Bruhat presentations for $*-$classical groups

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1 Introduction

One basic idea that has turned out to be fruitful in the presentation and representation theory of classical groups is to look at higher rank groups as "non commutative analogues" of related lower rank groups. A typical example is the case of the symplectic similitude group $GSp(2n, F)$ in $2n$ variables over a field F, which may be looked upon as a sort of $GL(2)$ with coefficients in the full matrix ring over F, that satisfy suitable commutation relations involving the transpose map $*$. Here the symplectic multiplier appears as a $*-$analogue of the classical 2×2 determinant.

The rationale behind this viewpoint is to extend to higher rank groups, methods that have been successful for lower rank groups. In particular, finding new presentations for higher rank classical groups by looking for non commutative versions of well known presentations for lower rank ones. Then these presentations may be used to construct remarkable linear representations for the higher rank groups, like (generalized) Weil representations, for instance.

In the case of our example above, recall that construction and decomposition of Weil representations associated to quadratic forms affords a uniform and universal solution to the problem of constructing all complex irreducible linear representations of the group $GL(2, F), F$ a finite field [10]. This method extends as well to the case of a local field (with the exception of the residual characteristic 2 in the non-archimedean case [7]). These Weil representations may be constructed in an elementary way with the help of a presentation of $GL(2, F)$ derived from its Bruhat decomposition, with generators for the Borel subgroup and the Weyl element ω. Now, it can be shown [10] that this presentation carries over to $GSp(2n, F)$, looked upon as a "non commutative" $GL(2)$, affording a presentation much simpler than the one found by Dickson at the turn of the century [1]. Then we can construct Weil representations by giving the linear

[1]Both authors were partially supported by Fondecyt grant 1060517. The first author was also partially supported by P. Universidad Católica de Valparaíso.

operators associated to the generators and checking the corresponding relations [10], an approach going back to P. Cartier.

This suggests to undertake a systematic study of non commutative generalizations of the classical groups, in the "tamely non commutative case" of a ring with involution as coefficient ring, with the aim of finding Bruhat - like presentations for them, to begin with. Recalling that our motivation stems from the coincidence of the classical 2 by 2 determinant with the multiplier associated to a non-degenerate *antisymmetric binary form*, a natural generalization for us is to consider any non-degenerate $\varepsilon-$ *hermitian form* H with respect to a given (anti-) involution $*$ on the base ring A. Here the sign $\varepsilon = \pm 1$ takes care simultaneously of the hermitian and anti hermitian case. In all cases we have then the corresponding unitary similitude monoid $MU_*^\varepsilon(H)$ and its central $\varepsilon-$symmetric valued unitary multiplier μ_H, but we will concentrate on the unitary similitude group $GU_*^\varepsilon(H)$, the group of invertible elements in $MU_*^\varepsilon(H)$. In the rank 2 case, the multiplier μ_H will afford a non commutative $*$ and $\varepsilon-$analogue of the determinant on suitable 2 by 2 matrices with coefficients in the involutive ring A. In this way, both even rank orthogonal and symplectic groups appear as particular cases of a general construction, based on forms over rings with involution in the sense of [2].

This construction gives back the classical linear groups, in various guises, for semi-simple A, but affords also other groups, for a non-semisimple ring A. The latter case is illustrated in [3] where A is taken to be a truncated polynomial ring over a finite field, a modular analogue of an algebra of k-jets in one variable, endowed with the canonical involution $X \mapsto -X$.

As said before, we are specially interested in the case of an ε - hermitian form of rank 2, because its unitary similitude group appears as a sort of non commutative involutive ε - analogue of the classical $GL(2, F)$. If we are able to extend the classical presentation of $GL(2, F)$ based on its Bruhat decomposition (see [8]) to our involutive ε - analogues $GL_*^\varepsilon(2, A)$ of $GL(2, F)$, we could try to construct Weil representations for all these groups. As mentioned above, these ideas were used already in [10] in the particular case $A = M(n, F)$, F a finite field, the transpose involution and $\varepsilon = -1$, to obtain a uniform construction of all irreducible complex representations of $GSp(4, F)$. Later, a Bruhat decomposition was obtained in the case of an artinian involutive base ring A in [9] and the classical presentation of $GL(2, F)$ was extended to the case of an artinian simple involutive A in [8]. Recently they have been applied to the nilpotent case (in the sense that the radical of the ring is nilpotent), see [3], with A a truncated polynomial ring over a finite field endowed with a non trivial involution, where an analogue of the classical presentation is obtained and a Weil representation is constructed.

We recall that in the trivial involution case for the ring $A = \mathbb{Z}/p^n\mathbb{Z}$ a Weil representation for $SL(2, A)$ has been constructed by a different method by Szechtman ([11]), who also solves the decomposition problem.

However this problem remains open for most of Weil representations so constructed, although their commuting algebras may be described as easily as in the reductive pairs approach to Weil representations [4].

It is a remarkable fact that an analogue of the classical Bruhat decomposition still holds when the involutive base ring is a full matrix ring, with the transpose as involution, as well as the ring $\mathbb{Z}$ with the trivial involution or a nilpotent polynomial ring with the canonical involution. As described below the existence of this sort of presentations seems to be closely related to the existence of a weak non commutative analogue of the euclidean algorithm in the base involutive ring.

Notice also that we consider here only coefficient rings A that are "tamely" non commutative, in the sense that non commutativity is "controlled" by an (anti-) involution $*$: We have the relation $(ab)^* = b^* a^*$, for all $a, b \in A$, which reduces to commutativity in case the anti-involution $*$ is the identity

We remark that the anti-involution $*$ plays a role analogous to the universal $R-$matrix that controls non-commutativity for quantum groups (more precisely, non-cocommutativity for a Hopf algebra H or just a bialgebra A). Recall that a bialgebra A is called quasi-cocommutative when it is endowed with a universal R-matrix, which is an invertible element of $A \otimes A$, equal to $1 \otimes 1$ in the co-commutative case (see [6]). This explains to some extent the striking analogy between our $GL_*(2, A)$ group with its $*$ -determinant and the quantum group $GL_q(2)$ with its q- determinant, described the way physicists do, as a "group" of matrices whose coefficients satisfy certain commutation relations.

In this note, after introducing in a general setting the aforementioned groups, we specialize to the case of an ε - hermitian form of rank 2 to obtain non commutative involutive ε determinants, and we show next how to recover this notion exploiting Grassman's approach to determinants in this case. Finally we present various examples of our groups and give some categorical properties of them.

We thank Phil Kutzko for the enriching conversations we had with him and for encouraging us to pursue on these topics.

2 The unitary similitude group $GU(H)$

In this section we present a general setup for what we call generalized classical groups. We rapidly concentrate ourselves on the case of rank 2 to study more closely involutive analogues of the general linear group and the special general linear groups in dimension 2. We present also a notion of generalized determinant for these groups.

The case of general dimension n will be considered elsewhere.

In what follows A will denote an involutive ring, with involution $*$.

We denote by $Z(A)$ the center of A, and by $Z_s(A)$ the subset of its $*$ symmetric elements, i.e., the set of elements $a \in Z(A)$ such that $a^* = a$.

Furthermore, $Z_s(A)^\times$ will denote the group of $*$ symmetric, central, invertible elements of A.

Rings with an involution form a category $\mathcal{U}$ whose objects are the pairs $(A, *)$ as above and the morphisms $\eta : (A, *) \to (B, *')$ are homomorphisms of rings

$\eta : A \to B$ such that $*' \circ \eta = \eta \circ *$. We will say that $\mathcal{I}$ is an ideal of the involutive ring $(A, *)$ if $\mathcal{I}$ is an ideal of the ring A such that $y^* \in \mathcal{I}, \quad$ for each $y \in \mathcal{I}$. It follows that the kernel of a homomorphism η of rings with involution is an ideal in this sense.

