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# Representation fields for quaternionic skew-hermitian forms

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**Abstract.** Classes of indefinite quadratic forms in a genus are in correspondence with the Galois group of an abelian extension called the spinor class field (Estes and Hsia, Japanese J. Math. **16**, 341–350 (1990)). Hsia has proved (Hsia et al., J. Reine Angew. Math. **494**, 129–140 (1998)) the existence of a representation field $F$ with the property that a lattice in the genus represents a fixed given lattice if and only if the corresponding element of the Galois group is trivial on $F$. This far, the corresponding result for skew-hermitian forms was known only in some special cases, e.g., when the ideal (2) is square free over the base field. In this work we prove the existence of representation fields for quaternionic skew-hermitian forms in complete generality.

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**1. Introduction.** Let $\mathfrak{A}$ denote a quaternion algebra over a number field $k$. A skew-hermitian space $(V, h)$ over $\mathfrak{A}$ is a free $\mathfrak{A}$-module provided with a skew-hermitian form $h : V \times V \to \mathfrak{A}$, i.e., $h$ is $\mathfrak{A}$-linear in the first variable and satisfies $h(v, w) = -\overline{h(w, v)}$, where $q \mapsto \bar{q}$ denotes the canonical involution of the quaternion algebra. In particular, the element $a = h(v, v) \in \mathfrak{A}$ is a pure quaternion, i.e., $\overline{a} = -a$. If $V = \mathfrak{A}^n$, the skew-hermitian forms on $V$ are the maps

$$h\left((x_1, \ldots, x_n), (y_1, \ldots, y_n)\right) = \sum_{i,j=1}^{n} x_i a_{i,j} \overline{y_j},$$

where $a_{i,j} = -\overline{a_{j,i}}$. The matrix $(a_{i,j})_{i,j} \in \mathbb{M}_n(\mathfrak{A})$ is called the Gram Matrix of $h$. An $\mathfrak{A}$-linear map $g : V \to V$ preserving $h$ is called an isometry. Skew-hermitian forms share many properties of quadratic forms. In fact, when $\mathfrak{A}$ is

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a matrix algebra, skew-hermitian forms in a rank-$n$ free $\mathfrak{A}$-module $V$ are naturally in correspondence with quadratic forms in the $2n$-dimensional $k$-vector space $PV$, for any idempotent matrix $P$ of rank 1 in $\mathfrak{A}$ (see [3]). The unitary group $G$ of every skew-hermitian form $h$, i.e., the group of isometries of $h$, is isomorphic to the orthogonal group of the corresponding quadratic form.

Let $S$ be a finite set of places in $k$ containing the archimedean places and let $\mathfrak{D}$ be a maximal $S$-order in $\mathfrak{A}$. A $\mathfrak{D}$-lattice in $V$ is an $S$-lattice $\Lambda$ satisfying $\mathfrak{D}\Lambda = \Lambda$. The group $G$ acts naturally on the set of $\mathfrak{D}$-lattices and the $G$-orbits are called classes. This action can be extended to the adelization $G_{\mathbb{A}}$ of $G$ [1, Section 2]. The $G_{\mathbb{A}}$-orbits are called genera. A genus of $\mathfrak{D}$-lattices can be defined as a set of lattices that are locally isometric. Between these two sets lies the spinor genus. Two lattices $L$ and $\Lambda$ are in the same spinor genus if, replacing each by an isometric lattice if needed, we can find a local isometry $\sigma_\wp \in G_\wp$ with trivial spinor norm satisfying $\sigma_\wp L_\wp = \Lambda_\wp$. The spinor norm $\theta_\wp : G_\wp \to k_\wp^*/k_\wp^{*2}$, or more generally, $\theta_L : G_L \to L^*/L^{*2} = H^1(L, F)$ for any field $L$ containing $k$ is the coboundary map defined from the universal cover $F \hookrightarrow \tilde{G} \twoheadrightarrow G$ [1, Section 2]. This concept is important because both the class and the spinor genus of a given lattice coincide whenever $G$ is non-compact at some place in $S$. Furthermore, the set of spinor genera in a genus is a principal homogeneous space for some abelian group, which we identify with the Galois group of an abelian extension called the spinor class field of the lattice. This makes it easier to compute the number of spinor genera in a genus.

