered the properties of possibly singular solutions of (4.5) in punctured domain . It is a well-known fact [100] that for $\alpha = 1$ the weak singular solutions of (4.5) in punctured domain are classified according the type of singularities they admits: either weak singularities with Dirac mass, or strong singularities which are the upper limit of solutions with weak singularities. One of our interests is to extend these properties to any $\alpha \in (0,1)$ and furthermore to consider general Radon measures.

In this chapter we study the existence and uniqueness of solutions of (4.1) in a measure framework. Before stating our main theorem we make precise the notion of weak solution used in this chapter.

Definition 4.1.1 *We say that u is a weak solution of (4.1), if $u \in L^1(\Omega)$, $g(u) \in L^1(\Omega, \rho^\alpha dx)$ and*

$$\int_\Omega [u(-\Delta)^\alpha \xi + g(u)\xi] dx = \int_\Omega \xi d\nu, \quad \forall \xi \in \mathbb{X}_\alpha, \tag{4.6}$$

where $\mathbb{X}_\alpha \subset C(\mathbb{R}^N)$ is the space of functions ξ satisfying:

(i) $\mathrm{supp}(\xi) \subset \bar{\Omega}$,

(ii) $(-\Delta)^\alpha \xi(x)$ exists for all $x \in \Omega$ and $|(-\Delta)^\alpha \xi(x)| \le C$ for some $C > 0$,

(iii) there exist $\varphi \in L^1(\Omega, \rho^\alpha dx)$ and $\epsilon_0 > 0$ such that $|(-\Delta)^\alpha_\epsilon \xi| \le \varphi$ a.e. in Ω, for all $\epsilon \in (0, \epsilon_0]$.

We notice that for $\alpha = 1$, the test space $\mathbb{X}_\alpha$ is used as $C_0^{1,L}(\Omega)$, which has similar properties like (i) and (ii). The counter part for the Laplacian of assumption (iii) would be that the difference quotient $\nabla_{x_j,h}[u](.) := h^{-1}[\partial_{x_j} u(. + h\mathbf{e}_j) - \partial_{x_j} u(.)]$ is bounded by an L^1-function, which is true since

$$\nabla_{x_j,h}[u](x) = h^{-1} \int_0^h \partial^2_{x_j,x_j} u(x + s\mathbf{e}_j) ds.$$

We denote by G_α the Green kernel of $(-\Delta)^\alpha$ in Ω and by $\mathbb{G}_\alpha[.]$ the Green operator defined by

$$\mathbb{G}_\alpha[f](x) = \int_\Omega G_\alpha(x,y)f(y)dy, \qquad \forall f \in L^1(\Omega, \rho^\alpha dx). \tag{4.7}$$

For $N \geq 2$, $0 < \alpha < 1$ and $\beta \in [0, \alpha]$, we define the critical exponent

$$k_{\alpha,\beta} = \begin{cases} \frac{N}{N-2\alpha}, & \text{if} \quad \beta \in [0, \frac{\mathrm{N}-2\alpha}{\mathrm{N}}\alpha], \\ \frac{N+\alpha}{N-2\alpha+\beta}, & \text{if} \quad \beta \in (\frac{\mathrm{N}-2\alpha}{\mathrm{N}}\alpha, \alpha]. \end{cases} \tag{4.8}$$

Our main result is the following:

Theorem 4.1.1 *Assume $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is an open bounded C^2 domain, $\alpha \in (0,1)$, $\beta \in [0, \alpha]$ and $k_{\alpha,\beta}$ is defined by (4.8). Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous, nondecreasing function, satisfying*

$$g(r)r \geq 0, \quad \forall r \in \mathbb{R} \quad \text{and} \quad \int_1^\infty (g(s) - g(-s))s^{-1-k_{\alpha,\beta}}ds < \infty. \tag{4.9}$$

Then for any $\nu \in \mathfrak{M}(\Omega, \rho^\beta)$ problem (4.1) admits a unique weak solution u. Furthermore, the mapping: $\nu \mapsto u$ is increasing and

$$-\mathbb{G}_\alpha[\nu_-] \leq u \leq \mathbb{G}_\alpha[\nu_+] \quad \text{a.e. in } \Omega \tag{4.10}$$

where ν_+ and ν_- are respectively the positive and negative part in the Jordan decomposition of ν.

We note that for $\alpha = 1$ and $\beta \in [0, 1)$, we have

$$k_{1,\beta} > \frac{N+\beta}{N-2+\beta}, \tag{4.11}$$

where $k_{1,\beta}$ is given in (4.8) and the number in right hand side of (4.11) is from Theorem 3.7 in [101]. Inspired by [62, 101], the existence of solution could be extended in assuming that $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies *the (N, α, β)-weak-singularity assumption*, that is, there exists $r_0 > 0$ such that

$$g(x,r)r \geq 0, \quad \forall (x,r) \in \Omega \times (\mathbb{R} \setminus (-r_0, r_0)),$$

and

$$|g(x,r)| \leq \tilde{g}(|r|), \quad \forall (x,r) \in \Omega \times \mathbb{R},$$

where $\tilde{g} : [0, \infty) \to [0, \infty)$ is continuous, nondecreasing and satisfies

$$\int_1^\infty \tilde{g}(s)s^{-1-k_{\alpha,\beta}}ds < \infty.$$

We also give a stability result which shows that problem (4.1) is weakly closed in the space of measures $\mathfrak{M}(\Omega,\rho^\beta)$. In the last section of this chapter we characterize the behaviour of the solution u of (4.1) when $\nu=\delta_a$ for some $a\in\Omega$. We also study the case where $g(r)=|r|^{k-1}r$ when $k\geq k_{\alpha,\beta}$, which doesn't satisfy (4.9). We show that a necessary and sufficient condition in order a weak solution to problem

$$\begin{aligned}(-\Delta)^\alpha u+|u|^{k-1}u=\nu \quad &\text{in} \quad \Omega,\\ u=0 \quad &\text{in} \quad \Omega^c,\end{aligned} \tag{4.12}$$

to exist where ν is a positive bounded measure is that ν vanishes on compact subsets K of Ω with zero $C_{2\alpha,k'}$ Bessel-capacity.

4.2. Linear estimates

4.2.1. The Marcinkiewicz spaces

We recall the definition and basic properties of the Marcinkiewicz spaces.

Definition 4.2.1 *Let $\Omega\subset\mathbb{R}^N$ be an open domain and μ be a positive Borel measure in Ω. For $\kappa>1$, $\kappa'=\kappa/(\kappa-1)$ and $u\in L^1_{loc}(\Omega,d\mu)$, we set*

$$\|u\|_{M^\kappa(\Omega,d\mu)}=\text{ínf}\{c\in[0,\infty]:\int_E|u|d\mu\leq c\left(\int_E d\mu\right)^{\frac{1}{\kappa'}},\ \forall E\subset\Omega \text{ Borel set}\} \tag{4.13}$$

and

$$M^\kappa(\Omega,d\mu)=\{u\in L^1_{loc}(\Omega,d\mu):\|u\|_{M^\kappa(\Omega,d\mu)}<\infty\}. \tag{4.14}$$

$M^\kappa(\Omega,d\mu)$ is called the Marcinkiewicz space of exponent κ or weak L^κ space and $\|.\|_{M^\kappa(\Omega,d\mu)}$ is a quasi-norm. The following property holds.

Proposition 4.2.1 *[11, 43] Assume $1\leq q<\kappa<\infty$ and $u\in L^1_{loc}(\Omega,d\mu)$. Then there exists $C(q,\kappa)>0$ such that*

$$\int_E|u|^q d\mu\leq C(q,\kappa)\|u\|_{M^\kappa(\Omega,d\mu)}\left(\int_E d\mu\right)^{1-q/\kappa},$$

for any Borel set E of Ω.

For $\alpha\in(0,1)$ and $\beta,\gamma\in[0,\alpha]$ we set

$$k_1(t)=\frac{\gamma}{\alpha}+\frac{N-(N-2\alpha)\frac{\gamma}{\alpha}}{N-2\alpha+t},\quad k_2(t)=\gamma+\frac{N-(N-2\alpha)\frac{\gamma}{\alpha}}{N-2\alpha+t}t \tag{4.15}$$

and

$$t_{\alpha,\beta,\gamma} = \text{mín}\{t \in [0,\alpha] : \frac{k_2(t)}{k_1(t)} \geq \beta\}. \tag{4.16}$$

Remark 4.2.1 *The quantity $t_{\alpha,\beta,\gamma}$ is well defined, since*

$$\frac{k_2(\alpha)}{k_1(\alpha)} = \frac{\gamma + \alpha\frac{N-(N-2\alpha)\frac{\gamma}{\alpha}}{N-\alpha}}{\frac{\gamma}{\alpha} + \frac{N-(N-2\alpha)\frac{\gamma}{\alpha}}{N-\alpha}} = \alpha \geq \beta.$$

Remark 4.2.2 *The function $t \mapsto k_1(t)$ is decreasing in $[0,\alpha]$ with the following bounds*

$$k_1(0) = \frac{N}{N-2\alpha} \quad \textit{and} \quad k_1(\alpha) = \frac{N+\gamma}{N-\alpha} > 1.$$

Remark 4.2.3 *The function $t \mapsto \frac{k_2(t)}{k_1(t)}$ is increasing in $[0,\alpha]$, since*

$$\left(\frac{k_2(t)}{k_1(t)}\right)' = \frac{[N-(N-2\alpha)\frac{\gamma}{\alpha}](N+\gamma)}{k_1^2(t)} > 0.$$

As a consequence (4.16) is equivalent to

$$t_{\alpha,\beta,\gamma} = \text{máx}\{0, t_\beta\}, \tag{4.17}$$

where

$$t_\beta = \frac{\beta N - (N-2\alpha)\gamma}{N-(N-3\alpha+\beta)\frac{\gamma}{\alpha}}. \tag{4.18}$$

is the solution of $\frac{k_2(t)}{k_1(t)} = \beta$.

Proposition 4.2.2 *Let $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ be an open bounded C^2 domain and $\nu \in \mathfrak{M}(\Omega, \rho^\beta)$ with $\beta \in [0,\alpha]$. Then*

$$\|\mathbb{G}_\alpha[\nu]\|_{M^{k_{\alpha,\beta,\gamma}}(\Omega,\rho^\gamma dx)} \leq C\|\nu\|_{\mathfrak{M}(\Omega,\rho^\beta)}, \tag{4.19}$$

where $\gamma \in [0,\alpha]$, $\mathbb{G}_\alpha[\nu](x) = \int_\Omega G_\alpha(x,y)d\nu(y)$ where G_α is Green's kernel of $(-\Delta)^\alpha$ and

$$k_{\alpha,\beta,\gamma} = \begin{cases} \frac{N+\gamma}{N-2\alpha+\beta}, & \text{if } \gamma \leq \frac{\mathrm{N}\beta}{\mathrm{N}-2\alpha}, \\ \frac{N}{N-2\alpha}, & \text{if not.} \end{cases} \tag{4.20}$$

Proof. For $\lambda > 0$ and $y \in \Omega$, we denote

$$A_\lambda(y) = \{x \in \Omega \setminus \{y\} : G_\alpha(x,y) > \lambda\} \text{ and } \quad m_\lambda(y) = \int_{A_\lambda(y)} \rho^\gamma(x)dx.$$

From [37], there exists $C > 0$ such that for any $(x, y) \in \Omega \times \Omega$, $x \neq y$,

$$G_\alpha(x, y) \leq C \min\left\{\frac{1}{|x-y|^{N-2\alpha}}, \frac{\rho^\alpha(x)}{|x-y|^{N-\alpha}}, \frac{\rho^\alpha(y)}{|x-y|^{N-\alpha}}\right\} \tag{4.21}$$

and

$$G_\alpha(x, y) \leq C \frac{\rho^\alpha(y)}{\rho^\alpha(x)|x-y|^{N-2\alpha}}. \tag{4.22}$$

Therefore, if $\gamma \in [0, \alpha]$ and $x \in A_\lambda(y)$, there holds

$$\rho^\gamma(x) \leq \frac{C\rho^\gamma(y)}{\lambda^{\frac{\gamma}{\alpha}}|x-y|^{(N-2\alpha)\frac{\gamma}{\alpha}}}. \tag{4.23}$$

Let $t \in [0, \alpha]$ be such that $\frac{k_2(t)}{k_1(t)} \geq \beta$, where $k_1(t)$ and $k_2(t)$ are given in (4.15), then

$$G_\alpha(x, y) \leq \left(\frac{C}{|x-y|^{N-2\alpha}}\right)^{1-\frac{t}{\alpha}} \left(\frac{C\rho^\alpha(y)}{|x-y|^{N-\alpha}}\right)^{\frac{t}{\alpha}} = \frac{C\rho^t(y)}{|x-y|^{N-2\alpha+t}}.$$

We observe that

$$A_\lambda(y) \subset \left\{x \in \Omega \setminus \{y\} : \frac{C\rho(y)^t}{|x-y|^{N-2\alpha+t}} > \lambda\right\} \subset D_\lambda(y)$$

where $D_\lambda(y) := \left\{x \in \Omega : |x-y| < (\frac{C\rho^t(y)}{\lambda})^{\frac{1}{N-2\alpha+t}}\right\}$; together with (4.23), this implies

$$m_\lambda(y) \leq \int_{D_\lambda(y)} \frac{C\rho^\gamma(y)}{\lambda^{\frac{\gamma}{\alpha}}|x-y|^{(N-2\alpha)\frac{\gamma}{\alpha}}} dx \leq C\rho(y)^{k_2(t)}\lambda^{-k_1(t)}.$$

For any Borel set E of Ω, we have

$$\int_E G_\alpha(x, y)\rho^\gamma(x)dx \leq \int_{A_\lambda(y)} G_\alpha(x, y)\rho^\gamma(x)dx + \lambda \int_E \rho^\gamma(x)dx$$

and

$$\begin{aligned} \int_{A_\lambda(y)} G_\alpha(x, y)\rho^\gamma(x)dx &= -\int_\lambda^\infty s dm_s(y) \\ &= \lambda m_\lambda(y) + \int_\lambda^\infty m_s(y)ds \\ &\leq C\rho(y)^{k_2(t)}\lambda^{1-k_1(t)}. \end{aligned}$$

Thus,

$$\int_E G_\alpha(x,y)\rho^\gamma(x)dx \le C\rho(y)^{k_2(t)}\lambda^{1-k_1(t)} + \lambda\int_E \rho^\gamma(x)dx.$$

By choosing $\lambda = [\rho(y)^{-k_2(t)}\int_E \rho^\gamma(x)dx]^{-\frac{1}{k_1(t)}}$, we have

$$\int_E G_\alpha(x,y)\rho^\gamma(x)dx \le C\rho(y)^{\frac{k_2(t)}{k_1(t)}}(\int_E \rho^\gamma(x)dx)^{\frac{k_1(t)-1}{k_1(t)}}.$$

Therefore,

$$\begin{aligned}\int_E \mathbb{G}_\alpha[|\nu|](x)\rho^\gamma(x)dx &= \int_\Omega\int_E G_\alpha(x,y)\rho^\gamma(x)dxd|\nu(y)| \\ &\le C\int_\Omega \rho(y)^{\frac{k_2(t)}{k_1(t)}}d|\nu(y)|\left(\int_E \rho^\gamma(x)dx\right)^{\frac{k_1(t)-1}{k_1(t)}} \\ &\le C\|\nu\|_{\mathfrak{M}(\Omega,\rho^\beta)}\left(\int_E \rho^\gamma(x)dx\right)^{\frac{k_1(t)-1}{k_1(t)}},\end{aligned}$$

since by our choice of t, $\frac{k_2(t)}{k_1(t)} \ge \beta$, which guarantees that

$$\int_\Omega \rho(y)^{\frac{k_2(t)}{k_1(t)}}d|\nu(y)| \le \max_\Omega \rho^{\frac{k_2(t)}{k_1(t)}-\beta}\int_\Omega \rho(y)^\beta d|\nu(y)|.$$

As a consequence,

$$\|\mathbb{G}_\alpha[\nu]\|_{M^{k_1(t)}(\Omega,\rho^\gamma dx)} \le C\|\nu\|_{\mathfrak{M}(\Omega,\rho^\beta)}.$$

Therefore,

$$k_{\alpha,\beta,\gamma} := \max\{k_1(t) : t\in[0,\alpha]\} = k_1(t_{\alpha,\beta,\gamma}),$$

where $t_{\alpha,\beta,\gamma}$ is defined by (4.16) and $k_{\alpha,\beta,\gamma}$ is given by (4.20). The proof complete. □

We choose the parameter γ in order to make $k_{\alpha,\beta,\gamma}$ the largest possible, and denote

$$k_{\alpha,\beta} = \max_{\gamma\in[0,\alpha]} k_{\alpha,\beta,\gamma}. \tag{4.24}$$

Since $\gamma \mapsto k_{\alpha,\beta,\gamma}$ is increasing, the following statement holds.

Proposition 4.2.3 *Let $N \ge 2$ and $k_{\alpha,\beta}$ be defined by (4.24), then*

$$k_{\alpha,\beta} = \begin{cases} \frac{N}{N-2\alpha}, & \text{if} \quad \beta\in[0,\frac{\mathrm{N}-2\alpha}{\mathrm{N}}\alpha], \\ \frac{N+\alpha}{N-2\alpha+\beta}, & \text{if} \quad \beta\in(\frac{\mathrm{N}-2\alpha}{\mathrm{N}}\alpha,\alpha]. \end{cases} \tag{4.25}$$

4.2.2. Non-homogeneous problem

In this subsection, we study some properties of the solution of the linear non-homogeneous, which will play a key role in the sequel. We assume that $\Omega \subset \mathbb{R}^N$, $N \geq 2$ is an open bounded domain with a C^2 boundary.

Lemma 4.2.1 (i) *There exists $C > 0$ such that for any $\xi \in \mathbb{X}_\alpha$ there holds*

$$\|\xi\|_{C^\alpha(\bar{\Omega})} \leq C\|(-\Delta)^\alpha \xi\|_{L^\infty(\Omega)} \tag{4.26}$$

and

$$\|\rho^{-\alpha}\xi\|_{C^\theta(\bar{\Omega})} \leq C\|(-\Delta)^\alpha \xi\|_{L^\infty(\Omega)}. \tag{4.27}$$

where $0 < \theta < \text{mín}\{\alpha, 1-\alpha\}$. In particular, for $x \in \Omega$

$$|\xi(x)| \leq C\|(-\Delta)^\alpha \xi\|_{L^\infty(\Omega)}\rho^\alpha(x). \tag{4.28}$$

(ii) *Let u be the solution of*

$$\begin{aligned} (-\Delta)^\alpha u &= f \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{in} \quad \Omega^c, \end{aligned} \tag{4.29}$$

where $f \in C^\gamma(\bar{\Omega})$ for $\gamma > 0$. Then $u \in \mathbb{X}_\alpha$.

Proof. (i). Estimates (4.26) and (4.29) are consequences of [88, Proposition 1.1] and [88, Theorem 1.2] respectively. Furthermore, if η_1 is the solution of (4.29) with $f \equiv 1$ in Ω, then $\eta_1 > 0$ in Ω and by follows [88, Theorem 1.2], there exists $C > 0$ such that

$$C^{-1} \leq \frac{\eta_1}{\rho^\alpha} \leq C \quad \text{in} \quad \Omega. \tag{4.30}$$

In this expression the right-side follows [88, Theorem 1.2] and the left-hand side inequality follows from the maximum principle and [37, Theorem 1.2]. Since

$$-\|(-\Delta)^\alpha \xi\|_{L^\infty(\Omega)} \leq (-\Delta)^\alpha \xi \leq \|(-\Delta)^\alpha \xi\|_{L^\infty(\Omega)} \quad \text{in } \Omega,$$

it follows by the comparison principle,

$$-\|(-\Delta)^\alpha \xi\|_{L^\infty(\Omega)}\eta_1(x) \leq \xi(x) \leq \|(-\Delta)^\alpha \xi\|_{L^\infty(\Omega)}\eta_1(x).$$

which, together with (4.30), implies (4.28).
(ii) For $r > 0$, we denote

$$\Omega_r = \{z \in \Omega : \; dist(z, \partial\Omega) > r\}.$$

Since $f \in C^\gamma(\bar{\Omega})$, then by Corollary 1.6 part (i) and Proposition 1.1 in [88], for $\theta \in [0, \text{mín}\{\alpha, 1-\alpha, \gamma\})$, there exists $C > 0$ such that for any $r > 0$, we have

$$\|u\|_{C^{2\alpha+\theta}(\Omega_r)} \le C r^{-\alpha-\theta}$$

and

$$\|u\|_{C^{\alpha}(\mathbb{R}^N)} \le C.$$

Then for $x \in \Omega$, letting $r = \rho(x)/2$,

$$|\delta(u,x,y)| \le C r^{-\alpha-\theta} |y|^{2\alpha+\theta}, \quad \forall y \in B_r(0) \tag{4.31}$$

and

$$|\delta(u,x,y)| \le C|y|^{\alpha}, \quad \forall y \in \mathbb{R}^N,$$

where $\delta(u,x,y) = u(x+y) + u(x-y) - 2u(x)$. Thus,

$$\begin{aligned} |(-\Delta)^\alpha_\epsilon u(x)| &\le \frac{1}{2}\int_{\mathbb{R}^N} \frac{|\delta(u,x,y)|}{|y|^{N+2\alpha}} \chi_\epsilon(|y|)dy \\ &\le \frac{1}{2}\int_{B_r(0)} \frac{|\delta(u,x,y)|}{|y|^{N+2\alpha}} dy + \frac{1}{2}\int_{B_r^c(0)} \frac{|\delta(u,x,y)|}{|y|^{N+2\alpha}} dy \\ &\le \frac{Cr^{-\alpha-\theta}}{2}\int_{B_r(0)} \frac{1}{|y|^{N-\theta}} dy + \frac{C}{2}\int_{B_r^c(0)} \frac{1}{|y|^{N+\alpha}} dy \\ &\le C\rho(x)^{-\alpha}, \quad x \in \Omega, \end{aligned}$$

for some $C > 0$ independent of ϵ. Moreover, $\rho^{-\alpha}$ is in $L^1(\Omega, \rho^\alpha dx)$. Finally, we prove $(-\Delta)^\alpha_\epsilon u \to (-\Delta)^\alpha u$ as $\epsilon \to 0^+$ pointwise. For $x \in \Omega$, choosing $\epsilon \in (0, \rho(x)/2)$, then by (4.31),

$$\begin{aligned} |(-\Delta)^\alpha u(x) - (-\Delta)^\alpha_\epsilon u(x)| &\le \frac{1}{2}\int_{B_\epsilon(0)} \frac{|\delta(u,x,y)|}{|y|^{N+2\alpha}} dy \\ &\le C\rho(x)^{-\alpha-\theta}\epsilon^\theta \\ &\to 0, \quad \epsilon \to 0^+. \end{aligned}$$

The proof is complete. □

The following Proposition is the Kato's type estimate for proving the uniqueness of the solution of (4.1).

Proposition 4.2.4 *If $\nu \in L^1(\Omega, \rho^\alpha dx)$, there exists a unique weak solution u of the problem*

$$\begin{aligned} (-\Delta)^\alpha u &= \nu \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{in} \quad \Omega^c. \end{aligned} \tag{4.32}$$

For any $\xi \in \mathbb{X}_\alpha$, $\xi \geq 0$, *we have*

$$\int_\Omega |u|(-\Delta)^\alpha \xi dx \leq \int_\Omega \xi \mathrm{sign}(u)\nu dx \tag{4.33}$$

and

$$\int_\Omega u_+(-\Delta)^\alpha \xi dx \leq \int_\Omega \xi \mathrm{sign}_+(u)\nu dx, \tag{4.34}$$

We note here that for $\alpha = 1$, the proof of Proposition 4.2.4 could be seen in [101, Theorem 2.4]. For $\alpha \in (0,1)$, we first prove some integration by parts formula.