Definition 1 *An involutive* (A, A)*-bimodule is an* $(A, A)-$ *bimodule* V *endowed with an additive involutive endomorphism, denoted also* $*$*, such that*

$$(avb)^* = b^* v^* a^*$$

for every $a, b \in A$ *and* $v \in V$.

We say that an involutive (A, A)*-bimodule* V *is free if there is a family of central symmetric elements in* V *that is simultaneously a basis for* V *as a left* $A-$*module and as a right* $A-$*module. We will call such a basis a bi-basis. We call central those elements* $v \in V$ *such that* $av = va$ *for all* $a \in A$*, and we call symmetric those* $v \in V$ *such that* $v^* = v$.

Notice that the involution $*$ of the bimodule V is antilinear as a map from the left $A-$ module V to the right $A-$ module V and also as a map from the right $A-$ module V to the left $A-$ module V.

Examples 1

i. *The involutive ring* A *itself is a free involutive* (A, A)*-bimodule, via left and right multiplication, the unit element* 1 *affording a bi-basis of* A.

ii. *More generally the direct sum of any family of copies of the free involutive bimodule* A *is a free involutive* (A, A)*-bimodule.*

iii. *The set* $\mathcal{F}_A(X)$ *of all finitely supported* $A-$*valued mappings defined on a set* X *becomes in a natural way an involutive* $(A, A)-$ *bimodule if we define*

$(af)(x) = af(x)$

$(fa)(x) = f(x)a$

$f^*(x) = (f(x))^*$

for all $x \in X, f \in \mathcal{F}_A(X), a \in A$.

The involutive bimodule $\mathcal{F}_A(X)$ *is also free, because it admits the bi-basis consisting of all Dirac's delta functions* $\delta_x \ (x \in X)$*, which take the value* 1 *at* $x \in X$ *and vanish elsewhere. Notice that with the help of this bi-basis we get immediately an isomorphism of involutive bimodules between* $\mathcal{F}_A(X)$ *and the direct sum* $\bigoplus_{x \in X} A_x$ *where* $A_x = A$ *for all* $x \in X$.

Definition 2 *A function* $H : V \times V \to A$ *is called a left* ε*-hermitian form if* H *is biadditive, left linear in the first variable and such that* $H(y, x) = \varepsilon H(x, y)^*$ *for all* $x, y \in V$*. Right* ε*-hermitian forms on* V *are defined analogously, so that they are right linear in the first variable.*

We say that the left ε-hermitian form H is *regular* iff its image contains a non left zero divisor of A, i. e. an element $a \in A$ such that $xa = 0$ implies $x = 0$, for $x \in A$.

We note that if $g \in End^l_A(V) = Hom^l_A(V,V)$ (g a left A-module homomorphism of V), then
$(H \circ (g \times g))(v,u) = H(g(v), g(u)) = \varepsilon H(g(u), g(v))^* = \varepsilon((H \circ (g \times g))(u,v))^*$,
for all $u, v \in V$ i.e., $H \circ (g \times g)$ is also left ε-hermitian

Definition 3 *The left unitary similitude monoid of a regular left ε-hermitian form H is*

$MU^l(H) = \{g \in End^l_A(V) : H \circ (g \times g) = \mu_g H, \ \ \mu_g \in A\}$.

The mapping $\mu : g \mapsto \mu_g$ is called the (left) hermitian multiplier on $MU^l(H)$.

We define mutatis mutandis the right unitary similitude monoid $MU^r(H)$ of a regular right ε-hermitian form H and the corresponding right hermitian multiplier.

Remark 1 *We notice here that since $H(v,u) = \varepsilon H(u,v)^*$ and $H(gu, gv) = \mu_g H(u,v)$ for all $u, v \in V$, applying the involution to this last equality we get $\mu_g H(v,u) = H(v,u)\mu_g^*$. If the form takes the value 1, and hence every possible value, then $\mu_g \in Z_s(A)$ (which is going to be the case below)*

We observe also that the hermitian multiplier is well defined since H is regular, and that

$\mu_{fg} = \mu_f \mu_g$, $\mu_{id} = 1$, so $MU^l(H)$ and $MU^r(H)$ are indeed (unitary) monoids and the left and right hermitian multipliers are homomorphisms of (unitary) monoids from $MU^l(H)$ and $MU^r(H)$, respectively, to $Z_s(A)$.

Remark 2 *Notice that if H is regular left $\varepsilon-$hermitian, then $\widetilde{H}$ defined by*

$$\widetilde{H}(u,v) = [H(u^*, v^*)]^*$$

for all $u, v \in V$, is regular right $\varepsilon-$hermitian. Moreover if $g \in End^l_A(V)$ is such that

$$H \circ (g \times g) = \mu_g H$$

for $\mu_g \in Z_s(A)$, then

$$\widetilde{H} \circ (\tilde{g} \times \tilde{g}) = \mu_g \widetilde{H}$$

for $\tilde{g} = * \circ g \circ *$.

It follows that the map $g \mapsto \tilde{g}$, which induces an isomorphisms of rings from $End^l_A(V)$ to $End^r_A(V)$, restricts to an isomorphism Ψ of monoids from $MU^l(H)$ to $MU^r(\widetilde{H})$ such that

$$\tilde{\mu} \circ \Psi = \mu,$$

where $\tilde{\mu}$ denotes the $\varepsilon-$hermitian multiplier associated to $\widetilde{H}$

3 Some matrix descriptions

3.1 Matrix description of left and right linear endomorphisms

We consider now the case where V is a free involutive $(A,A)-$ bimodule of dimension n, endowed with a fixed bi-basis $\mathbf{e} : e_1, ..., e_n$.

To any left or right linear endomorphism g of V we may associate its matrix $[g] = (g_{ij})_{1\leq i,j\leq n} = (g_{ij})$ with respect to the bi-basis $\mathbf{e}$, defined by

$$g(e_j) = \sum_i g_{ji}e_i \qquad \text{or} \qquad g(e_j) = \sum_i e_i g_{ij},$$

respectively.

We will eventually write $[g]^l$ or $[g]^r$, to avoid confusions, in case g is both left and right linear

We recover then our endomorphism g as the left or right canonical linear extension of its restriction to the bi-basis $\mathbf{e}$, given by

$$g(v) = \sum_j v_j g(e_j) = \sum_{i,j} v_j g_{ji} e_i$$

or

$$g(v) = \sum_j g(e_j) v_j = \sum_{i,j} e_i g_{ij} v_j$$

respectively, for $v = \sum_j v_j e_j = \sum_j e_j v_j$.

On the other hand, we may associate to any $n \times n$ matrix $c = (c_{ij})$ a left $A-$linear endomorphism $g_c^{(l)}$ and a right $A-$linear endomorphism $g_c^{(r)}$ of V, defined by

$$g_c^{(l)}(v) = \sum_{i,j} v_j c_{ji} e_i \qquad \text{and} \qquad g_c^{(r)}(v) = \sum_{i,j} e_i c_{ij} v_j,$$

for $v = \sum_j v_j e_j = \sum_j e_j v_j$.

We have then

$$[g_c^{(l)}] = c = [g_c^{(r)}]$$

and also

$$g_{[g]}^{(l)} = g \qquad \text{or} \qquad g_{[g]}^{(r)} = g$$

according to our previous endomorphism g being left or right $A-$linear. So the maps $c \mapsto g_c^{(l)}$ and $c \mapsto g_c^{(r)}$ afford isomorphisms from the full matrix ring $M(n,A)$ onto the rings $End_A^l(V)$ and $End_A^r(V)$ respectively. Moreover we have

$$(g_c^{(l)})\tilde{} = g_{c^*}^{(r)} \qquad \text{and} \qquad [\tilde{g}] = [g]^*$$

where we have extended the involution $*$ to an involutive anti-automorphism $*$ of the ring $M(n,A)$

by $(c_{ij})^* = (c_{ji}^*)$ and $\tilde{g} = * \circ g \circ *$ as introduced in section 2.