Hsia, Shao, and Xu proved in [5] that for any sub-lattice $M$ of a quadratic lattice $\Lambda$ there exists a subfield $F$ of the spinor class field $\Sigma$ such that, for any spinor genus $X$ in the genus of $\Lambda$, there exists a lattice $L$ in $X$ representing $M$ if and only if the unique element of $\mathrm{Gal}(\Sigma/k)$ sending the spinor genus of $\Lambda$ to $X$ was trivial on $F$. This field is called the representation field for $\Lambda|M$. When $\mathfrak{A} = \mathbb{M}_2(k)$ is a matrix algebra, the maximal order $\mathfrak{D} = \mathbb{M}_2(\mathcal{O})$ is the ring of matrices with coefficients in the ring $\mathcal{O}$ of $S$-integers in $k$, and $P = \mathrm{diag}(0, 1)$, then there exists a natural correspondence between $\mathfrak{D}$-lattices in $V$ and $S$-lattices in $PV$ [3, Section 3]. This correspondence respects classes, genera, and spinor genera. In particular the representation field always exists in this case. More generally, for any quaternion algebra $\mathfrak{A}$, and any maximal order $\mathfrak{D}$ in $\mathfrak{A}$, to determine the spinor class field or representation fields we must compute local images of the spinor norm, which reduce to spinor norm computations for quadratic forms at any place splitting $\mathfrak{A}$ [3]. We are therefore reduced to compute images of the spinor norm at non-split places. The existence of representation fields for quadratic forms follows from local considerations that also apply at split places for skew-hermitian forms. In order to prove the existence of representation fields for skew-hermitian forms we need to prove that the image of the set of local generators (see Section 2 below) is a group at non-split places. The local computations in [2] are insufficient to accomplish this in all cases. In Section 2 below we prove the result:

**Theorem 1.** *The representation field always exists for skew-hermitian forms over quaternion algebras.*

In Section 3 we show how the techniques used in the proof can be used to expand the tables in [2]. In particular, we are now able to compute the image of the spinor norm for all non-diagonalizable lattices.

**2. Proof of Theorem 1.** Let $\mathfrak{A}$ be a quaternion algebra and let $V$ be a free left $\mathfrak{A}$-module provided with a skew-hermitian form $h$. Let $G$ be the unitary group of the skew-hermitian space $(V, h)$. Assume $M$ is a sublattice of the maximal rank lattice $\Lambda \subseteq V$. As usual [1], we call an element $u \in G_{\mathbb{A}}$ a generator for $\Lambda|M$ if $M \subseteq u\Lambda$. A local generator $u \in G_{\wp}$ is defined analogously. We denote by $X$ the set of generators and by $\Theta : G_{\mathbb{A}} \to J_k/J_k^2$ the spinor norm on adeles [1]. We recall that the representation field for $\Lambda|M$ exists if and only if the image $\Theta(X)$ is a subgroup of $\Theta(G_{\mathbb{A}})$. This can be computed locally, i.e., it suffices to check that $\theta_{\wp}(X_{\wp})$ is a subgroup of $\theta_{\wp}(G_{\wp}) \subseteq k_{\wp}^*/k_{\wp}^{*2}$ [1]. Note that $X_{\wp} = X_{\wp}G_{\wp}^{\Lambda}$, where $G_{\wp}^{\Lambda}$ is the local stabilizer of $\Lambda$, whence it suffices to prove the statement if $[\theta_{\wp}(G_{\wp}) : \theta_{\wp}(G_{\wp}^{\Lambda})] > 2$. On the other hand, whenever $\mathfrak{A}_{\wp}$ is a matrix algebra and $P \in \mathfrak{D}_{\wp}$ is an idempotent of rank 1, then the set $\theta_{\wp}(X_{\wp})$ equals the spinor image of the set of generators for the pair $P\Lambda_{\wp}|PM_{\wp}$ of quadratic lattices [3, Section 3] and the result follows from the local computations in [5]. It suffices therefore to prove that $[\theta_{\wp}(G_{\wp}) : \theta_{\wp}(G_{\wp}^{\Lambda})] \leq 2$ for any skew-hermitian lattice $\Lambda$ over a local quaternion division algebra. For simplicity, in all that follows we denote by $H_{\wp}(\Lambda)$ the subgroup of $k_{\wp}^*$ satisfying $H_{\wp}(\Lambda)/k_{\wp}^{*2} = \theta_{\wp}(G_{\wp}^{\Lambda})$. Theorem 1 follows therefore from next lemma:

**Lemma 2.1.** *For any skew-hermitian lattice $\Lambda$ over a local quaternion division algebra $\mathfrak{A}_{\wp}$ over a local field $k_{\wp}$ we have $[k_{\wp}^* : H_{\wp}(\Lambda)] \leq 2$.*

In order to prove Lemma 2.1 we need the following result, which is also needed in the next section. For definitions see [2].