Lemma 4.2.2 *Assume* $u, \xi \in \mathbb{X}_\alpha$, *then*

$$\int_\Omega u(-\Delta)^\alpha \xi dx = \int_\Omega \xi(-\Delta)^\alpha u dx. \tag{4.35}$$

Proof. Denote

$$(-\Delta)^\alpha_{\Omega,\epsilon} u(x) = -\int_\Omega \frac{u(z)-u(x)}{|z-x|^{N+2\alpha}}\chi_\epsilon(|x-z|)dz. \tag{4.36}$$

By the definition of $(-\Delta)^\alpha_\epsilon$, we have

$$\begin{aligned}(-\Delta)^\alpha_\epsilon u(x) &= -\int_\Omega \frac{u(z)-u(x)}{|z-x|^{N+2\alpha}}\chi_\epsilon(|x-z|)dz + u(x)\int_{\Omega^c}\frac{\chi_\epsilon(|x-z|)}{|z-x|^{N+2\alpha}}dz \\ &= (-\Delta)^\alpha_{\Omega,\epsilon}u(x) + u(x)\int_{\Omega^c}\frac{\chi_\epsilon(|x-z|)}{|z-x|^{N+2\alpha}}dz.\end{aligned}$$

We claim that

$$\int_\Omega \xi(x)(-\Delta)^\alpha_{\Omega,\epsilon}u(x)dx = \int_\Omega u(x)(-\Delta)^\alpha_{\Omega,\epsilon}\xi(x)dx, \quad \text{for } u,\xi \in \mathbb{X}_\alpha. \tag{4.37}$$

By using the fact of

$$\int_\Omega\int_\Omega \frac{[u(z)-u(x)]\xi(x)}{|z-x|^{N+2\alpha}}\chi_\epsilon(|x-z|)dzdx = \int_\Omega\int_\Omega \frac{[u(x)-u(z)]\xi(z)}{|z-x|^{N+2\alpha}}\chi_\epsilon(|x-z|)dzdx,$$

we have

$$\begin{aligned}&\int_\Omega \xi(x)(-\Delta)^\alpha_{\Omega,\epsilon}u(x)dx \\ &= -\frac{1}{2}\int_\Omega\int_\Omega [\frac{(u(z)-u(x))\xi(x)}{|z-x|^{N+2\alpha}} + \frac{(u(x)-u(z))\xi(z)}{|z-x|^{N+2\alpha}}]\chi_\epsilon(|x-z|)dzdx \\ &= \frac{1}{2}\int_\Omega\int_\Omega \frac{[u(z)-u(x)][\xi(z)-\xi(x)]}{|z-x|^{N+2\alpha}}\chi_\epsilon(|x-z|)dzdx.\end{aligned}$$

Similarly, by the fact that $u \in \mathbb{X}_\alpha$,

$$\int_\Omega u(x)(-\Delta)^\alpha_{\Omega,\epsilon}\xi(x)dx = \frac{1}{2}\int_\Omega\int_\Omega \frac{[u(z)-u(x)][\xi(z)-\xi(x)]}{|z-x|^{N+2\alpha}}\chi_\epsilon(|x-z|)dzdx.$$

Then (4.37) holds. In order to prove (4.35), we first notice that by (4.37),

$$\begin{aligned}
&\int_\Omega \xi(x)(-\Delta)^\alpha_\epsilon u(x)dx \\
&= \int_\Omega \xi(x)(-\Delta)^\alpha_{\Omega,\epsilon} u(x)dx + \int_\Omega u(x)\xi(x)\int_{\Omega^c}\frac{\chi_\epsilon(|x-z|)}{|z-x|^{N+2\alpha}}dzdx \\
&= \int_\Omega u(x)(-\Delta)^\alpha_{\Omega,\epsilon} \xi(x)dx + \int_\Omega u(x)\xi(x)\int_{\Omega^c}\frac{\chi_\epsilon(|x-z|)}{|z-x|^{N+2\alpha}}dzdx \\
&= \int_\Omega u(x)(-\Delta)^\alpha_\epsilon \xi(x)dx. \qquad (4.38)
\end{aligned}$$

Since u and ξ belongs to $\mathbb{X}_\alpha$, $(-\Delta)^\alpha_\epsilon\xi \to (-\Delta)^\alpha\xi$ and $(-\Delta)^\alpha_\epsilon u \to (-\Delta)^\alpha u$ and $|u(-\Delta)^\alpha_\epsilon\xi| + |\xi(-\Delta)^\alpha_\epsilon u| \le C\varphi$ for some $C > 0$ and $\varphi \in L^1(\Omega, \rho^\alpha dx)$. It follows by the Dominated Convergence Theorem

$$\lim_{\epsilon\to 0^+}\int_\Omega \xi(x)(-\Delta)^\alpha_\epsilon u(x)dx = \int_\Omega \xi(x)(-\Delta)^\alpha u(x)dx$$

and

$$\lim_{\epsilon\to 0^+}\int_\Omega (-\Delta)^\alpha_\epsilon \xi(x)u(x)dx = \int_\Omega (-\Delta)^\alpha \xi(x)u(x)dx.$$

Letting $\epsilon \to 0^+$ of (4.38) we conclude that (4.35) holds. □

For $1 \le p < \infty$ and $0 < s < 1$, $W^{s,p}(\Omega)$ is the set of $\xi \in L^p(\Omega)$ such that

$$\int_\Omega\int_\Omega \frac{|\xi(x)-\xi(y)|^p}{|x-y|^{N+sp}}dydx < \infty. \qquad (4.39)$$

This space is endowed with the norm

$$\|\xi\|_{W^{s,p}(\Omega)} = \left(\int_\Omega |\xi(x)|^p dx + \int_\Omega\int_\Omega \frac{|\xi(x)-\xi(y)|^p}{|x-y|^{N+sp}}dydx\right)^{\frac{1}{p}}. \qquad (4.40)$$

Furthermore, if Ω is bounded, the following Poincaré inequality holds [95, p 134].

$$\left(\int_\Omega |\xi(x)|^p dx\right)^{\frac{1}{p}} \le C\left(\int_\Omega\int_\Omega \frac{|\xi(x)-\xi(y)|^p}{|x-y|^{N+sp}}dydx\right)^{\frac{1}{p}}, \quad \forall \xi \in C_c^\infty(\Omega). \qquad (4.41)$$

Lemma 4.2.3 *Let $u \in \mathbb{X}_\alpha$ and γ be C^2 in the interval $u(\bar{\Omega})$ and satisfy $\gamma(0) = 0$,* *then* $u \in W^{\alpha,2}(\Omega)$, $\gamma \circ u \in \mathbb{X}_\alpha$ *and for all* $x \in \Omega$, *there exists* $z_x \in \bar{\Omega}$ *such that*

$$(-\Delta)^\alpha(\gamma \circ u)(x) = (\gamma' \circ u)(x)(-\Delta)^\alpha u(x) - \frac{\gamma'' \circ u(z_x)}{2}\int_\Omega \frac{(u(y)-u(x))^2}{|y-x|^{N+2\alpha}}dy. \quad (4.42)$$

Proof. Since $u \in C(\bar{\Omega})$ vanishes in Ω^c, $\gamma \circ u$ shares the same properties. By (4.26), for any x and y in Ω

$$(u(x)-u(y))^2 \leq C|x-y|^{2\alpha}\|(-\Delta)^\alpha u\|^2_{L^\infty(\Omega)}.$$

Then $u \in W^{\alpha,2}(\Omega)$. Similarly $\gamma \circ u \in W^{\alpha,2}(\Omega)$. Furthermore

$$(\gamma \circ u)(y) - (\gamma \circ u)(x) = (\gamma' \circ u)(x)\,(u(y)-u(x)) + \int_{u(x)}^{u(y)}(u(y)-t)\gamma''(t)dt.$$

By the mean value theorem, there exists some $\tau \in [0,1]$ such that

$$\int_{u(x)}^{u(y)}(u(y)-t)\gamma''(t)dt = \frac{\gamma''(\tau u(y)+(1-\tau)u(x))}{2}(u(y)-u(x))^2.$$

Since γ'' is continuous and u is continuous in $\bar{\Omega}$,

$$\left|\int_{u(x)}^{u(y)}(u(y)-t)\gamma''(t)dt\right| \leq \frac{\|\gamma'' \circ u\|_{L^\infty(\bar{\Omega})}}{2}(u(y)-u(x))^2$$

and by (4.26),

$$\begin{aligned}\left|\int_{|y-x|>\epsilon}\int_{u(x)}^{u(y)}(u(y)-t)\gamma''(t)dt\frac{dy}{|y-x|^{N+2\alpha}}\right| & \\ \leq \frac{\|\gamma'' \circ u\|_{L^\infty}}{2}\int_\Omega (u(y)-u(x))^2\frac{dy}{|y-x|^{N+2\alpha}}.\end{aligned}$$

Notice also that $\tau u(y)+(1-\tau)u(x) \in u(\bar{\Omega}) := I$, therefore

$$\min_{t\in I}\gamma''(t) \leq \gamma''(\tau u(y)+(1-\tau)u(x)) \leq \max_{t\in I}\gamma''(t),$$

thus

$$\begin{aligned}\frac{\min_{t\in I}\gamma''(t)}{2}\int_\Omega \frac{(u(y)-u(x))^2}{|y-x|^{N+2\alpha}}dy &\leq \int_\Omega\int_{u(x)}^{u(y)}(u(y)-t)\gamma''(t)dt\frac{dy}{|y-x|^{N+2\alpha}} \\ &\leq \frac{\max_{t\in I}\gamma''(t)}{2}\int_\Omega \frac{(u(y)-u(x))^2}{|y-x|^{N+2\alpha}}dy.\end{aligned}$$

Since γ'' is continuous, there exists $t_0 \in I$ such that

$$\int_\Omega \int_{u(x)}^{u(y)} (u(y)-t)\gamma''(t)dt \frac{dy}{|y-x|^{N+2\alpha}} = \frac{\gamma''(t_0)}{2}\int_\Omega \frac{(u(y)-u(x))^2}{|y-x|^{N+2\alpha}}dy$$

and since u is continuous in $\mathbb{R}^N$ and vanishes in Ω^c, there exists $z_x \in \bar{\Omega}$ such that $t_0 = u(z_x)$, which ends the proof. $\square$

Proof of Proposition 4.2.4. *Uniqueness.* Let w be a weak solution of

$$\begin{aligned} (-\Delta)^\alpha w = 0 \qquad & \text{in} \quad \Omega \\ w = 0 \qquad & \text{in} \quad \Omega^c. \end{aligned} \tag{4.43}$$

If ω is a Borel subset of Ω and $\eta_{\omega,n}$ the solution of

$$\begin{aligned} (-\Delta)^\alpha \eta_{\omega,n} = \zeta_n \qquad & \text{in} \quad \Omega \\ \eta_{\omega,n} = 0 \qquad & \text{in} \quad \Omega^c, \end{aligned} \tag{4.44}$$

where $\zeta_n : \bar{\Omega} \mapsto [0,1]$ is a $C^1(\bar{\Omega})$ function such that

$$\zeta_n \to \chi_\omega \quad \text{in } L^\infty(\bar{\Omega}) \quad \text{as } n \to \infty.$$

Then by Lemma 4.2.1 part (ii), $\eta_{\omega,n} \in \mathbb{X}_\alpha$ and

$$\int_\Omega w\zeta_n dx = 0.$$

Then passing the limit of $n \to \infty$, we have

$$\int_\omega w dx = 0.$$

This implies $w = 0$.

Existence and estimate (4.33). For $\delta > 0$ we define an even convex function ϕ_δ by

$$\phi_\delta(t) = \begin{cases} |t| - \frac{\delta}{2}, & \text{if} \quad |\mathrm{t}| \geq \delta, \\ \frac{t^2}{2\delta}, & \text{if} \quad |\mathrm{t}| < \delta/2. \end{cases} \tag{4.45}$$

Then for any $t, s \in \mathbb{R}$, $|\phi_\delta'(t)| \leq 1$, $\phi_\delta(t) \to |t|$ and $\phi_\delta'(t) \to$ cap5sign(t) when $\delta \to 0^+$. Moreover

$$\phi_\delta(s) - \phi_\delta(t) \geq \phi_\delta'(t)(s-t). \tag{4.46}$$

Let $\{\nu_n\}$ be a sequence functions in $C^1(\bar{\Omega})$ such that

$$\lim_{n\to\infty} \int_\Omega |\nu_n - \nu|\rho^\alpha dx = 0.$$

Let u_n be the corresponding solution to (4.32) with right-hand side ν_n, then by Lemma 4.2.1, $u_n \in \mathbb{X}_\alpha$ and by Lemmas 4.2.2, 4.2.3, for any $\delta > 0$ and $\xi \in \mathbb{X}_\alpha,\ \xi \geq 0$,

$$\begin{aligned}\int_\Omega \phi_\delta(u_n)(-\Delta)^\alpha \xi dx &= \int_\Omega \xi(-\Delta)^\alpha \phi_\delta(u_n) dx \\ &\leq \int_\Omega \xi \phi_\delta'(u_n)(-\Delta)^\alpha u_n dx \\ &= \int_\Omega \xi \phi_\delta'(u_n) \nu_n dx.\end{aligned} \tag{4.47}$$

Letting $\delta \to 0$, we obtain

$$\int_\Omega |u_n|(-\Delta)^\alpha \xi dx \leq \int_\Omega \xi \mathrm{sign}(u_n)\nu_n dx \leq \int_\Omega \xi |\nu_n| dx. \tag{4.48}$$

If we take $\xi = \eta_1$, we derive from Lemma 4.2.1,

$$\int_\Omega |u_n| dx \leq C \int_\Omega |\nu_n| \rho^\alpha dx. \tag{4.49}$$

Similarly

$$\int_\Omega |u_n - u_m| dx \leq C \int_\Omega |\nu_n - \nu_m| \rho^\alpha dx. \tag{4.50}$$

Therefore, $\{u_n\}$ is a Cauchy sequence in L^1 and its limit u is a weak solution of (4.32). Letting $n \to \infty$ in (4.48) we obtain (4.33). Inequality (4.34) is proved by replacing ϕ_δ by $\tilde{\phi}_\delta$ which is zero on $(-\infty, 0]$ and ϕ_δ on $[0, \infty)$. □

The next result is a higher order regularity result

Proposition 4.2.5 *Let the assumptions of Proposition 4.2.2 be fulfilled and* $0 \leq \beta \leq \alpha$. *Then for* $p \in (1, \frac{N}{N+\beta-2\alpha})$ *there exists* $c_p > 0$ *such that for any* $\nu \in L^1(\Omega, \rho^\beta dx)$

$$\|\mathbb{G}_\alpha[\nu]\|_{W^{2\alpha-\gamma,p}(\Omega)} \leq c_p \|\nu\|_{L^1(\Omega,\rho^\beta dx)} \tag{4.51}$$

where $\gamma = \beta + \frac{N}{p'}$ *if* $\beta > 0$ *and* $\gamma > \frac{N}{p'}$ *if* $\beta = 0$.

Proof. We use Stampacchia's duality method [93] and put $u = \mathbb{G}_\alpha[\nu]$. If $\psi \in C_c^\infty(\bar{\Omega})$, then

$$\begin{aligned}\left|\int_\Omega \psi(-\Delta)^\alpha u dx\right| &\leq \int_\Omega |\nu||\psi| dx \\ &\leq \sup_\Omega |\rho^{-\beta}\psi| \int_\Omega |\nu| \rho^\beta dx \\ &\leq \|\psi\|_{C^\beta(\bar{\Omega})} \|\nu\|_{L^1(\Omega,\rho^\beta dx)}.\end{aligned} \tag{4.52}$$

By Sobolev-Morrey embedding type theorem (see e.g. [81, Theorem 8.2]), for any $p \in (1, \frac{N}{N+\beta-2\alpha})$ and $p' = \frac{p}{p-1}$,

$$\|\psi\|_{C^\beta(\bar{\Omega})} \le C\|\psi\|_{W^{\gamma,p'}(\Omega)}$$

with $\gamma = \beta + \frac{N}{p'}$ if $\beta > 0$ and $\gamma > \frac{N}{p'}$ if $\beta = 0$. Therefore,

$$\left|\int_\Omega \psi(-\Delta)^\alpha u dx\right| \le C\|\psi\|_{W^{\gamma,p'}(\Omega)}\|\nu\|_{L^1(\Omega,\rho^\beta dx)}, \tag{4.53}$$

which implies that the mapping $\psi \mapsto \int_\Omega \psi(-\Delta)^\alpha u dx$ is continuous on $W^{\gamma,p'}(\Omega)$ and thus

$$\|(-\Delta)^\alpha u\|_{W^{-\gamma,p}(\Omega)} \le C\|\nu\|_{L^1(\Omega,\rho^\beta dx)}. \tag{4.54}$$

Since $(-\Delta)^{-\alpha}$ is an isomorphism from $W^{-\gamma,p}(\Omega)$ into $W^{2\alpha-\gamma,p}(\Omega)$, it follows that

$$\|u\|_{W^{2\alpha-\gamma,p}(\Omega)} \le C\|\nu\|_{L^1(\Omega,\rho^\beta dx)}. \tag{4.55}$$

□

Proposition 4.2.6 *Under the assumptions of Proposition 4.2.5 the mapping* $\nu \mapsto \mathbb{G}[\nu]$ *is compact from* $L^1(\Omega, \rho^\beta dx)$ *into* $L^q(\Omega)$ *for any* $q \in [1, \frac{N}{N+\beta-2\alpha})$.

Proof. By [81, Theorem 6.5] the embedding of $W^{2\alpha-\gamma,p}(\Omega)$ into $L^q(\Omega)$ is compact, this ends the proof. □

4.3. Proof of Theorem 4.1.1

Before proving the main we give a general existence result in $L^1(\Omega, \rho^\alpha dx)$.

Proposition 4.3.1 *Suppose that* Ω *is an open bounded* C^2 *domain of* $\mathbb{R}^N$ $(N \ge 2)$, $\alpha \in (0,1)$ *and the function* $g : \mathbb{R} \to \mathbb{R}$ *is continuous, nondecreasing and* $rg(r) \ge 0$ *for all* $r \in \mathbb{R}$. *Then for any* $f \in L^1(\Omega, \rho^\alpha dx)$ *there exists a unique weak solution* u *of (4.1) with* $\nu = f$. *Moreover the mapping* $f \mapsto u$ *is increasing.*

Proof. *Step 1: Variational solutions.* If $w \in L^2(\Omega)$, we denote by $\underline{w}$ its extension by 0 in Ω^c and by $W_c^{\alpha,2}(\Omega)$ the set of function in $L^2(\Omega)$ such that

$$\|w\|^2_{W_c^{\alpha,2}(\Omega)} := \int_{\mathbb{R}^N} |\hat{\underline{w}}|^2(1+|x|^\alpha)dx < \infty,$$

where $\hat{\underline{w}}$ is the Fourier transform of $\underline{w}$. For $\epsilon > 0$ we set

$$J(w) = \frac{1}{2}\int_{\mathbb{R}^N} \left((-\Delta)^{\frac{\alpha}{2}}\underline{w}\right)^2 dx + \int_\Omega (j(w) + \epsilon w^2)dx,$$

with domain $D(J)=\{w\in W_c^{\alpha,2}(\mathbb{R}^N)$ s.t. $j(w)\in L^1(\Omega)\}$ and $j(s)=\int_0^s g(t)dt$. Furthermore since there holds $J(w)\geq\sigma\|w\|^2_{W_c^{\alpha,2}}$ for some $\sigma>0$, the subdifferential ∂J of J is a maximal monotone in the sense of Browder-Minty (see [17] and the references therein) which satisfies $R(\partial J)=L^2(\Omega)$. Then for any $f\in L^2(\Omega)$ there exists a unique u_ϵ in the domain $D(\partial J)$ such that $\partial J(u_\epsilon)=f$. Since for any $\psi\in W_c^{\alpha,2}(\Omega)$

$$\int_{\mathbb{R}^N}(-\Delta)^{\frac{\alpha}{2}}\underline{w}\,(-\Delta)^{\frac{\alpha}{2}}\underline{\psi}dx=(4\pi)^\alpha\int_{\mathbb{R}^N}\hat{\underline{w}}\,\hat{\underline{\psi}}|x|^{2\alpha}dx=\int_\Omega\psi(-\Delta)^\alpha\underline{w}dx,$$

$$\partial J(u_\epsilon)=(-\Delta)^\alpha u_\epsilon+g(u_\epsilon)+2\epsilon u=f,$$

with $u_\epsilon\in W_c^{2\alpha,2}(\Omega)$ such that $g(u_\epsilon)\in L^2(\Omega)$. This is also a consequence of [17, Corollary 2.11]. If f is assumed to be bounded, then $u\in C^\alpha(\overline{\Omega})$ by [88, Proposition 1.1].

Step 2: L^1 solutions. For $n\in\mathbb{N}^*$ we denote by $u_{n,\epsilon}$ the solution of

$$\begin{aligned}(-\Delta)^\alpha u_{n,\epsilon}+g(u_{n,\epsilon})+2\epsilon u_{n,\epsilon}&=f_n &&\text{in }\Omega\\ u_{n,\epsilon}&=0 &&\text{in }\Omega^c\end{aligned}\tag{4.56}$$

where $f_n=\operatorname{sgn}(f)\operatorname{mín}\{n,|f|\}$. By (4.48) with $\xi=\eta_1$,

$$\int_\Omega\left(|u_{n,\epsilon}|+(2\epsilon|u_{n,\epsilon}|+|g(u_{n,\epsilon})|)\eta_1\right)dx\leq\int_\Omega|f_n|\eta_1dx\leq\int_\Omega|f|\eta_1dx,\tag{4.57}$$

and for $\epsilon'>0$ and $m\in\mathbb{N}^*$,

$$\begin{aligned}&\int_\Omega\left(|u_{n,\epsilon}-u_{m,\epsilon'}|+|g(u_{n,\epsilon})-g(u_{m,\epsilon'})|\eta_1\right)dx\\&\qquad\qquad\leq\int_\Omega\left(|f_n-f_m|+2\epsilon|u_{n,\epsilon}|+2\epsilon'|u_{m,\epsilon'}|\right)\eta_1dx.\end{aligned}\tag{4.58}$$

Since $f_n\to f$ in $L^1(\Omega,\rho^\alpha dx)$, $\{u_{n,\epsilon}\}$ and $\{g\circ u_{n,\epsilon}\}$ are Cauchy filters in $L^1(\Omega)$ and $L^1(\Omega,\rho^\alpha dx)$ respectively. Set $u=\operatorname{lím}_{n\to\infty,\epsilon\to0}u_{n,\epsilon}$, we derive from the following identity valid for any $\xi\in\mathbb{X}_\alpha$,

$$\int_\Omega\left(u_{n,\epsilon}(-\Delta)^\alpha\xi+g(u_{n,\epsilon})\xi\right)dx=\int_\Omega\left(f_n-\epsilon u_{n,\epsilon}\right)\xi dx$$

that u is a solution of (4.1). Uniqueness follows from (4.48)-(4.58), since for any f and f' in $L^1(\Omega,\rho^\alpha dx)$, the any couple (u,u') of weak solutions with respective right-hand side f and f' satisfies

$$\int_\Omega\left(|u-u'|+|g(u)-g(u')|\eta_1\right)dx\leq\int_\Omega|f-f'|\eta_1dx.\tag{4.59}$$

Finally, the monotonicity of the mapping $f \mapsto u$ follows from (4.34) thanks to which (4.59) is transformed into

$$\int_{\Omega} ((u-u')_+ + (g(u)-g(u'))_+\eta_1)\, dx \leq \int_{\Omega} (f-f')_+\eta_1 dx. \tag{4.60}$$

$\square$

Proof of Theorem 4.1.1. Uniqueness follows from (4.59). For existence we define

$$C_\beta(\bar{\Omega}) = \{\zeta \in C(\bar{\Omega}) : \rho^{-\beta}\zeta \in C(\bar{\Omega})\}$$

endowed with the norm

$$\|\zeta\|_{C_\beta(\bar{\Omega})} = \|\rho^{-\beta}\zeta\|_{C(\bar{\Omega})}.$$

We consider a sequence $\{\nu_n\} \subset C^1(\bar{\Omega})$ such that $\nu_{n,\pm} \to \nu_\pm$ in the duality sense with $C_\beta(\bar{\Omega})$, which means

$$\lim_{n\to\infty} \int_{\bar{\Omega}} \zeta\nu_{n,\pm} dx = \int_{\bar{\Omega}} \zeta d\nu_\pm$$

for all $\zeta \in C_\beta(\bar{\Omega})$. It follows from the Banach-Steinhaus theorem that $\|\nu_n\|_{\mathfrak{M}(\Omega,\rho^\beta)}$ is bounded independently of n, therefore

$$\int_{\Omega} (|u_n| + |g(u_n)|\eta_1)\, dx \leq \int_{\Omega} |\nu_n|\eta_1 dx \leq C. \tag{4.61}$$

Therefore $\|g(u_n)\|_{\mathfrak{M}(\Omega,\rho^\alpha)}$ is bounded independently of n. For $\epsilon > 0$, set $\xi_\epsilon = (\eta_1 + \epsilon)^{\frac{\beta}{\alpha}} - \epsilon^{\frac{\beta}{\alpha}}$, which is concave in the interval $\eta(\bar{\omega})$. Then, by Lemma 4.2.3 part (ii),

$$\begin{aligned}(-\Delta)^\alpha \xi_\epsilon &= \frac{\beta}{\alpha}(\eta_1+\epsilon)^{\frac{\beta-\alpha}{\alpha}}(-\Delta)^\alpha\eta_1 - \frac{\beta(\beta-\alpha)}{\alpha^2}(\eta_1+\epsilon)^{\frac{\beta-2\alpha}{\alpha}} \int_{\Omega} \frac{(\eta_1(y)-\eta_1(x))^2}{|y-x|^{N+2\alpha}} dy \\ &\geq \frac{\beta}{\alpha}(\eta_1+\epsilon)^{\frac{\beta-\alpha}{\alpha}},\end{aligned}$$

and $\xi_\epsilon \in \mathbb{X}_\alpha$. Since

$$\int_{\Omega} (|u_n|(-\Delta)^\alpha\xi_\epsilon + |g(u_n)|\xi_\epsilon)\, dx \leq \int_{\Omega} \xi_\epsilon d|\nu_n|,$$

we obtain

$$\int_{\Omega} \left(|u_n|\frac{\beta}{\alpha}(\eta_1+\epsilon)^{\frac{\beta-\alpha}{\alpha}} + |g(u_n)|\xi_\epsilon\right) dx \leq \int_{\Omega} \xi_\epsilon d|\nu_n|.$$

If we let $\epsilon \to 0$, we obtain

$$\int_{\Omega} \left(|u_n|\frac{\beta}{\alpha}\eta_1^{\frac{\beta-\alpha}{\alpha}} + |g(u_n)|\eta_1^{\frac{\beta}{\alpha}}\right) dx \leq \int_{\Omega} \eta_1^{\frac{\beta}{\alpha}} d|\nu_n|.$$

By Lemma 4.2.3, we derive the estimate

$$\int_{\Omega} \left(|u_n|\rho^{\beta-\alpha} + |g(u_n)|\rho^{\beta}\right) dx \le C\|\nu_n\|_{\mathfrak{M}(\Omega,\rho^\beta)} \le C'. \tag{4.62}$$

Since $u_n = \mathbb{G}_\alpha[\nu_n - g(u_n)]$, it follows by (4.19), that

$$\|u_n\|_{M^{k_{\alpha,\beta}}(\Omega,\rho^\beta dx)} \le \|\nu_n - g(u_n)\|_{\mathfrak{M}(\Omega,\rho^\beta)}, \tag{4.63}$$

where $k_{\alpha,\beta}$ is defined by (4.25). By Corollary 4.2.6 the sequence $\{u_n\}$ is relatively compact in the $L^q(\Omega)$ for $1 \le q < \frac{N}{N+\beta-2\alpha}$. Therefore there exist a sub-sequence $\{u_{n_k}\}$ and some $u \in L^1(\Omega) \cap L^q(\Omega)$ such that $u_{n_k} \to u$ in $L^q(\Omega)$ and almost every where in Ω. Furthermore $g(u_{n_k}) \to g(u)$ almost every where. Put $\tilde{g}(r) = g(|r|) - g(-|r|)$ and we note that $|g(r)| \le \tilde{g}(|r|)$ for $r \in \mathbb{R}$ and $\tilde{g}$ is nondecreasing. For $\lambda > 0$, we set $S_\lambda = \{x \in \Omega : |u_{n_k}(x)| > \lambda\}$ and $\omega(\lambda) = \int_{S_\lambda} \rho^\beta dx$. Then for any Borel set $E \subset \Omega$, we have

$$\begin{aligned}
\int_E |g(u_{n_k})|\rho^\beta dx &= \int_{E\cap S_\lambda^c} |g(u_{n_k})|\rho^\beta dx + \int_{E\cap S_\lambda} |g(u_{n_k})|\rho^\beta dx \\
&\le \tilde{g}(\lambda)\int_E \rho^\beta dx + \int_{S_\lambda} \tilde{g}(|u_{n_k}|)\rho^\beta dx \\
&\le \tilde{g}(\lambda)\int_E \rho^\beta dx - \int_\lambda^\infty \tilde{g}(s)d\omega(s).
\end{aligned}$$

But

$$\int_\lambda^\infty \tilde{g}(s)d\omega(s) = \lim_{T\to\infty}\int_\lambda^T \tilde{g}(s)d\omega(s).$$

Since $u_{n_k} \in M^{k_{\alpha,\beta}}(\Omega,\rho^\beta dx)$, $\omega(s) \le cs^{-k_{\alpha,\beta}}$ and

$$\begin{aligned}
-\int_\lambda^T \tilde{g}(s)d\omega(s) &= -\left[\tilde{g}(s)\omega(s)\right]_{s=\lambda}^{s=T} + \int_\lambda^T \omega(s)d\tilde{g}(s) \\
&\le \tilde{g}(\lambda)\omega(\lambda) - \tilde{g}(T)\omega(T) + c\int_\lambda^T s^{-k_{\alpha,\beta}}d\tilde{g}(s) \\
&\le \tilde{g}(\lambda)\omega(\lambda) - \tilde{g}(T)\omega(T) + c\left(T^{-k_{\alpha,\beta}}\tilde{g}(T) - \lambda^{-k_{\alpha,\beta}}\tilde{g}(\lambda)\right) \\
&\qquad + \frac{c}{k_{\alpha,\beta}+1}\int_\lambda^T s^{-1-k_{\alpha,\beta}}\tilde{g}(s)ds.
\end{aligned}$$

By assumption (4.9) there exists $\{T_n\} \to \infty$ such that $T_n^{-k_{\alpha,\beta}}\tilde{g}(T_n) \to 0$ when $n \to \infty$. Furthermore $\tilde{g}(\lambda)\omega(\lambda) \le c\lambda^{-k_{\alpha,\beta}}\tilde{g}(\lambda)$, therefore

$$-\int_\lambda^\infty \tilde{g}(s)d\omega(s) \le \frac{c}{k_{\alpha,\beta}+1}\int_\lambda^\infty s^{-1-k_{\alpha,\beta}}\tilde{g}(s)ds.$$

Notice that the above quantity on the right-hand side tends to 0 when $\lambda \to \infty$. The conclusion follows: for any $\epsilon > 0$ there exists $\lambda > 0$ such that

$$\frac{c}{k_{\alpha,\beta}+1}\int_\lambda^\infty s^{-1-k_{\alpha,\beta}}\tilde{g}(s)ds \le \frac{\epsilon}{2}$$

and $\delta > 0$ such that

$$\int_E \rho^\beta dx \le \delta \Longrightarrow \tilde{g}(\lambda)\int_E \rho^\beta dx \le \frac{\epsilon}{2}.$$

This proves that $\{g\circ u_{n_k}\}$ is uniformly integrable in $L^1(\Omega,\rho^\beta dx)$. Then $g\circ u_{n_k} \to g\circ u$ in $L^1(\Omega,\rho^\beta dx)$ by Vitali convergence theorem. Letting $n_k \to \infty$ in the identity

$$\int_\Omega (u_{n_k}(-\Delta)^\alpha\xi + \xi g\circ u_{n_k})\, dx = \int_\Omega \nu_{n_k}\xi dx$$

where $\xi \in \mathbb{X}_\alpha$, it infers that u is a weak solution of (4.1).

