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Dichotomy and existence of periodic solutions of quasilinear functional differential equations

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Using Krasnoselskii's fixed point theorem, functional analysis methods and dichotomy theory, we study the existence and uniqueness of the periodic solutions of integro-differential equations with bounded and unbounded delays.

1. Introduction

The importance of the dichotomy theory of a linear differential system

$$x'(t) = A(t)x \tag{1}$$

in the study of qualitative properties of solutions of general differential equations is known. The most used types of dichotomies are the ordinary dichotomy and the exponential dichotomy, see [1–7]. They allow us to characterize the bounded solutions of the non-homogeneous linear differential system

$$x'(t) = A(t)x + f(t). \tag{2}$$

A remarkable solution of system (2) is given by

$$x(t) = \int_{-\infty}^{\infty} G(t, s)f(s)\mathrm{d}s, \tag{3}$$

where $G(t, s)$ is a Green matrix. For example, if $f(t) \in L^1(\mathrm{R})$ and (1) has an ordinary dichotomy, i.e. $|G(t, s)| \leq c$, where c is a constant, for all $t, s \in \mathrm{R}$, then $x(\cdot)$ is a bounded solution.

If f is a bounded function on R, then x is a bounded solution of (2) if, for example, system (1) has an exponential dichotomy, i.e. $|G(t, s)| \leq c\mathrm{e}^{-\alpha|t-s|}$, for $t, s \in \mathrm{R}$ and $\alpha > 0$. Really, a sufficient condition is

$$\sup_{t\in \mathrm{R}} \int_{-\infty}^{\infty} |G(t, s)|\, \mathrm{d}s = \mu < \infty \tag{4}$$

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i.e. the integrability of the dichotomy. This is the only condition that allows us to obtain important uniqueness facts such as: the trivial solution is the unique bounded solution of the linear system (1) and x, given by (3), is the unique bounded solution of the non-homogeneous linear system (2). Moreover, if in addition $G(t, t)$ is bounded for $t \in \mathrm{R}$, then the projection matrix defining Green's matrix G is also unique. Finally, when A is periodic, $G(t, t)$ is also periodic and x, given by (3), is the unique periodic solution of the non-homogeneous linear system (2). See Propositions 1–5 in Section 2. All this will allow us to establish the existence and uniqueness of periodic solutions of general integro-differential delayed systems

$$y'(t) = A(t)y(t) + F\left(t, y_t, y((-\infty, t])\right), \tag{5}$$

where F is a general functional including y_t as bounded delay and $y((-\infty, t])$ denoting a functional with unbounded delay, particularly, cases as (7)–(9) below. See Burton [8–11], Corduneanu [4–6,12–15]. The existence of periodic solutions of functional differential equations has been extensively studied in theory and in practice (for example, see [8,9,16–21] and the references cited therein), but few papers have considered integrable dichotomies.

In nonlinear Volterra equations with infinite delay, the existence of periodic solutions has been extensively developed by Burton and others under the boundedness or stability conditions (see [8,9]). The introduction of special Banach spaces as $BC\,(-\infty, \rho]$ (see Section 3) combined with Lyapunov function (functional) and fixed point theory have allowed us to obtain sufficient conditions which guarantee the existence of periodic solutions of general infinite delay systems

$$y'(t) = f(t, y_t). \tag{6}$$

Several works are treated on all these advances, Burton [8–11], Corduneanu [4,5], Fink [6], Hale–Verduyn [7], Gopalsamy [12], Lakshmikantham, Sivasundaram and Kaymakçalan [22],Yoshizawa [14,15], etc. In the theory of boundary value problems for systems of functional differential equations, the modern level is determined by the work of Kiguradze and Puza [23–26], summarized in the interesting monograph [23]. They directly consider the positivity and negativity of Green's matrices.

Also, the integro-differential equations

$$y'(t) = A(t)y(t) + \int_{-\infty}^{t} C(t,s)y(s)\mathrm{d}s + f(t) \tag{7}$$

have been successively studied and sufficient conditions which guarantee the existence of periodic solution of system (7) are obtained. As an example, Chen [27] consider a kind of integro-differential equation with infinite delay

$$y'(t) = A(t)y(t) + \int_{-\infty}^{t} C(t,s)y(s)\mathrm{d}s + g(t, y(t)) + f(t) \tag{8}$$

and using exponential dichotomy and fixed point theorem, discusses the existence, uniqueness and stability of periodic solutions of (8). Besides its theoretical interest, the study of these problems has great importance in applications. For these reasons the theory of integro-differential equations with delay has drawn the attention of several authors (see, for example, [8–11,27,4,6,12,16,28,17–20,29–31,21]).