3.2 Matrix description of $\varepsilon-$hermitian forms

Now we define the matrix $[H]$ of the (left or right) $\varepsilon-$hermitian form H to be the $n \times n$ matrix whose i,j entry is $H(e_i, e_j)$.

Then we have $[H]^* = \varepsilon[H]$ and

$$H(u,v) = \sum_{i,j} u_i[H]_{ij}v_j^* \qquad \text{or} \qquad H(u,v) = \sum_{i,j} v_j^*[H]_{ij}u_i$$

for $u = \sum_i u_i e_i = \sum_i e_i u_i, \quad v = \sum_i v_i e_i = \sum_i e_i v_i$, according to H being left or right hermitian, respectively.

Conversely, suppose that you have a matrix $M = (m_{ij})$ such that $M^* = \varepsilon M$, i.e., such that $m_{ji}^* = \varepsilon m_{ij}$; then we get a left $\varepsilon-$hermitian form $H_M^{(l)}$ as well as a right $\varepsilon-$hermitian form $H_M^{(r)}$ by setting

$$H_M^{(l)}(u,v) = \sum_{i,j} u_i m_{ij} v_j^*$$

and

$$H_M^{(r)}(u,v) = \sum_{i,j} v_j^* m_{ij} u_i.$$

3.3 Matrix description of the unitary similitude monoid $MU(H)$ and of the unitary similitude group $GU(H)$

We can give now a convenient matrix description of the unitary similitude monoid $MU(H)$.

Proposition 1 *With the notations introduced in the previous subsection, we have*

$$MU(H) = \{q \in M_n(A) : q[H]q^* = \mu_q[H], \ \ \mu_q \in Z_s(A)\}.$$

We have again a triangle of isomorphisms between the right monoid, the left monoid and the matrix monoid The correspondence $g \mapsto g^*$ sends isomorphically the monoid $MU^l(H)$ onto the corresponding right monoid $MU^r(H)$

Proof. Our description follows from the fact that

$$[H \circ (g \times g)] = [g][H][g]^*$$

for a left $\varepsilon-$ hermitian form H and a left $A-$ endomorphism g of V. ∎

Definition 4 *The unitary similitude group of an $\varepsilon-$hermitian form H is the group of all invertible elements in the monoid $MU(H)$. We will denote it by $GU(H)$.*

Proposition 2 *Assume that the $\varepsilon-$hermitian form H is non degenerate, i.e. its matrix $[H]$ is invertible in $M_n(A)$. Then the unitary similitude group $GU(H)$ consists of all $q \in MU(H)$ that are invertible as matrices in $M_n(A)$.*

Proof. Assume that $g \in MU(H)$ is invertible in $M_n(A)$. Then, since $[H]$ is invertible, the multiplier μ_g must also be invertible. Since
$H(u,v) = H(gg^{-1}u, gg^{-1}v) = \mu_g H(g^{-1}u, g^{-1}v), \quad u,v \in V$ we get
$H(g^{-1}u, g^{-1}v) = \mu_g^{-1} H(u,v)$
from where it follows that
$g^{-1} \in MU(H)$. ∎

Definition 5 *In what follows, given an $n \times n$ $\varepsilon-$hermitian matrix K we set*
$MU(K) = MU(H)$ *and* $GU(K) = GU(H)$
where H denotes the left $\varepsilon-$hermitian form defined by the matrix K on the free A^n-bimodule $V = A^n \times A^n = (A^n)^2$ endowed with its canonical bi-basis.

4 Involutive analogues of The General Linear Group and the Special General Linear Group and the non commutative $\varepsilon-$determinant

We specialize now to the study of unitary similitude groups in the case of $\varepsilon-$hermitian forms of rank 2. More precisely we consider the $\varepsilon-$hermitian form on A^2 defined by the matrix $J_\varepsilon = \begin{pmatrix} 0 & 1 \\ \varepsilon 1 & 0 \end{pmatrix}$, which we denote shortly by K in what follows. Notice that $K^* = \varepsilon K$.

We look at the unitary monoid $MU(K)$ associated to K as a non commutative (involutive) twisted analogue of the classical linear endomorphism monoid $ML(2,A)$ of A^2 for commutative A.

Definition 6 *We set $ML_*^\varepsilon(2,A) = MU(K)$ and $det_*^\varepsilon = \mu$.*

We call these objects the $* - \varepsilon-$ analogues of the rank 2 linear monoid $ML(2,A)$ and the $\varepsilon-$determinant of $GL(2,A)$, respectively.

Remark 3 *We could characterize $ML_*^\varepsilon(2,A)$ as*

$$ML_*^\varepsilon(2,A) = \{g \in M(2,A) : gKg^*K^{-1} \in Z_s(A)\, I_2\},$$

and define then the non-commutative (involutive) $\varepsilon-$determinant $\det_*^\varepsilon$ by

$$det_*^\varepsilon(g) = gKg^*K^{-1}$$

Since $\det_*^\varepsilon(g)$ is the multiplier μ_g of g, it is clear that it takes central symmetric values and that $\det_*^\varepsilon(gh) = \det_*^\varepsilon(g)\det_*^\varepsilon(h)$, for all $g,h \in MU(H)$. Furthermore

Lemma 1 *Let $g \in ML_*^\varepsilon(2,A)$. Then:*

1. *If g is invertible as a 2×2 matrix, then also* $g^{-1},\ g^{*} \in ML_{*}^{\varepsilon}(2, A)$ *and we have*

   $\det_{*}^{\varepsilon}(g) = \det_{*}^{\varepsilon}(g^{*})$

2. *If* $\det_{*}^{\varepsilon}(g) = \det_{*}^{\varepsilon}(g^{*}) \in Z_s(A)^{\times}$ *, then g is invertible*

Proof. We give a direct proof of 2 (1 follows from above, or can be proved as the corresponding statement of [9])

We have that $\det_{*}^{\varepsilon}(g)\, K = gKg^{*}$ and that $\det_{*}^{\varepsilon}(g^{*})\, K = g^{*}Kg$. But $\mu = \det_{*}^{\varepsilon}(g) = \det_{*}^{\varepsilon}(g^{*}) \in Z_s(A)^{\times}$. Using the fact that $K^{-1} = K^{*} = \varepsilon K$, we get $(\varepsilon\mu^{-1})Kg^{*}Kg = I_2 = g(\varepsilon\mu^{-1})Kg^{*}K$, from where our result. ■

We define now

Definition 7 *Let $GL_{*}^{\varepsilon}(2, A) = GU(K)$, i. e. $GL_{*}^{\varepsilon}(2, A)$ is the group of all invertible elements in the monoid $ML_{*}^{\varepsilon}(2, A)$.*

Proposition 3 *The group $GL_{*}^{\varepsilon}(2, A)$ may be described in several equivalent ways, as:*

1. *the set of all $g \in ML_{*}^{\varepsilon}(2, A)$ that are invertible as 2×2 matrices ;*

2. *the set of all $g \in ML_{*}^{\varepsilon}(2, A)$ such that*

$$det_{*}^{\varepsilon}(g) = det_{*}^{\varepsilon}(g^{*}) \in Z_s(A)^{\times};$$

3. *the set of all matrices* $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $a, b, c, d \in A$ *such that*

   $ab^{*} = -\varepsilon ba^{*}, cd^{*} = -\varepsilon dc^{*}, a^{*}c = -\varepsilon c^{*}a, b^{*}d = -\varepsilon d^{*}b,$

   $ad^{*} + \varepsilon bc^{*} = a^{*}d + \varepsilon c^{*}b \in Z_s(A)^{\times}.$

Proof. This follows immediately from the previous lemma ■

Remark 4 *Notice the analogy of the relations in 3. of Prop. 3. with the physicist's approach to quantum groups as groups of matrices whose coefficients satisfy suitable $q-$commuting relations (like $ab = qba$ for example)*

The function $\det_{*}^{\varepsilon} : GL_{*}^{\varepsilon}(2, A) \to Z_s(A)^{\times}$ is clearly a group epimorphism.