The right-hand side of estimate (4.9) follows from the fact that $v_{n,+} := \mathbb{G}_\alpha[\nu_{n,+}]$ satisfies

$$(-\Delta)^\alpha v_{n,+} + g(v_{n,+}) = \nu_{n,+} + g(v_{n,+}) \ge \nu_n$$

Therefore $v_{n,+} \ge u_n$ by Proposition 4.3.1. Letting $n \to \infty$ yields to (4.10). The left-hand side is proved similarly.

To prove the mapping $\nu \mapsto u$ is increasing. Let $\nu_1, \nu_2 \in \mathfrak{M}(\Omega,\rho^\beta)$ and $\nu_1 \ge \nu_2$, then there exist two sequences $\{\nu_{1,n}\}$ and $\{\nu_{2,n}\}$ in $C^\infty(\bar{\Omega})$ such that $\nu_{1,n} \ge \nu_{2,n}$ and

$$\nu_{i,n} \to \nu_i \quad \text{as } n \to \infty, \quad i = 1,2.$$

Let $u_{i,n}$ be the unique solution of (4.1) with $\nu_{i,n}$ and u_i be the unique solution of (4.1) with ν_i where $i = 1,2$. Then $u_{1,n} \ge u_{2,n}$. Moveover, by uniqueness $u_{i,n}$ convergence to u_i in $L^1(\Omega)$ for $i = 1$ and $i = 2$. Then we have $u_1 \ge u_2$. □

Corollary 4.3.1 *Under the hypotheses of Theorem 4.1.1, we further assume that $\{\nu_n\}$ is a sequence of measures in $\mathfrak{M}(\Omega,\rho^\beta)$ and $\nu \in \mathfrak{M}(\Omega,\rho^\beta)$ such that for any $\xi \in C_\beta(\bar{\Omega})$,*

$$\int_\Omega \xi d\nu_n \to \int_\Omega \xi d\nu \quad \text{as} \quad n \to \infty.$$

Then the sequence $\{u_n\}$ of weak solutions to

$$\begin{aligned} (-\Delta)^\alpha u_n + g\circ u_n = \nu_n \quad & \text{in} \quad \Omega, \\ u_n = 0 \quad & \text{in} \quad \Omega^c, \end{aligned} \tag{4.64}$$

converges to the solution u of (4.1) in $L^q(\Omega)$ for $1 \le q < \frac{N}{N+\beta-2\alpha}$ and $\{g\circ u_n\}$

converges to $g \circ u$ in $L^1(\Omega, \rho^\beta dx)$.

Proof. The method is an adaptation of [102]. Since $\nu_n \to \nu$ in the duality sense of $C_\beta(\overline{\Omega})$, there exists $M > 0$ such that

$$\|\nu_n\|_{\mathfrak{M}(\Omega,\rho^\beta)} \leq M, \qquad \forall n \in \mathbb{N}.$$

Therefore (4.62) and (4.63) hold (but with u_n solution of (4.64). The above proof shows that $\{g \circ u_n\}$ is uniformly integrable in $L^1(\Omega, \rho^\beta dx)$ and $\{u_n\}$ relatively compact in $L^q(\Omega)$ for $1 \leq q < \frac{N}{N+\beta-2\alpha}$. Thus, up to a subsequence $\{u_{n_k}\} \subset \{u_n\}$, $u_{n_k} \to u$, and u is the weak solution of (4.1). Since u is unique, $u_n \to u$ as $n \to \infty$.
□

Remark 4.3.1 *Under the hypotheses of Theorem 4.1.1, we assume $\nu \geq 0$, then*

$$\mathbb{G}_\alpha[\nu] - \mathbb{G}_\alpha[g(\mathbb{G}_\alpha[\nu])] \leq u \leq \mathbb{G}_\alpha[\nu]. \tag{4.65}$$

Indeed, since g is nondecreasing and $u \leq \mathbb{G}_\alpha[\nu]$, then

$$\begin{aligned} u &= \mathbb{G}_\alpha[\nu] - \mathbb{G}_\alpha[g(u)] \\ &\geq \mathbb{G}_\alpha[\nu] - \mathbb{G}_\alpha[g(\mathbb{G}_\alpha[\nu])]. \end{aligned}$$

4.4. Applications

4.4.1. The case of Dirac mass

In this subsection we characterize the asymptotic behavior of a solution near a singularity created by a Dirac mass.

Theorem 4.4.1 *Assume that Ω is an open, bounded and C^2 domain of $\mathbb{R}^N$ $(N \geq 2)$ with $0 \in \Omega$, $\alpha \in (0,1)$, $\nu = \delta_0$ and the function $g : [0,\infty) \to [0,\infty)$ is continuous, nondecreasing and (4.9) holds for*

$$k_{\alpha,0} = \frac{N}{N-2\alpha}. \tag{4.66}$$

Then problem (4.1) admits a unique positive weak solution u such that

$$\lim_{x\to 0} u(x)|x|^{N-2\alpha} = C, \tag{4.67}$$

for some $C > 0$.

Remark 4.4.1 *We note here that a weak solution u of (4.1) with $\nu = \delta_0$ satisfies*

$$\begin{aligned} (-\Delta)^\alpha u + g(u) = 0 \quad &\text{in} \quad \Omega \setminus \{0\}, \\ u = 0 \quad &\text{in} \quad \mathbb{R}^{\mathrm{N}} \setminus \Omega. \end{aligned} \tag{4.68}$$

The asymptotic behavior (4.67) is one of the possible singular behaviors of solutions of (4.68) given in [39].

Lemma 4.4.1 *Assume that $g : [0,\infty) \to [0,\infty)$ is continuous, nondecreasing and (4.9) holds with $k_{\alpha,\beta} > 1$. Then*

$$\lim_{s\to\infty} g(s)s^{-k_{\alpha,\beta}} = 0. \tag{4.69}$$

Proof. Since

$$\int_s^{2s} g(t)t^{-1-k_{\alpha,\beta}}dt \geq g(s)(2s)^{-1-k_{\alpha,\beta}} \int_s^{2s} dt = 2^{-1-k_{\alpha,\beta}} g(s)s^{-k_{\alpha,\beta}}$$

and by (4.9), we have that $\lim_{s\to\infty} \int_s^{2s} g(t)t^{-1-k_{\alpha,\beta}}dt = 0$. Then (4.69) holds. □

Proof of Theorem 4.4.1. Existence, uniqueness and positiveness follow from Theorem 4.1.1 with $\beta = 0$. For (4.67), we shall use (4.10). From [38] there holds,

$$0 < \frac{C}{|x|^{N-2\alpha}} - G_\alpha(x,0) < \frac{C}{\rho(0)^{N-2\alpha}}, \quad x \in \Omega \setminus \{0\}. \tag{4.70}$$

for some $C > 0$ dependent of N and α. Since

$$\mathbb{G}_\alpha[\delta_0](x) = G_\alpha(x,0) < \frac{C}{|x|^{N-2\alpha}}, \quad x \in \Omega \setminus \{0\},$$

then for $x \in \Omega \setminus \{0\}$,

$$\begin{aligned} 0 \quad &\leq \quad \mathbb{G}_\alpha[g(\mathbb{G}_\alpha[\delta_0])](x)|x|^{N-2\alpha} \leq \int_\Omega \frac{1}{|x-y|^{N-2\alpha}} g(\frac{C}{|y|^{N-2\alpha}})dy|x|^{N-2\alpha} \\ &\leq \quad \int_\Omega \frac{1}{|e_x - y|^{N-2\alpha}} g(\frac{C}{(|x||z|)^{N-2\alpha}})dz|x|^N \\ &= \quad |x|^N \int_{\Omega\cap B_{1/2}(e_x)} \frac{1}{|e_x - y|^{N-2\alpha}} g(\frac{C}{(|x||z|)^{N-2\alpha}})dz \\ &\quad\ +|x|^N \int_{\Omega\cap B^c_{1/2}(e_x)} \frac{1}{|e_x - y|^{N-2\alpha}} g(\frac{C}{(|x||z|)^{N-2\alpha}})dz \\ &=: \quad A_1(x) + A_2(x), \end{aligned}$$

where $e_x = x/|x|$. By Lemma 4.4.1,

$$\begin{aligned} A_1(x) &\leq |x|^N g(\frac{2^{N-2\alpha}C}{|x|^{N-2\alpha}}) \int_{B_{1/2}(e_x)} \frac{1}{|e_x - y|^{N-2\alpha}} dz \\ &\to 0 \quad \text{as} \quad |x| \to 0, \end{aligned}$$

and by (4.9),

$$\begin{aligned} A_2(x) &\leq \bar{C}|x|^N \int_{B_R(0)} g(\frac{C}{(|x||z|)^{N-2\alpha}}) dz \\ &\leq \bar{C} \int_{\frac{R^{1/(N-2\alpha)}}{|x|}}^{\infty} g(Cs) s^{-1-\frac{N}{N-2\alpha}} ds \\ &\to 0 \quad \text{as} \quad |x| \to 0, \end{aligned}$$

where $R > 0$ such that $B_R(0) \supset \Omega$. That is

$$\lim_{|x|\to 0} \mathbb{G}_\alpha[g(\mathbb{G}_\alpha[\delta_0])](x)|x|^{N-2\alpha} = 0. \tag{4.71}$$

We plug (4.70) and (4.71) into (4.65), then (4.67) holds. □

4.4.2. The power case

If $g(s) = |s|^{k-1}s$ with $k \geq 1$, then (4.9) is satisfied if $1 \leq k < k_{\alpha,\beta}$ where $k_{\alpha,\beta}$ defined by (4.25) is called the *critical exponent* with limit values $k_{\alpha,0} = \frac{N}{N-2\alpha}$ and $k_{\alpha,\alpha} = \frac{N+\alpha}{N-\alpha}$. If we consider the problem

$$\begin{aligned} (-\Delta)^\alpha u + |u|^{k-1}u = \nu \qquad &\text{in} \quad \Omega, \\ u = 0 \qquad &\text{in} \quad \Omega^c, \end{aligned} \tag{4.72}$$

then if $1 < k < k_{\alpha,\beta}$ it is solvable for any $\nu \in \mathfrak{M}(\Omega, \rho^\beta)$, but it may not be the case if $k \geq k_{\alpha,\beta}$. As in the case $\alpha = 1$, the sharp solvability of (4.72) is associated to a concentration property of the measure ν and this concentration is expressed by the mean of Bessel capacities. If $k > 1$ and $k' = \frac{k}{k-1}$, we define for any compact set $K \subset \Omega$,

$$C^{\Omega}_{2\alpha,k'}(K) = \inf\{\|\phi\|^{k'}_{W^{2\alpha,k'}(\Omega)} : \phi \in C^\infty_c(\Omega), 0 \leq \phi \leq 1, \phi \equiv 1 \text{ on } K\}. \tag{4.73}$$

Then $C_{2\alpha,k'}$ is an outer measure or capacity in Ω extended to Borel sets by standard processes. Our result is the following in the case of bounded measures

Theorem 4.4.2 *Assume Ω is an open bounded C^2 domain in $\mathbb{R}^N$ and $k > 1$. Then problem (4.72) can be solved with a nonnegative bounded measure ν if and only if ν*

satisfies on compact subsets $K \subset \Omega$

$$C^{\Omega}_{2\alpha,k'}(K) = 0 \Longrightarrow \nu(K) = 0. \tag{4.74}$$

Proof. *1-The condition is necessary.* Assume u is a weak solution and let $K \subset \Omega$ be compact. Let $\phi \in C_c^{\infty}(\mathbb{R}^N)$ such that $0 \le \phi \le 1$ and $\phi(x) = 1$ for all $x \in K$, and set $\xi = \phi^{k'}$, then $\xi \in \mathbb{X}_\alpha$ and

$$\int_\Omega \left(u(-\Delta)^\alpha \xi + u^k \xi\right) dx = \int_\Omega \xi d\nu.$$

Since $\xi \ge \chi_K$ it follows from (4.42) that

$$\int_\Omega \left(k'\phi^{k'-1}u(-\Delta)^\alpha \phi + \phi^{k'}u^k\right) dx \ge \nu(K). \tag{4.75}$$

By Hölder's inequality

$$\left|\int_\Omega \phi^{k'-1}u(-\Delta)^\alpha \phi dx\right| \le \left(\int_\Omega \phi^{k'}u^k dx\right)^{\frac{1}{k}} \left(\int_\Omega |(-\Delta)^\alpha \phi|^{k'} dx\right)^{\frac{1}{k'}} \tag{4.76}$$

By [81, Theorem 5.4], there exists $\tilde{\phi} \in W^{2\alpha,k'}(\mathbb{R}^N)$ such that $\tilde{\phi}|_\Omega = \phi$ and

$$\|\tilde{\phi}\|_{W^{2\alpha,k'}(\mathbb{R}^N)} \le C\|\phi\|_{W^{2\alpha,k'}(\Omega)}$$

Then, by standard regularity result on the Riesz potential $(-\Delta)^{-\alpha}$ in $\mathbb{R}^N$,

$$\begin{aligned}
\left|\int_\Omega \phi^{k'-1}u(-\Delta)^\alpha \phi dx\right| &\le \left(\int_\Omega \phi^{k'}u^k dx\right)^{\frac{1}{k}} \left(\int_{\mathbb{R}^N} |(-\Delta)^\alpha \phi|^{k'} dx\right)^{\frac{1}{k'}} \\
&\le C'\left(\int_\Omega \phi^{k'}u^k dx\right)^{\frac{1}{k'}} \|\tilde{\phi}\|_{W^{2\alpha,k'}(\mathbb{R}^N)} \\
&\le C'\left(\int_\Omega \phi^{k'}u^k dx\right)^{\frac{1}{k'}} \|\phi\|_{W^{2\alpha,k'}(\Omega)}.
\end{aligned} \tag{4.77}$$

Therefore, (4.77) yields to

$$C\|\phi\|_{W^{2\alpha,k'}(\Omega)}\left(\int_\Omega \phi^{k'}u^k dx\right)^{\frac{1}{k}} + \int_\Omega \phi^{k'}u^k dx \ge \nu(K). \tag{4.78}$$

If $C^{\Omega}_{2\alpha,k'}(K) = 0$, there exists a sequence $\{\phi_n\} \subset C_c^\infty(\Omega)$ such that $0 \le \phi_n \le 1$ and $\phi_n = 1$ on K and $\|\phi_n\|_{W^{2\alpha,k'}(\Omega)} \to 0$ as $n \to \infty$. Furthermore K has zero Lebesgue measure and $\phi_n \to 0$ almost everywhere. If we replace ϕ by ϕ_n in (4.78) and let $n \to \infty$ we obtain $\nu(K) = 0$.

2-The condition is sufficient. We first assume that $\nu \in W^{-2\alpha,k}(\Omega) \cap \mathfrak{M}_+^b(\Omega)$; for $n \in \mathbb{N}$, we denote by u_n the solution of

$$\begin{aligned} (-\Delta)^\alpha u + |T_n(u)|^{k-1}T_n(u) = \nu \qquad &\text{in} \quad \Omega \\ u = 0 \qquad &\text{in} \quad \Omega^c \end{aligned} \tag{4.79}$$

where $T_n(r) = \text{sign}(r)\,\text{mín}\{n, |r|\}$. Such a solution exists by Theorem 4.1.1, is nonnegative and the sequence $\{u_n\}$ is decreasing and converges to some nonnegative u since $\{T_n(r)\}$ is increasing on $\mathbb{R}_+$. Furthermore, by (4.10),

$$0 \le u_n \le \mathbb{G}_\alpha[\nu].$$

This implies that the convergence holds in $L^1(\Omega)$. Since $\nu \in W^{-2\alpha,k}(\Omega)$, $\mathbb{G}_\alpha[\nu] \in L^k(\Omega)$, it infers that

$$|T_n(u_n)|^{k-1}T_n(u_n) = (T_n(u_n))^k \le (\mathbb{G}_\alpha[\nu])^k.$$

Since for any $\xi \in \mathbb{X}_\alpha$ there holds

$$\int_\Omega \left(u_n(-\Delta)^\alpha \xi + (T_n(u_n))^k \xi\right) dx = \int_\Omega \xi d\nu \tag{4.80}$$

we can let $n \to \infty$ and conclude that u is a solution of (4.72), unique by (4.59). Next we assume that (4.74) holds. By a result of Feyel and de la Pradelle [58] (see also [46]), there exists an increasing sequence $\{\nu_n\} \subset W^{-2\alpha,k}(\Omega) \cap \mathfrak{M}_+^b(\Omega)$ which converges to ν in the weak sense of measures. This implies that the sequence $\{u_n\}$ of weak solutions of

$$\begin{aligned} (-\Delta)^\alpha u_n + u_n^k = \nu_n \qquad &\text{in} \quad \Omega, \\ u_n = 0 \qquad &\text{in} \quad \Omega^c \end{aligned} \tag{4.81}$$

is increasing with limit u. Taking $\eta_1 := \mathbb{G}_\alpha[1]$ as a test function in the weak formulation, we have

$$\int_\Omega \left(u_n + u_n^k \eta_1\right) dx = \int_\Omega \eta_1 d\nu_n \le \int_\Omega \eta_1 d\nu.$$

So $u_n \to u$ in $L^1(\Omega) \cap L^k(\Omega, \rho^\alpha dx)$. Let $n \to \infty$, then u satisfies (4.72). $\square$

Remark 4.4.2 *If ν is a signed bounded measure a sufficient condition for solving (4.72) is*

$$C^\Omega_{2\alpha,k'}(K) = 0 \Longrightarrow |\nu|(K) = 0. \tag{4.82}$$

This can be obtained by using the fact that the solutions of (4.72) with right-hand side ν_+ and $-\nu_-$ are respectively a supersolution and a subsolution of (4.72). It is not clear whether it is also a necessary condition.

Capítulo 5

Weakly and strongly singular solutions of semilinear fractional elliptic equations

Abstract: in this chapter [1], let $p \in (0, \frac{N}{N-2\alpha})$, $\alpha \in (0,1)$, $k > 0$ and $\Omega \subset \mathbb{R}^N$ is an open bounded C^2 domain containing 0 and δ_0 is the Dirac measure at 0, we prove that the weak solution of $(E)_k$ $(-\Delta)^\alpha u + u^p = k\delta_0$ in Ω which vanishes in Ω^c is a weakly singular solution of $(E)_\infty$ $(-\Delta)^\alpha u + u^p = 0$ in $\Omega \setminus \{0\}$ with the same outer data. Furthermore, we study the limit of weak solutions of $(E)_k$ when $k \to \infty$. For $p \in (0, 1 + \frac{2\alpha}{N}]$, the limit is infinity in Ω. For $p \in (1 + \frac{2\alpha}{N}, \frac{N}{N-2\alpha})$, the limit is a strongly singular solution of $(E)_\infty$.

5.1. Introduction

Let Ω be a bounded C^2 domain of $\mathbb{R}^N (N \geq 2)$ containing 0, $\alpha \in (0,1)$ and δ_0 denote the Dirac mass at 0. In this chapter, we study the properties of the weak solution to problem

$$\begin{aligned} (-\Delta)^\alpha u + u^p = k\delta_0 \quad &\text{in} \quad \Omega, \\ u = 0 \quad &\text{in} \quad \Omega^c, \end{aligned} \tag{5.1}$$

where $k > 0$, $p \in (0, \frac{N}{N-2\alpha})$ and $(-\Delta)^\alpha$ is the fractional Laplacian defined by

$$(-\Delta)^\alpha u(x) = \lim_{\epsilon \to 0^+} (-\Delta)^\alpha_\epsilon u(x),$$

[1] This chapter is based on the paper: H. Chen and L. Véron, *Weak and strong singular solutions of semilinear fractional elliptic equations,* accepted by Asymptotic Analysis (2013).

where for $\epsilon > 0$,

$$(-\Delta)^\alpha_\epsilon u(x) = -\int_{\mathbb{R}^N} \frac{u(z)-u(x)}{|z-x|^{N+2\alpha}}\chi_\epsilon(|x-z|)dz$$

and

$$\chi_\epsilon(t) = \begin{cases} 0, & \text{if} \quad \mathrm{t} \in [0,\epsilon] \\ 1, & \text{if} \quad \mathrm{t} > \epsilon. \end{cases}$$

In 1980, Brezis in[16] (also see [10]) obtained that the problem

$$\begin{aligned} -\Delta u + u^q &= k\delta_0 \quad \text{in} \quad \Omega, \\ u &= 0 \qquad \text{on} \quad \partial\Omega \end{aligned} \tag{5.2}$$

admits a unique solution u_k for $1 < q < N/(N-2)$, while no solution exists when $q \geq N/(N-2)$. Later on, Brezis and Véron in [18] proved that the problem

$$\begin{aligned} -\Delta u + u^q &= 0 \quad \text{in} \quad \Omega \setminus \{0\}, \\ u &= 0 \quad \text{on} \quad \partial\Omega \end{aligned} \tag{5.3}$$

admits only the zero solution when $q \geq N/(N-2)$. When $1 < q < N/(N-2)$, Véron in [100] described all the possible singular behaviour of positive solutions of (5.3). In particular he proved that this behaviour is always isotropic (when $(N+1)/(N-1) \leq q < N/(N-2)$ the assumption of positivity is unnecessary) and that two types of singular behaviour occur:
(i) either $u(x) \sim c_N k|x|^{2-N}$ as $x \to 0$ and k can take any positive value; u is said to have a *weak singularity* at 0, and actually $u = u_k$,
(ii) or $u(x) \sim c_{N,q}|x|^{-\frac{2}{q-1}}$ as $x \to 0$; u is said to have a *strong singularity* at 0, and $u = u_\infty := \mathrm{lím}_{k\to\infty} u_k$.

In a recent work, Chen and Véron [39] derived that for $1 + \frac{2\alpha}{N} < p < \frac{N}{N-2\alpha}$, the problem

$$\begin{aligned} (-\Delta)^\alpha u + u^p &= 0 \quad \text{in} \quad \Omega \setminus \{0\}, \\ u &= 0 \quad \text{in} \quad \Omega^c \end{aligned} \tag{5.4}$$

admits a solution u_s satisfying

$$\operatorname*{lím}_{x\to 0} u_s(x)|x|^{\frac{2\alpha}{p-1}} = c_p, \tag{5.5}$$

for some $c_p > 0$. Moreover u_s is the unique positive solution of (5.4) in the class set of

$$0 < \operatorname*{lím\,inf}_{x\to 0} u(x)|x|^{\frac{2\alpha}{p-1}} \leq \operatorname*{lím\,sup}_{x\to 0} u(x)|x|^{\frac{2\alpha}{p-1}} < +\infty. \tag{5.6}$$

We say that u is a *weakly singular solution* of (5.4) if $\limsup_{x\to 0}|u(x)||x|^{N-2\alpha}<+\infty$, or *strongly singular solution* if $\lim_{x\to 0}|u(x)||x|^{N-2\alpha}=+\infty$.

We also in [40] obtained that there exists a unique weak solution to the problem

$$\begin{aligned}(-\Delta)^\alpha u+g(u)=\nu \quad &\text{in} \quad \Omega,\\ u=0 \quad &\text{in} \quad \Omega^c,\end{aligned} \tag{5.7}$$

where g is a subcritical nonlinearity, ν is a Radon measure in Ω. In the fractional framework, the definition of weak solution is given as follows.

Definition 5.1.1 *A function $u\in L^1(\Omega)$ is a weak solution of (5.7) if $g(u)\in L^1(\Omega,\rho^\alpha dx)$ and*

$$\int_\Omega [u(-\Delta)^\alpha\xi+g(u)\xi]dx=\int_\Omega \xi d\nu,\quad \forall \xi\in\mathbb{X}_\alpha, \tag{5.8}$$

where $\rho(x)=dist(x,\Omega^c)$ and $\mathbb{X}_\alpha\subset C(\mathbb{R}^N)$ is the space of functions ξ satisfying:

(i) $\mathrm{supp}(\xi)\subset\bar{\Omega}$,

(ii) $(-\Delta)^\alpha\xi(x)$ exists for all $x\in\Omega$ and $|(-\Delta)^\alpha\xi(x)|\le C$ for some $C>0$,

(iii) there exist $\varphi\in L^1(\Omega,\rho^\alpha dx)$ and $\epsilon_0>0$ such that $|(-\Delta)^\alpha_\epsilon\xi|\le\varphi$ a.e. in Ω, for all $\epsilon\in(0,\epsilon_0]$.