**Lemma 2.2.** *Let $\Lambda$ be an irreducible binary prime-modular skew-hermitian $\mathfrak{D}_{\wp}$-lattice, where $\mathfrak{D}_{\wp}$ is the maximal order of a quaternion division algebra over a dyadic local field $k_{\wp}$. Then there exists a unimodular $\mathfrak{D}_{\wp}$-lattice $L$ containing $\Lambda$. Any such lattice $L$ contains an element $t$ such that $a = h(t,t) \in \mathfrak{D}_{\wp}^*$. For any such element $t$ we have $N[k_{\wp}^*(a)] \subseteq H_{\wp}(\Lambda)$.*

*Proof.* Let $\{v, w\}$ be a basis of $\Lambda$. Then both $h(v, v)$ and $h(w, w)$ are in $\mathcal{M}^2$, where $\mathcal{M}$ is the unique maximal two-sided ideal of $\mathfrak{D}_{\wp}$, since none of the lattices $\mathfrak{D}_{\wp}v$ or $\mathfrak{D}_{\wp}w$ has an orthogonal complement in the lattice. It follows that $h(v, w)$ is a quaternion $q$ generating $\mathcal{M}$. We conclude that the lattice $L = \langle q^{-1}v, w\rangle$ is unimodular, and therefore it has an orthogonal basis [2, Lemma 5.2]. Note that $\Lambda = \mathfrak{D}_{\wp}w + qL$. Any $t$ as above is part of an orthogonal basis $\{s, t\}$ of $L$. Write $w = xt + ys$ with $x, y \in \mathfrak{D}_{\wp}$. Since $H_{\wp}(\mathfrak{D}_{\wp}t) = N[k_{\wp}(a)^*]$ [2, Proposition 6.1], for any $c \in N[k_{\wp}(a)^*]$ we can find an isometry $g : \mathfrak{D}_{\wp}t \to \mathfrak{D}_{\wp}t$ satisfying $\theta_{\wp}(g) = ck_{\wp}^{*2}$. We extend $g$ to $\tilde{g} : L \to L$ by setting $\tilde{g}s = s$. Note that the spinor norm of $\tilde{g}$ is $\theta_{\wp}(\tilde{g}) = \theta_{\wp}(g) = ck_{\wp}^{*2}$. By definition $\tilde{g}t = gt = rt$ for some $r \in \mathfrak{D}^*$. If $r \equiv 1 \mod \mathcal{M}$, the automorphism $g$ of $L$ coincides with the identity modulo $q$, whence $g\Lambda = g(\mathfrak{D}_{\wp}w + qL) =$

$\mathfrak{D}_\wp w+qL=\Lambda$. In the general case, we can replace $\tilde{g}$ by $\tilde{g}^f$ where $f=\sharp(\mathfrak{D}_\wp/\mathcal{M})^* = \sharp(\mathfrak{D}_\wp/\mathcal{M})-1$ is odd since $k_\wp$ is dyadic. It follows that

$$\theta_\wp(\tilde{g}^f)=c^f k_\wp^{*2}=c\left(c^{\frac{f-1}{2}}\right)^2 k_\wp^{*2}=ck_\wp^{*2},$$

where $\tilde{g}^f t=r^f t$ with $r^f\equiv 1 \bmod \mathcal{M}$, so that the preceding reasoning applies. □

*Proof of Lemma 2.1.* Any skew-hermitian lattice $\Lambda$ over a local quaternion division algebra $\mathfrak{A}_\wp$ has an orthogonal decomposition

$$\Lambda=\perp_{t=1}^{n}\Lambda_t$$

where every modular irreducible lattice $\Lambda_t$ has rank 1 or 2 [2]. It suffices therefore to prove the lemma when $\Lambda$ is an indecomposable lattice of rank 1 or 2.

A lattice of rank 1 is of the form $\mathfrak{D}_\wp v$ where $\mathfrak{D}_\wp$ is the unique maximal order in $\mathfrak{A}_\wp$ and $q=h(v,v)$ is a pure quaternion. In this case the image of the spinor norm is $H(\mathfrak{D}_\wp v)=N\left[k_\wp(q)^*\right]$ ([2, Proposition 6.1]). Since $k_\wp(q)/k_\wp$ is a quadratic extension, the result follows.

Assume now that the lattice $\Lambda_t$ is indecomposable of rank 2. In particular $k_\wp$ is dyadic. It follows from Section 5 in [2] that $\Lambda_t$ is a re-scaling of a prime-modular binary lattice. We can therefore assume that $\Lambda_t$ is prime modular. Now Lemma 2.1 follows from Lemma 2.2. □

**3. Some spinor norm computations.** Following [2], in all of this section we let $i$ and $j$ denote pure quaternions in $\mathfrak{D}_\wp$ satisfying the following conditions:

1. $i$ generates the maximal two sided ideal $\mathcal{M}$ of $\mathfrak{D}_\wp$,
2. $j^2=\Delta\in k_\wp$ is a unit of maximal quadratic defect,
3. $ij=-ji$.