Explicitly, if $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then

$$\det_{*}^{\varepsilon}(g) = ad^{*} + \varepsilon bc^{*} = a^{*}d + \varepsilon c^{*}b.$$

Recall that we call $\det_{*}^{\varepsilon}(g)$ the involutive (non commutative) ε-determinant of g. Next, we consider the kernel of this epimorphism

Definition 8 *Let* $SL_{*}^{\varepsilon}(2,A)$ *the subgroup of* $GL_{*}^{\varepsilon}(2,A)$ *consisting of the matrices of* ε*-determinant* 1.

Remark 5 *Notice that if the ring of coefficients is a field* F *and the involution is the identity, then it follows that the above map sending a matrix* g *to* $ad+bc$ *is "determinant like". The point is, of course, that this map is an homomorphism on a small subgroup* $GL_{id}^{+}(2,F)$ *of* $GL(2,F)$*, to wit the well known orthogonal similitude group* $GO^{+}(2,F)$ *of the hyperbolic quadratic form of rank* 2

5 $\varepsilon *-$euclidean rings

We will study in this section a notion on involutive rings that has its origin in the integers $\mathbb{Z}$, and that implies the existence of Bruhat generators for our matrix groups $SL_{*}^{\varepsilon}(2,A)$ and $GL_{*}^{\varepsilon}(2,A)$, as it does in the classical case for $SL(2,\mathbb{Z})$.

Definition 9 *A unitary ring with involution* $*$ *is called a* $\varepsilon *-$*euclidean ring (or just* $\varepsilon-$*euclidean ring) if given* $a,c\in A$ *such* $a^{*}c=-\varepsilon c^{*}a$ *and* $Aa+Ac=A$*, then there is a finite sequence* $s_{0},s_{1},...,s_{n-1}\in A_{s}=\{s\in A: s^{*}+\varepsilon s=0\}$ *and* $r_{1},r_{2},...,r_{n}\in A$*, with* $r_{n}\in A^{\times}$ *such that*

$a=s_{0}c+r_{1}$
$c=s_{1}r_{1}+r_{2}$
.
.
.
$r_{n-2}=s_{n-1}r_{n-1}+r_{n}$

Remark 6 *We should observe at this point that*

1. *If* $g=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *belongs to* $SL_{*}^{\varepsilon}(2,A)$ *or* $GL_{*}^{\varepsilon}(2,A)$*, then* $a^{*}c=-\varepsilon c^{*}a$ *and* $Aa+Ac=A$
2. $\mathbb{Z}$ *is an* $(-1)-$*euclidean ring (with* $*=id$*)*

We set $h_{t}=\begin{pmatrix} t & 0 \\ 0 & t^{*-1} \end{pmatrix}$ $(t\in A^{\times})$, $h_{r}^{'}=\begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix}$ $(r\in Z_{s}(A)^{\times})$, $\omega=\omega_{\varepsilon}=\begin{pmatrix} 0 & 1 \\ \varepsilon 1 & 0 \end{pmatrix}$ and $u_{s}=\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$ $(s\in A_{s})$

We observe that $h_{t},\omega,u_{s}\in SL_{*}^{\varepsilon}(2,A)$ and that $h_{r}^{'}\in GL_{*}^{\varepsilon}(2,A)$

Lemma 2 *Let* $g=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in SL_{*}^{\varepsilon}(2,A)$ *with* $c\in A^{\times}$*. Then*

$g=h_{\varepsilon c^{*-1}}u_{c^{*}a}\omega u_{c^{-1}d}$

Proof. By Proposition 3, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = h_t u_s \omega u_l$ if and only if
$a = \varepsilon ts$
$b = t + \varepsilon tsl$
$c = \varepsilon t^{*^{-1}}$
$d = \varepsilon t^{*^{-1}} l$
These equations have solution $t = \varepsilon c^{*^{-1}}, s = c^* a, l = c^{-1} d$
from which the result follows. ■

Proposition 4 *Let A be an $\varepsilon *-$euclidean ring. Then the elements h_t, ω, u_s generate the group $SL_*^{\varepsilon}(2, A)$*

Proof. If $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_*^{\varepsilon}(2, A)$,

we noticed before that $a^* c = -\varepsilon c^* a$ and $Aa + Ac = A$. There is then a finite sequence $s_0, s_1, ..., s_{n-1} \in A_s$ and $r_1, r_2, ..., r_n \in A$, with $r_n \in A^{\times}$ such that
$a = s_0 c + r_1$
$c = s_1 r_1 + r_2$
.
.
.
$r_{n-2} = s_{n-1} r_{n-1} + r_n$
Then
$$\begin{pmatrix} 1 & -s_0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a - s_0 c & b - s_0 d \\ c & d \end{pmatrix} = \begin{pmatrix} r_1 & b - s_0 d \\ c & d \end{pmatrix}$$
We have then
$$\begin{pmatrix} 0 & 1 \\ \varepsilon 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -s_0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \varepsilon 1 & 0 \end{pmatrix}\begin{pmatrix} r_1 & b - s_0 d \\ c & d \end{pmatrix}$$
$$= \begin{pmatrix} c & d \\ \varepsilon r_1 & \varepsilon(b - s_0 d) \end{pmatrix}$$
We multiply now on the left this last matrix by
$\begin{pmatrix} 0 & 1 \\ \varepsilon 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -\varepsilon s_1 \\ 0 & 1 \end{pmatrix}$ to get
$$\begin{pmatrix} \varepsilon r_1 & \varepsilon(b - s_0 d) \\ \varepsilon r_2 & \varepsilon(d - s_1(b - s_0 d)) \end{pmatrix}$$
We continue with this process until we get in the position $(2, 1)$ the element r_n. We apply lemma above to this matrix and we solve for $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. From this the result follows. ■

Now, the following corollary follows

Corollary 1 *Let A be a $\varepsilon-$euclidean ring. Then the elements h'_r, h_t, ω, u_s generate the group $GL_*^{\varepsilon}(2, A)$*

Proposition 5 *The above generators, called Bruhat generators, satisfy the following universal relations*

1. $h'_r h'_{r'} = h_{rr'}$, $h_t h_{t'} = h_{tt'}$, $u_b u_{b'} = u_{b+b'}$, $\omega^2 = h_\varepsilon$
2. $h'_r h_t = h_t h'_r$
3. $h'_r u_b = u_{br^{-1}} h'_r$
4. $h_t u_b = u_{tbt^*} h_t$
5. $\omega h_t = h_{t^{*-1}} \omega$
6. $h_r h'_r \omega = \omega h'_r$
7. $\omega u_{t^{-1}} \omega u_{-\varepsilon t} \omega u_{t^{-1}} = h_{-\varepsilon t}$

6 A Bruhat presentation for $GL_*^\varepsilon(2, A)$ and $SL_*^\varepsilon(2, A)$

The results of this section generalize previous work of the authors on generators of $SL_*^-(2, A)$ and $GL_*^-(2, A)$ (see [9]) and of the first author on a presentation of $SL_*^-(2, A)$ (see [8])

We assume now that A is a simple artinian ring. By Jacobson [5], we have that: A is (isomorphic to) $End_D V$ where V is a finite dimensional vector space over a division ring D. If A has an involution, then D has an involution $-$, and there exists a non-degenerate hermitian form $<,>$ on V with respect to $-$ such that the involution coincides with the adjoint map, or D is a field, $-$ is the identity map and there exists an anti-symmetric form $<,>$ on V such that the involution coincides with the corresponding adjoint map.