In the present paper, we consider a general system (5) including

$$F(t, y_t, y((-\infty, t])) = \int_{-\infty}^{t} C(t, s, y(s))\mathrm{d}s + \sum_{i=1}^{l} g_i(t, y(t - \tau_i(t))) + f(t, y(t)). \tag{9}$$

Recently, in the interesting paper [32], Agarwal et al. reduce a general system of nonlinear integro-differential equations to a system of ordinary differential equations. A Floquet theory and exponential stability results are obtained. This opens a new manner to use our technique.

The rest of the paper is organized as follows. In the next section, some definitions and preliminary results are introduced. We show some interesting properties about integrable dichotomies. Any integrable (h, k)-dichotomy is applicable. Section 3 is devoted to establishing some criteria for the existence and uniqueness of periodic solutions of system (5). Integrable dichotomy and Krasnoselskii's Theorem A below are fundamental to obtain the main results.

Now, we state a fixed point theorem due to Krasnoselskii [10,11].

Theorem A. *Let S be a closed, bounded convex, non-empty subset of a Banach space E. Suppose that Γ_1 and Γ_2 map S into E and that* (i) *$\Gamma_1 x + \Gamma_2 y \in S$ for all $x, y \in S$* (ii) *Γ_1 is completely continuous on S and* (iii) *Γ_2 is a contraction on S. Then, there exists $z \in S$ such that $\Gamma_1 z + \Gamma_2 z = z$.*

2. Integrable dichotomy and periodicity

Let C^n, R^n denote the sets of complex and real n-vectors, and $|x|$ any convenient norm for $x \in \mathrm{C}^n$, also let $\mathrm{C} = \mathrm{C}^1$, $\mathrm{R} = \mathrm{R}^1$ and $\mathrm{R}_+ = (0, \infty)$.

Now, we recall (see [1,33,34,23,35–39]), the notions of integrable dichotomy and (h, k)-dichotomy for linear nonautonomous ordinary differential equations. A solution matrix $\Phi(t)$ of system (1) is said to be a fundamental matrix, if $\Phi(0) = I$. We define a Green matrix $G = G_P$ as:

$$G(t,s) = \begin{cases} \Phi(t)P\Phi^{-1}(s), & \text{for } t \geq s \\ -\Phi(t)(I-P)\Phi^{-1}(s), & \text{for } s > t, \end{cases} \tag{10}$$

where P is a projection matrix.

Definition 1. System (1) is said to have an integrable dichotomy, if there exist a projection P and $\mu > 0$ such that its Green matrix $G = G_P$ satisfies:

$$\sup_{t\in\mathbf{R}} \int_{-\infty}^{\infty} |G(t,s)|\,\mathrm{d}s = \mu. \tag{11}$$

As examples of integrable dichotomies, we have the integrable (h, k)-dichotomies.

Definition 2. Let $h, k : \mathrm{R} \to \mathrm{R}_+$ be two positive continuous functions. The linear system (1) is said to possess an (h, k)-dichotomy, if there are a projection matrix P and a positive constant c such that its Green matrix $G = G_P$ satisfies:

$$|G(t,s)| \leq g_{h,k}(t,s), \quad t, s \in \mathrm{R},$$

where

$$g_{h,k}(t,s) = \begin{cases} c\,h(t)h(s)^{-1}, & \text{if } t \geq s, \\ c\,k(s)k(t)^{-1}, & \text{if } t \leq s \end{cases}$$

and $h(t)^{-1}$ denotes $1/h(t)$.

Definition 3. We say that the (h, k)-dichotomy is integrable if there exists $\mu_{h,k} > 0$ such that

$$\sup_{t\in\mathbf{R}} \int_{-\infty}^{\infty} g_{h,k}(t,s)\mathrm{d}s = \mu_{h,k}.$$

Definition 4. The system (1) is said to have a h-dichotomy, if it has a (h, h)-dichotomy and a (h, k)-dichotomy is said to fulfill a compensation law if there exists a positive constant $C_{h,k}$ such that

$$h(t)h(s)^{-1} \leq C_{h,k}k(s)k(t)^{-1}, \quad t \geq s.$$

Our main condition on the linear system (1) will be:

(D) System (1) possesses an integrable dichotomy with projection P for which $\Phi(t)P\Phi^{-1}(t)$ is bounded.

