METAPHORS AND COGNITIVE MODES IN THE TEACHING-LEARNING OF MATHEMATICS

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The didactic role of metaphors and cognitive modes as well as their interplay is discussed, based on examples. Transition from one cognitive mode to another is illustrated, in case studies with students and in-service teachers. Its relevance to the learning process is appraised.

INTRODUCTION

In this paper we intend to continue the research undertaken in Soto-Andrade (2006), presenting further examples of didactical uses of metaphors and cognitive modes, as well as exploring the interplay between them, as they emerge in the didactical praxis.

This exploration requires a first-person approach, in the sense of Varela & Shear (1999). Indeed, metaphors having a deeper cognitive thrust are usually those that entail a switch in the cognitive mode of the subject perceiving them. For instance, when you approach solving linear equations with the help of the "scales metaphor", you switch from a verbal cognitive mode to a non verbal one: Instead of checking an equality by an arithmetic or algebraic calculation, you put and take out weights on both pans of a scale, trying to preserve balance.

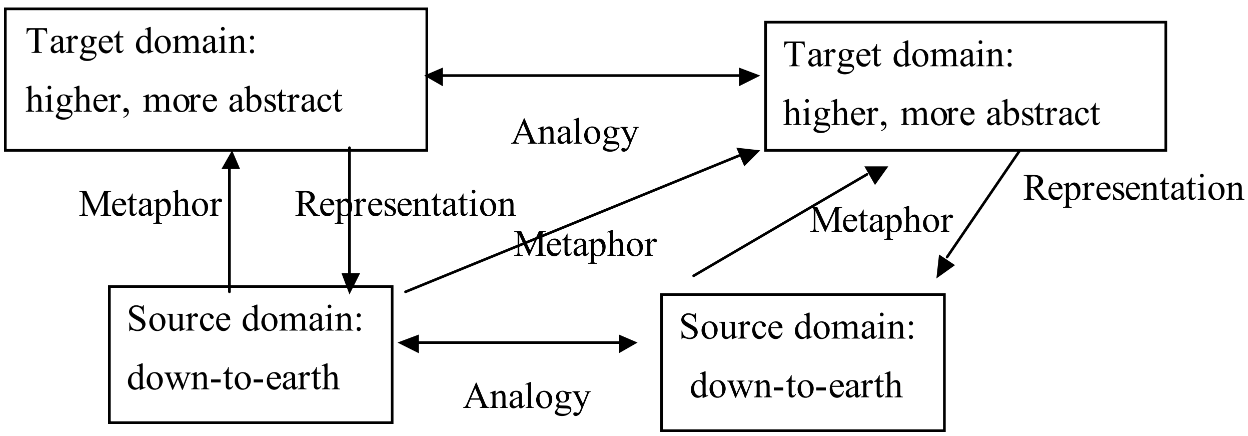
After setting up our tentative theoretical framework, we set down our main research hypotheses, related to our teaching experiments, and proceed to report on some specific examples of metaphors and cognitive modes in action, that give preliminary experimental evidence to support our hypotheses and suggest further research along these lines, that we discuss in the final section.

THEORETICAL FRAMEWORK

Nature and Role of Metaphors

It has been progressively recognized during the last decade (Araya, 2000; Bills, 2003; Edward, 2005; English, 1997; Ferrara, 2003; Johnson & Lakoff, 2003; Lakoff & Núñez, 2000; Parzysz et al., 2003; Pouilloux, 2004; Presmeg, 1997; Seitz, 2001; Sfard, 1994, 1997, and many others) that metaphors are not just rhetorical devices, but powerful cognitive tools, that help us in building or grasping new concepts, as well as in solving problems in an efficient and friendly way: "metaphors we calculate by" (Bills, 2003). We meet conceptual metaphors (Lakoff & Núñez, 2002), that appear as mappings from a "source domain" into a "target domain", carrying the inferential structure of the first domain into the one of the second, and enable us to understand the latter, usually more abstract and opaque, in terms of the former, more down-to-earth and transparent.