We remark that we may as well assume that the form is antihermitian with respect to an involution $\sim$ on D and that the involution on A coincides with the adjoint map: Indeed, suppose that $-$ is not the identity map of D. Then there exists an element $\alpha \in D$ such that $\overline{\alpha} \neq \alpha$. Let $\beta = \alpha - \overline{\alpha}$. Then $\overline{\beta} = -\beta$. We may consider then a new form $< x, y >_1 = < x, y > \beta$ together with an involution $\sim$ on D given by $(\lambda)^\sim = \beta^{-1}\overline{\lambda}\beta$. A computation shows that the form $< \, , \, >_1$ is sesquilinear. Furthermore, $< \, , \, >_1$ is antihermitian with respect to$\sim$: $(< x, y >_1)^\sim = (< x, y > \beta)^\sim = (\beta)^\sim (< x, y >)^\sim = (\beta)^\sim \beta^{-1}\overline{< x, y >}\beta = -\beta\beta^{-1} < y, x > \beta$
$= - < y, x > \beta = - < y, x >_1$

From here the result follows.

In [8] the first case i.e., the hermitian case was considered. Also the case where the scalars constitute a field and the form is anti-symmetric was treated in loc.cit.. So we will be interested mainly in the case where the form is anti-hermitian.

Let V be a finite dimensional left D-module, D a division ring with involution $-$.

Let $<,>$ be a non-degenerate *sign*-hermitian form with respect to $-$(here *sign* is $+1$ or -1 according to the form being hermitian or anti-hermitian) , i.e. $<,>$: $V \times V \to D$ is bi-additive,

$$\overline{< v, w >} = sign < w, v >,$$
$$< v, wa > = < v, w > a.$$

Let $F : V \times V \to D$, be bi-additive, linear in the first variable, and such that $\overline{F(x,y)} = F(y,x)$

We are going to construct now an element $s \in End_D(V)$, which depends on F, such that $s^* = sign\, s$(recall that the involution $*$ coincides with the adjoint map of the form $<,>$ on V).

This construction will be used next for certain forms F.

Let then F be as above, and let $<,>$ be a non-degenerate *sign*-hermitian form with respect to $-$. If $v_1, ..., v_n$ is a basis for V, then the forms $< \,, v_i >$: $x \mapsto < x, v_i >$ provide a basis for V°(the dual of V)

If $v \in V$, the form $F(\,, v)$ is an element of V° (which is a right module) and there exist $\alpha_i \in D, \;\; 1 \leq i \leq n$, so that $F(\,, v) = \sum < \,, v_i > \alpha_i$.

Then $F(v', v) = \sum < v', v_i > \alpha_i$
$=< v', \sum \overline{\alpha_i} v_i >$.

If we set $s(v) = \sum \overline{\alpha_i} v_i$, then we have that given $v \in V$ there exists a unique element $s(v) \in V$ such that $F(v', v) =< v', s(v) >$ for all $v' \in V$. It is clear that s is additive, also $< v', s(\lambda v) >= F(v', \lambda v) = F(v', v)\overline{\lambda} =< v', s(v) > \overline{\lambda} =< v', \lambda s(v) >$, which proves that $s \in End_D(V)$.

Furthermore, $< sign \;\; s(v'), v >= \overline{< v, s(v') >} = \overline{F(v, v')} = F(v', v) =< v', s(v) >$ and then $s^* = sign\ s$.

Proposition 6 *(Transversality lemma).*

Let V be as above. Let $W < V$. Then there exists $s \in End_D(V)$ such that

1)$s^* = sign\ s$

2)The restriction of s to W is 1-1

3)$\operatorname{Im} s = s(W)$

4)$s(W) \cap W^\perp = 0$

Proof. Let T be a supplement of W,i.e., $V = W \oplus T$

Let C be a non-degenerate hermitian pairing on W with respect to $-$.

Let F be the extension of C to an hermitian form on V given by $F(w + t, w' + t') = C(w, w')$

Let s be the element defined as above. Then $F(v', v) =< v', s(v) >$and $s^* = sign\ s$

Noting that $\ker s = T$, we get 2) and 3). Finally, 4) follows from the fact that if $s(w) \in W^\perp$, then

$0 =< s(w), w' >= F(w, w') = C(w, w') \; \forall w' \in W$ implies that $w = 0$ ■

Now, the same argument used in [8] shows

Proposition 7 *(Co-prime lemma). Let $a, c \in A$ be such that $a^* c = sign\ c^* a$. Then the following are equivalent*

1) $Aa + Ac = A$

2) $\ker a \cap \ker c = (0)$

3) There exits an element $s \in A$ with $s^* = sign\ s$ such that $a + sc \in A^\times$.

Corollary 2 *Let A be as above. Then A is a $\varepsilon-$euclidean ring.*

Suppose now that we have a non degenerate hermitian form $<,>$ on V together with an involution $-$ on D, and a non degenerate anti-hermitian form $[,]$ on V together with an involution $\sim$ on D, so that $*$ is the adjoint map for both forms. If $a \in A$ is symmetric (anti-symmetric), we will show the existence of an anti-symmetric (symmetric) element $s \in A$ such that $< av, v' >= [v, sv']$. To this end, let us consider the function $F(v, v') =< av, v' >$. Then, by the above construction, there exists an anti-symmetric (symmetric) element s such that $F(v, v') = [v, sv']$, i.e., such that $< av, v' >= [v, sv']$. From here the claim follows.

We observe that the element a is invertible if and only if $< a(\ ), >$ is non degenerate, if and only if $[\ , s(\ )]$ is non degenerate, if and only if s is invertible.

We have in this way a correspondence $a \mapsto s = s_a$ which is injective since $s_a = s_{a'}$ implies $< (a - a')v, v' >= 0$ and then $a - a' = 0$ because of the non degeneracy of the form.

Also, the above correspondence is surjective since, by the above, given s there is an a such that $[sv', v] = \langle v', av \rangle$ (reverting the roles of $<\ ,\ >$ and $[\ ,\ ]$). From which $\langle av, v' \rangle = [v, av']$.

The above says then that there is a bijection between symmetric elements and anti-symmetric elements that restricts to a bijection on invertible elements.

We will prove now that in fact the Bruhat generators together with the universal relations listed in the above section define a presentation of the groups $GL_*^{\varepsilon}(2, A)$ and $SL_*^{\varepsilon}(2, A)$, for A artinian simple, extending the results of [8] (where the result was proved in the case of $SL_*^{-}(2, A)$).

Lemma 3 *If F is a finite field of q elements, then* $\frac{|A^\times \cap A_s|}{|A_s|} > 1 - \frac{q}{q^2-1}$

Proof. We have shown that there is a bijection between symmetric and anti-symmetric elements that restricts to a bijection between invertible symmetric and antisymmetric elements.

On the other hand the inequality of the lemma was proved to be true for symmetric elements in lemma 11 of [8]. From this, the result follows. ■

Now, the above lemma was needed for symmetric elements to prove lemma 12 in [8] by a counting argument.

Since the lemma is now known to be true, mutatis mutandis, also for anti-symmetric elements, lemma 12 of [8] shows

Lemma 4 *Let A be a simple artinian ring with involution that is either infinite or isomorphic to the full matrix ring over $\mathbb{F}_q$ with $q > 3$. Let $a, b \in A_s$ be such that $a, b \notin A^\times$. Then there exists $u \in A^\times \cap A_s$ such that $a + u, b + \varepsilon u^{-1} \in A^\times$*

Definition 10 *Let H be the abstract group generated by the symbols h'_r, h_t, u_s, ω, parametrized by $r \in Z_s(A)^\times, t \in A^\times, s \in A_s$, subject to the relations*

1. $h'_r h'_{r'} = h_{rr'}$, $h_t h_{t'} = h_{tt'}$, $u_b u_{b'} = u_{b+b'}$, $\omega^2 = h_\varepsilon$
2. $h'_r h_t = h_t h'_r$
3. $h'_r u_b = u_{br^{-1}} h'_r$
4. $h_t u_b = u_{tbt^*} h_t$

5. $\omega h_t = h_{t^{*-1}}\omega$
6. $h_r h'_r \omega = \omega h'_r$
7. $\omega u_{t^{-1}} \omega u_{-\varepsilon t} \omega u_{t^{-1}} = h_{-\varepsilon t}$