Remark 1. Clearly, a system having a (h, k)-dichotomy with compensation law is a system with a h-dichotomy. Obviously, the case $h(t) = \mathrm{e}^{-\beta t}, k(t) = \mathrm{e}^{-\alpha t}, \alpha, \beta > 0$ constants, yields an exponential dichotomy, but (h, k)-dichotomic systems are more general than these ones. If system (1) has an integrable (h, k)-dichotomy, then condition (D) is satisfied. Even if the projection P is the identity, the exponential stability does not follow from the integrable dichotomy. See, for example [2, page 73;12,30]. Exponential and ordinary dichotomy can be characterized in terms of the bounded solutions, with respect to some admissible Banach spaces. See for example [1,9,2,3]. The big generality of an integrable dichotomy does not allow this.

However, the functions h and k have an exponential domination:

Lemma 1. *Let $\varphi : \mathrm{R} \to (0, \infty)$ and $\psi : \mathrm{R} \to (0, \infty)$ be two locally integrable functions, satisfying for $\mu > 0$ constant*

$$\varphi(t) \int_{-\infty}^{t} \varphi(s)^{-1}\mathrm{d}s \leq \mu, \quad t \in \mathrm{R}, \tag{12}$$

$$\psi(t) \int_{t}^{\infty} \psi(s)^{-1}\mathrm{d}s \leq \mu, \quad t \in \mathrm{R}. \tag{13}$$

Then for any $t_0 \in \mathrm{R}$, $\varphi(t) \leq c\mathrm{e}^{-\mu^{-1}t}$, $t \geq t_0$, and $\psi(t) \leq c\,\mathrm{e}^{\mu^{-1}t}$ for $t \leq t_0$, where $c > 0$.

Proof. If $u(t) = \int_{-\infty}^{t} \varphi(s)^{-1}\mathrm{d}s$ then $u' = \varphi^{-1} \geq \mu^{-1}u$ by (12). So, $u(t) \geq u(t_0)\mathrm{e}^{\mu^{-1}(t-t_0)}$ for $t \geq t_0$. Therefore $\varphi(t) \leq \mu u(t)^{-1} \leq \mu u(t_0)^{-1}\mathrm{e}^{-\mu^{-1}(t-t_0)}$. To solve (13), let $v(t) = \int_{t}^{\infty} \psi(s)^{-1}\mathrm{d}s$. We have $v \leq -\mu v'$, *i.e.*$(v\mathrm{e}^{\mu^{-1}t})' \geq 0$ or $v(t_0) - v(t)\mathrm{e}^{\mu^{-1}(t-t_0)} \leq 0$. By (13), $\psi(t) \leq \mu v(t_0)^{-1}\mathrm{e}^{\mu^{-1}(t-t_0)}$ for $t \leq t_0$. □

Corollary 1. *For every integrable (h,k)-dichotomy, there exist constants $M,\alpha>0$ such that $h(t)\le M\mathrm{e}^{-\alpha t}$, for all $t\ge 0$ and $k(t)^{-1}\le M\mathrm{e}^{\alpha t}$, for all $t\le 0$.*

Proposition 1. *If system (1) has an integrable dichotomy, then $x(t)=0$ is the unique bounded solution of system (1).*

Proof. Define $B_0\subset \mathrm{C}^n$ to be the set of initial conditions $\xi\in\mathrm{C}^n$ pertaining to bounded solutions of Eq. (1). Assume first that $(I-P)\xi\neq 0$. Define $\phi(t)^{-1}=|\Phi(t)(I-P)\xi|$, by using $(I-P)^2=I-P$ we may write

$$\int_t^\infty \phi(s)\Phi(t)(I-P)\xi\,\mathrm{d}s=\int_t^\infty \Phi(t)(I-P)\Phi^{-1}(s)\Phi(s)(I-P)\xi\phi(s)\mathrm{d}s.$$

So that upon taking norms and using the integrability of the dichotomy, we have

$$\int_t^\infty \phi(s)\mathrm{d}s\le\mu\phi(t),\quad \text{uniformly in } t.$$

Then $\liminf_{s\in[t,\infty)}\phi(s)=0$, which means that $|\Phi(t)(I-P)\xi|$ must be unbounded.