The term “metaphor” is often nowadays taken in a loose sense, as a synonym of “representation”, “analogy”, “model”, “image”, etc. (Parzysz et al., 2003). We intend nevertheless to be more precise: the following diagram may be helpful to clarify our viewpoint on the difference among metaphors, representations and analogies.



So, as indicated, in operational terms, conceptual metaphors “go up”, representations “go down” and analogies, “go horizontally” both ways. Notice that we take analogy in a rather narrow sense, kin to a “simile” (that draws an explicit comparison between two different things), symmetric in nature, and not as an “umbrella” concept embracing metaphors, representations, similes, etc. So our viewpoint is closer to Sfard’s (1997) than to Presmeg’s (1997). We may have, moreover, metaphors going up from different source domains to the same target domain and also from the same source domain to different target domains.

Notice however that this scheme doesn’t impair the subjective aspects of the difference between metaphor and representation. For instance, if probability is a new concept or a concept under construction for us, then “probabilities are masses or weights” is clearly a metaphor for us, that helps us to grasp the concept of probability, or better, to build it. On the other hand, if we are to some extent already familiar, albeit not quite comfortable, with probabilities, we may realize that probabilities may be represented by masses, we feel more at home with.

Others might want to say that there is an analogy between probabilities and masses, because they see analogy as a symmetrical relationship and they see probabilities and masses on the same footing.

Anyway, be metaphor, representation or analogy, we gather that to solve probabilistic problems we may just solve mass or weight problems, where we can take advantage of our physical intuition, in static or dynamic settings.

Example: Metaphors for multiplication

Both addition and multiplication of numbers are commutative. It is however an interesting metaphorical insight of Lakoff & Núñez (2000) that while addition is a priori commutative, multiplication is only a posteriori commutative, we might say. For instance, addition of vectors, complex numbers, quaternions or matrices is as commutative as the addition of real numbers. However, multiplication of quaternions or matrices is not in general commutative. So, in a “good” metaphor for addition commutativity should be built-in, but not so in a good one for multiplication.

This may be compared with Soto-Andrade (2006), where the “product is area” metaphor and the “multiplication is concatenated branching” metaphor are presented, as two ways of “seeing” that 2 x 3 = 3 x 2, as illustrated below:

In the first one, commutativity of multiplication is perceived as invariance of area under rotation in one fourth of a turn. So you “see” that 2 x 3 = 3 x 2, without even knowing that it is 6, like the Amazonian Indians in Dehaene (2004). In the second one, commutativity of multiplication is less obvious: it is perceived as the fact that the order in which we concatenate branchings is irrelevant for the final harvest. If one has played around a lot with tree diagrams, this metaphor may become a “met-before” in the sense of Tall (2005). But otherwise one would rather count… This suggest indeed that multiplication is not a priori commutative.

Cognitive styles and modes

The concept of cognitive styles emerged from work by Neisser (1967), Luria (1973) and de La Garanderie (1989) and was further developed by Flessas (1997) and Flessas & Lussier (2005), who pointed out to their impact on the teaching-learning process.

A cognitive style is defined nowadays as one’s preferred way to think, perceive and recall, in short, to cognize. It reveals itself, for instance, in problem solving.

The term “cognitive mode” is often used as a synonym to “cognitive style”. It suggests however a way of cognizing that is usually more transient and not as stable, or even rigid, as a cognitive style is. Since one of our theses is that the ability to switch from one way of cognizing to another is trainable, we will rather say “cognitive mode” instead of “cognitive style” in this paper, so that Flessas and Lussier’s “styles cognitifs” will become “cognitive modes” from now on.

To generate what they call the 4 basic cognitive modes, Flessas and Lussier (2005) combine 2 dichotomies: verbal – non verbal and sequential – non sequential (or simultaneous), closely related to the left – right hemisphere and frontal – parietal

The 4 cognitive modes	VERBAL	NON - VERBAL
SEQUENTIAL	S-V	S-NV
NON - SEQUENTIAL	NS-V	NS-NV

Example: Solving a problem through different cognitive modes.