According to Theorem 4.1.1 in chapter 4 with $g(s)=|s|^{p-1}s$ and $\nu=k\delta_0$, we have following result for problem (5.1).

Proposition 5.1.1 *Assume that $p\in(0,\frac{N}{N-2\alpha})$. Then for any $k>0$, problem (5.1) admits a unique weak solution u_k satisfying*

$$\mathbb{G}_\alpha[k\delta_0]-\mathbb{G}_\alpha[(\mathbb{G}_\alpha[k\delta_0])^p]\le u_k\le\mathbb{G}_\alpha[k\delta_0] \quad \text{in} \quad \Omega. \tag{5.9}$$

Moreover, (i) u_k is positive in Ω;
(ii) $\{u_k\}_k$ is a sequence increasing functions, i.e.

$$u_k(x)\le u_{k+1}(x),\quad \forall x\in\Omega. \tag{5.10}$$

Here $\mathbb{G}_\alpha[\cdot]$ is the Green operator defined by

$$\mathbb{G}_\alpha[\nu](x)=\int_\Omega G_\alpha(x,y)d\nu(y),\quad \forall\nu\in\mathfrak{M}(\Omega,\rho^\alpha), \tag{5.11}$$

where G_α is the Green kernel of $(-\Delta)^\alpha$ in $\Omega\times\Omega$. By monotonicity of $\{u_k\}_k$,

$$u_\infty(x):=\lim_{k\to\infty}u_k(x),\quad \forall x\in\mathbb{R}^N\setminus\{0\} \tag{5.12}$$

and then $u_\infty(x) \in \mathbb{R}_+ \cup \{+\infty\}$ for $x \in \mathbb{R}^N \setminus \{0\}$.

Our purpose in this chapter is to do further study on the properties of u_k, including the regularity and the limit of u_k, which is the unique weak solution of (5.1).

Theorem 5.1.1 *Assume that* $1+\frac{2\alpha}{N} \ge \frac{2\alpha}{N-2\alpha}$, $p \in (0, \frac{N}{N-2\alpha})$, u_k *is the weak solution of (5.1) and* u_∞ *is given by (5.12).*

Then u_k *is a classical solution of (5.4). Furthermore,*
(i) if $p \in (0, 1+\frac{2\alpha}{N})$,

$$u_\infty(x) = \infty, \quad \forall x \in \Omega; \tag{5.13}$$

(ii) if $p \in (1+\frac{2\alpha}{N}, \frac{N}{N-2\alpha})$,

$$u_\infty = u_s,$$

where u_s *is the solution of (5.4) satisfying (5.5).*

The result of part (i) indicates that there is no strongly singular solution to problem (5.4) for $p \in (0, 1+\frac{2\alpha}{N})$, which is different from the result for Laplacian case. This phenomenon comes from the fact that the fractional Laplacian is a nonlocal operator, which requires the solution to belong to $L^1(\Omega)$, therefore no barrier can be constructed for $p < 1+\frac{2\alpha}{N}$. On the contrary, part (ii) points out that u_∞ is the least strongly singular solution of (5.4).

Next we consider the case $1+\frac{2\alpha}{N} < \frac{2\alpha}{N-2\alpha}$. It occurs only when

$$\frac{\sqrt{5}-1}{4}N < \alpha < 1, \quad N = 2, 3.$$

In this situation, it is obvious that $\frac{N}{2\alpha} < 1+\frac{2\alpha}{N}$. Now we state our second theorem as following.

Theorem 5.1.2 *Assume that* $1+\frac{2\alpha}{N} < \frac{2\alpha}{N-2\alpha}$, $p \in (0, \frac{N}{N-2\alpha})$, u_k *is the weak solution of (5.1) and* u_∞ *is given by (5.12).*

Then u_k *is a classical solution of (5.4). Furthermore,*
(i) if $p \in (0, \frac{N}{2\alpha})$, *then*

$$u_\infty(x) = \infty, \qquad \forall x \in \Omega;$$

(ii) if $p \in (1+\frac{2\alpha}{N}, \frac{2\alpha}{N-2\alpha})$, *then* u_∞ *is a classical solution of (5.4) and there exist* $\rho_0 > 0$ *and* $c_0 > 0$ *such that*

$$c_0|x|^{-\frac{(N-2\alpha)p}{p-1}} \le u_\infty \le u_s, \qquad \forall x \in B_{\rho_0}(0) \setminus \{0\}; \tag{5.14}$$

(iii) if $p = \frac{2\alpha}{N-2\alpha}$, *then* u_∞ *is a classical solution of (5.4) and there exist* $\rho_0 > 0$ *and*

$c_1 > 0$ *such that*

$$c_1 \frac{|x|^{-\frac{(N-2\alpha)p}{p-1}}}{(1+|\log(|x|)|)^{\frac{1}{p-1}}} \le u_\infty \le u_s, \qquad \forall x \in B_{\rho_0}(0)\setminus\{0\}; \tag{5.15}$$

(iv) *if* $p \in (\frac{2\alpha}{N-2\alpha}, \frac{N}{N-2\alpha})$, *then*

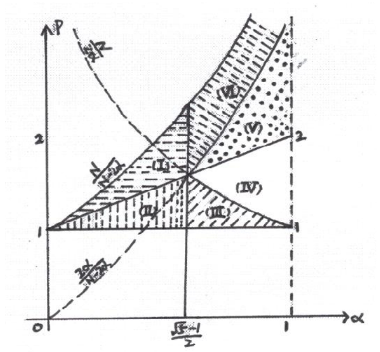
$$u_\infty = u_s,$$

where u_s *is the solution of (5.4) satisfying (5.5)*

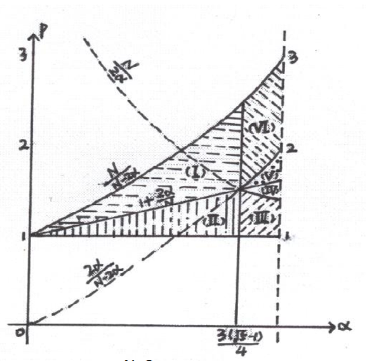
We note that Theorem 5.1.1 and Theorem 5.1.2 do not provide description of u_∞ in the region

$$\mathcal{U} := \left\{(\alpha,p) \in (0,1)\times(1,\tfrac{N}{N-2}) : 1+\tfrac{2\alpha}{N} < \tfrac{2\alpha}{N-2\alpha},\ \tfrac{N}{2\alpha} \le p \le 1+\tfrac{2\alpha}{N}\right\}$$
$$\bigcup \left\{(\alpha,p) \in (0,1)\times(1,\tfrac{N}{N-2}) : 1+\tfrac{2\alpha}{N} \ge \tfrac{2\alpha}{N-2\alpha},\ p = 1+\tfrac{2\alpha}{N}\right\},$$

which is region (IV) and the segment $p = 1+\frac{2\alpha}{N}$, see the pictures $N=2$ and $N=3$.



N=2



N=3

5.2. Preliminaries

The purpose of this section is to give the estimates for $\mathbb{G}_\alpha[(\mathbb{G}_\alpha[\delta_0])^p]$, comparison principle and stability theorem. We denote by $B_r(x)$ the ball centered at x with radius r and $B_r := B_r(0)$.

Lemma 5.2.1 *Assume that* Ω *is a bounded* C^2 *domain of* $\mathbb{R}^N$ *containing* 0 *and* $r = \frac{1}{4}\min\{1, dist(0,\partial\Omega)\}$. *Then there exists* $c_2 > 1$ *such that*
(i) *for* $p \in (0, \frac{2\alpha}{N-2\alpha})$,

$$\mathbb{G}_\alpha[(\mathbb{G}_\alpha[\delta_0])^p] \le c_2 \quad \text{in } B_r\setminus\{0\};$$

(ii) for $p = \frac{2\alpha}{N-2\alpha}$,

$$\mathbb{G}_\alpha[(\mathbb{G}_\alpha[\delta_0])^p] \le -c_2 \ln|x| \quad \text{in} \;\; B_r \setminus \{0\};$$

(iii) for $p \in (\frac{2\alpha}{N-2\alpha}, \frac{N}{N-2\alpha})$,

$$\mathbb{G}_\alpha[(\mathbb{G}_\alpha[\delta_0])^p] \le c_2 |x|^{2\alpha-(N-2\alpha)p} \quad \text{in} \;\; B_r \setminus \{0\}.$$

Proof. We observe that there exists $c_3 > 1$ such that

$$\mathbb{G}_\alpha[\delta_0](x) \le c_3 |x|^{2\alpha-N} \chi_\Omega(x), \quad x \in \mathbb{R}^N \setminus \{0\} \tag{5.16}$$

and for all $x, y \in \mathbb{R}^N$ with $x \ne y$,

$$G_\alpha(x,y) \le c_3 |x-y|^{2\alpha-N} \chi_\Omega(x)\chi_\Omega(y).$$

Then we derive that for $x \in B_r \setminus \{0\}$,

$$\begin{aligned} \mathbb{G}_\alpha[(\mathbb{G}_\alpha[\delta_0])^p](x) &\le c_3^{p+1} \int_{B_R} \tfrac{1}{|y-x|^{N-2\alpha}} \tfrac{1}{|y|^{(N-2\alpha)p}} dy \\ &\le c_3^{p+1} |x|^{N-(N-2\alpha)(p+1)} \int_{B_{\frac{R}{|x|}}} \tfrac{1}{|z-e_x|^{N-2\alpha}} \tfrac{1}{|z|^{(N-2\alpha)p}} dz \\ &\le c_4 |x|^{2\alpha-(N-2\alpha)p} (\int_2^{\frac{R}{|x|}} s^{-1+2\alpha-(N-2\alpha)p} ds + 1), \end{aligned} \tag{5.17}$$

where $c_4 > 1$, $e_x = \frac{x}{|x|}$ and $R = \text{máx}_{z \in \partial\Omega} |z|$.

For $p \in (0, \frac{2\alpha}{N-2\alpha})$, we observe that $2\alpha - (N-2\alpha)p > 0$ and it follows by (5.17) that for $x \in B_r \setminus \{0\}$,

$$\begin{aligned} \mathbb{G}_\alpha[(\mathbb{G}_\alpha[\delta_0])^p](x) &\le c_4 |x|^{2\alpha-(N-2\alpha)p} [\tfrac{1}{2\alpha-(N-2\alpha)p} (\tfrac{R}{|x|})^{2\alpha-(N-2\alpha)p} + 1] \\ &\le c_5, \qquad \text{for some} \;\; c_5 > 1. \end{aligned}$$

For $p = \frac{2\alpha}{N-2\alpha}$, we observe that $2\alpha - (N-2\alpha)p = 0$ and it follows by (5.17) that for $x \in B_r \setminus \{0\}$,

$$\begin{aligned} \mathbb{G}_\alpha[(\mathbb{G}_\alpha[\delta_0])^p](x) &\le c_4 (\int_2^{\frac{R}{|x|}} s^{-1} ds + 1) \\ &\le -c_4 \ln|x| + c_4 \ln R + c_4. \end{aligned}$$

For $p \in (\frac{2\alpha}{N-2\alpha}, \frac{N}{N-2\alpha})$, we observe that $2\alpha - (N-2\alpha)p < 0$ and it derives by (5.17) that for $x \in B_r \setminus \{0\}$,

$$\begin{aligned} \mathbb{G}_\alpha[(\mathbb{G}_\alpha[\delta_0])^p](x) &\le c_4 |x|^{2\alpha-(N-2\alpha)p} (\int_2^{\infty} s^{-1+2\alpha-(N-2\alpha)p} ds + 1) \\ &\le c_6 |x|^{2\alpha-(N-2\alpha)p}, \end{aligned}$$

for some $c_6 > 1$. The proof is completed. $\square$

Theorem 5.2.1 *Suppose that O is a bounded domain of $\mathbb{R}^N$, $p > 0$, the functions u_1, u_2 are continuous in $\bar{O}$ and satisfy*

$$(-\Delta)^\alpha u_1 + |u_1|^{p-1}u_1 \geq 0 \text{ in } O \quad \text{and} \quad (-\Delta)^\alpha u_2 + |u_2|^{p-1}u_2 \leq 0 \text{ in } O.$$

Assume more that $u_1 \geq u_2$ a.e. in O^c.

Then $u_1 > u_2$ in O or $u_1 \equiv u_2$ a.e. in $\mathbb{R}^N$.

Proof. The proof refers to [33, Theorem 2.3] (see also [26, Theorem 5.2]). □

The following stability result is given by Theorem 2.2 in [33].

Theorem 5.2.2 *Suppose that $\mathcal{O}$ is a bounded C^2 domain and $h : \mathbb{R} \to \mathbb{R}$ is continuous. Assume that $\{u_n\}$ is a sequence of functions, uniformly bounded in $L^1(\mathcal{O}^c, \frac{dy}{1+|y|^{N+2\alpha}})$, satisfying*

$$(-\Delta)^\alpha u_n + h(u_n) \geq f_n \text{ (resp } (-\Delta)^\alpha u_n + h(u_n) \leq f_n \text{)} \quad \text{in } \mathcal{O}$$

in the viscosity sense, where $\{f_n\}$ are continuous functions in $\mathcal{O}$. If there holds
(i) $u_n \to u$ locally uniformly in $\mathcal{O}$,
(ii) $u_n \to u$ in $L^1(\mathbb{R}^N, \frac{dy}{1+|y|^{N+2\alpha}})$,
(iii) $f_n \to f$ locally uniformly in $\mathcal{O}$,
then

$$(-\Delta)^\alpha u + h(u) \geq f \text{ (resp } (-\Delta)^\alpha u + h(u) \leq f \text{)} \quad \text{in } \mathcal{O}$$

in the viscosity sense.

5.3. Regularity

In this section, we prove that any weak solution of (5.1) is a classical solution of (5.4). To this end, we introduce an auxiliary lemma.

Lemma 5.3.1 *Assume that $w \in C^{2\alpha+\epsilon}(\bar{B}_1)$ with $\epsilon > 0$ satisfies*

$$(-\Delta)^\alpha w = h \quad \text{in} \quad B_1,$$

where $h \in C^1(\bar{B}_1)$. Then for $\beta \in (0, 2\alpha)$, there exists $c_7 > 0$ such that

$$\|w\|_{C^\beta(\bar{B}_{1/4})} \leq c_7(\|w\|_{L^\infty(B_1)} + \|h\|_{L^\infty(B_1)} + \|(1+|\cdot|)^{-N-2\alpha}w\|_{L^1(\mathbb{R}^N)}). \tag{5.18}$$

Proof. We denote $v = w\eta$, where $\eta : \mathbb{R}^N \to [0, 1]$ is a C^∞ function such that

$$\eta = 1 \quad \text{in} \quad B_{\frac{3}{4}} \quad \text{and} \quad \eta = 0 \quad \text{in} \quad B_1^c.$$

Then $v \in C^{2\alpha+\epsilon}(\mathbb{R}^N)$ and for any $x \in B_{\frac{1}{2}}$, $\epsilon \in (0, \frac{1}{4})$,

$$\begin{aligned}(-\Delta)^\alpha_\epsilon v(x) &= -\int_{\mathbb{R}^N \setminus B_\epsilon} \frac{v(x+y) - v(x)}{|y|^{N+2\alpha}} dy \\ &= (-\Delta)^\alpha_\epsilon w(x) + \int_{\mathbb{R}^N \setminus B_\epsilon} \frac{(1-\eta(x+y))w(x+y)}{|y|^{N+2\alpha}} dy.\end{aligned}$$

Together with the fact of $\eta(x+y) = 1$ for $y \in B_\epsilon$, we derive that

$$\int_{\mathbb{R}^N \setminus B_\epsilon} \frac{(1-\eta(x+y))w(x+y)}{|y|^{N+2\alpha}} dy = \int_{\mathbb{R}^N} \frac{(1-\eta(x+y))w(x+y)}{|y|^{N+2\alpha}} dy =: h_1(x),$$

thus,

$$(-\Delta)^\alpha v = h + h_1 \quad \text{in} \quad B_{\frac{1}{2}}.$$

For $x \in B_{\frac{1}{2}}$ and $z \in \mathbb{R}^N \setminus B_{\frac{3}{4}}$, there holds

$$|z - x| \geq |z| - |x| \geq |z| - \frac{1}{2} \geq \frac{1}{16}(1 + |z|),$$

which implies that

$$\begin{aligned}|h_1(x)| = \left| \int_{\mathbb{R}^N} \frac{(1-\eta(z))w(z)}{|z-x|^{N+2\alpha}} dz \right| &\leq \int_{\mathbb{R}^N \setminus B_{\frac{3}{4}}} \frac{|w(z)|}{|z-x|^{N+2\alpha}} dz \\ &\leq 16^{N+2\alpha} \int_{\mathbb{R}^N} \frac{|w(z)|}{(1+|z|)^{N+2\alpha}} dz \\ &= 16^{N+2\alpha} \|(1+|\cdot|)^{-N-2\alpha} w\|_{L^1(\mathbb{R}^N)}.\end{aligned}$$

By [91, Proposition 2.1.9], for $\beta \in (0, 2\alpha)$, there exists $c_8 > 0$ such that

$$\begin{aligned}\|v\|_{C^\beta(\bar{B}_{1/4})} &\leq c_8(\|v\|_{L^\infty(\mathbb{R}^N)} + \|h + h_1\|_{L^\infty(B_{1/2})}) \\ &\leq c_8(\|w\|_{L^\infty(B_1)} + \|h\|_{L^\infty(B_1)} + \|h_1\|_{L^\infty(B_{1/2})}) \\ &\leq c_9(\|w\|_{L^\infty(B_1)} + \|h\|_{L^\infty(B_1)} + \|(1+|\cdot|)^{-N-2\alpha} w\|_{L^1(\mathbb{R}^N)}),\end{aligned}$$

where $c_9 = 16^{N+2\alpha} c_8$. Combining with $w = v$ in $B_{\frac{3}{4}}$, we obtain (5.18). □

Theorem 5.3.1 *Let $\alpha \in (0,1)$ and $0 < p < \frac{N}{N-2\alpha}$, then the weak solution of (5.1) is a classical solution of (5.4).*

Proof. Let u_k be the weak solution of (5.1). By [40, Theorem 1.1], we have

$$0 \leq u_k = \mathbb{G}_\alpha[k\delta_0] - \mathbb{G}_\alpha[u_k^p] \leq \mathbb{G}_\alpha[k\delta_0]. \tag{5.19}$$

We observe that $\mathbb{G}_\alpha[k\delta_0] = k\mathbb{G}_\alpha[\delta_0] = kG_\alpha(\cdot,0)$ is $C^2_{loc}(\Omega\setminus\{0\})$. Denote by O an open set satisfying $\bar{O}\subset\Omega\setminus B_r$ with $r>0$. Then $\mathbb{G}_\alpha[k\delta_0]$ is uniformly bounded in $\Omega\setminus B_{r/2}$, so is u_k^p by (5.19).

Let $\{g_n\}$ be a sequence nonnegative functions in $C_0^\infty(\mathbb{R}^N)$ such that $g_n\to\delta_0$ in the distribution sense and let w_n be the solution of

$$\begin{aligned} (-\Delta)^\alpha u + u^p &= kg_n \quad \text{in} \quad \Omega,\\ u &= 0 \qquad \text{in} \quad \Omega^c. \end{aligned} \tag{5.20}$$

From [40], we obtain that

$$u_k = \lim_{n\to\infty} w_n \quad \text{a.e. in} \quad \Omega. \tag{5.21}$$

We observe that $0\le w_n = \mathbb{G}_\alpha[kg_n] - \mathbb{G}_\alpha[w_n^p] \le k\mathbb{G}_\alpha[g_n]$ and $\mathbb{G}_\alpha[g_n]$ converges to $\mathbb{G}_\alpha[\delta_0]$ uniformly in any compact set of $\Omega\setminus\{0\}$ and in $L^1(\Omega)$, then there exists $c_{10}>0$ independent of n such that

$$\|w_n\|_{L^\infty(\Omega\setminus B_{r/2})}\le c_{10}k \quad \text{and} \quad \|w_n\|_{L^1(\Omega)}\le c_{10}k.$$

By [88, Corollary 2.4] and Lemma 5.3.1, there exist $\epsilon>0$, $\beta\in(0,2\alpha)$ and positive constants $c_{11},c_{12},c_{13}>0$ independent of n and k, such that

$$\begin{aligned} \|w_n\|_{C^{2\alpha+\epsilon}(O)} &\le c_{11}(\|w_n\|^p_{L^\infty(\Omega\setminus B_{\frac{r}{2}})} + \|kg_n\|_{L^\infty(\Omega\setminus B_{\frac{r}{2}})} + \|w_n\|_{C^\beta(\Omega\setminus B_{\frac{3r}{4}})})\\ &\le c_{12}(\|w_n\|^p_{L^\infty(\Omega\setminus B_{\frac{r}{2}})} + \|w_n\|_{L^\infty(\Omega\setminus B_{\frac{r}{2}})} + \|kg_n\|_{L^\infty(\Omega\setminus B_{\frac{r}{2}})} + \|w_n\|_{L^1(\Omega)})\\ &\le c_{13}(k+k^p). \end{aligned}$$

Therefore, together with (5.21) and the Arzela-Ascoli Theorem, it follows that $u_k\in C^{2\alpha+\frac{\epsilon}{2}}(O)$. This implies that u_k is $C^{2\alpha+\frac{\epsilon}{2}}$ locally in $\Omega\setminus\{0\}$. Therefore, $w_n\to u_k$ and $g_n\to 0$ uniformly in any compact subset of $\Omega\setminus\{0\}$ as $n\to\infty$. We conclude that u_k is a classical solution of (5.4) by Theorem 5.2.2. □

Corollary 5.3.1 *Let u_k be the weak solution of (5.1) and O be an open set satisfying $\bar{O}\subset\Omega\setminus B_r$ with $r>0$. Then there exist $\epsilon>0$ and $c_{14}>0$ independent of k such that*

$$\|u_k\|_{C^{2\alpha+\epsilon}(O)}\le c_{14}(\|u_k\|^p_{L^\infty(\Omega\setminus B_{\frac{r}{2}})} + \|u_k\|_{L^\infty(\Omega\setminus B_{\frac{r}{2}})} + \|u_k\|_{L^1(\Omega)}). \tag{5.22}$$

Proof. By Theorem 5.3.1, u_k is a solution of (5.4). By [88, Corollary 2.4] and Lemma 5.3.1, there exist $\epsilon>0$, $\beta\in(0,2\alpha)$ and $c_{15},c_{16}>0$ independent of k such that

$$\begin{aligned} \|u_k\|_{C^{2\alpha+\epsilon}(O)} &\le c_{15}(\|u_k\|^p_{L^\infty(\Omega\setminus B_{\frac{r}{2}})} + \|u_k\|_{C^\beta(\Omega\setminus B_{\frac{3r}{4}})})\\ &\le c_{16}(\|u_k\|^p_{L^\infty(\Omega\setminus B_{\frac{r}{2}})} + \|u_k\|_{L^\infty(\Omega\setminus B_{\frac{r}{2}})} + \|u_k\|_{L^1(\Omega)}), \end{aligned}$$

which ends the proof. $\square$

Theorem 5.3.2 *Suppose that $p \in (1+\frac{2\alpha}{N}, \frac{N}{N-2\alpha})$ and u_∞ is given by (5.12). Then u_∞ is a classical solution of (5.4) and $u_\infty \le u_s$ in $\Omega \setminus \{0\}$.*

Proof. For $p \in (1+\frac{2\alpha}{N}, \frac{N}{N-2\alpha})$, there exists the solution u_s of (5.4) satisfying (5.5). Then for any $k > 0$, there exists $\sigma > 0$ such that

$$u_k < u_s \quad \text{in} \quad B_\sigma \setminus \{0\}, \tag{5.23}$$

where u_k is the solution of (5.1). By Theorem 5.3.1, u_k is a classical solution of (5.4). It derives by Theorem 5.2.1 that

$$u_k < u_s \quad \text{in} \quad \Omega \setminus \{0\}. \tag{5.24}$$

It infers that $u_\infty \le u_s$ in $\Omega \setminus \{0\}$.

Let O be an open set satisfying $\bar{O} \subset \Omega \setminus B_r$ for $0 < r < dist(0, \partial\Omega)$. We observe that $u_s \in L^1(\Omega)$ and u_s is a continuous in $\Omega \setminus \{0\}$. By (5.22) and (5.24), there exist $c_{17}, c_{18} > 0$ independent of k such that

$$\|u_k\|_{L^1(\Omega)} \le c_{17} \quad \text{and} \quad \|u_k\|_{L^\infty(\Omega \setminus B_r)} \le c_{18}.$$

Thus, there exist $\epsilon > 0$ and $c_{19} > 0$ independent of k such that

$$\|u_k\|_{C^{2\alpha+\epsilon}(O)} \le c_{19}.$$

Together with (5.12) and the Arzela-Ascoli Theorem, it implies that u_∞ belongs to $C^{2\alpha+\frac{\epsilon}{2}}(O)$. Then u_∞ is $C^{2\alpha+\frac{\epsilon}{2}}$ locally in $\Omega \setminus \{0\}$. Therefore, by Theorem 5.2.2, we conclude that u_∞ is a classical solution of (5.4). $\square$

5.4. The limit of u_k as $k \to \infty$

5.4.1. Basic estimates

Let $d = \min\{1, dist(0, \partial\Omega)\}$ and $\{r_k\} \subset (0, \frac{d}{2}]$ be a strictly decreasing sequence of numbers satisfying $\lim_{k\to\infty} r_k = 0$. Denote by $\{z_k\}$ the sequence of functions defined by

$$z_k(x) = \begin{cases} -d^{-N}, & x \in B_{r_k}, \\ |x|^{-N} - d^{-N}, & x \in B^c_{r_k}. \end{cases} \tag{5.25}$$

Lemma 5.4.1 *Let $\{\rho_k\}$ be a strictly decreasing sequence of numbers such that $\frac{r_k}{\rho_k} < \frac{1}{2}$ and $\lim_{k\to\infty} \frac{r_k}{\rho_k} = 0$. Then*

$$(-\Delta)^\alpha z_k(x) \le -c_{1,k}|x|^{-N-2\alpha}, \quad \forall x \in B^c_{\rho_k},$$

where $c_{1,k} = -c_{20}\ln(\frac{r_k}{\rho_k})$ with $c_{20} > 0$ independent of k.