Now if we assume that $P\xi\neq 0$, then defining $\phi(t)^{-1}=|\Phi(t)P\xi|$, we perform the same procedure, with the integral over the interval $(-\infty,t]$; we conclude that $\liminf_{s\in(-\infty,t]}\phi(s)=0$, which means $|\Phi(t)P\xi|$ must be unbounded. Thus the only possibility for boundedness of the solutions of system (1) is that $B_0=\{0\}$ i.e., $x(t)=0$. □

Proposition 2. *If the homogeneous system (1) possesses an integrable dichotomy, then system (2) has exactly one bounded solution which can be represented by (3).*

Proof. Let x be given by (3). Since $|x(t)|\le\mu\sup_{t\in\mathrm{R}}|f(t)|$, $x(t)$ is a bounded solution of (2). If there exists another bounded solution $x_1(t)$ of (2), obviously, $x(t)-x_1(t)$ is a bounded solution of the homogeneous linear system (1). By Proposition 1, $x(t)\equiv x_1(t)$. The uniqueness of the bounded solution of (2) is proved. □

Proposition 3. *If the linear system (1) satisfies condition* (D)*, then the projector P is unique, i.e., P is decided uniquely by the integrable dichotomy.*

Proof. Firstly, prove that for an integrable dichotomy we have that for every $t_0\in\mathrm{R}$:

$$|\Phi(t)P|\quad\text{is bounded for } t\ge t_0\quad\text{and}\quad |\Phi(t)(I-P)|\quad\text{is bounded for } t\le t_0. \tag{14}$$

Let $\varphi(t)=|\Phi(t)P|$. We have

$$\int_{-\infty}^t \Phi(t)P\varphi(s)^{-1}\mathrm{d}s=\int_{-\infty}^t \Phi(t)P\Phi^{-1}(s)\Phi(s)P\varphi(s)^{-1}\mathrm{d}s.$$

If follows from (11) that $\int_{-\infty}^t\varphi(t)\varphi(s)^{-1}\mathrm{d}s\le\mu$. By (11), $\psi(t)=|\Phi(t)P|$ similarly satisfies $\int_t^\infty\psi(t)\psi(s)^{-1}\mathrm{d}s\le\mu$. So, Lemma 1 implies (14). Now assume that there exists another projector $\tilde{P}$ satisfying the integrability condition (11), i.e.,

$$\int_{-\infty}^t\left|\Phi(t)\tilde{P}\Phi^{-1}(s)\right|\mathrm{d}s+\int_t^\infty\left|\Phi(t)(I-\tilde{P})\Phi^{-1}(s)\right|\mathrm{d}s\le\tilde{\mu}.$$

Similarly to the above discussion, there exists a constant $\tilde{M}>0$ such that

$$|\Phi(t)\tilde{P}|\le\tilde{M},\quad\text{for all } t\ge 0,\qquad |\Phi(t)(I-\tilde{P})|\le\tilde{M},\quad\text{for all } t\le 0. \tag{15}$$

Take any $\xi\in\mathrm{C}^n$, for $t\ge 0$, it follows from (14) that

$$\begin{aligned}|\Phi(t)P(I-\tilde{P})\xi|&=|\Phi(t)P\Phi^{-1}(0)\Phi(0)(I-\tilde{P})\xi|\\&\le|\Phi(t)P\Phi^{-1}(0)||\Phi(0)(I-\tilde{P})\xi|\\&\le M|(I-\tilde{P})\xi|,\quad(t\ge 0),\end{aligned} \tag{16}$$

where M is constant. On the other hand, for $t\le 0$, it follows from (15) and (D) that

$$\begin{aligned}|\Phi(t)P(I-\tilde{P})\xi|&=|\Phi(t)P\Phi^{-1}(t)\Phi(t)(I-\tilde{P})\Phi^{-1}(0)\Phi(0)(I-\tilde{P})\xi|\\&\le|\Phi(t)P\Phi^{-1}(t)||\Phi(t)(I-\tilde{P})\Phi^{-1}(0)||\Phi(0)(I-\tilde{P})\xi|\\&\le M|(I-\tilde{P})\xi|,\quad(t\le 0).\end{aligned} \tag{17}$$

It follows from (16) and (17) that for any $\xi\in\mathrm{C}^n$, $x(t)=\Phi(t)P(I-\tilde{P})\xi$ is the bounded solution of system (1). By Proposition 1, we have $P(I-\tilde{P})\xi=0$, which implies $P=P\tilde{P}$. Similarly to the above discussion, we also have $(I-P)\tilde{P}=0$, i.e., $\tilde{P}=P\tilde{P}$. Therefore, $P=P\tilde{P}=\tilde{P}$. This shows that the projection P is unique. □

Consider the system (2), where $A(t)$ is a T-periodic matrix function and $f(t)$ is a T-periodic bounded function.