Definition 11 *Let G be the subgroup of $GL_*^{\varepsilon}(2,A)$ given by $\bigcup_{j=0}^{\infty}(B\omega B)^j$ (with $(B\omega B)^0 = B$), where B is the subgroup of $GL_*^{\varepsilon}(2,A)$ consisting of the upper triangular matrices.*

We have that $B = EDN$ where E,D and N are the subgroups of $GL_*^{\varepsilon}(2,A)$ defined as

$E = \{h'_r : r \in Z_s(A)^{\times}\},\ D = \{h_t : t \in A^{\times}\},\ N = \{u_a : a \in A_s\}$. Here $h_t = \begin{pmatrix} t & 0 \\ 0 & t^{*-1} \end{pmatrix}$ $(t \in A^{\times})$, $h'_r = \begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix}$ $(r \in Z_s(A)^{\times})$, $\omega = \omega_{\varepsilon} = \begin{pmatrix} 0 & 1 \\ \varepsilon & 0 \end{pmatrix}$ and $u_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$ $(s \in A_s)$.Note that we use the same symbols (h_t, h'_r, ω, u_s) to denote the above matrices and the generators of H

Definition 12 *The ω-length of an element $g \in H$ is the minimal j such that $g \in (B\omega B)^j$ $(B = B_H$, the subgroup of H generated by the elements h'_r,h_t,u_s)). We define in the same way the ω-length of an element of G*

Proposition 8 *We have $G = GL_*^{\varepsilon}(2,A)$.*

Moreover, the length of an element of G is at most 2.

Proof. Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$. Suppose first that $c = 0$. Then $g = h'_{ad^*} h_a u_{a^{-1}b}$

If $c \in A^{\times}$, let $\delta = \det_* g$. Then $g = h'_{\delta} h_{\varepsilon\delta c^{*-1}} u_{\delta^{-1}c^*a} \omega u_{c^{-1}d}$

Finally, if $c \in A - \{A^{\times} \cup \{0\}\}$, let $s \in A_s$ be such that $a + s\delta^{-1}c = y \in A^{\times}$. In this case we have $g = h'_{\delta} u_s h_{\varepsilon} \omega h_{y^{*-1}} u_{\varepsilon y^* c} \omega u_{y^{-1}(b+sd)}$

Then the result follows. ■

It follows from the relations that satisfy the elements of H (relations that also satisfy the corresponding elements of G) that $(B\omega B)^j = ED(N\omega N)^j$ for any $j > 0$, where E, D, N are the subgroups of H generated by, respectively, the h'_r, h_t, u_s. Since the defining relations of H are also satisfied by the corresponding elements in the group G, a similar situation holds in G.

Now, a computation shows

Lemma 5 *In the group G we have*

1. $h'_r h_t u_a \omega u_b \omega u_c = 1 \Rightarrow r = 1, t = \varepsilon, a = -c, b = 0$
2. $1 \notin B\omega B$
3. $h'_r h_t u_a = 1 \Rightarrow r = t = 1, a = 0$

There is a natural epimorphism $\varphi : H \to G$. We will prove that in fact, φ is an isomorphism. This gives then a presentation for the group G, that we call Bruhat presentation of G.

The key thing is to show that the elements of H have $\omega-$length bounded by 2.

This is the content of the next proposition.

Proposition 9 *Let H be the group defined above, where the parameters A of the definition of H satisfy the hypothesis in lemma 4. Then every element of H has length at most* 2.

Proof. Let us consider an expression of the form $g_1 g_2 z = g_3 z'$ where $g_i \in B\omega B \cup \{1\} \quad (i = 1, 2, 3)$ and $z, z' \in H$ are arbitrary. Using the defining relations of H, this expression is equivalent to the expression

($\boxminus$): $\omega u_a \omega y = u_b \omega$

where y is certain element of H.

If at least one of the elements a or b is invertible, we can use the defining relation

($\boxplus$): $\omega u_{t^{-1}} \omega u_{-\varepsilon t} \omega u_{t^{-1}} = h_{-\varepsilon t}$

to lower by one the number of ω in ($\boxminus$). We turn then to the case where a and b are not invertible. We apply lemma 4 to get an element $x \in A^\times \cap A_s$ such that $b + x$ and $a + \varepsilon x^{-1}$ are invertible.

We multiply ($\boxminus$) by u_x to get

$$\begin{aligned} u_{x+b}\omega &= u_x \omega u_a \omega y \\ &= \varepsilon\omega(\omega u_x \omega) u_a \omega y \\ &= \varepsilon\omega(u_{\varepsilon x^{-1}} \varepsilon\omega h_x u_{\varepsilon x^{-1}}) u_a \omega y \\ &= \omega u_{\varepsilon x^{-1}} \omega h_x u_{\varepsilon x^{-1}+a} \omega y \\ &= \omega u_{\varepsilon x^{-1}} h_{-\varepsilon x^{-1}} (\omega u_{a+\varepsilon x^{-1}} \omega) y \\ &= \omega u_{\varepsilon x^{-1}} h_{-\varepsilon x^{-1}} u_{\varepsilon(a+\varepsilon x^{-1})^{-1}} \varepsilon\omega h_{a+\varepsilon x^{-1}} u_{e(a+\varepsilon x^{-1})^{-1}} y \end{aligned}$$

From which $\omega u_{x+b}\omega = u_{\varepsilon x^{-1}} h_{-\varepsilon x^{-1}} u_{\varepsilon(a+\varepsilon x^{-1})^{-1}} \omega h_{a+\varepsilon x^{-1}} u_{e(a+\varepsilon x^{-1})^{-1}} y$

Now, we use once more ($\boxplus$) to get

$\varepsilon\omega u_{e(x+b)^{-1}} h_{x+b} u_{\varepsilon(x+b)^{-1}} = u_{\varepsilon x^{-1}} h_{-\varepsilon x^{-1}} u_{\varepsilon(a+\varepsilon x^{-1})^{-1}} \omega h_{a+\varepsilon x^{-1}} u_{e(a+\varepsilon x^{-1})^{-1}} y$

Again we were able to lower the by one the number of ω in a expression equivalent to ($\boxminus$)

The result now follows by induction. ■

We are ready to prove

Theorem 1 *With the hypothesis of lemma 4, the group G has a Bruhat presentation, i.e., $G =< h'_r, h_t, u_s, \omega : r \in Z_s(A)^\times, t \in A^\times, s \in A_s, \mathcal{R}\}$ where $\mathcal{R}$ is the set of relations*

1. $h'_r h'_{r'} = h_{rr'}$, $h_t h_{t'} = h_{tt'}$, $u_b u_{b'} = u_{b+b'}$, $\omega^2 = h_\varepsilon$
2. $h'_r h_t = h_t h'_r$
3. $h'_r u_b = u_{br^{-1}} h'_r$
4. $h_t u_b = u_{tbt^*} h_t$
5. $\omega h_t = h_{t^{*-1}} \omega$
6. $h_r h'_r \omega = \omega h'_r$
7. $\omega u_{t^{-1}} \omega u_{-\varepsilon t} \omega u_{t^{-1}} = h_{-\varepsilon t}$

Proof. Keeping the notations as above, we show that the natural epimorphism $\varphi : H \to G$ is an isomorphism.

By proposition 9, any element of H is of one of the forms

i. $h'_r h_t u_a$

ii. $h'_r h_t u_a \omega u_b$

iii. $h'_r h_t u_a \omega u_b \omega u_c$

If we take an element in the kernel of φ, the result follows applying lemma 5 ■

7 Rank 2 non commutative $\varepsilon-$exterior calculus

The classical homomorphism called determinant arises from the one dimensionality and functoriality of the n-th exterior power of an n-dimensional vector space. We extend this, for $n = 2$, in the setting of function bimodules, to the tamely non commutative case of a base ring with involution.
Let A be a ring with involution $*$ and let V be the free involutive $(A, A)-$bimodule of all A valued functions on a finite set X (see Example 1.iii. in section 2 above). Taking advantage of the involution, we may extend the classical formula for the anti-symmetrization of the tensor product as follows.