How can you check that you have the same number of fingers in your hands?

You can approach this problem with different cognitive modes.

You can count the fingers in your left hand first: one, two, three, four, five. Then, you do the same with the fingers in your right hand and you discover that you have the same number of fingers in both hands, indeed. This a typical *verbal and sequential* cognitive mode approach.

But you can also, in a single gesture, put into a one to one correspondence the fingers in both your hands, in a natural way. This is a typical *non verbal and non sequential* approach. You don't name the numbers, you don't count, you don't write any formulae. Moreover your checking is simultaneous, non sequential, because you can make your homologous fingers touch in just one simultaneous gesture.

If you make each finger of your right hand touch one finger of your left hand, one by one, you would be using a *non-verbal, sequential* cognitive mode.

Flessas and Lussier emphasize that effective teaching of a group of students, who may display a high degree of cognitive diversity, needs teachers supple enough to be able to tune easily to the different cognitive modes of the students. This necessary competence has a neurological correlate that can be imaged and monitored in contemporary neuroscience (Dehaene, 1997, 2004; Varela & Shear, 1999).

In what follows we adhere mainly to the framework laid by Lakoff & Núñez (2000) for metaphors and Flessas & Lussier (2005) for cognitive modes.

PROBLEMATICS

We claim that most in-service teachers are not familiar with either metaphors or cognitive modes, in relation with their teaching of mathematics. More precisely:

Most teachers are frozen in just one cognitive mode, unaware of it to begin with, and so unable to switch to another one. They are also unaware that their teaching is shaped by unconscious and misleading metaphors, like the container-filling metaphor or the gastronomic metaphor (called "métaphore alimentaire" in Soto-Andrade (2005)).

Moreover, their metaphor quiver is poorly furnished: they would rarely have more than one metaphor for each mathematical concept or process and they have trouble creating "unlocking metaphors" for their students.

Students, on the other hand, are not stimulated to work in more than one cognitive mode and they have very often the feeling that tackling a problem in a different cognitive mode that the one it came wrapped up, is definitely bad manners.

RESEARCH HYPOTHESES

Our main research hypothesis is that metaphors and cognitive modes are key ingredients in a meaningful teaching-learning process. Moreover the deepest impact on this process is usually attained by metaphors that involve a switch in the previous cognitive mode of the subject.

We also claim that competences regarding multi-mode cognition and use and creation of metaphors and representations are trainable and that measurable progress can be achieved in a one semester course or even in a one week workshop. This, in spite of the fact that most teachers report that their initial training included no metaphors and privileged just one cognitive mode: the usually dominant verbal-sequential one.

We conjecture that on the average primary school teachers will be more successful in learning to switch cognitive modes and evoking and using metaphors than secondary school teachers.

Regarding students, our working hypothesis is that they would significantly improve their learning if they were able to approach problems with more than one cognitive mode and to draw from a suitable spectrum of metaphors.

We present below some examples, tested in teaching mathematics courses to various audiences of students, to illustrate the didactical use of different cognitive modes for approaching mathematical objects and their interplay with the use of metaphors.

RESEARCH BACKGROUND AND METHODOLOGY

The background for our experimental research consisted in several courses, to wit:

- Mathematics 0: A one semester, general mathematics course, given to first year students of the Bachelor in Humanities and Social Sciences Program at the University of Chile. Its formal aim is to teach the students "all the mathematics" they will need during their university studies, besides statistics. Its real aim is to introduce them to the mathematical way of thinking and to the cognitive attitudes of mathematics. Classes have 35 students, lessons are 3 hrs a week. Experiments are carried out during the lessons (90 minutes each) and during exams (2 hours each, 4 in a semester). Lessons are interactive, questions and activities are suggested and students propose ways to tackle them. Students engage often in "horizontal" discussions but not so much in group work. Gleaned knowledge is periodically "harvested" and recorded in more formal language.