Proof. For any $x \in B^c_{\rho_k}$, there holds

$$\begin{aligned}
(-\Delta)^\alpha z_k(x) &= -\frac{1}{2}\int_{\mathbb{R}^N} \frac{z_k(x+y) + z_k(x-y) - 2z_k(x)}{|y|^{N+2\alpha}}dy \\
&= -\frac{1}{2}\int_{\mathbb{R}^N} \frac{|x+y|^{-N}\chi_{B^c_{r_k}(-x)}(y) + |x-y|^{-N}\chi_{B^c_{r_k}(x)}(y) - 2|x|^{-N}}{|y|^{N+2\alpha}}dy \\
&= -\frac{1}{2}|x|^{-N-2\alpha}\int_{\mathbb{R}^N} \frac{\delta(x,z,r_k)}{|z|^{N+2\alpha}}dz,
\end{aligned}$$

where $\delta(x,z,r_k) = |z+e_x|^{-N}\chi_{B^c_{\frac{r_k}{|x|}}(-e_x)}(z) + |z-e_x|^{-N}\chi_{B^c_{\frac{r_k}{|x|}}(e_x)}(z) - 2$ and $e_x = \frac{x}{|x|}$.

We observe that $\frac{r_k}{|x|} \le \frac{r_k}{\rho_k} < \frac{1}{2}$ and $|z \pm e_x| \ge 1 - |z| \ge \frac{1}{2}$ for $z \in B_{\frac{1}{2}}$, then there exists $c_{21} > 0$ such that

$$|\delta(x,z,r_k)| = ||z+e_x|^{-N} + |z-e_x|^{-N} - 2| \le c_{21}|z|^2.$$

Therefore,

$$\begin{aligned}
|\int_{B_{\frac{1}{2}}(0)} \frac{\delta(x,z,r_k)}{|z|^{N+2\alpha}}dz| &\le \int_{B_{\frac{1}{2}}(0)} \frac{|\delta(x,z,r_k)|}{|z|^{N+2\alpha}}dz \\
&\le c_{21}\int_{B_{\frac{1}{2}}(0)} |z|^{2-N-2\alpha}dz \le c_{22},
\end{aligned}$$

where $c_{22} > 0$ is independent of k.

For $z \in B_{\frac{1}{2}}(-e_x)$, there holds

$$\begin{aligned}
\int_{B_{\frac{1}{2}}(-e_x)} \frac{\delta(x,z,r_k)}{|z|^{N+2\alpha}}dz &\ge \int_{B^c_{\frac{1}{2}}(-e_x)} \frac{|z+e_x|^{-N}\chi_{B_{\frac{r_k}{|x|}}(-e_x)}(z) - 2}{|z|^{N+2\alpha}}dz \\
&\ge c_{23}\int_{B_{\frac{1}{2}}(0)\setminus B_{\frac{r_k}{|x|}}(0)} (|z|^{-N} - 2)dz \\
&\ge -c_{24}\ln\left(\frac{r_k}{|x|}\right) \ge -c_{24}\ln\left(\frac{r_k}{\rho_k}\right),
\end{aligned}$$

where $c_{23}, c_{24} > 0$ are independent of k.
For $z \in B_{\frac{1}{2}}(e_x)$, we have that

$$\int_{B_{\frac{1}{2}}(e_x)} \frac{\delta(x,z,r_k)}{|z|^{N+2\alpha}} dz = \int_{B_{\frac{1}{2}}(-e_x)} \frac{\delta(x,z,r_k)}{|z|^{N+2\alpha}} dz.$$

Finally, for $z \in O := \mathbb{R}^N \setminus (B_{\frac{1}{2}}(0) \cup B_{\frac{1}{2}}(-e_x) \cup B_{\frac{1}{2}}(e_x))$, we obtain that

$$|\int_O \frac{\delta(x,z,r_k)}{|z|^{N+2\alpha}} dz| \leq c_{25} \int_{B^c_{\frac{1}{2}}(0)} \frac{|z|^{-N}+1}{|z|^{N+2\alpha}} dz \leq c_{26},$$

where $c_{25}, c_{26} > 0$ are independent of k.
Combining these inequalities we obtain that there exists $c_{20} > 0$ independent of k such that for $x \in B^c_{\rho_k}$.

$$(-\Delta)^\alpha z_k(x)|x|^{N+2\alpha} \leq c_{20} \ln\left(\frac{r_k}{\rho_k}\right) := -c_{1,k},$$

which ends the proof. $\square$

Proposition 5.4.1 *Assume that*

$$\frac{2\alpha}{N-2\alpha} < 1 + \frac{2\alpha}{N}, \quad \text{máx}\{1, \frac{2\alpha}{N-2\alpha}\} < p < 1 + \frac{2\alpha}{N} \tag{5.26}$$

and z_k is defined by (5.25) with $r_k = k^{-\frac{p-1}{N-(N-2\alpha)p}} (\ln k)^{-2}$. Then there exists $k_0 > 3$ such that for any $k \geq k_0$,

$$u_k \geq c_{2,k}^{\frac{1}{p-1}} z_k \quad \text{in} \quad B_d, \tag{5.27}$$

where $c_{2,k} = c_{20} \ln\ln k$ and the constant c_{20} is from Lemma 5.4.1.

Proof. For $p \in (\text{máx}\{1, \frac{2\alpha}{N-2\alpha}\}, 1 + \frac{2\alpha}{N})$, it follows by (**??**) and Lemma 5.2.1(iii) that there exist $\rho_0 \in (0,d)$ and $c_{27}, c_{28} > 0$ independent of k such that, for $x \in \bar{B}_{\rho_0} \setminus \{0\}$,

$$\begin{aligned} u_k(x) &\geq k\mathbb{G}_\alpha[\delta_0](x) - k^p \mathbb{G}_\alpha[(\mathbb{G}_\alpha[\delta_0])^p](x) \\ &\geq c_{27} k|x|^{-N+2\alpha} - c_{28} k^p |x|^{-(N-2\alpha)p+2\alpha} \\ &= c_{27} k|x|^{-N+2\alpha}(1 - \frac{c_{28}}{c_{27}} k^{p-1} |x|^{N-(N-2\alpha)p}). \end{aligned}$$

We choose

$$\rho_k = k^{-\frac{p-1}{N-(N-2\alpha)p}} (\ln k)^{-1}, \tag{5.28}$$

then there exists $k_1 > 3$ such that for $k \geq k_1$,

$$\begin{aligned} u_k(x) &\geq c_{27}k|x|^{-N+2\alpha}(1 - \frac{c_{28}}{c_{27}}k^{p-1}\rho_k^{N-(N-2\alpha)p}) \\ &\geq \frac{c_{27}}{2}k|x|^{-N+2\alpha}, \quad x \in \bar{B}_{\rho_k} \setminus \{0\}. \end{aligned} \tag{5.29}$$

Since $p < 1 + \frac{2\alpha}{N}$, then $1 - \frac{2\alpha(p-1)}{N-(N-2\alpha)p} > 0$ and there exists $k_0 \geq k_1$ such that

$$\frac{c_{27}}{2}kr_k^{2\alpha} \geq (c_{20}\ln\ln k)^{\frac{1}{p-1}}, \tag{5.30}$$

for $k \geq k_0$. Thus,

$$\frac{c_{27}}{2}k|x|^{2\alpha} \geq (c_{20}\ln\ln k)^{\frac{1}{p-1}}, \quad x \in \bar{B}_{\rho_k} \setminus B_{r_k}.$$

Together with (5.25) and (5.29), we derive that

$$u_k(x) \geq (c_{20}\ln\ln k)^{\frac{1}{p-1}}z_k(x), \quad x \in \bar{B}_{\rho_k} \setminus B_{r_k},$$

for $k \geq k_0$. Furthermore, it is clear that

$$(c_{20}\ln\ln k)^{\frac{1}{p-1}}z_k \leq 0 \leq u_k \quad \text{in} \quad B_{r_k} \cup B_d^c.$$

Set $c_{2,k} = c_{20}\ln\ln k$, then by Lemma 5.4.1,

$$(-\Delta)^\alpha c_{2,k}^{\frac{1}{p-1}}z_k(x) + c_{2,k}^{\frac{p}{p-1}}z_k(x)^p \leq c_{2,k}^{\frac{p}{p-1}}|x|^{-N-2\alpha}(-1 + |x|^{N+2\alpha-Np}) \leq 0,$$

for any $x \in B_d \setminus B_{\rho_k}$, since $N + 2\alpha - Np > 0$ and $d \leq 1$. Applying Theorem 5.2.1, we infer that

$$c_{2,k}^{\frac{1}{p-1}}z_k(x) \leq u_k(x), \quad \forall x \in B_d,$$

which ends the proof. □

Proposition 5.4.2 *Assume that*

$$1 < \frac{2\alpha}{N-2\alpha} < 1 + \frac{2\alpha}{N} \;\; \textit{and} \;\; p = \frac{2\alpha}{N-2\alpha} \tag{5.31}$$

and z_k is defined by (5.25) with $r_k = k^{-\frac{2\alpha}{N(N-2\alpha)}}(\ln k)^{-3}$. Then there exists $k_0 > 3$ such that (5.27) holds for all $k \geq k_0$.

Proof. By (5.9) and Lemma 5.2.1(ii), there exist $\rho_0 \in (0, d)$ and $c_{30}, c_{31} > 0$ independent of k such that for $x \in \bar{B}_{\rho_0} \setminus \{0\}$,

$$u_k(x) \geq c_{30}k|x|^{-N+2\alpha} + c_{31}k^p \ln|x| = c_{30}k|x|^{-N+2\alpha}[1 + \frac{c_{31}}{c_{30}}k^{p-1}|x|^{N-2\alpha}\ln|x|].$$

Choosing $\rho_k = k^{-\frac{2\alpha}{N(N-2\alpha)}}(\ln k)^{-2}$, there exists $k_1 > 3$ such that for $k \geq k_1$, $1 + \frac{c_{31}}{c_{30}} k^{p-1} \rho_k^{N-2\alpha} \ln \rho_k \geq \frac{1}{2}$ and

$$u_k(x) \geq \frac{c_{30}}{2} k|x|^{-N+2\alpha}, \qquad \forall x \in \bar{B}_{\rho_k} \setminus \{0\}. \tag{5.32}$$

Since $\frac{2\alpha}{N-2\alpha} < 1 + \frac{2\alpha}{N}$, there holds $1 - \frac{4\alpha^2}{N(N-2\alpha)} > 0$ and there exists $k_0 \geq k_1$ such that

$$\frac{c_{30}}{2} k r_k^{2\alpha} = \frac{c_{30}}{2} k^{1-\frac{4\alpha^2}{N(N-2\alpha)}} (\ln k)^{-6\alpha} \geq (c_{20} \ln \ln k)^{\frac{1}{p-1}}$$

for $k \geq k_0$. The remaining of the proof is the same as in Proposition 5.4.1. $\square$

Proposition 5.4.3 *Assume that*

$$1 < \frac{2\alpha}{N-2\alpha} \leq 1 + \frac{2\alpha}{N} \ \ \textit{and} \ \ 1 < p < \frac{2\alpha}{N-2\alpha}, \tag{5.33}$$

or

$$1 + \frac{2\alpha}{N} < \frac{2\alpha}{N-2\alpha} \ \ \textit{and} \ \ 1 < p < \frac{N}{2\alpha}, \tag{5.34}$$

and z_k is defined by (5.25) with $r_k = k^{-\frac{p-1}{N-2\alpha}}(\ln k)^{-1}$. Then there exists $k_0 > 3$ such that (5.27) holds for all $k \geq k_0$.

Proof. By (5.9) and Lemma 5.2.1(i), there exist $\rho_0 \in (0, d)$ and $c_{33}, c_{34} > 0$ independent of k such that for $x \in \bar{B}_{\rho_0} \setminus \{0\}$,

$$\begin{aligned} u_k(x) &\geq c_{33} k|x|^{-N+2\alpha} - c_{34} k^p \\ &= c_{33} k|x|^{-N+2\alpha} \left(1 - \frac{c_{34}}{c_{33}} k^{p-1} |x|^{N-2\alpha}\right). \end{aligned}$$

Choosing $\rho_k = k^{-\frac{p-1}{N-2\alpha}}$, there exists $k_1 > 3$ such that for $k \geq k_1$, $1 - \frac{c_{34}}{c_{33}} k^{p-1} \rho_k^{N-2\alpha} \geq \frac{1}{2}$ and

$$u_k(x) \geq \frac{c_{33}}{2} k|x|^{-N+2\alpha}, \quad \forall x \in \bar{B}_{\rho_k} \setminus \{0\}. \tag{5.35}$$

We observe that if $\frac{2\alpha}{N-2\alpha} \leq 1 + \frac{2\alpha}{N}$, then $1 + \frac{2\alpha}{N} \leq \frac{N}{2\alpha}$. It infers by (5.33), (5.34) that $p < \frac{N}{2\alpha}$, thus $1 - (p-1)\frac{2\alpha}{N-2\alpha} > 0$. Therefore there exists $k_0 \geq k_1$ such that

$$\frac{c_{33}}{2} k r_k^{2\alpha} = \frac{c_{33}}{2} k^{1-(p-1)\frac{2\alpha}{N-2\alpha}} (\ln k)^{-2\alpha} \geq (c_{20} \ln \ln k)^{\frac{1}{p-1}} = c_{2,k}^{\frac{1}{p-1}}$$

for $k \geq k_0$. The remaining of the proof is the same as in Proposition 5.4.1. $\square$

5.4.2. u_∞ blows up in whole Ω

Proof of Theorem 5.1.1(i) **and Theorem 5.1.2**(i). We first prove the case $p \in (0,1]$. We observe that $\mathbb{G}_\alpha[\delta_0], \mathbb{G}_\alpha[(\mathbb{G}_\alpha[\delta_0])^p] > 0$ in Ω. It derives by (5.9) that

$$u_k \geq k\mathbb{G}_\alpha[\delta_0] - k^p \mathbb{G}_\alpha[(\mathbb{G}_\alpha[\delta_0])^p].$$

Then $\lim_{k\to\infty} u_k = \infty$ in Ω for $p \in (0,1)$. For $p = 1$, we see that $u_k = ku_1$. Then $\lim_{k\to\infty} u_k = \infty$ in Ω by the fact that $u_1 > 0$ in Ω.

We next prove $u_\infty = \infty$ in Ω when $p \in (1, 1+\frac{2\alpha}{N})$ if $1+\frac{2\alpha}{N} \geq \frac{2\alpha}{N-2\alpha}$ and $p \in (1, \frac{N}{2\alpha})$ if $1 + \frac{2\alpha}{N} < \frac{2\alpha}{N-2\alpha}$. The proof is divided into two steps.

Step 1: We claim that $u_\infty = \infty$ in B_d. We observe that for $1 + \frac{2\alpha}{N} > \frac{2\alpha}{N-2\alpha}$, Propositions 5.4.1, 5.4.2, 5.4.3 cover the region $p \in (\max\{1, \frac{2\alpha}{N-2\alpha}\}, 1 + \frac{2\alpha}{N})$, the region $1 < \frac{2\alpha}{N-2\alpha} < 1 + \frac{2\alpha}{N}$ along with $p = \frac{2\alpha}{N-2\alpha}$ and the region $1 < \frac{2\alpha}{N-2\alpha} < 1 + \frac{2\alpha}{N}$ along with $p \in (1, \frac{2\alpha}{N-2\alpha})$ respectively. For $\frac{2\alpha}{N-2\alpha} = 1 + \frac{2\alpha}{N}$, Proposition 5.4.3 covers the region $p \in (1, \frac{2\alpha}{N-2\alpha})$. So it covers $p \in (1, 1 + \frac{2\alpha}{N}]$ in Theorem 5.1.1 part (i). When $\frac{2\alpha}{N-2\alpha} > 1 + \frac{2\alpha}{N}$, Proposition 5.4.3 covers $p \in (1, \frac{N}{2\alpha})$ in Theorem 5.1.2 part (i). Therefore, we have that

$$u_\infty \geq c_{2,k}^{\frac{1}{p-1}} z_k \quad \text{in} \quad B_d$$

and since for any $x \in B_d \setminus \{0\}$, $\lim_{k\to\infty} c_{2,k}^{\frac{1}{p-1}} z_k(x) = \infty$, we derive that

$$u_\infty = \infty \quad \text{in} \quad B_d. \tag{5.36}$$

Step 2: We claim that $u_\infty = \infty$ in Ω. By the fact of $u_\infty = \infty$ in B_d and $u_{k+1} \geq u_k$ in Ω, then for any $n > 1$ there exists $k_n > 0$ such that $u_{k_n} \geq n$ in B_d. For any $x_0 \in \Omega \setminus B_d$, there exists $\rho > 0$ such that $\bar{B}_\rho(x_0) \subset \Omega \cap B^c_{d/2}$. We denote by w_n the solution of

$$\begin{aligned} (-\Delta)^\alpha u + u^p = 0 \quad & \text{in} \quad B_\rho(x_0), \\ u = 0 \quad & \text{in} \quad B^c_\rho(x_0) \setminus B_{d/2}, \\ u = n \quad & \text{in} \quad B_{d/2}. \end{aligned} \tag{5.37}$$

Then by Theorem 5.2.1, we have that

$$u_{k_n} \geq w_n. \tag{5.38}$$

Let $v_n = w_n - n\chi_{B_{d/2}}$, then $v_n = w_n$ in $B_\rho(x_0)$ and

$$\begin{aligned} (-\Delta)^\alpha v_n(x) + v_n^p(x) &= (-\Delta)^\alpha w_n(x) - n(-\Delta)^\alpha \chi_{B_{d/2}}(x) + w_n^p(x) \\ &= n\int_{B_{d/2}} \frac{dy}{|y-x|^{N+2\alpha}}, \qquad \forall x \in B_\rho(x_0), \end{aligned}$$

that is, v_n is a solution of

$$\begin{aligned} (-\Delta)^\alpha u + u^p = n\int_{B_{d/2}}\frac{dy}{|y-x|^{N+2\alpha}} \quad & \text{in} \quad B_\rho(x_0), \\ u = 0 \quad & \text{in} \quad B^c_\rho(x_0). \end{aligned} \tag{5.39}$$

By direct computation,

$$\frac{1}{c_{35}} \le \int_{B_{d/2}}\frac{dy}{|y-x|^{N+2\alpha}} \le c_{35}, \quad \forall x\in B_\rho(x_0),$$

for some $c_{35} > 1$.

Let η_1 be the solution of

$$\begin{aligned} (-\Delta)^\alpha u = 1 \quad & \text{in} \quad B_\rho(x_0), \\ u = 0 \quad & \text{in} \quad B^c_\rho(x_0) \end{aligned}$$

and then $(\frac{n}{2c_{35}\,\text{máx}\,\eta_1})^{\frac{1}{p}}\eta_1$ is sub solution of (5.39) for n large enough. Then it infers by Theorem 5.2.1 that

$$v_n \ge (\frac{n}{2c_{35}\,\text{máx}\,\eta_1})^{\frac{1}{p}}\eta_1, \qquad \forall x\in B_\rho(x_0),$$

which implies that

$$w_n \ge (\frac{n}{2c_{35}\,\text{máx}\,\eta_1})^{\frac{1}{p}}\eta_1, \qquad \forall x\in B_\rho(x_0).$$

Then by (5.38),

$$\lim_{n\to\infty} u_{k_n}(x_0) \ge \lim_{n\to\infty} w_n(x_0) = \infty.$$

Since x_0 is arbitrary in $\Omega\setminus B_d$ and combine with (5.36), it implies that $u_\infty = \infty$ in Ω. □

5.4.3. u_∞ is a strongly singular solution

Proposition 5.4.4 *Let $r_0 = dist(0,\partial\Omega)$. Then*
(i) if $\text{máx}\{1+\frac{2\alpha}{N}, \frac{2\alpha}{N-2\alpha}\} < p < \frac{N}{N-2\alpha}$, *there exist $R_0\in(0,r_0)$ and $c_{36} > 0$ such that*

$$u_\infty(x) \ge c_{36}|x|^{-\frac{2\alpha}{p-1}}, \qquad \forall x\in B_{R_0}\setminus\{0\}; \tag{5.40}$$

(ii) if $\frac{2\alpha}{N-2\alpha} > 1+\frac{2\alpha}{N}$ and $p = \frac{2\alpha}{N-2\alpha}$, there exist $R_0\in(0,r_0)$ and $c_{37} > 0$ such that

$$u_\infty(x) \ge \frac{c_{37}}{(1+|\log(|x|)|)^{\frac{1}{p-1}}}|x|^{-\frac{p(N-2\alpha)}{p-1}}, \quad \forall x\in B_{R_0}\setminus\{0\}; \tag{5.41}$$

(iii) if $\frac{2\alpha}{N-2\alpha} > 1+\frac{2\alpha}{N}$ *and* $p\in(1+\frac{2\alpha}{N},\frac{2\alpha}{N-2\alpha})$*, there exist* $R_0\in(0,r_0)$ *and* $c_{38}>0$ *such that*

$$u_\infty(x)\geq c_{38}|x|^{-\frac{p(N-2\alpha)}{p-1}},\qquad \forall x\in B_{R_0}\setminus\{0\}. \tag{5.42}$$

Proof. (i) Using (5.9) and Lemma 5.2.1(i) with $\max\{1+\frac{2\alpha}{N},\frac{2\alpha}{N-2\alpha}\}<p<\frac{N}{N-2\alpha}$, then there exist $\rho_0\in(0,r_0)$ and $c_{39},c_{40}>0$ such that

$$u_k(x)\geq c_{39}k|x|^{-N+2\alpha}-c_{40}k^p|x|^{-(N-2\alpha)p+2\alpha},\qquad \forall x\in B_{\rho_0}\setminus\{0\}. \tag{5.43}$$

Set

$$\rho_k=\left(2^{(N-2\alpha)p-2\alpha-1}\frac{c_{40}}{c_{39}}k^{p-1}\right)^{\frac{1}{(N-2\alpha)(p-1)-2\alpha}}. \tag{5.44}$$

Since $(N-2\alpha)(p-1)-2\alpha<0$, there holds $\lim_{k\to\infty}\rho_k=0$. Let $k_0>0$ such that $\rho_{k_0}\leq\rho_0$, then for $x\in B_{\rho_k}\setminus B_{\frac{\rho_k}{2}}$, we have that

$$\begin{aligned} c_{40}k^p|x|^{-(N-2\alpha)p+2\alpha} &\leq c_{40}k^p\left(\frac{\rho_k}{2}\right)^{-(N-2\alpha)p+2\alpha}\\ &= \frac{c_{39}}{2}k\rho_k^{-N+2\alpha}\leq\frac{c_{39}}{2}k|x|^{-N+2\alpha}\end{aligned}$$

and

$$k=\left(2^{(N-2\alpha)p-2\alpha-1}\frac{c_{40}}{c_{39}}\right)^{-\frac{1}{p-1}}\rho_k^{N-2\alpha-\frac{2\alpha}{p-1}}\geq c_{41}|x|^{N-2\alpha-\frac{2\alpha}{p-1}},$$

where $c_{41}=\left(2^{(N-2\alpha)p-2\alpha-1}\frac{c_{40}}{c_{39}}\right)^{-\frac{1}{p-1}}2^{(N-2\alpha)(p-1)-2\alpha-1}$. Combining with (5.40), we obtain that

$$\begin{aligned} u_k(x) &= c_{39}k|x|^{-N+2\alpha}-c_{40}k^p|x|^{-(N-2\alpha)p+2\alpha}\\ &\geq \frac{c_{39}}{2}k|x|^{-N+2\alpha}\geq c_{42}|x|^{-\frac{2\alpha}{p-1}},\end{aligned} \tag{5.45}$$

for $x\in B_{\rho_k}\setminus B_{\frac{\rho_k}{2}}$, where $c_{42}=c_{39}c_{41}/2$ is independent of k. By (5.44), we can choose a sequence $\{k_n\}\subset[1,+\infty)$ such that

$$\rho_{k_{n+1}}\geq\frac{1}{2}\rho_{k_n}.$$

For any $x\in B_{\rho_{k_0}}\setminus\{0\}$, there exists k_n such that $x\in B_{\rho_{k_n}}\setminus B_{\frac{\rho_{k_n}}{2}}$, then by (5.45),

$$u_{k_n}(x)\geq c_{42}|x|^{-\frac{2\alpha}{p-1}}.$$

Together with $u_{k+1} > u_k$, we derive that

$$u_\infty(x) \geq c_{42}|x|^{-\frac{2\alpha}{p-1}}, \quad x \in B_{\rho_{k_0}} \setminus \{0\}.$$

(ii) By (5.9) and Lemma 5.2.1(ii) with $p = \frac{2\alpha}{N-2\alpha}$, there exist $\rho_0 \in (0, r_0)$ and $c_{43}, c_{44} > 0$ such that

$$u_k(x) \geq c_{43}k|x|^{-N+2\alpha} - c_{44}k^p|\ln|x||, \quad x \in B_{\rho_0} \setminus \{0\}. \tag{5.46}$$

Let $\{\rho_k\}$ be a sequence of real numbers with value in $(0,1)$ and such that

$$c_{44}k^{p-1}|\ln\left(\frac{\rho_k}{2}\right)| = \frac{c_{43}}{2}\rho_k^{-N+2\alpha}. \tag{5.47}$$

Then $\lim_{k\to\infty}\rho_k = 0$ and there exists $k_0 > 0$ such that $\rho_{k_0} \leq \rho_0$. Thus, for any $x \in B_{\rho_k} \setminus B_{\frac{\rho_k}{2}}$ and $k \geq k_0$,

$$c_{43}k^p|\ln|x|| \leq c_{44}k^p|\ln\left(\frac{\rho_k}{2}\right)| = \frac{c_{43}}{2}k\rho_k^{-N+2\alpha} \leq \frac{c_{43}}{2}k|x|^{-N+2\alpha}.$$

For any $x \in B_{\rho_k} \setminus B_{\frac{\rho_k}{2}}$, we derive from (5.47) that

$$k = \left(\frac{c_{44}}{2c_{43}}\right)^{-\frac{1}{p-1}}\left(\frac{\rho_k^{-N+2\alpha}}{1+|\ln\rho_k|}\right)^{\frac{1}{p-1}} \geq c_{45}\frac{|x|^{-\frac{N-2\alpha}{p-1}}}{(1+|\ln|x||)^{\frac{1}{p-1}}},$$

where $c_{45} = 2^{-\frac{N-2\alpha}{p-1}}(\frac{c_{44}}{2c_{43}})^{-\frac{1}{p-1}}$. As a consequence,

$$\begin{aligned} u_k(x) &\geq c_{43}k|x|^{-N+2\alpha} - c_{44}k^p|\ln|x|| \\ &\geq \frac{c_{43}}{2}k|x|^{-N+2\alpha} \geq c_{46}\frac{|x|^{-\frac{p(N-2\alpha)}{p-1}}}{(1+|\ln|x||)^{\frac{1}{p-1}}}, \end{aligned} \tag{5.48}$$

where $c_{46} = c_{43}c_{45}/2$ is independent of k.