Definition 13 *If $f, g \in V$, then $f \wedge g$ is the function on $X \times X$ given by* $(f \wedge g)(x, y) = f(x)^* g(y) - g(x)^* f(y)$, *i.e.* $f \wedge g = f^* \otimes g - g^* \otimes f$

The following properties are then readily verified.

Proposition 10 *For $f, g \in V, a \in A$ we have*

i. $g \wedge f = -f \wedge g$

ii. $f \wedge f = 0$

iii. $af \wedge g = f \wedge a^* g$,

iv. *If* $a = a^*$, *then* $af \wedge f = 0$

v. *If* $a \in Z_s(A)$, *then* $af \wedge g = f \wedge ag = a(f \wedge g)$

Remark 7 *If we symmetrize instead of antisymmetrizing the tensor product, i.e., if we set $f \,\overline{\wedge}\, g = f^* \otimes g + g^* \otimes f$ then it is not longer true that $f \,\overline{\wedge}\, f = 0$, but $af \,\overline{\wedge}\, f = 0$ for every* antisymmetric a

We will prove now that in the case of 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, whose coefficients satisfy suitable commutation relations, $(af + bg) \wedge (cf + dg)$ is a scalar multiple of $f \wedge g$. In the case of the product $f \,\overline{\wedge}\, g$ we have an analogous result that provides a symmetric analogue of the classical determinant. To embrace both the symmetric and antisymmetric cases, we define a general $\varepsilon-$symmetric binary product $\curlywedge_\varepsilon$ which gives back $\wedge$ for $\varepsilon = -1$ and $\overline{\wedge}$ for $\varepsilon = 1$.

Definition 14 *We set* $f \curlywedge_\varepsilon g = f^* \otimes g + \varepsilon g^* \otimes f$ *and*
$\curlywedge^2_\varepsilon V = \{h : X \times X \to A \mid h(y,x) = \varepsilon h(x,y)^*\}$
In what follows we will write for short $\curlywedge_\varepsilon = \curlywedge$.

Then $f \curlywedge g$ is clearly ε-hermitian (with respect to $*$), i. e. $f \curlywedge g \in \curlywedge^2 V$, and the following properties are readily verified.

Proposition 11 *We have, for all* $a \in A, f, g \in V$,
i. $g \curlywedge f = \varepsilon f \curlywedge g$
ii. $f \curlywedge f = (1+\varepsilon) f \otimes f$
iii. $af \curlywedge g = f \curlywedge a^* g$
iv. If $a = -\varepsilon a^*$, *then* $af \curlywedge f = f \curlywedge af = 0$
v. If $a \in Z_s(A)$, *then* $af \curlywedge g = f \curlywedge ag = a(f \curlywedge g)$

7.1 Linear action of $ML^\varepsilon_*(2, A)$ in $\curlywedge^2 V$

A matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M(2, A)$ acts naturally on the left by right endomorphisms on the A-module $\curlywedge^2_\varepsilon V$ by $(af + bg) \curlywedge (cf + dg)$. It also acts on the right through $\begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}$ by $(a^* f + c^* g) \curlywedge (b^* f + d^* g)$. Now,

$(af + bg) \curlywedge (cf + dg) = f \curlywedge a^* cf + f \curlywedge a^* dg + c^* bg \curlywedge f + g \curlywedge b^* dg.$

If we ask for $a^* c$ and $b^* d$ to be ε-symmetric, then $f \curlywedge a^* cf = g \curlywedge b^* dg = 0$ and we get

$(af + bg) \curlywedge (cf + dg) = f \curlywedge a^* dg + c^* bg \curlywedge f = f \curlywedge (a^* d + \varepsilon c^* b) g$

If we impose now that $a^* d + \varepsilon c^* b$ be central and symmetric, we obtain

$(af + bg) \curlywedge (cf + dg) = (a^* d + \varepsilon c^* b)(f \curlywedge g).$

Similarly, if we take ab^* and cd^* to be symmetric and $ad^* + \varepsilon bc^*$ to be central and symmetric we get

$(a^* f + c^* g) \curlywedge (b^* f + d^* g) = (ad^* + \varepsilon bc^*)(f \curlywedge g)$

It follows that the set of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $a^* c, b^* d, ab^*, cd^*$ are ε-symmetric and $a^* d + \varepsilon c^* b = ad^* + \varepsilon bc^* \in Z_s(A)$, i.e., the monoid $ML^\varepsilon_*(2, A)$, acts on $\curlywedge^2_* V$ by multiplication by a scalar (the ε-determinant) and that both actions of matrices described before are equal.

So we see that we may construct in a self-contained and independent way the monoid $ML^\varepsilon_*(2, A)$ and the $\varepsilon-$determinant. The key thing is to notice that the monoid just "found" acts on the right and on the left on $\curlywedge^2_\varepsilon V$ by scalars.

8 Examples

We give several specific examples of rings with involution and also of groups $SL^\varepsilon_*(2, A)$ and $GL^\varepsilon_*(2, A)$ for different choices of the involutive ring.

We will always denote by $*$ the corresponding involution of A

1. Let A be a commutative ring, and let $*$ be the identity map.

   Then, of course, $GL_*^-(2,A) = GL(2,A)$, if $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $\det_*^{\varepsilon}(g) = ad - bc$, $\quad SL_*^-(2,A) = SL(2,A)$

   In particular,

   - If m is a positive integer and $A = \mathbf{Z}/m\mathbf{Z}$, then $GL_*^-(2,A) = GL(2,\mathbf{Z}/m\mathbf{Z})$

   On the other hand, if A is an integral domain of characteristic different from 2, we have

   $GL_*^+(2,A) = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} : ad \neq 0 \right\} \cup \left\{ \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} : bc \neq 0 \right\}$ If $g = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ then $\det_*^{\varepsilon}(g) = ad$, and if $g = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$ then $\det_*^{\varepsilon}(g) = bc$; and

   $SL_*^+(2,A) = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} : ad = 1 \right\} \cup \left\{ \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} : bc = 1 \right\}$

2. Let F be a field, $A = M(n,F)$, $*$ the transposition of matrices.

   In this case,

   (a) $SL_*^-(2,A) = Sp(2n,F)$, the corresponding symplectic group

   (b) $SL_*^+(2,A) = O(n,n)(F)$ the split orthogonal group

3. Let G be a finite group, F a field, $A = F[G]$, $*$ the involution on A defined by $g^* = g^{-1}$

   - An interesting case of this example of a ring with involution is the modular case (see [3]):

     If $F[G]$ is not semi-simple, then our involution ring A does not reduce to a direct sum of copies of full matrix rings.

     Let $F = \mathbf{F}_q$ be the field of $q = p^n$ elements and $G = C_m$ be the cyclic group of $m = p^r$ elements.

     Then, $A \simeq \mathbf{F}_q[x] \diagup \langle x^m \rangle$

     and the involution $*$ is given, by

     $x \mapsto \frac{-x}{1-x}$

     In this case, as in the case of a simple artinian ring with involution (see [8], [9]), the group $SL_*^-(2,A)$ is the group defined by the presentation

     $\left\langle h_t = \begin{pmatrix} t & 0 \\ & t^{*-1} \end{pmatrix}, u_l = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}, \omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} : \ h_t, u_l, \omega \; satisfy \; R \right\rangle$

     where R is the set of relations: $h_t h_{t'} = htt', u_r u_l = u_{r+l}, \omega^2 = h_{-1} = -1, h_t u_b = u_{tbt^*} h_t, \omega h_t = h_{t^{*-1}} \omega, \omega u_{t^{-1}} \omega u_t \omega u_{t^{-1}} = h_t$

4. Another interesting example of a ring with involution is given by A a Clifford Algebra, in particular an exterior algebra with a suitable involution.