- Random walks in "Metaphorland" (Paseos al azar en el país de las metáforas): A one semester optional mathematics course, addressed to all students of the University of Chile. The aim of the course is to introduce them to the power of metaphorical thinking in mathematics, while performing a "random walk" through several key topics, like randomness, symmetry, infinity and the systemic approach. The class had 70 students in 2006. Lessons consist mainly of group work, in small groups of 4 to 5 students. Activities and problematic situations are proposed, to be tackled by the students, each group working on its own first, then putting together their findings.

- Numbers: One yearly module (220 hrs approx.), for 2 classes of 30 primary school teachers, in the post-graduate program of the University of Chile, for in-service teachers who didn't major in mathematics in their initial formation. The aim of this module is to review the mathematics as well as the didactics of numbers, specially fractions, ratios, decimal and binary description of numbers. The teachers usually work in interactive sessions, forming small groups of 3 to 4 teachers.

The methodology consisted in observing the students and teachers, as they carried out various activities, as in the examples described below, that were proposed during lessons, group work sessions and as a part of exams and diagnostics. Records of this observation comprised the written and drawn production of the students and some transcriptions.

EXPERIMENTAL ACTIVITIES AND PRELIMINARY RESULTS:

Example 1: Who has more marbles?

John and Mary have a bag of marbles each, all of the same size. How can they decide who has more marbles?

They could take the marbles out of each bag, one by one, count them and compare the resulting numbers. This is a verbal-sequential approach, the most frequent one.

Working with two separate classes of 30 in-service primary teachers, organized in small groups (3 to 4 each), we invited them to figure out other approaches, which would involve non-verbal or non-sequential modes. They were also asked how they would work this problem with their students. In a few minutes, they came out with:

- John pulls out his marbles and Mary hers, one by one, and without counting them, they put them side by side, in pairs, sequentially, until one of them, or both simultaneously, runs out of marbles. They recognized this as non verbal – sequential.

- To weigh the bags in one's hands to assess which is heavier. If it is hard to tell, weigh then in a scale, one bag in each pane (non verbal – non sequential approach).

- After some 15 min. discussion on the non sequential – verbal approach, 3 teachers got the idea of weighing the bags simultaneously in 2 digital scales and compare the readings.

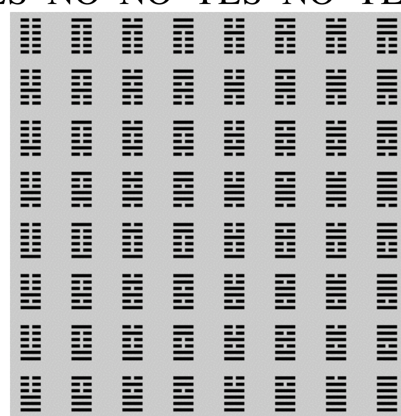
However, roughly 80% of the teachers reported that they had never before tried to employ more than one cognitive mode to solve this type of problem.

Example 2: The number sequence, otherwise…

Is it possible to represent the numerical sequence 0, 1, 2, 3, …. up to 63, let us say, in a non verbal and non sequential way?

We presented this challenge to the courses described above, suggesting to try first representing the sequence in non verbal - sequential way. As a preliminary, we proposed to the students in the classroom to try to get the binary description of their number without counting themselves first, as in Soto-Andrade (2006).