By (5.47), we can choose a sequence $\{k_n\} \subset [1, +\infty)$ such that

$$\rho_{k_{n+1}} \geq \frac{1}{2}\rho_{k_n}.$$

Then for any $x \in B_{\rho_{k_0}} \setminus \{0\}$, there exists k_n such that $x \in B_{\rho_{k_n}} \setminus B_{\frac{\rho_{k_n}}{2}}$. By (5.48), there holds

$$u_{k_n}(x) \geq c_{46}\frac{|x|^{-\frac{p(N-2\alpha)}{p-1}}}{(1+|\ln|x||)^{\frac{1}{p-1}}}.$$

Together with $u_{k+1} > u_k$, we infer

$$u_\infty(x) \geq c_{46} \frac{|x|^{-\frac{p(N-2\alpha)}{p-1}}}{(1+|\ln|x||)^{\frac{1}{p-1}}}, \qquad \forall x \in B_{\rho_{k_0}} \setminus \{0\}.$$

(iii) By (5.9) and Lemma 5.2.1(iii) with $p \in (1+\frac{2\alpha}{N}, \frac{2\alpha}{N-2\alpha})$, there exist $\rho_0 \in (0, r_0)$ and $c_{47}, c_{48} > 0$ such that

$$u_k(x) \geq c_{47} k|x|^{-N+2\alpha} - c_{48} k^p, \qquad \forall x \in B_{\rho_0} \setminus \{0\}. \tag{5.49}$$

Set

$$\rho_k = \left(\frac{c_{48}}{2c_{47}} k^{p-1}\right)^{-\frac{1}{N-2\alpha}}, \tag{5.50}$$

then $\lim_{k\to\infty} \rho_k = 0$ and there exists $k_0 > 0$ such that $\rho_{k_0} \leq \rho_0$. Therefore, for $x \in B_{\rho_k} \setminus B_{\frac{\rho_k}{2}}$ and $k \geq k_0$, there holds

$$c_{48} k^p = \frac{c_{47}}{2} k \rho_k^{-N+2\alpha} \leq \frac{c_{47}}{2} k|x|^{-N+2\alpha},$$

which, along with (5.50), yields

$$k = \left(\frac{c_{48}}{2c_{47}}\right)^{-\frac{1}{p-1}} \rho_k^{-\frac{N-2\alpha}{p-1}} \geq c_{49}|x|^{-\frac{N-2\alpha}{p-1}},$$

where $c_{49} = 2^{-\frac{N-2\alpha}{p-1}} (\frac{c_{48}}{2c_{47}})^{-\frac{1}{p-1}}$. Thus,

$$\begin{aligned} u_k(x) \geq c_{47} k|x|^{-N+2\alpha} - c_{48} k^p \ &\geq \ \frac{c_{47}}{2} k|x|^{-N+2\alpha} \\ &\geq \ c_{50}|x|^{-\frac{p}{p-1}(N-2\alpha)}, \end{aligned} \tag{5.51}$$

where $c_{50} = c_{47}c_{49}/2$ is independent of k.

By (5.50), we can choose a sequence $\{k_n\} \subset [1, +\infty)$ such that

$$\rho_{k_{n+1}} \geq \frac{1}{2} \rho_{k_n},$$

Then for any $x \in B_{\rho_{k_0}} \setminus \{0\}$, there exists k_n such that $x \in B_{\rho_{k_n}} \setminus B_{\frac{\rho_{k_n}}{2}}$ and then by (5.51),

$$u_{k_n}(x) \geq c_{50}|x|^{-\frac{p(N-2\alpha)}{p-1}}.$$

Together with $u_{k+1} > u_k$, we have

$$u_\infty(x) \geq c_{50}|x|^{-\frac{p(N-2\alpha)}{p-1}}, \qquad \forall x \in B_{\rho_{k_0}} \setminus \{0\},$$

which ends the proof. $\square$

Proof of Theorem 5.1.1 (ii) **and Theorem 5.1.2** (iv)**.** By Theorem 5.3.2, we obtain that u_∞ is a classical solution of (5.4) and $u_\infty \leq u_s$ in $\Omega \setminus \{0\}$. By Proposition 5.4.4(i), there exist $c_{36}, R_0 > 0$ such that

$$c_{36}|x|^{-\frac{2\alpha}{p-1}} \leq u_\infty(x) \leq u_s(x), \quad x \in B_{R_0} \setminus \{0\}.$$

Then $u_\infty = u_s$, since u_s is unique in the class of solutions satisfying (5.6).

$\square$

Proof of Theorem 5.1.2 (ii) **and** (iii)**.** By Theorem 5.3.2, u_∞ is a classical solution of (**??**) and it satisfies

$$u_\infty \leq u_s \quad \text{in} \quad \Omega \setminus \{0\}.$$

Then (5.15) and (5.14) follow by Proposition 5.4.4 (ii) and (iii), respectively.

$\square$

Capítulo 6

Semilinear fractional elliptic equations with gradient nonlinearity involving measures

Abstract: in this chapter [1], we study the existence of solutions to the fractional elliptic equation (E1) $(-\Delta)^\alpha u + \epsilon g(|\nabla u|) = \nu$ in a bounded regular domain Ω of $\mathbb{R}^N (N \geq 2)$, subject to the condition (E2) $u = 0$ in Ω^c, where $\epsilon = 1$ or -1, $(-\Delta)^\alpha$ denotes the fractional Laplacian with $\alpha \in (1/2, 1)$, ν is a Radon measure and $g : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is a continuous function. We prove the existence of weak solutions for problem (E1)-(E2) when g is subcritical. Furthermore, the asymptotic behavior and uniqueness of solutions are described when $\epsilon = 1$, ν is a Dirac mass and $g(s) = s^p$ with $p \in (0, \frac{N}{N-2\alpha+1})$.

6.1. Introduction

Let $\Omega \subset \mathbb{R}^N (N \geq 2)$ be an open bounded C^2 domain and $g : \mathbb{R}_+ \mapsto \mathbb{R}_+$ be a continuous function. The purpose of this chapter is to study the existence of weak solutions to the semilinear fractional elliptic problem with $\alpha \in (1/2, 1)$,

$$\begin{aligned} (-\Delta)^\alpha u + \epsilon g(|\nabla u|) &= \nu \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{in} \quad \Omega^c, \end{aligned} \tag{6.1}$$

[1] This chapter is based on the paper: H. Chen and L. Véron, *Semilinear fractional elliptic equations with gradient nonlinearity involving measures,* accepted by Journal of Functional Analysis (2013).

where $\epsilon = 1$ or -1 and $\nu \in \mathfrak{M}(\Omega, \rho^\beta)$ with $\beta \in [0, 2\alpha - 1)$. Here $\rho(x) = dist(x, \Omega^c)$ and $\mathfrak{M}(\Omega, \rho^\beta)$ is the space of Radon measures in Ω satisfying

$$\int_\Omega \rho^\beta d|\nu| < +\infty. \tag{6.2}$$

In particular, we denote $\mathfrak{M}^b(\Omega) = \mathfrak{M}(\Omega, \rho^0)$. The associated positive cones are respectively $\mathfrak{M}_+(\Omega, \rho^\beta)$ and $\mathfrak{M}_+^b(\Omega)$. According to the value of ϵ, we speak of an absorbing nonlinearity the case $\epsilon = 1$ and a source nonlinearity the case $\epsilon = -1$. The operator $(-\Delta)^\alpha$ is the fractional Laplacian defined as

$$(-\Delta)^\alpha u(x) = \lim_{\varepsilon \to 0^+} (-\Delta)_\varepsilon^\alpha u(x),$$

where for $\varepsilon > 0$,

$$(-\Delta)_\varepsilon^\alpha u(x) = -\int_{\mathbb{R}^N} \frac{u(z) - u(x)}{|z - x|^{N+2\alpha}} \chi_\varepsilon(|x - z|) dz \tag{6.3}$$

and

$$\chi_\varepsilon(t) = \begin{cases} 0, & \text{if} \quad \mathrm{t} \in [0, \varepsilon], \\ 1, & \text{if} \quad \mathrm{t} > \varepsilon. \end{cases}$$

In a pioneering work, Brezis [16] (also see Bénilan and Brezis [10]) studied the existence and uniqueness of the solution to the semilinear Dirichlet elliptic problem

$$\begin{aligned} -\Delta u + h(u) = \nu \quad & \text{in} \quad \Omega, \\ u = 0 \quad & \text{on} \quad \partial\Omega, \end{aligned} \tag{6.4}$$

where ν is a bounded measure in Ω and the function h is nondecreasing, positive on $(0, +\infty)$ and satisfies that

$$\int_1^{+\infty} (h(s) - h(-s)) s^{-2\frac{N-1}{N-2}} ds < +\infty.$$

Later on, Véron [101] improved this result in replacing the Laplacian by more general uniformly elliptic second order differential operator, where $\nu \in \mathfrak{M}(\Omega, \rho^\beta)$ with $\beta \in [0, 1]$ and h is a nondecreasing function satisfying

$$\int_1^{+\infty} (h(s) - h(-s)) s^{-2\frac{N+\beta-1}{N+\beta-2}} ds < +\infty.$$

The general semilinear elliptic problems involving measures such as the equations involving boundary measures have been intensively studied; it was initiated by Gmira and Véron [62] and then this subject has being extended in various ways, see

[13, 14, 76, 77, 78, 79] for details and [80] for a general panorama. In a recent work, Nguyen-Phuoc and Véron [82] obtained the existence of solutions to the viscous Hamilton-Jacobi equation

$$\begin{aligned} -\Delta u + h(|\nabla u|) &= \nu \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{6.5}$$

when $\nu \in \mathfrak{M}^b(\Omega)$, h is a continuous nondecreasing function vanishing at 0 which satisfies

$$\int_1^{+\infty} h(s)s^{-\frac{2N-1}{N-1}}ds < +\infty.$$

More recently, Bidaut-Véron, García-Huidobro and Véron in [12] studied the existence of solutions to the Dirichlet problem

$$\begin{aligned} -\Delta_p u + \epsilon|\nabla u|^q &= \nu, \quad \text{in} \quad \Omega, \\ u &= 0, \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{6.6}$$

with $1 < p \leq N$, $\epsilon = 1$ or -1, $q > 0$ and $\nu \in \mathfrak{M}^b(\Omega)$.

During the last years there has also been a renewed and increasing interest in the study of linear and nonlinear integro-differential operators, especially, the fractional Laplacian, motivated by great applications in physics and by important links on the theory of Lévy processes, refer to [26, 39, 40, 33, 54, 88, 91, 92]. Many estimates of its Green kernel and generation formula can be found in the references [15, 37]. Recently, Chen and Véron [40] studied the semilinear fractional elliptic equation

$$\begin{aligned} (-\Delta)^\alpha u + h(u) &= \nu \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{in} \quad \Omega^c, \end{aligned} \tag{6.7}$$

where $\nu \in \mathfrak{M}(\Omega, \rho^\beta)$ with $\beta \in [0, \alpha]$. We proved the existence and uniqueness of the solution to (6.7) when the function h is nondecreasing and satisfies

$$\int_1^{+\infty} (h(s) - h(-s))s^{-1-k_{\alpha,\beta}}ds < +\infty,$$

where

$$k_{\alpha,\beta} = \begin{cases} \frac{N}{N-2\alpha}, & \text{if} \quad \beta \in [0, \frac{\mathrm{N}-2\alpha}{\mathrm{N}}\alpha], \\ \frac{N+\alpha}{N-2\alpha+\beta}, & \text{if} \quad \beta \in (\frac{\mathrm{N}-2\alpha}{\mathrm{N}}\alpha, \alpha]. \end{cases} \tag{6.8}$$

Our interest in this chapter is to investigate the existence of weak solutions to fractional equations involving nonlinearity in the gradient term and with Radon measure. In order the fractional Laplacian be the dominant operator in terms of order of differentiation, it is natural to assume that $\alpha \in (1/2, 1)$.

Definition 6.1.1 *We say that u is a weak solution of (6.1), if $u \in L^1(\Omega)$, $|\nabla u| \in L^1_{loc}(\Omega)$, $g(|\nabla u|) \in L^1(\Omega, \rho^\alpha dx)$ and*

$$\int_\Omega [u(-\Delta)^\alpha \xi + \epsilon g(|\nabla u|)\xi] dx = \int_\Omega \xi d\nu, \qquad \forall\, \xi \in \mathbb{X}_\alpha, \tag{6.9}$$

where $\mathbb{X}_\alpha \subset C(\mathbb{R}^N)$ is the space of functions ξ satisfying:

(i) $\mathrm{supp}(\xi) \subset \bar{\Omega}$,

(ii) $(-\Delta)^\alpha \xi(x)$ exists for all $x \in \Omega$ and $|(-\Delta)^\alpha \xi(x)| \le C$ for some $C > 0$,

(iii) there exist $\varphi \in L^1(\Omega, \rho^\alpha dx)$ and $\varepsilon_0 > 0$ such that $|(-\Delta)^\alpha_\varepsilon \xi| \le \varphi$ a.e. in Ω, for all $\varepsilon \in (0, \varepsilon_0]$.

We denote by G_α the Green kernel of $(-\Delta)^\alpha$ in Ω and by $\mathbb{G}_\alpha[.]$ the associated Green operator defined by

$$\mathbb{G}_\alpha[\nu](x) = \int_\Omega G_\alpha(x,y) d\nu(y), \qquad \forall\, \nu \in \mathfrak{M}(\Omega, \rho^\alpha). \tag{6.10}$$

Using bounds of $\mathbb{G}_\alpha[\nu]$, we obtain in section 6.2 some crucial estimates which will play an important role in our construction of weak solutions. Our main result in the case $\epsilon = 1$ is the following.

Theorem 6.1.1 *Assume that $\epsilon = 1$ and $g : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is a continuous function verifying $g(0) = 0$ and*

$$\int_1^{+\infty} g(s) s^{-1-p^*_\alpha} ds < +\infty, \tag{6.11}$$

where

$$p^*_\alpha = \frac{N}{N - 2\alpha + 1}. \tag{6.12}$$

Then for any $\nu \in \mathfrak{M}_+(\Omega, \rho^\beta)$ with $\beta \in [0, 2\alpha - 1)$, problem (6.1) admits a nonnegative weak solution u_ν which satisfies

$$u_\nu \le \mathbb{G}_\alpha[\nu]. \tag{6.13}$$

As in the case $\alpha = 1$, uniqueness remains an open question. We remark that the critical value p^*_α is independent of β. A similar fact was first observed when dealing with problem (6.7) where the critical value $k_{\alpha,\beta}$ defined by (6.8) does not depend on β when $\beta \in [0, \frac{N-2\alpha}{N}\alpha]$.

When $\epsilon = -1$, we have to consider the critical value $p^*_{\alpha,\beta}$ which depends truly on β and is expressed by

$$p^*_{\alpha,\beta} = \frac{N}{N - 2\alpha + 1 + \beta}. \tag{6.14}$$

We observe that $p^*_{\alpha,0} = p^*_\alpha$ and $p^*_{\alpha,\beta} < p^*_\alpha$ when $\beta > 0$. In the source case, the assumptions on g are of a different nature from in the absorption case, namely
(G) $g : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is a continuous function which satisfies

$$g(s) \le c_1 s^p + \sigma_0, \quad \forall s \ge 0, \tag{6.15}$$

for some $p \in (0, p^*_{\alpha,\beta})$, where $c_1 > 0$ and $\sigma_0 > 0$.

Our main result concerning the source case is the following.

Theorem 6.1.2 *Assume that $\epsilon = -1$, $\nu \in \mathfrak{M}(\Omega, \rho^\beta)$ with $\beta \in [0, 2\alpha - 1)$ is non-negative, g satisfies (G) and*

(i) $p \in (0, 1)$, or

(ii) $p = 1$ and c_1 is small enough, or

(iii) $p \in (1, p^*_{\alpha,\beta})$, σ_0 and $\|\nu\|_{\mathfrak{M}(\Omega,\rho^\beta)}$ are small enough.

Then problem (6.1) admits a weak nonnegative solution u_ν which satisfies

$$u_\nu \ge \mathbb{G}_\alpha[\nu]. \tag{6.16}$$

In the last section of this chapter, we assume that Ω contains 0 and give pointwise estimates of the positive solutions

$$\begin{aligned} (-\Delta)^\alpha u + |\nabla u|^p = \delta_0 \quad &\text{in} \quad \Omega, \\ u = 0 \quad &\text{in} \quad \Omega^c, \end{aligned} \tag{6.17}$$

when $0 < p < p^*_\alpha$. Combining properties of the Riesz kernel with a bootstrap argument, we prove that any weak solution of (6.17) is regular outside 0 and is actually a classical solution of

$$\begin{aligned} (-\Delta)^\alpha u + |\nabla u|^p = 0 \quad &\text{in} \quad \Omega \setminus \{0\}, \\ u = 0 \quad &\text{in} \quad \Omega^c. \end{aligned} \tag{6.18}$$

These pointwise estimates are quite easy to establish in the case $\alpha = 1$, but much more delicate when the diffusion operator is non-local. We give sharp asymptotics of the behaviour of u near 0 and prove that the solution of (6.17) is unique in the class of positive solutions.

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6.2. Preliminaries

6.2.1. Marcinkiewicz type estimates

In this subsection, we recall some definitions and properties of Marcinkiewicz spaces.

Definition 6.2.1 *Let $\Theta \subset \mathbb{R}^N$ be a domain and μ be a positive Borel measure in Θ. For $\kappa > 1$, $\kappa' = \kappa/(\kappa - 1)$ and $u \in L^1_{loc}(\Theta, d\mu)$, we set*

$$\|u\|_{M^\kappa(\Theta,d\mu)} = \text{ínf}\left\{c \in [0,\infty] : \int_E |u|d\mu \leq c\left(\int_E d\mu\right)^{\frac{1}{\kappa'}},\ \forall E \subset \Theta,\ E \text{ Borel}\right\} \tag{6.19}$$

and

$$M^\kappa(\Theta, d\mu) = \{u \in L^1_{loc}(\Theta, d\mu) : \|u\|_{M^\kappa(\Theta,d\mu)} < \infty\}. \tag{6.20}$$

$M^\kappa(\Theta, d\mu)$ is called the Marcinkiewicz space of exponent κ, or weak L^κ-space and $\|.\|_{M^\kappa(\Theta,d\mu)}$ is a quasi-norm.

Proposition 6.2.1 *[11, 43] Assume that $1 \leq q < \kappa < \infty$ and $u \in L^1_{loc}(\Theta, d\mu)$. Then there exists $c_3 > 0$ dependent of q, κ such that*

$$\int_E |u|^q d\mu \leq c_3 \|u\|_{M^\kappa(\Theta,d\mu)} \left(\int_E d\mu\right)^{1-q/\kappa},$$

for any Borel set E of Θ.

The next estimate is the key-stone in the proof of Theorem 6.1.1.

Proposition 6.2.2 *Let $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ be a bounded C^2 domain and $\nu \in \mathfrak{M}(\Omega, \rho^\beta)$ with $\beta \in [0, 2\alpha - 1]$. Then there exists $c_2 > 0$ such that*

$$\|\nabla\mathbb{G}_\alpha[|\nu|]\|_{M^{p^*_\alpha}(\Omega,\rho^\alpha dx)} \leq c_2 \|\nu\|_{\mathfrak{M}(\Omega,\rho^\beta)}, \tag{6.21}$$

where $\nabla\mathbb{G}_\alpha[|\nu|](x) = \displaystyle\int_\Omega \nabla_x G_\alpha(x, y)d|\nu(y)|$ and p^*_α is given by (6.12).

Proof. For $\lambda > 0$ and $y \in \Omega$, we set

$$\omega_\lambda(y) = \{x \in \Omega \setminus \{y\} : |\nabla_x G_\alpha(x, y)|\rho^\alpha(x) > \lambda\},\ m_\lambda(y) = \int_{\omega_\lambda(y)} dx.$$

From [37], there exists $c_4 > 0$ such that for any $(x, y) \in \Omega \times \Omega$ with $x \neq y$,

$$G_\alpha(x, y) \leq c_4 \operatorname{mín} \left\{ \frac{1}{|x - y|^{N-2\alpha}}, \frac{\rho^\alpha(x)}{|x - y|^{N-\alpha}}, \frac{\rho^\alpha(y)}{|x - y|^{N-\alpha}} \right\}, \tag{6.22}$$

$$G_\alpha(x, y) \leq c_4 \frac{\rho^\alpha(y)}{\rho^\alpha(x)|x - y|^{N-2\alpha}},$$

and by Corollary 3.3 in [15], we have

$$|\nabla_x G_\alpha(x, y)| \leq N G_\alpha(x, y) \operatorname{máx} \left\{ \frac{1}{|x - y|}, \frac{1}{\rho(x)} \right\}. \tag{6.23}$$

This implies that for any $\tau \in [0, 1]$

$$G_\alpha(x, y) \leq c_4 (\frac{\rho^\alpha(y)}{|x - y|^{N-\alpha}})^\tau (\frac{\rho^\alpha(x)}{|x - y|^{N-\alpha}})^{1-\tau} = c_4 \frac{\rho^{\alpha\tau}(y)\rho^{\alpha(1-\tau)}(x)}{|x - y|^{N-\alpha}},$$

and then

$$|\nabla_x G_\alpha(x, y)| \leq c_5 \operatorname{máx} \left\{ \frac{\rho^\alpha(y)}{\rho^\alpha(x)|x - y|^{N-2\alpha+1}}, \frac{\rho^{\alpha\tau}(y)\rho^{\alpha(1-\tau)-1}(x)}{|x - y|^{N-\alpha}} \right\}. \tag{6.24}$$

Letting $\tau = \frac{2\alpha-1}{\alpha} \frac{N-\alpha}{N-2\alpha+1} \in (0, 1)$, we derive

$$|\nabla_x G_\alpha(x, y)|\rho^\alpha(x) \leq c_5 \operatorname{máx} \left\{ \frac{\rho^{2\alpha-1}(y)\rho_\Omega^{1-\alpha}}{|x - y|^{N-2\alpha+1}}, \frac{\rho^{\frac{(2\alpha-1)(N-\alpha)}{N-2\alpha+1}}(y)\rho_\Omega^{\frac{(2\alpha-1)(1-\alpha)}{N-2\alpha+1}}}{|x - y|^{N-\alpha}} \right\}.$$

where $\rho_\Omega = \sup_{z\in\Omega} \rho(z)$. There exists some $c_6 > 0$ such that

$$\omega_\lambda(y) \subset \left\{ x \in \Omega : |x - y| \leq c_6 \rho^{\frac{2\alpha-1}{N-2\alpha+1}}(y) \operatorname{máx}\{\lambda^{-\frac{1}{N-2\alpha+1}}, \lambda^{-\frac{1}{N-\alpha}}\} \right\}.$$

By $N - 2\alpha + 1 > N - \alpha$, we deduce that for any $\lambda > 1$, there holds

$$\omega_\lambda(y) \subset \{x \in \Omega : |x - y| \leq c_6 \rho^{\frac{2\alpha-1}{N-2\alpha+1}}(y)\lambda^{-\frac{1}{N-2\alpha+1}}\}. \tag{6.25}$$

As a consequence,

$$m_\lambda(y) \leq c_7 \rho^{(2\alpha-1)p_\alpha^*}(y)\lambda^{-p_\alpha^*},$$

where $c_7 > 0$ independent of y and λ.

Let $E \subset \Omega$ be a Borel set and $\lambda > 1$, then

$$\int_E |\nabla_x G_\alpha(x, y)|\rho^\alpha(x)dx \leq \int_{\omega_\lambda(y)} |\nabla_x G_\alpha(x, y)|\rho^\alpha(x)dx + \lambda \int_E dx.$$

Noting that

$$\begin{aligned}
\int_{\omega_\lambda(y)} |\nabla_x G_\alpha(x,y)|\rho^\alpha(x)dx &= -\int_\lambda^\infty s dm_s(y) \\
&= \lambda m_\lambda(y) + \int_\lambda^\infty m_s(y)ds \\
&\le c_8 \rho^{(2\alpha-1)p_\alpha^*}(y)\lambda^{1-p_\alpha^*},
\end{aligned}$$

for some $c_8 > 0$, we derive

$$\int_E |\nabla_x G_\alpha(x,y)|\rho^\alpha(x)dx \le c_8\rho^{(2\alpha-1)p_\alpha^*}(y)\lambda^{1-p_\alpha^*} + \lambda\int_E dx.$$

Choosing $\lambda = \rho^{2\alpha-1}(y)(\int_E dx)^{-\frac{1}{p_\alpha^*}}$ yields

$$\int_E |\nabla_x G_\alpha(x,y)|\rho^\alpha(x)dx \le (c_8+1)\rho^{2\alpha-1}(y)(\int_E dx)^{\frac{p_\alpha^*-1}{p_\alpha^*}}, \quad \forall y \in \Omega.$$

Therefore,

$$\begin{aligned}
\int_E |\nabla \mathbb{G}_\alpha[|\nu|](x)|\rho^\alpha(x)dx &= \int_\Omega\int_E |\nabla_x G_\alpha(x,y)|\rho^\alpha(x)dxd|\nu(y)| \\
&\le \int_\Omega \rho^{2\alpha-1}(y)\left(\rho^{1-2\alpha}(y)\int_E |\nabla_x G_\alpha(x,y)|\rho^\alpha(x)dx\right)d|\nu(y)| \\
&\le (c_8+1)\int_\Omega \rho^\beta(y)\rho^{2\alpha-1-\beta}(y)d|\nu(y)|\left(\int_E dx\right)^{\frac{p_\alpha^*-1}{p_\alpha^*}} \\
&\le (c_8+1)\rho_\Omega^{2\alpha-1-\beta}\|\nu\|_{\mathfrak{M}(\Omega,\rho^\beta)}\left(\int_E dx\right)^{\frac{p_\alpha^*-1}{p_\alpha^*}}.
\end{aligned} \tag{6.26}$$

As a consequence,

$$\|\nabla\mathbb{G}_\alpha[|\nu|]\|_{M^{p_\alpha^*}(\Omega,\rho^\alpha dx)} \le c_2\|\nu\|_{\mathfrak{M}(\Omega,\rho^\beta)},$$

which ends the proof. □

Proposition 6.2.3 *[40] Assume that $\nu \in L^1(\Omega,\rho^\beta dx)$ with $0 \le \beta \le \alpha$. Then for $p \in (1, \frac{N}{N-2\alpha+\beta})$, there exists $c_9 > 0$ such that for any $\nu \in L^1(\Omega,\rho^\beta dx)$*

$$\|\mathbb{G}_\alpha[\nu]\|_{W^{2\alpha-\gamma,p}(\Omega)} \le c_9\|\nu\|_{L^1(\Omega,\rho^\beta dx)}, \tag{6.27}$$

where $p' = \frac{p}{p-1}$, $\gamma = \beta + \frac{N}{p'}$ if $\beta > 0$ and $\gamma > \frac{N}{p'}$ if $\beta = 0$.

Proposition 6.2.4 *If $0 \leq \beta < 2\alpha - 1$, then the mapping $\nu \mapsto |\nabla \mathbb{G}_\alpha[\nu]|$ is compact from $L^1(\Omega, \rho^\beta dx)$ into $L^q(\Omega)$ for any $q \in [1, p^*_{\alpha,\beta})$ and there exists $c_{10} > 0$ such that*

$$\left(\int_\Omega |\nabla \mathbb{G}_\alpha[\nu](x)|^q dx\right)^{\frac{1}{q}} \leq c_{10} \int_\Omega |\nu(x)|\rho^\beta(x)dx, \tag{6.28}$$

where $p^*_{\alpha,\beta}$ is given by (6.14).