In particular, $\pi=i^2$ is a uniformizing parameter of the ring of integers in $k_\wp$. We let $q\mapsto|q|$ denote any absolute value defining the usual topology on $k_\wp$.

**Theorem 2.** *For any irreducible binary skew-hermitian lattice $\Lambda$ over a dyadic local field $k_\wp$, the local image of the spinor norm is $H_\wp(\Lambda)=k_\wp^*$.*

*Proof.* Assume first that $\Lambda$ is anisotropic. Let $L$ and $t$ be as in Lemma 2.2. Then $t$ is part of an orthogonal basis $\{s,t\}$ of $L$. It follows that if $a=h(t,t)$ and $b=h(s,s)$, then

$$N\left(k_\wp(a)^*\right)N\left(k_\wp(b)^*\right)\subseteq H_\wp(\Lambda).$$

As $\Lambda$ is anisotropic, the fields $k_\wp(a)$ and $k_\wp(b)$ are not isomorphic, whence the corresponding norm groups are different and the result follows.

Assume next that $\Lambda$ is isotropic. Let $\{v,w\}$ be a basis of $\Lambda$ such that $h(v,v)=0$. Replacing $w$ by a multiple if needed we may assume $h(v,w)=h(w,v)=i$. Since $\Lambda$ is irreducible, we have $|h(w,w)|\leq|i|^2$. Let $L$ be the lattice generated by $v$ and $i^{-1}w$. A straightforward computation shows that the Gram matrix of $L$ with respect to the basis $\{v, i^{-1}w\}$ has the form $\left(\begin{smallmatrix} 0 & -1 \\ 1 & a \end{smallmatrix}\right)$ for some $a \in \mathfrak{D}_\wp$, in particular $L$ is unimodular. Furthermore, $a$ is a pure quaternion, whence it is of the form $a = nj + \chi i$ where $n \in \mathcal{O}_k$ and $\chi \in \mathcal{O}_{k(j)}$. Assume first that $|\chi| > |2|$. Let $s' = n\omega v + i^{-1}w$, where $\omega - \bar{\omega} = j$, which exists since $j$ is even [2, Section 4]. Note that $is' \in \Lambda$. Since $h(s', s') = \chi i$, the rotation $\left(s' : \frac{i\chi}{2}\right)$ is well defined and its spinor norm is $-\pi N(\chi)$ [2, Section 3]. Furthermore, since $\chi$ belongs to the unramified extension $k(j)/k$, the inequality $|\chi| > |2|$ implies $|\frac{2}{\pi}\chi^{-1}| \leq |1|$, whence

$$\left(s' : \frac{i\chi}{2}\right)(v) = v - 2h(v, s')\chi^{-1}i^{-1}s' = v + \frac{2}{\pi}\chi^{-1}(is') \in \Lambda$$

and

$$\left(s' : \frac{i\chi}{2}\right)(w) = w - 2h(w, s')\chi^{-1}i^{-1}s' = w - \frac{2}{\pi}ih(i^{-1}w, s')\chi^{-1}(is') \in \Lambda.$$

The last contention follows since both $i^{-1}w$ and $s'$ are in $L$, whence $h(i^{-1}w, s')$ belongs to $\mathfrak{D}_\wp$. It follows that $\left(s' : \frac{i\chi}{2}\right)$ is in the stabilizer of $\Lambda$. On the other hand, $L$ contains the element $s = (n-1)\omega v + i^{-1}w$. Note that $b = h(s, s) = j + \chi i$ is a unit, and therefore $s$ is part of a diagonal basis. In particular Lemma 2.2 implies $N[k_\wp(b)^*] \subseteq H_\wp(\Lambda)$. It suffices to show that $-\pi N(\chi)$ does not belong to this norm group, but this follows as in the proof of Proposition 6.6 in [2].

It remains the case $|\chi| \leq |2|$. In this case $a$ is an even pure quaternion, i.e., there exists an integral quaternion $\eta$ satisfying $\eta - \bar{\eta} = a$ [2, Section 4]. In particular, the element $s' = \eta v + i^{-1}w$ satisfies $h(s', s') = 0$. Furthermore, $\Lambda$ is generated by $v$ and $w' = is' = w + i\eta v$. It follows that $\Lambda$ is a prime hyperbolic plane. Now the result follows as in Proposition 6.2 of [2]. □

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