   - For instance:

     Let k be a field, and let V be a n-dimensional vector space over k.

     Fix a basis $e_1, e_2, \dots, e_n$ of V. Let $A = \Lambda V$ be the exterior algebra of V.

     We can define several involutions on A by considering their action on the given basis of V.

     In order to do this, let us take $I \subset \{1, 2, \dots, n\}$ with an even number of elements; we partition the set I in pairs (i, j), and define then $*$ by $e_i^* = e_j$ and $e_j^* = e_i$ for every such pair, and $e_t^* = \pm e_t$ for $t \in \{1, 2, \dots, n\} - I$.

     Then, $Z(A) = \Lambda^0 V \oplus \Lambda^2 V \oplus \dots \oplus \Lambda^{2\left[\frac{n}{2}\right]} V$. For $z \in Z(A)$ we write $z = \sum_{i=0}^{\left[\frac{n}{2}\right]} z_{2i}$. We have

     $Z(A)^{\times} = \{z \in Z(A) : z_0 \neq 0\}$.

     Notice that the Grassmann algebra considered as a supercommutative algebra, i.e., endowed with its canonical $\mathbb{Z}/2\mathbb{Z}$ grading, does not afford an example of an involutive ring, if we take the involution to be the identity Id on even elements and -Id on odd elements.

5. Let k be a finite field and let K be a quadratic extension of k.

   Let $A = K$ and let $*$ be the function defined by $\alpha^* = F(\alpha)$, where F is the Frobenius automorphism of K.

   In this case, $GL_*^-(2, K) = K^{\times} GL(2, k)$,and

   $SL_*^-(2, K) = \left\{\lambda d : \lambda \in K^*, d \in GL(2, k), N_{K/k}(\lambda) = \frac{1}{\det d}\right\}$

   If $char(k) \neq 2$, let $\mathrm{Im}(K) = \{\alpha - \overline{\alpha} : \alpha \in K\}$. Then

   $GL_*^+(2, K) = K^{\times} \left(\left\{ \begin{pmatrix} 1 & i \\ j & t \end{pmatrix} : i, j \in \mathrm{Im}(K), \quad t \in k, \quad ij \neq t \right\} \cup \right.$
   $\left. \left\{ \begin{pmatrix} 0 & t \\ 1 & i \end{pmatrix} : i \in \mathrm{Im}(K), \quad t \in k, \quad t \neq 0 \right\} \right)$

   $SL_*^+(2, K) =$
   $\left\{ x \begin{pmatrix} 1 & i \\ j & t \end{pmatrix} : x \in K^{\times}\ i, j \in \mathrm{Im}(K), t \neq ij, N_{K/k}(x) = \frac{1}{t - ij} \right\} \cup$
   $\left\{ x \begin{pmatrix} 0 & t \\ 1 & i \end{pmatrix} : x \in K^{\times}\ i \in \mathrm{Im}(K), \quad t \in k, \quad t \neq 0, N_{K/k}(x) = -\frac{1}{t} \right\}$

9 The additive category $\mathcal{SL}$

Let $\mathfrak{A}$ be the category of unitary rings with an involution, i.e.:

the objects are the unitary rings with involution $(A, *)$ and

the morphisms are the homomorphisms f of rings with identity such that $f(a^*) = f(a)^*$.

Let $\mathfrak{G}$ be the category of groups

Definition 15 *We define three functors $\mathcal{G} = \mathcal{G}_\varepsilon, \mathcal{H} = \mathcal{H}_\varepsilon, \mathcal{O} = \mathcal{O}_\varepsilon$ from $\mathfrak{A}$ to $\mathfrak{S}$ as follows:*

1. *The functor $\mathcal{G}$*

   (a) *For A an object of $\mathfrak{A}$, set $\mathcal{G}(A) = GL_*^\varepsilon(2, A)$ and*

   (b) *If $f : A \to B$ is a morphism in $\mathfrak{A}$, set $\mathcal{G}(f) : GL_*^\varepsilon(2, A) \to GL_*^\varepsilon(2, B)$ by $\mathcal{G}(f)\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} f(a) & f(b) \\ f(c) & f(d) \end{bmatrix}$*

2. *The functor $\mathcal{H}$:*

   (a) *For A an object of $\mathfrak{A}$, set $\mathcal{H}(A) = SL_*^\varepsilon(2, A)$ and*

   (b) *If $f : A \to B$ is a morphism in $\mathfrak{A}$, set $\mathcal{H}(f) = \mathcal{G}(f)$*

3. *The functor $\mathcal{O}$*

   (a) *For A an object of $\mathfrak{A}$, set $\mathcal{O}(A) = A_s^\times$ and*

   (b) *If $f : A \to B$ is a morphism in $\mathfrak{A}$, set $\mathcal{O}(f) = f$*

The following proposition is instrumental in working in the categories above:

Proposition 12 *The $\varepsilon*$-determinant induces a natural transformation between the functors $\mathcal{G}$ and $\mathcal{O}$*

Proof. It suffices to notice that the diagram

$$\begin{array}{ccc} GL_*^\varepsilon(2, A) & \xrightarrow{\det_*^\varepsilon} & A_s^\times \\ \downarrow \mathcal{G}(f) & & \downarrow f \\ GL_*^\varepsilon(2, B) & \xrightarrow{\det_*^\varepsilon} & B_s^\times \end{array}$$

is commutative ■

Proposition 13 *$\mathcal{U}$ is an additive category. Then if $(A, *), (B, *') \in \mathcal{U}$, we have $(A \oplus B; * \oplus *') \in \mathcal{U}$*

Proof. straightforward ■

Definition 16 *We denote by $\mathcal{SL} = \mathcal{SL}_\varepsilon$ the category whose objects are the groups $SL_*^\varepsilon(2, A)$ where $(A, *) \in \mathcal{U}$, and the morphisms are the group homomorphisms.*

Proposition 14 *If* $(A_1, *_1)$ *and* $(A_2, *_2)$ *are objects of* $\mathcal{U}$*, then* $SL^{\varepsilon}_{*_1}(2, A_1) \oplus SL^{\varepsilon}_{*_2}(2, A_2) \simeq SL^{\varepsilon}_{*}(2, A_1 \oplus A_2)$*, where* $* = *_1 \oplus *_2$*. It follows that the category* $\mathcal{SL}$ *is an additive category*

Proof. The function $\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}, \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \right) \mapsto \begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{bmatrix}$ provides the desired isomorphism. ■

10 Bibliography

[1] Dickson, L. E., *Linear groups: With an exposition of the Galois field theory*, Dover Publications Inc., New York, 1958

[2] Fröhlich, A. and McEvett, A.M., Forms over Rings with Involution, Journal of Algebra 12, 79-104 (1969)

[3] Gutierrez, L., *Representaciones de Weil del Grupo* $SL_*(2, \mathbb{F}_q[x]/(x^m))$, Ph. D. Thesis, Universidad de Chile, Santiago, 2006.

[4] Howe, R., L2-Duality for Stable Dual Pairs, preprint.

[5] Jacobson, N., *Finite Dimensional Division Algebras over Fields*, Springer-Verlarg, Berlin, 1996.

[6] Kassel, C., *Quantum groups*, Springer-Verlag, Berlin, 2000.

[7] Kutzko,P., The exceptional representations of Gl_2, Compositio Math 51, 3-14 (1984)

[8] Pantoja, J., A presentation of the group $SL_*(2, A)$, A a simple artinian ring with involution, Manuscripta math. 121, 97-104 (2006)

[9] Pantoja, J. and Soto-Andrade, J., A Bruhat decomposition of the group $SL_*(2, A)$, J. of Algebra 262, 401-412 (2003)

[10] Soto-Andrade, J., *Représentations de certains groupes symplectiques finis*, Bull.Soc. Math. France, Mém. 55-56 (1978).

[11] Szechtman, F., Weil representations of symplectic groups over non-principal rings, Rocky Mountains, J. of Math. 35, 309-329 (2005).