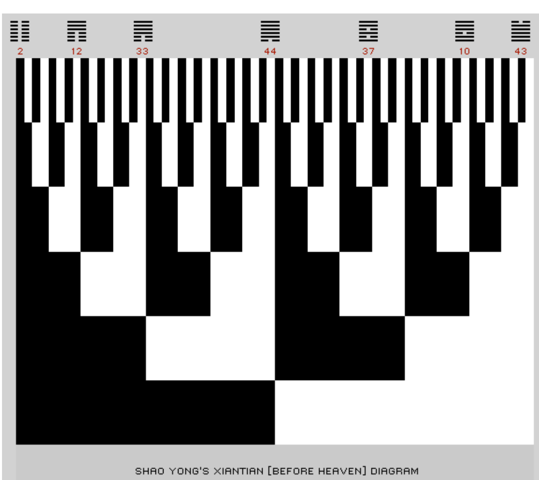
To do this, they just stand up, trying to match up in pairs, checking whether there was one "odd man out". Then the pairs did the same, and so on. When the pairing game was over, we asked: Is there an unmatched person? An unmatched pair? An unmatched quadruplet? and so on. They answered: YES–NO–NO–YES–NO–YES.

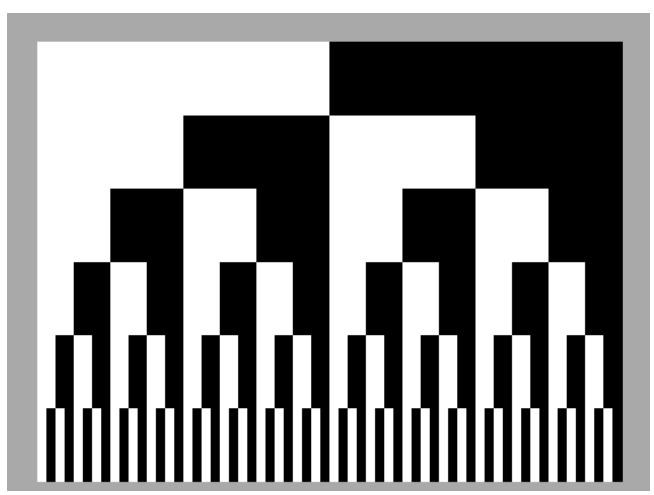


When prompted to codify this in a non verbal way, they eventually rediscovered the I Ching (Yi Jing) codification: a broken line for NO, a continuous line for YES, or something equivalent. Reading hexagrams from top to bottom, they got the one in the 2nd column, 6th row, in the figure to the right (i. e. number 41). Notice that this square arrangement due to Chinese philosopher and mathematician Shao Yong (1011-1077), displays the binary sequence of numbers 0 to 63, in their natural order. Students in all 3 courses successfully completed this activity and reported later having understood for the first time the binary description of a number or quantity. After this non-verbal approach to the binary description of numbers, we suggested the harder challenge:

Approach the binary hexagram sequence in a non verbal and simultaneous way, by encapsulating it in a single image that can be reconstructed from just a glimpse of it.

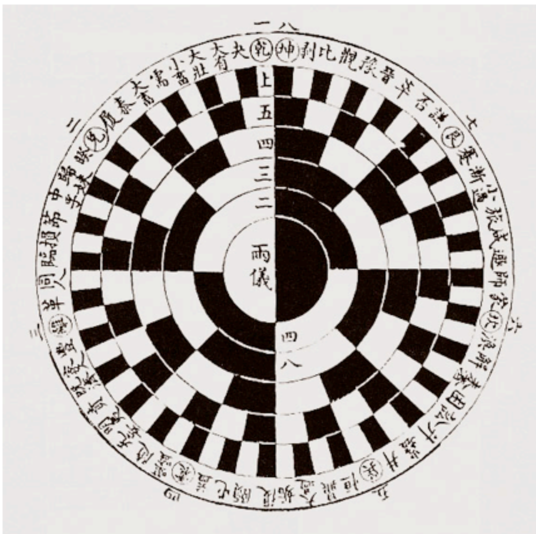
In a first class of 30 primary school teachers, after 30 min. work in small groups, 5 of them came up with diagrams equivalent to famous Shao Yong's Xiantian ("Before Heaven") diagram or its inverted form (Marshall, 2006), illustrated below.