Proof. For $\nu \in L^1(\Omega, \rho^\beta dx)$ with $0 \leq \beta < 2\alpha - 1 < \alpha$, we obtain from Proposition 6.2.3 that

$$\mathbb{G}_\alpha[\nu] \in W^{2\alpha-\gamma,p}(\Omega),$$

where $p \in (1, p^*_{\alpha,\beta})$ and $2\alpha - \gamma > 1$. Therefore, $|\nabla \mathbb{G}_\alpha[\nu]| \in W^{2\alpha-\gamma-1,p}(\Omega)$ and

$$\|\nabla \mathbb{G}_\alpha[\nu]\|_{W^{2\alpha-\gamma-1,p}(\Omega)} \leq c_9 \|\nu\|_{L^1(\Omega,\rho^\beta dx)}. \tag{6.29}$$

By [49, Corollary 7.2], the embedding of $W^{2\alpha-\gamma-1,p}(\Omega)$ into $L^q(\Omega)$ is compact for $q \in [1, \frac{Np}{N-(2\alpha-\gamma-1)p})$. When $\beta > 0$,

$$\begin{aligned}\frac{Np}{N-(2\alpha-\gamma-1)p} &= \frac{Np}{N-(2\alpha-\beta-N\frac{p-1}{p}-1)p} \\ &= \frac{N}{N-2\alpha+1+\beta} = p^*_{\alpha,\beta}.\end{aligned}$$

When $\beta = 0$,

$$\begin{aligned}\lim_{\gamma \to (\frac{N}{p'})^+} \frac{Np}{N-(2\alpha-\gamma-1)p} &= \frac{Np}{N-(2\alpha-N\frac{p-1}{p}-1)p} \\ &= \frac{N}{N-2\alpha+1} = p^*_{\alpha,0}.\end{aligned}$$

Then the mapping $\nu \mapsto |\nabla \mathbb{G}_\alpha[\nu]|$ is compact from $L^1(\Omega, \rho^\beta dx)$ into $L^q(\Omega)$ for any $q \in [1, p^*_{\alpha,\beta})$. Inequality (6.28) follows by (6.29) and the continuity of the embedding of $W^{2\alpha-\gamma-1,p}(\Omega)$ into $L^q(\Omega)$. $\square$

Remark 6.2.1 *If $\nu \in L^1(\Omega, \rho^\beta dx)$ with $0 \leq \beta < 2\alpha - 1$ and u is the solution of*

$$\begin{aligned}(-\Delta)^\alpha u &= \nu \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{in} \quad \Omega^c,\end{aligned}$$

then for any $q \in [1, p^*_{\alpha,\beta})$,

$$\left(\int_\Omega |\nabla u|^q dx\right)^{\frac{1}{q}} \leq c_{10} \int_\Omega |\nu(x)|\rho^\beta(x)dx.$$

6.2.2. Classical solutions

In this subsection we consider the question of existence of classical solutions to problem

$$\begin{aligned} (-\Delta)^\alpha u + h(|\nabla u|) = f \quad &\text{in} \quad \Omega, \\ u = 0 \quad &\text{in} \quad \Omega^c. \end{aligned} \tag{6.30}$$

Theorem 6.2.1 *Assume $h \in C^\theta(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$ for some $\theta \in (0,1]$ and $f \in C^\theta(\bar{\Omega})$. Then problem (6.30) admits a unique classical solution u. Moreover,*
(i) if $f - h(0) \geq 0$ in Ω, then $u \geq 0$;
(ii) the mappings $h \mapsto u$ and $f \mapsto u$ are respectively nonincreasing and nondecreasing.

Proof. We divide the proof into several steps.
Step 1. Existence. We define the operator T by

$$Tu = \mathbb{G}_\alpha\left[f - h(|\nabla u|)\right], \quad \forall u \in W_0^{1,1}(\Omega).$$

Using (6.24) with $\tau = 0$ yields

$$\begin{aligned} \|Tu\|_{W^{1,1}(\Omega)} &\leq \|\mathbb{G}_\alpha[f]\|_{W^{1,1}(\Omega)} + \|\mathbb{G}_\alpha[h(|\nabla u|)]\|_{W^{1,1}(\Omega)} \\ &\leq \left(\|f\|_{L^\infty(\Omega)} + \|h(|\nabla u|)\|_{L^\infty(\Omega)}\right) \|\int_\Omega G_\alpha(\cdot, y)dy\|_{W^{1,1}(\Omega)} \\ &= c_{11}\left(\|f\|_{L^\infty(\Omega)} + \|h\|_{L^\infty(\mathbb{R}_+)}\right), \end{aligned} \tag{6.31}$$

where $c_{11} = \|\int_\Omega G_\alpha(\cdot, y)dy\|_{W^{1,1}(\Omega)}$. Thus T maps $W_0^{1,1}(\Omega)$ into itself. Clearly, if $u_n \to u$ in $W_0^{1,1}(\Omega)$ as $n \to \infty$, then $h(|\nabla u_n|) \to h(|\nabla u|)$ in $L^1(\Omega)$, thus T is continuous. We claim that T is a compact operator. In fact, for $u \in W_0^{1,1}(\Omega)$, we see that $f - h(|\nabla u|) \in L^1(\Omega)$ and then, by Proposition 6.2.3, it implies that $Tu \in W_0^{2\alpha-\gamma,p}(\Omega)$ where $\gamma \in (\frac{N(p-1)}{p}, 2\alpha - 1)$ and $2\alpha - 1 > \frac{N(p-1)}{p} > 0$ for $p \in (1, \frac{N}{N-2\alpha+1})$. Since the embedding $W_0^{2\alpha-\gamma,p}(\Omega) \hookrightarrow W_0^{1,1}(\Omega)$ is compact, T is a compact operator.

Let $\mathcal{O} = \{u \in W_0^{1,1}(\Omega) : \|u\|_{W^{1,1}(\Omega)} \leq c_{10}(\|f\|_{L^\infty(\Omega)} + \|h\|_{L^\infty(\mathbb{R}_+)})\ \}$, which is a closed and convex set of $W_0^{1,1}(\Omega)$. Combining with (6.31), there holds

$$T(\mathcal{O}) \subset \mathcal{O}.$$

It follows by Schauder's fixed point theorem that there exists some $u \in W_0^{1,1}(\Omega)$ such that $Tu = u$.

Next we show that u is a classical solution of (6.30). Let open set O satisfy $O \subset \bar{O} \subset \Omega$. By Proposition 2.3 in [88], for any $\sigma \in (0, 2\alpha)$, there exists $c_{12} > 0$ such that

$$\|u\|_{C^\sigma(O)} \leq c_{12}\{\|h(|\nabla u|)\|_{L^\infty(\Omega)} + \|f\|_{L^\infty(\Omega)}\},$$

and by choosing $\sigma = \frac{2\alpha+1}{2} \in (1, 2\alpha)$, then

$$\||\nabla u|\|_{C^{\sigma-1}(O)} \leq c_{12}\{\|h(|\nabla u|)\|_{L^\infty(\Omega)} + \|f\|_{L^\infty(\Omega)}\},$$

and then applied [88, Corollary 2.4], u is $C^{2\alpha+\epsilon_0}$ locally in Ω for some $\epsilon_0 > 0$. Then u is a classical solution of (6.30). Moreover, from [40], we have

$$\int_\Omega [u(-\Delta)^\alpha \xi + h(|\nabla u|)\xi] dx = \int_\Omega \xi f dx, \quad \forall \xi \in \mathbb{X}_\alpha. \tag{6.32}$$

Step 2. Proof of (i). If u is not nonnegative, then there exists $x_0 \in \Omega$ such that

$$u(x_0) = \min_{x\in\Omega} u(x) < 0,$$

then $\nabla u(x_0) = 0$ and $(-\Delta)^\alpha u(x_0) < 0$. Since u is the classical solution of (6.30), $(-\Delta)^\alpha u(x_0) = f(x_0) - h(0) \geq 0$, which is a contradiction.

Step 3. Proof of (ii). We just give the proof of the first argument, the proof of the second being similar. Let h_1 and h_2 satisfy our hypotheses for h and $h_1 \leq h_2$. Denote u_1 and u_2 the solutions of (6.30) with h replaced by h_1 and h_2 respectively. If there exists $x_0 \in \Omega$ such that

$$(u_1 - u_2)(x_0) = \min_{x\in\Omega}\{(u_1 - u_2)(x)\} < 0.$$

Then

$$(-\Delta)^\alpha(u_1 - u_2)(x_0) < 0, \quad \nabla u_1(x_0) = \nabla u_2(x_0).$$

This implies

$$(-\Delta)^\alpha(u_1 - u_2)(x_0) + h_1(|\nabla u_1(x_0)|) - h_2(|\nabla u_2(x_0)|) < 0. \tag{6.33}$$

However,

$$(-\Delta)^\alpha(u_1 - u_2)(x_0) + h_1(|\nabla u_1(x_0)|) - h_2(|\nabla u_2(x_0)|) = f(x_0) - f(x_0) = 0,$$

contradiction. Then $u_1 \geq u_2$.

Uniqueness follows from Step 3. □

6.3. Proof of Theorems 6.1.1 and 6.1.2

6.3.1. The absorption case

In this subsection, we prove the existence of a weak solution to (6.1) when $\epsilon = 1$. To this end, we give below an auxiliary lemma.

Lemma 6.3.1 *Assume that $g : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is continuous and (6.11) holds with p^*_α. Then there is a sequence real positive numbers $\{T_n\}$ such that*

$$\lim_{n\to\infty} T_n = \infty \quad \text{and} \quad \lim_{n\to\infty} g(T_n)T_n^{-p^*_\alpha} = 0.$$

Proof. Let $\{s_n\}$ be a sequence of real positive numbers converging to ∞. We observe

$$\begin{aligned} \int_{s_n}^{2s_n} g(t)t^{-1-p^*_\alpha}dt &\geq \min_{t\in[s_n,2s_n]} g(t)(2s_n)^{-1-p^*_\alpha}\int_{s_n}^{2s_n} dt \\ &= 2^{-1-p^*_\alpha}s_n^{-p^*_\alpha}\min_{t\in[s_n,2s_n]} g(t) \end{aligned}$$

and by (6.11),

$$\lim_{n\to\infty}\int_{s_n}^{2s_n} g(t)t^{-1-p^*_\alpha}dt = 0.$$

Then we choose $T_n \in [s_n, 2s_n]$ such that $g(T_n) = \min_{t\in[s_n,2s_n]} g(t)$ and then the claim follows. $\square$

Proof of Theorem 6.1.1. Let $\beta \in [0, 2\alpha - 1)$, we define the space

$$C_\beta(\bar\Omega) = \{\zeta \in C(\bar\Omega) : \rho^{-\beta}\zeta \in C(\bar\Omega)\}$$

endowed with the norm

$$\|\zeta\|_{C_\beta(\bar\Omega)} = \|\rho^{-\beta}\zeta\|_{C(\bar\Omega)}.$$

Let $\{\nu_n\} \subset C^1(\bar\Omega)$ be a sequence of nonnegative functions such that $\nu_n \to \nu$ in sense of duality with $C_\beta(\bar\Omega)$, that is,

$$\lim_{n\to\infty}\int_{\bar\Omega}\zeta\nu_n dx = \int_{\bar\Omega}\zeta d\nu, \qquad \forall\zeta \in C_\beta(\bar\Omega). \tag{6.34}$$

By the Banach-Steinhaus Theorem, $\|\nu_n\|_{\mathfrak{M}(\Omega,\rho^\beta)}$ is bounded independently of n. We consider a sequence $\{g_n\}$ of C^1 nonnegative functions defined on $\mathbb{R}_+$ such that $g_n(0) = 0$ and

$$g_n \leq g_{n+1} \leq g, \quad \sup_{s\in\mathbb{R}_+} g_n(s) = n \quad \text{and} \quad \lim_{n\to\infty}\|g_n - g\|_{L^\infty_{loc}(\mathbb{R}_+)} = 0. \tag{6.35}$$

By Theorem 6.2.1, there exists a unique *nonnegative* solution u_n of (6.1) with data ν_n and g_n instead of ν and g, and there holds

$$\int_\Omega (u_n + g_n(|\nabla u_n|)\eta_1)\,dx = \int_\Omega \nu_n\eta_1 dx \leq C\|\nu\|_{\mathfrak{M}(\Omega,\rho^\beta)}, \tag{6.36}$$

where $\eta_1 = \mathbb{G}_\alpha[1]$. Therefore, $\|g_n(|\nabla u_n|)\|_{\mathfrak{M}(\Omega,\rho^\alpha)}$ is bounded independently of n. For $\varepsilon > 0$ and $\xi_\varepsilon = (\eta_1+\varepsilon)^{\frac{\beta}{\alpha}} - \varepsilon^{\frac{\beta}{\alpha}} \in \mathbb{X}_\alpha$ which is concave in the interval $[0, \eta_1(\bar{\omega})]$, where $\eta_1(\bar{\omega}) = \text{máx}_{x\in\Omega}\,\eta_1(x)$. By [40, Lemma 2.3 (ii)], we see that

$$\begin{aligned}(-\Delta)^\alpha \xi_\varepsilon &= \frac{\beta}{\alpha}(\eta_1+\varepsilon)^{\frac{1}{\alpha}}(-\Delta)^\alpha\eta_1 - \frac{\beta(\beta-\alpha)}{\alpha^2}(\eta_1+\varepsilon)^{\frac{\beta-2\alpha}{\alpha}}\int_\Omega \frac{(\eta_1(y)-\eta_1(x))^2}{|y-x|^{N+2\alpha}}dy\\ &\geq \frac{\beta}{\alpha}(\eta_1+\varepsilon)^{\frac{\beta-\alpha}{\alpha}},\end{aligned}$$

and $\xi_\varepsilon \in \mathbb{X}_\alpha$. Since

$$\int_\Omega (u_n(-\Delta)^\alpha\xi_\varepsilon + g_n(|\nabla u_n|)\xi_\varepsilon)\,dx = \int_\Omega \xi_\varepsilon \nu_n dx,$$

we obtain

$$\int_\Omega \left(\frac{\beta}{\alpha}u_n(\eta_1+\varepsilon)^{\frac{\beta-\alpha}{\alpha}} + g_n(|\nabla u_n|)\xi_\varepsilon\right)dx \leq \int_\Omega \xi_\varepsilon\nu_n dx.$$

If we let $\varepsilon \to 0$, it yields

$$\int_\Omega \left(\frac{\beta}{\alpha}u_n\eta_1^{\frac{\beta-\alpha}{\alpha}} + g_n(|\nabla u_n|)\eta_1^{\frac{\beta}{\alpha}}\right)dx \leq \int_\Omega \eta_1^{\frac{\beta}{\alpha}}\nu_n dx.$$

Using [40, Lemma 2.3], we derive the estimate

$$\int_\Omega \left(u_n\rho^{\beta-\alpha} + g_n(|\nabla u_n|)\rho^\beta\right)dx \leq c_{13}\|\nu_n\|_{\mathfrak{M}(\Omega,\rho^\beta)} \leq c_{14}\|\nu\|_{\mathfrak{M}(\Omega,\rho^\beta)}. \tag{6.37}$$

Thus $\{g_n(|\nabla u_n|)\}$ is uniformly bounded in $L^1(\Omega,\rho^\beta dx)$. Since $u_n = \mathbb{G}[\nu_n - g_n(|\nabla u_n|)]$, there holds

$$\begin{aligned}\||\nabla u_n|\|_{M^{p^*_\alpha}(\Omega,\rho^\alpha dx)} &\leq \|\nu_n\|_{\mathfrak{M}(\Omega,\rho^\beta)} + \|g_n(|\nabla u_n|)\|_{\mathfrak{M}(\Omega,\rho^\beta)}\\ &\leq c_{15}\|\nu\|_{\mathfrak{M}(\Omega,\rho^\beta)}.\end{aligned}$$

Since $\nu_n - g_n(|\nabla u_n|)$ is uniformly bounded in $L^1(\Omega,\rho^\beta dx)$, we use Proposition 6.2.4 to obtain that the sequences $\{u_n\}$, $\{|\nabla u_n|\}$ are relatively compact in $L^q(\Omega)$ for $q \in [1, \frac{N}{N-2\alpha+\beta})$ and $q \in [1, p^*_{\alpha,\beta})$, respectively. Thus, there exist a sub-sequence $\{u_{n_k}\}$ and some $u \in L^q(\Omega)$ with $q \in [1, \frac{N}{N-2\alpha+\beta})$ such that

(i) $u_{n_k} \to u$ a.e. in Ω and in $L^q(\Omega)$ with $q \in [1, \frac{N}{N-2\alpha+\beta})$;
(ii) $|\nabla u_{n_k}| \to |\nabla u|$ a.e. in Ω and in $L^q(\Omega)$ with $q \in [1, p^*_{\alpha,\beta})$.

Therefore, $g_{n_k}(|\nabla u_{n_k}|) \to g(|\nabla u|)$ a.e. in Ω. For $\lambda > 0$, we denote

$$S_\lambda = \{x \in \Omega : |\nabla u_{n_k}(x)| > \lambda\} \quad \text{and} \quad \omega(\lambda) = \int_{S_\lambda}\rho^\alpha(x)dx.$$

Then for any Borel set $E \subset \Omega$, we have that

$$\begin{aligned}\int_E g_{n_k}(|\nabla u_{n_k}|)|\rho^\alpha(x)dx &\le \int_E g(|\nabla u_{n_k}|)|\rho^\alpha(x)dx \\ &= \int_{E\cap S_\lambda^c} g(|\nabla u_{n_k}|)\rho^\alpha(x)dx + \int_{E\cap S_\lambda} g(|\nabla u_{n_k}|)\rho^\alpha(x)dx \\ &\le \tilde{g}(\lambda)\int_E \rho^\alpha(x)dx + \int_{S_\lambda} g(|\nabla u_{n_k}|)\rho^\alpha(x)dx \\ &\le \tilde{g}(\lambda)\int_E \rho^\alpha(x)dx - \int_\lambda^\infty g(s)d\omega(s),\end{aligned}$$

where $\tilde{g}(s) = \text{máx}_{t\in[0,s]}\{g(t)\}$. But

$$\int_\lambda^\infty g(s)d\omega(s) = \lim_{n\to\infty}\int_\lambda^{T_n} g(s)d\omega(s).$$

where $\{T_n\}$ is given by Lemma 6.3.1. Since $|\nabla u_{n_k}| \in M^{p_\alpha^*}(\Omega, \rho^\alpha dx)$, $\omega(s) \le c_{16}s^{-p_\alpha^*}$ and

$$\begin{aligned}-\int_\lambda^{T_n} g(s)d\omega(s) &= -\Big[g(s)\omega(s)\Big]_{s=\lambda}^{s=T_n} + \int_\lambda^{T_n}\omega(s)dg(s) \\ &\le g(\lambda)\omega(\lambda) - g(T_n)\omega(T_n) + c_{16}\int_\lambda^{T_n} s^{-p_\alpha^*}dg(s) \\ &\le g(\lambda)\omega(\lambda) - g(T_n)\omega(T_n) + c_{16}\left({T_n}^{-p_\alpha^*}g(T_n) - \lambda^{-p_\alpha^*}g(\lambda)\right) \\ &\qquad + \frac{c_{16}}{p_\alpha^*+1}\int_\lambda^{T_n} s^{-1-p_\alpha^*}g(s)ds.\end{aligned}$$

By assumption (6.11) and Lemma 6.3.1, it follows

$$\lim_{n\to\infty} T_n^{-p_\alpha^*}g(T_n) = 0. \tag{6.38}$$

Along with $g(\lambda)\omega(\lambda) \le c_{16}\lambda^{-p_\alpha^*}g(\lambda)$, we have

$$-\int_\lambda^\infty g(s)d\omega(s) \le \frac{c_{16}}{p_\alpha^*+1}\int_\lambda^\infty s^{-1-p_\alpha^*}g(s)ds.$$

Notice that the above quantity on the right-hand side tends to 0 when $\lambda \to \infty$. It implies that for any $\epsilon > 0$ there exists $\lambda > 0$ such that

$$\frac{c_{16}}{p_\alpha^*+1}\int_\lambda^\infty s^{-1-p_\alpha^*}g(s)ds \le \frac{\epsilon}{2},$$

and $\delta > 0$ such that

$$\int_E \rho^\alpha(x)dx \le \delta \Longrightarrow \tilde{g}(\lambda)\int_E dx \le \frac{\epsilon}{2}.$$

This proves that $\{g_{n_k}(|\nabla u_{n_k}|)\}$ is uniformly integrable in $L^1(\Omega,\rho^\alpha dx)$. Then $g_{n_k}(|\nabla u_{n_k}|) \to g(|\nabla u|)$ in $L^1(\Omega,\rho^\alpha dx)$ by Vitali convergence theorem. Letting $n_k \to \infty$ in the identity

$$\int_\Omega (u_{n_k}(-\Delta)^\alpha \xi + g_{n_k}(|\nabla u_{n_k}|)\xi)\,dx = \int_\Omega \nu_{n_k}\xi dx, \quad \forall \xi \in \mathbb{X}_\alpha,$$

it infers that u is a weak solution of (6.1). Since u_{n_k} is nonnegative, so is u.

Estimate (6.13) is a consequence of positivity and

$$u_{n_k} = \mathbb{G}_\alpha[\nu_{n_k}] - \mathbb{G}_\alpha[g_{n_k}(|\nabla u_{n_k}|)] \le \mathbb{G}_\alpha[\nu_{n_k}].$$

Since $\text{lím}_{n_k\to\infty} u_{n_k} = u$, (6.13) follows. □

6.3.2. The source case

In this subsection we study the existence of solutions to problem (6.1) when $\epsilon = -1$.

Proof of Theorem 6.1.2. Let $\{\nu_n\}$ be a sequence of C^2 nonnegative functions converging to ν in the sense of (6.34), $\{g_n\}$ an increasing sequence of C^1, nonnegative bounded functions defined on $\mathbb{R}_+$ satisfying (6.35) and converging to g. We set $p_0 = \frac{p+p^*_{\alpha,\beta}}{2} \in (p, p^*_{\alpha,\beta})$, where $p^*_{\alpha,\beta}$ is given by (6.14) and $p < p^*_{\alpha,\beta}$ is the maximal growth rate of g which satisfies (6.15), and

$$M(v) = \left(\int_\Omega |\nabla v|^{p_0} dx\right)^{\frac{1}{p_0}}.$$

We may assume that $\|\nu_n\|_{L^1(\Omega,\rho^\beta dx)} \le 2\|\nu\|_{\mathfrak{M}(\Omega,\rho^\beta)}$ for all $n \ge 1$.
Step 1. To prove that for $n \ge 1$,

$$\begin{aligned} (-\Delta)^\alpha u_n &= g_n(|\nabla u_n|) + \nu_n \quad &&\text{in} \quad \Omega,\\ u_n &= 0 &&\text{in} \quad \Omega^c \end{aligned}$$

admits a solution u_n such that

$$M(u_n) \le \bar{\lambda},$$

where $\bar{\lambda} > 0$ independent of n.

To this end, we define the operators $\{T_n\}$ by

$$T_n u = \mathbb{G}_\alpha\left[g_n(|\nabla u|) + \nu_n\right], \quad \forall u \in W^{1,p_0}_0(\Omega).$$

On the one hand, using (6.24) with $\tau = 0$ yields

$$\begin{aligned} \|T_n u\|_{W^{1,1}(\Omega)} &\leq \|\mathbb{G}_\alpha[\nu_n]\|_{W^{1,1}(\Omega)} + \|\mathbb{G}_\alpha[g_n(|\nabla u|)]\|_{W^{1,1}(\Omega)} \\ &\leq c_{11}\left(\|\nu_n\|_{L^\infty(\Omega)} + \|g_n\|_{L^\infty(\mathbb{R}_+)}\right), \end{aligned}$$

where $c_{11} = \| \int_\Omega G_\alpha(\cdot, y)dy\|_{W^{1,1}(\Omega)}$. On the other hand, by (6.15) and Proposition 6.2.4, we have

$$\begin{aligned} \left(\int_\Omega |\nabla(T_n u)|^{p_0} dx\right)^{\frac{1}{p_0}} &\leq c_2\|g_n(|\nabla u|) + \nu_n\|_{L^1(\Omega,\rho^\beta dx)} \\ &\leq c_2[\|g_n(|\nabla u|)\|_{L^1(\Omega,\rho^\beta dx)} + 2\|\nu\|_{\mathfrak{M}(\Omega,\rho^\beta)}] \qquad (6.39) \\ &\leq c_2 c_1 \int_\Omega |\nabla u|^p \rho^\beta dx + c_{17}\sigma_0 + 2c_2\|\nu\|_{\mathfrak{M}(\Omega,\rho^\beta)}, \end{aligned}$$

where $c_{17} = c_2 \int_\Omega \rho^\beta dx$. Then we use Hölder inequality to obtain that

$$\left(\int_\Omega |\nabla u|^p \rho^\beta dx\right)^{\frac{1}{p}} \leq (\int_\Omega \rho^{\frac{\beta p_0}{p_0 - p}} dx)^{\frac{1}{p} - \frac{1}{p_0}} \left(\int_\Omega |\nabla u|^{p_0} dx\right)^{\frac{1}{p_0}}, \qquad (6.40)$$

where $\int_\Omega \rho^{\frac{\beta p_0}{p_0 - p}} dx$ is bounded, since $\frac{\beta p_0}{p_0 - p} \geq 0$. Along with (6.39) and (6.40), we derive

$$M(T_n u) \leq c_{18} M(u)^p + c_{19}\|\nu\|_{\mathfrak{M}(\Omega,\rho^\beta)} + c_{17}\sigma_0, \qquad (6.41)$$

where $c_{18} = c_2 c_1 (\int_\Omega \rho^{\frac{\beta p_0}{p_0 - p}} dx)^{\frac{1}{p} - \frac{1}{p_0}} > 0$ and $c_{19} > 0$ independent of n. Therefore, if we assume that $M(u) \leq \lambda$, inequality (6.41) implies

$$M(T_n u) \leq c_{18}\lambda^p + c_{19}\|\nu\|_{\mathfrak{M}(\Omega,\rho^\beta)} + c_{17}\sigma_0. \qquad (6.42)$$

Let $\bar{\lambda} > 0$ be the largest root of the equation

$$c_{18}\lambda^p + c_{19}\|\nu\|_{\mathfrak{M}(\Omega,\rho^\beta)} + c_{17}\sigma_0 = \lambda, \qquad (6.43)$$

This root exists if one of the following condition holds:

(i) $p \in (0, 1)$, in which case (6.43) admits only one root;

(ii) $p = 1$ and $c_{17} < 1$, and again (6.43) admits only one root;

(iii) $p \in (1, p_\alpha^*)$ and there exists $\varepsilon_0 > 0$ such that $\text{máx}\left\{\|\nu\|_{\mathfrak{M}(\Omega,\rho^\beta)}, \sigma_0\right\} \leq \varepsilon_0$. In that case (6.43) admits usually two positive roots.