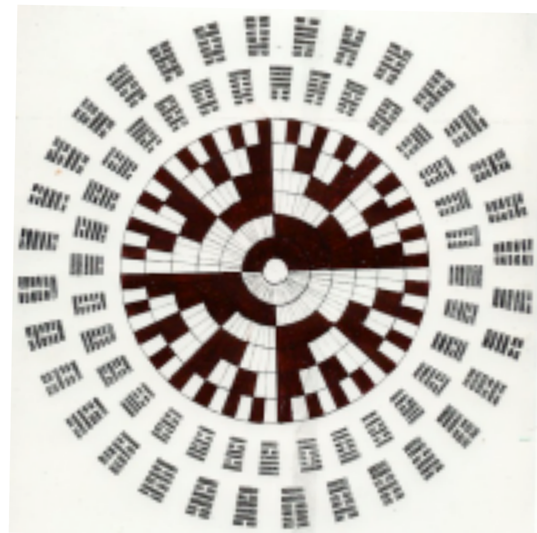


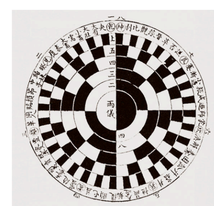


Notice the underlying binary tree! In a second group of 30 teachers, 6 rediscovered Xiantian and, most remarkably, one of them, Ofelia, draw all by herself a circular version of the Xiantian diagram, that is a rotation of the classical one, unrolling counterclockwise. See below the classical version (left) and Ofelia's version (right).

Yijing Dao - A 'compass rose' star in the circular Xiantian diagram http://www.biroco.com/yijing/xiantianstar.htm







They also noticed, as pointed out by S. J. Marshall (2006), that this circular diagram reveals a compass rose when looked at from a distance. When interviewed, they recurrently reported:

- I had just learned by heart a recipe to transform usual numbers to binary form, but now (after playing the pairing-off game) I understand it for the first time!
- I would have never thought of this way of approaching binary numbers!
- This is really new to me! I have some trouble in getting used to it, because I am too structured and used to seeing things always from the same viewpoint.

Approx. 5% of the students in the Random Walks course, when exposed to Xiantian (without the 7 examples above the square), quickly realized how to recover the whole sequence from this image. The circular Xiantian was tested in Maths. 0 exams, as an optional question: out of 40 students, 50% did choose this question and 78% of them reconstructed the circular diagram after two 2 second glimpses of it. Out of the latter, 40% explained correctly how to recover the binary sequence from circular Xiantian.

DISCUSSION

We have shown, through several activities carried out in the classroom, how classical mathematical objects and problematic situations may be unexpectedly approached with cognitive modes different from the usually dominant verbal-sequential one.

We have seen how to facilitate the activation of these less usual cognitive modes, even for in-service teachers who never had this sort of experience before. After some prompting, a high percentage of students and teachers were able to switch from their dominant verbal-sequential cognitive mode to a non verbal or non sequential one. In this way, some of them rediscovered representations of - or metaphors for - familiar mathematical objects, developed in other cultures (like the ancient Chinese, for instance), that favoured more than ours non verbal and non sequential cognitive modes: "One image is worth 1000 words…" they said.

According to their reports, taking advantage of more than one cognitive mode fostered their understanding of important mathematical objects and processes, like the binary description of numbers.

Our observations show that the ability to approach the same object through various cognitive modes and transiting from one cognitive mode to other, is trainable, in students as well as in teachers. Experimentation suggests that this is facilitated by group work. First person reports by students and teachers bear witness of the impact and meaningfulness that this sort of cognitive experience had for them.

The experiences carried out reveal that often the activation of a different cognitive mode, when approaching a mathematical object, entails the emergence of a metaphor, or a representation, depending on the previous background of the subject. It remains to be checked that, vice versa, students and teachers looking for a useful metaphor to get hold of or to build a new concept, will learn to switch to another cognitive mode.

It would be interesting to test and measure the depth of learning that students may achieve when taught with the help of a broad spectrum of metaphors and various cognitive modes and to undertake the design of unblocking metaphors.

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