If we suppose that one of the above conditions holds, the definition of $\bar{\lambda} > 0$ implies that it is the largest $\lambda > 0$ such that

$$c_{18}\lambda^p + c_{19}\|\nu\|_{\mathfrak{M}(\Omega,\rho^\beta)} + c_{17}\sigma_0 \leq \lambda, \qquad (6.44)$$

For $M(u) \le \bar{\lambda}$, we obtain that

$$M(T_n u) \le c_{18}\bar{\lambda}^p + c_{19}\|\nu\|_{\mathfrak{M}(\Omega,\rho^\beta)} + c_{17}\sigma_0 = \bar{\lambda}.$$

By the assumptions of Theorem 6.1.2, $\bar{\lambda}$ exists and it is larger than $M(u_n)$. Therefore,

$$\int_\Omega |\nabla(T_n u)|^{p_0} dx \le \bar{\lambda}^{p_0}. \tag{6.45}$$

Thus T_n maps $W_0^{1,p_0}(\Omega)$ into itself. Clearly, if $u_n \to u$ in $W_0^{1,p_0}(\Omega)$ as $n \to \infty$, then $g_n(|\nabla u_n|) \to g_n(|\nabla u|)$ in $L^1(\Omega)$, thus T is continuous. We claim that T is a compact operator. In fact, for $u \in W_0^{1,p_0}(\Omega)$, we see that $\nu_n - g_n(|\nabla u|) \in L^1(\Omega)$ and then, by Proposition 6.2.3, it implies that $T_n u \in W_0^{2\alpha-\gamma,p}(\Omega)$ where $\gamma \in (\frac{N(p-1)}{p}, 2\alpha - 1)$ and $2\alpha - 1 > \frac{N(p-1)}{p} > 0$ for $p \in (1, \frac{N}{N-2\alpha+1})$. Since the embedding $W_0^{2\alpha-\gamma,p}(\Omega) \hookrightarrow W_0^{1,p_0}(\Omega)$ is compact, T_n is a compact operator.

Let

$$\mathcal{G} = \{u \in W_0^{1,p_0}(\Omega) : \|u\|_{W^{1,1}(\Omega)} \le c_{11}(\|\nu_n\|_{L^\infty(\Omega)} + \|g_n\|_{L^\infty(\mathbb{R}_+)}) \\ \text{and} \quad M(u) \le \bar{\lambda}\},$$

which is a closed and convex set of $W_0^{1,p_0}(\Omega)$. Combining with (6.31), there holds

$$T_n(\mathcal{G}) \subset \mathcal{G}.$$

It follows by Schauder's fixed point theorem that there exists some $u_n \in W_0^{1,p_0}(\Omega)$ such that $T_n u_n = u_n$ and $M(u_n) \le \bar{\lambda}$, where $\bar{\lambda} > 0$ independent of n. By the same arguments as in Theorem 6.2.1, u_n belongs to $C^{2\alpha+\epsilon_0}$ locally in Ω, and

$$\int_\Omega u_n(-\Delta)^\alpha \xi = \int_\Omega g_n(|\nabla u_n|)\xi dx + \int_\Omega \xi \nu_n dx, \quad \forall \xi \in \mathbb{X}_\alpha. \tag{6.46}$$

Step 2: Convergence. By (6.45) and (6.40), $g_n(|\nabla u_n|)$ is uniformly bounded in $L^1(\Omega, \rho^\beta dx)$. By Proposition 6.2.3, $\{u_n\}$ is bounded in $W_0^{2\alpha-\gamma,q}(\Omega)$ where $q \in (1, p^*_{\alpha,\beta})$ and $2\alpha - \gamma > 1$. By Proposition 6.2.4, there exist a subsequence $\{u_{n_k}\}$ and u such that $u_{n_k} \to u$ a.e. in Ω and in $L^1(\Omega)$, and $|\nabla u_{n_k}| \to |\nabla u|$ a.e. in Ω and in $L^q(\Omega)$ for any $q \in [1, p^*_{\alpha,\beta})$. By assumption (G), $g_{n_k}(|\nabla u_{n_k}|) \to g(|\nabla u|)$ in $L^1(\Omega)$. Letting $n_k \to \infty$ to have that

$$\int_\Omega u(-\Delta)^\alpha \xi = \int_\Omega g(|\nabla u|)\xi dx + \int_\Omega \xi d\nu, \quad \forall \xi \in \mathbb{X}_\alpha,$$

thus u is a weak solution of (6.1) which is nonnegative as $\{u_n\}$ are nonnegative. Furthermore, (6.16) follows from the positivity of $g(|\nabla u_n])$. $\square$

6.4. The case of Dirac mass

In this section we assume that Ω is an open, bounded and C^2 domain containing 0 and u a nonnegative weak solution of

$$\begin{aligned} (-\Delta)^\alpha u + |\nabla u|^p &= \delta_0 \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{in} \quad \Omega^c, \end{aligned} \tag{6.47}$$

where $p \in (0, p_\alpha^*)$ and δ_0 is the Dirac mass at 0. We recall the following result dealing with the convolution operator $*$ in Lorentz spaces $L^{p,q}(\mathbb{R}^N)$ (see [83]).

Proposition 6.4.1 *Let $1 \le p_1, q_1, p_2, q_2 \le \infty$ and suppose $\frac{1}{p_1} + \frac{1}{p_2} > 1$. If $f \in L^{p_1,q_1}(\mathbb{R}^N)$ and $g \in L^{p_2,q_2}(\mathbb{R}^N)$, then $f*g \in L^{r,s}(\mathbb{R}^N)$ with $\frac{1}{r} = \frac{1}{p_1} + \frac{1}{p_2} - 1$, $\frac{1}{q_1} + \frac{1}{q_2} \ge \frac{1}{s}$ and there holds*

$$\|f * g\|_{L^{r,s}(\mathbb{R}^N)} \le 3r \|f\|_{L^{p_1,q_1}(\mathbb{R}^N)} \|g\|_{L^{p_2,q_2}(\mathbb{R}^N)}. \tag{6.48}$$

In the particular case of Marcinkiewicz spaces $L^{p,\infty}(\mathbb{R}^N) = M^p(\mathbb{R}^N)$, the result takes the form

$$\|f * g\|_{M^r(\mathbb{R}^N)} \le 3r \|f\|_{M^{p_1}(\mathbb{R}^N)} \|g\|_{M^{p_2}(\mathbb{R}^N)}. \tag{6.49}$$

Proposition 6.4.2 *Assume that $0 < p < p_\alpha^*$ and u is a nonnegative weak solution of (6.47). Then*

$$0 \le u \le \mathbb{G}_\alpha[\delta_0], \tag{6.50}$$

$|\nabla u| \in L^\infty_{loc}(\Omega \setminus \{0\})$ and u is a classical solution of

$$\begin{aligned} (-\Delta)^\alpha u + |\nabla u|^p &= 0 \quad \text{in} \quad \Omega \setminus \{0\}, \\ u &= 0 \quad \text{in} \quad \Omega^c. \end{aligned} \tag{6.51}$$

Proof. Since $0 < p < p_\alpha^*$, (6.47) admits a solution. Estimate (6.50) is a particular case of (6.13). We pick a point $a \in \Omega \setminus \{0\}$ and consider a finite sequence $\{r_j\}_{j=0}^{\kappa}$ such that $0 < r_\kappa < r_{\kappa-1} < ... < r_0$ and $\bar{B}_{r_0}(a) \subset \Omega \setminus \{0\}$. We set $d_j = r_{j-1} - r_j$, $j = 1, ...\kappa$. By (6.37) with $\beta = 0$, it follows that

$$\int_\Omega (u + |\nabla u|^p)\, dx \le c_{20}. \tag{6.52}$$

Let $\{\eta_n\} \subset C_0^\infty(\mathbb{R}^N)$ be a sequence of radially decreasing and symmetric mollifiers such that $\mathrm{supp}(\eta_n) \subset B_{\varepsilon_n}(0)$ and $\varepsilon_n \le \frac{1}{2} \text{mín}\{\rho(a) - r_0, |a| - r_0\}$ and $u_n = u * \eta_n$. Since

$$\eta_n * (-\Delta)^\alpha \xi = (-\Delta)^\alpha (\xi * \eta_n)$$

by Fourier analysis and

$$\int_{\mathbb{R}^N}(u(-\Delta)^\alpha(\xi*\eta_n)+\xi*\eta_n|\nabla u|^p)dx=\int_{\mathbb{R}^N}(u*\eta_n(-\Delta)^\alpha\xi+\eta_n*|\nabla u|^p\xi)dx$$

because η_n is radially symmetric, it follows that u_n is a classical solution of

$$\begin{aligned} (-\Delta)^\alpha u_n+|\nabla u|^p*\eta_n=\eta_n \qquad & \text{in } \ \Omega_n,\\ u_n=0 \qquad & \text{in } \ \Omega_n^c, \end{aligned} \tag{6.53}$$

where $\Omega_n=\{x\in\mathbb{R}^N:\operatorname{dist}(x,\Omega)<\varepsilon_n\}$. We denote by $G_{\alpha,n}(x,y)$ the Green kernel of $(-\Delta)^\alpha$ in Ω_n and by $\mathbb{G}_{\alpha,n}$ the Green operator. Set $f_n=\eta_n-|\nabla u|^p*\eta_n$, then $u_n=\mathbb{G}_{\alpha,n}[f_n]$. If we set $f_{n,r_0}=f_n\chi_{B_{r_0}(a)}$, $\tilde f_{n,r_0}=f_n-f_{n,r_0}$, we have

$$\begin{aligned}\partial_{x_i}u_n(x)&=\int_{\Omega_n}\partial_{x_i}G_{\alpha,n}(x,y)f_n(y)dy\\ &=\int_{\Omega_n}\partial_{x_i}G_{\alpha,n}(x,y)f_{n,r_0}(y)dy+\int_{\Omega_n}\partial_{x_i}G_{\alpha,n}(x,y)\tilde f_{n,r_0}(y)dy\\ &=v_{n,r_0}(x)+\tilde v_{n,r_0}(x),\end{aligned}$$

where

$$v_{n,r_0}(x)=\int_{B_{r_0}(a)}\partial_{x_i}G_{\alpha,n}(x,y)f_n(y)dy=-\int_{B_{r_0}(a)}\partial_{x_i}G_{\alpha,n}(x,y)|\nabla u|^p*\eta_n(y)dy$$

and

$$\tilde v_{n,r_0}(x)=\int_{\Omega_n\setminus B_{r_0}(a)}\partial_{x_i}G_{\alpha,n}(x,y)f_n(y)dy.$$

We set $\rho_n(x)=\operatorname{dist}(x,\Omega_n^c)$, then by (6.23) and (6.23), we have

$$|\partial_{x_i}G_{\alpha,n}(x,y)|\le c_4N\,\text{máx}\left\{\frac{1}{|x-y|^{N-2\alpha+1}},\frac{\rho_n^{-1}(x)}{|x-y|^{N-2\alpha}}\right\}.$$

Thus, if $x\in B_{r_1}(a)$ and $y\in\Omega_n\setminus B_{r_0}(a)$, then $\rho_n(x)>d_1$ and $|x-y|>d_1$,

$$|\tilde v_{n,r_0}(x)|\le c_{21}\int_{\Omega_n\setminus B_{r_0}(a)}f_n(y)dy\le c_{20}c_{21}, \tag{6.54}$$

where $c_{21}>0$ depends on $d_1^{-N+2\alpha-1}$, N and α. Furthermore, if $x\in B_{r_1}(a)$ and $y\in B_{r_0}(a)$,

$$|\partial_{x_i}G_{\alpha,n}(x,y)|\le\frac{c_4N}{|x-y|^{N-2\alpha+1}}. \tag{6.55}$$

We have already use the fact that $y \mapsto |y|^{2\alpha-N-1} \in L^{q_1}_{loc}(\mathbb{R}^N)$ with $q_1 \in (\text{máx}\{1,p\}, p^*_\alpha)$. Since f_n is uniformly bounded in $L^1(\Omega)$, there exists $c_{22} > 0$ such that

$$\|v_{n,r_0}\|_{M^{q_1}(B_{r_1}(a))} \leq c_{22}. \tag{6.56}$$

Combined with (6.54), it yields

$$\||\nabla u|^p * \eta_n\|_{M^{\frac{q_1}{p}}(B_{r_1}(a))} \leq c_{23}. \tag{6.57}$$

Next we set $f_{n,r_1} = f_n \chi_{B_{r_1}(a)}$ and $\tilde{f}_{n,r_1} = f_n - f_{n,r_1}$. Then

$$\partial_{x_i} u_n = v_{n,r_1} + \tilde{v}_{n,r_1},$$

where

$$v_{n,r_1}(x) = \int_{B_{r_1}(a)} \partial_{x_i} G_\alpha(x,y) f_n(y) dy = -\int_{B_{r_1}(a)} \partial_{x_i} G_\alpha(x,y) |\nabla u|^p * \eta_n(y) dy$$

and

$$\tilde{v}_{n,r_1}(x) = \int_{\Omega_n \setminus B_{r_1}(a)} \partial_{x_i} G_\alpha(x,y) f_n(y) dy$$

Clearly $\tilde{v}_{n,r_1}(x)$ is uniformly bounded in $B_{r_2}(a)$ by a constant c_{24} depending on the structural constants and $d_2 = r_1 - r_2$. Estimate (6.55) holds if we assume $x \in B_{r_2}(a)$ and $y \in B_{r_1}(a)$. Therefore

$$|v_{n,r_1}(x)| \leq c_4 N \int_{B_{r_1}(a)} \frac{|\nabla u|^p * \eta_n(y)}{|x-y|^{N-2\alpha+1}} dy.$$

We derive from Proposition 6.4.1

$$\|v_{n,r_1}\|_{M^{q_2}(B_{r_2}(a))} \leq c_{24} \||\nabla u|^p * \eta_n\|_{M^{\frac{q_1}{p}}(B_{r_1}(a))},$$

with

$$\frac{1}{q_2} = \frac{p}{q_1} + \frac{1}{q_1} - 1. \tag{6.58}$$

Notice that $q_2 > q_1$. Therefore

$$\||\nabla u|^p * \eta_n\|_{M^{\frac{q_2}{p}}(B_{r_2}(a))} \leq c_{25}. \tag{6.59}$$

We iterate this construction and obtain the existence of constants c_j such that

$$\||\nabla u|^p * \eta_n\|_{M^{\frac{q_j}{p}}(B_{r_j}(a))} \leq \bar{c}_j, \qquad \forall j = 1, 2, ... \tag{6.60}$$

We pick $q_1 = \frac{1}{2}(p^*_\alpha + p)$ if $p > 1$ or $q_1 = \frac{1}{2}(p^*_\alpha + 1)$ if $p \in (0, 1]$

$$\frac{1}{q_{j+1}} = \frac{p}{q_j} + \frac{1}{q_1} - 1. \tag{6.61}$$

If $p = 1$, there exists $j_0 \in \mathbb{N}$ such that $q_{j_0} > 0$ and $q_{j_0+1} \leq 0$.
If $p \in (0, p^*_\alpha) \setminus \{1\}$, let $\ell = \frac{q_1-1}{q_1(p-1)}$, then $\ell = p\ell + \frac{1}{q_1} - 1$, thus

$$\frac{1}{q_{j+1}} = \ell + p^j \left(\frac{1}{q_1} - \ell\right) = \ell - p^j \frac{q_1 - p}{q_1(p-1)}. \tag{6.62}$$

Therefore there exists j_0 such that $q_{j_0} > 0$ and $q_{j_0+1} \leq 0$. This implies

$$\||\nabla u|^p * \eta_n\|_{L^s(B_{r_{j_0+1}}(a))} \leq c_{26}, \qquad \forall s < \infty \tag{6.63}$$

and

$$\||\nabla u|^p * \eta_n\|_{L^\infty(B_{r_{j_0+2}}(a))} \leq c_{27}, \tag{6.64}$$

with c_{27} independent of n. Letting $n \to \infty$ infers

$$\|\nabla u\|_{L^\infty(B_{r_{j_0+2}}(a))} \leq c_{27}^{\frac{1}{p}}. \tag{6.65}$$

Combining this estimate with (6.50) and using [88, Corollary 2.5] which states

$$\begin{aligned} \|u\|_{C^\beta(B_{r_{j_0+3}}(a))} \leq c \Bigg(& \|u\|_{L^1(\mathbb{R}^N, \frac{dx}{1+|x|^{N+2\alpha}})} \\ & + \|u\|_{L^\infty(B_{r_{j_0+2}}(a))} + \|\nabla u\|_{L^\infty(B_{r_{j_0+2}}(a))} \Bigg), \end{aligned} \tag{6.66}$$

for any $\beta < 2\alpha$, we obtain that u remains bounded in $C^{1+\varepsilon}(K)$ for any compact set $K \subset \Omega \setminus \{0\}$ and some $\varepsilon > 0$. Using now [88, Corollary 2.4], we obtain that $C^{2\alpha+\varepsilon'}(\Omega \setminus \{0\})$ for $0 < \varepsilon' < \varepsilon$. Furthermore u is continuous up to $\partial\Omega$. As a consequence it is a strong solution in $\Omega \setminus \{0\}$. □

In the next result we give a pointwise estimate of ∇u for a positive solution u of (6.47).

Proposition 6.4.3 *Assume that $R = \frac{1}{2}dist(0, \partial\Omega)$, $p \in (0, p^*_\alpha)$ and u is a nonnegative weak solution of (6.47). Then there exists $c_{28} > 0$ depending on R, p and α such that*

$$|\nabla u(x)| \leq c_{28}|x|^{2\alpha-N-1}, \qquad \forall x \in \bar{B}_{R/4}(0) \setminus \{0\}. \tag{6.67}$$

Proof. Up to a change of variable we can assume that $R = 1$. For $0 < |x| \leq 1$, there exists $b \in (0, 1)$ such that $b/2 \leq |x| \leq b$. We set

$$u_b(y) = b^{N-2\alpha}u(by).$$

Then

$$(-\Delta)^\alpha u_b + b^{N+p(2\alpha-N-1)}|\nabla u_b|^p = 0 \qquad \text{in } \ \Omega_b := b^{-1}\Omega.$$

Using [88, Corollary 2.5] with $\beta < 2\alpha$, for any $a \in \Omega_b$ such that $|a| = 3/4$, there holds

$$\begin{aligned}\|u_b\|_{C^\beta(B_{\frac{3}{16}}(a))} \le c_{29}\Bigg(&\|u_b\|_{L^1(\mathbb{R}^N,\frac{dx}{1+|y|^{N+2\alpha}})} + \|u_b\|_{L^\infty(B_{\frac{3}{8}}(a))} \\ &+b^{N+p(2\alpha-N-1)}\||\nabla u_b|^p\|_{L^\infty(B_{\frac{3}{8}}(a))}\Bigg).\end{aligned} \tag{6.68}$$

Furthermore, by the same argument as in Proposition 6.4.2,

$$\||\nabla u_b|^p\|_{L^\infty(B_{\frac{3}{8}}(a))} \le c_{30}\int_{\Omega_b}|\nabla u_b(y)|^p dy = c_{30}b^{p(N+1-2\alpha)-N}\int_\Omega |\nabla u(x)|^p dx, \tag{6.69}$$

and from (6.50) and (6.23)

$$u(x) \le G_\alpha(x,0) \le \frac{c_4}{|x|^{N-2\alpha}} \Longrightarrow u_b(y) \le \frac{c_4}{|y|^{N-2\alpha}}.$$

Then

$$\|u_b\|_{L^1(\mathbb{R}^N,\frac{dy}{1+|y|^{N+2\alpha}})} \le c_4\int_{\mathbb{R}^N}\frac{dy}{|y|^{N-2\alpha}(1+|y|)^{N+2\alpha}} = c_{31}.$$

If we take $\beta = 1$, which is possible since $\alpha > 1/2$, we derive

$$|\nabla u_b(a)| \le c_{32} \Longrightarrow |\nabla u(ba)| \le c_{32}^{-1}b^{2\alpha-N-1}$$

In particular, with $|b| = 4|x|/3$ we derive (6.67) with $c_{28} = c_{32}^{-1}(\frac{4}{3})^{2\alpha-N-1}$. □

We denote

$$c_{N,\alpha} = \lim_{x\to 0}|x|^{N-2\alpha}G_\alpha(x,0). \tag{6.70}$$

It is well known that $c_{N,\alpha} > 0$ does not depend on the domain Ω and, by the maximum principle, $G_\alpha(x,0) \le c_{N,\alpha}|x|^{2\alpha-N}$ in $\Omega\setminus\{0\}$.

Theorem 6.4.1 *Let Ω be an open bounded C^2 domain containing 0, $\alpha \in (\frac{1}{2},1)$ and $0 < p < p^*_\alpha$. If u is a positive solution of problem (6.47) and $\bar{B}_R(0) \subset \Omega$, it satisfies*

(i) if $\frac{2\alpha}{N-2\alpha+1} < p < p^*_\alpha$,

$$0 < \frac{c_{N,\alpha}}{|x|^{N-2\alpha}} - u(x) \le \frac{c_{33}}{|x|^{(N-2\alpha+1)p-2\alpha}}, \quad x \in B_{R/4}(0)\setminus\{0\};$$

(ii) if $p = \frac{2\alpha}{N-2\alpha+1}$,

$$0 < \frac{c_{N,\alpha}}{|x|^{N-2\alpha}} - u(x) \le -c_{33}\ln(|x|), \quad x \in B_{R/4}(0)\setminus\{0\};$$

(iii) if $0 < p < \frac{2\alpha}{N-2\alpha+1}$,

$$0 < \frac{c_{N,\alpha}}{|x|^{N-2\alpha}} - u(x) \le c_{33}, \quad x \in B_{R/4}(0) \setminus \{0\},$$

where c_{33} *depends on* N, p, α, R *and is independent of* u.
Furthermore, if $1 \le p < p^*_\alpha$, *this solution is unique.*

Proof. The existence of a nonnegative weak solution is a consequence of the subriticality assumption; the fact that this solution is a classical solution in $\Omega \setminus \{0\}$ derives from Proposition 6.4.2. It follows by (6.50) and (6.52) that for any $x \in \Omega \setminus \{0\}$,

$$\begin{aligned}
\frac{c_{N,\alpha}}{|x|^{N-2\alpha}} - u(x) &\le \int_\Omega G_\alpha(x,y)|\nabla u(y)|^p dy \\
&\le c_{28}^p c_4 \int_{B_{\frac{R}{4}}(0)} |x-y|^{2\alpha-N}|y|^{p(2\alpha-N-1)}dy + c_{34}\|\nabla u\|_{L^p(\Omega)} \\
&\le c_{35}\left[\int_{B_{\frac{R}{4}}(0)} |x-y|^{2\alpha-N}|y|^{p(2\alpha-N-1)}dy + 1\right]
\end{aligned} \tag{6.71}$$

where $c_{34}, c_{35} > 0$ depend on N, p and α. Next we assume $0 < |x| \le \frac{R}{16}$.
Case 1: $\frac{2\alpha}{N-2\alpha+1} < p < p^*_\alpha$. We can write

$$\int_{B_{\frac{R}{4}}(0)} |x-y|^{2\alpha-N}|y|^{p(2\alpha-N-1)}dy = E_1 + E_2$$

with

$$E_1 = \int_{B_{\frac{R}{4}(0)} \setminus B_{\frac{R}{8}}(0)} |x-y|^{2\alpha-N}|y|^{p(2\alpha-N-1)}dy \le c_{36},$$

and

$$E_2 = \int_{B_{\frac{R}{8}}(0)} |x-y|^{2\alpha-N}|y|^{p(2\alpha-N-1)}dy,$$

where $c_{36} > 0$ depends on N, α, p and R. Let $\xi = x/|x|$, then

$$\begin{aligned}
E_2 &= |x|^{2\alpha-p(N+1-2\alpha)}\int_{B_{\frac{R}{8|x|}}(0)} |\xi-\zeta|^{2\alpha-N}|\zeta|^{p(2\alpha-N-1)}d\zeta \\
&\le \int_{|\zeta|>2} |\xi-\zeta|^{2\alpha-N}|\zeta|^{p(2\alpha-N-1)}d\zeta \\
&\le c_N \int_2^\infty (r-1)^{2\alpha-N} r^{p(2\alpha-N-1)+N-1}dr = c_{37},
\end{aligned}$$

where the last inequality holds by the fact of $2\alpha-N<0$, $|\xi-\zeta|^{2\alpha-N}\le(|\zeta|-1)^{2\alpha-N}$. Thus (i) follows.

Case 2: $\frac{2\alpha}{N-2\alpha+1}=p$. We see that

$$E_2=\int_{B_{\frac{R}{8|x|}}(0)}|\xi-\zeta|^{2\alpha-N}|\zeta|^{-2\alpha}d\zeta,=-\ln|x|+o(1)\quad\text{as}\quad|x|\to 0.$$

Thus (ii) follows.

Case 3: $0<p<\frac{2\alpha}{N-2\alpha+1}$. We have that

$$E_2=\int_{B_{\frac{R}{8|x|}}(0)}|\xi-\zeta|^{2\alpha-N}|\zeta|^{-2\alpha}d\zeta=c_{29}|x|^{p(N+1-2\alpha)-2\alpha}+o(1)\quad\text{when}\quad|x|\to 0.$$

Thus (iii) follows.

Uniqueness in the case $1\le p<p^*_\alpha$, is very standard, since if u_1 and u_2 are two positive solutions of (6.47), they satisfies

$$\lim_{x\to 0}\frac{u_1(x)}{u_2(x)}=1.$$

Then, for any $\varepsilon>0$, $u_{1,\varepsilon}:=(1+\varepsilon)u_1$ is a supersolution which dominates u_2 near 0, it follows by the maximum principle that $w:=u_2-(1+\varepsilon)u_1$ satisfies

$$(-\Delta)^\alpha w+|\nabla u_2|^p-|\nabla u_{1,\varepsilon}|^p\le 0$$

since w is negative near 0 and vanishes on $\partial\Omega$, if it is not always negative, there would exists $x_0\in\Omega\setminus\{0\}$ such that $w(x_0)$ reaches a maximum and $|\nabla u_2(x_0)|=|\nabla u_{1,\varepsilon}(x_0)|$, thus $(-\Delta)^\alpha w(x_0)\le 0$, contradiction. $\square$

Remark 6.4.1 *If $0<p<1$, the nonlinearity is not convex and uniqueness does hold only if two solutions u_1 and u_2 satisfy*

$$\lim_{x\to 0}(u_1(x)-u_2(x))=0.$$

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