The bounded matrix $\Phi(t)P\Phi^{-1}(t)$ is periodic if $A(\cdot)$ is so.

Proposition 4. *Let the linear system (1) satisfy condition (D). If we further assume that $A(t+T)=A(t)$, then $\Phi(t)P\Phi^{-1}(t)$ is also a T-periodic function.*

Proof. By the periodicity, we note that $\Phi(t+T)$ is also a solution matrix of (1). Then, $\Phi(t+T)=\Phi(t)C=\Phi(t)\Phi(T)$, since $\Phi(0)=I$. Note that $\tilde{P}=\Phi(T)P\Phi^{-1}(T)$ is also a projection. Since $\Phi(t)\tilde{P}\Phi^{-1}(s)=\Phi(t+T)P\Phi^{-1}(s+T)$, the dichotomy is also integrable with $\tilde{P}$. By Proposition 3, the projection P is unique. Thus, $\Phi(T)P\Phi^{-1}(T)=P$. Therefore, $\Phi(t+T)P\Phi^{-1}(t+T)=\Phi(t)P\Phi^{-1}(t)$,i.e.,$\Phi(t)P\Phi^{-1}(t)$ is T-periodic function. □

Proposition 5. *Let the conditions in Proposition 4 hold and further assume that f is T-periodic. Then system (2) has exactly one T-periodic solution, which can be represented as (3).*

Proof. By Proposition 4, it is not difficult to check that $x(t)$, given by (3), is a T-periodic solution. Then, by Proposition 2, Proposition 5 is proved immediately. □

3. Existence and uniqueness of periodic solutions

We will study the existence and uniqueness of periodic solutions of distributed and discrete delays of the form

$$y'(t)=A(t)y(t)+F_1(t,y_t)+F_2(t,y_t),\tag{18}$$

where F_1 involves unbounded delays and F_2 bounded delays. For us, (18) has a form as (9), specifically

$$F_1(t,y_t)=\int_{-\infty}^{t}c(t,s,y(s))\mathrm{d}s \quad \text{and} \quad F_2(t,y(t),y_t)=g(t,y(t),y(t-r(t))).\tag{19}$$

For this, a natural vectorial space for the initial conditions is

$$BC(-\infty,t_0]=\left\{\varphi:(-\infty,t_0]\to \mathrm{C}^n|\varphi \text{ is a bounded continuous function}\right\}$$

with the supremum norm $\|\varphi\|=\sup_{t\in(-\infty,t_0]}|\varphi(t)|$.

Consider $A=A(t)$ as a continuous matrix on R, $g:\mathrm{R}\times\mathrm{C}^n\times\mathrm{C}^n\to\mathrm{C}^n$ and $c:\mathrm{R}\times\mathrm{R}\times\mathrm{C}^n\to\mathrm{C}^n$ are continuous functions.

Moreover, we will refer to the following specific conditions.

(D) The linear system (1) possesses an integrable dichotomy (11) such that $\Phi(t)P\Phi^{-1}(t)$ is bounded for $t\in\mathrm{R}$.

(P) $A(t+T)=A(t)$, $r(t+T)=r(t)$, and for $x,y\in\mathrm{C}^n$ fixed, $g(t+T,x,y)=g(t,x,y)$, $c(t+T,s+T,y)=c(t,s,y)$.

Lipschitz conditions:

(L1) For $t,s\in\mathrm{R}$ and $y_1,y_2\in\mathrm{C}^n$, there exists a function $\lambda:\mathrm{R}\times\mathrm{R}\to[0,\infty)$ such that

$$|c(t,s,y_1)-c(t,s,y_2)|\le\lambda(t,s)\,|y_1-y_2|$$

and $\sup_{t\in\mathrm{R}}\int_{-\infty}^{t}\lambda(t,s)\mathrm{d}s=L_1$, $L_1<\mu^{-1}$.

(L2) For any $t\in[0,T]$, $x_1,x_2,y_1,y_2\in\mathrm{C}^n$, there is a positive constant L_2 such that

$$|g(t,x_1,y_1)-g(t,x_2,y_2)|\le L_2\left(|x_1-x_2|+|y_1-y_2|\right),\quad 2L_2<\mu^{-1}.$$

Continuity conditions:

(C1) F_1 is a continuous functional, say: let $r>0$, $t,s\in\mathrm{R}$ and $y_1,y_2\in\mathrm{C}^n$,$|y_i|\le r$. For any $\varepsilon>0$, there exist $\delta>0$ and $\gamma:\mathrm{R}\times\mathrm{R}\to[0,\infty)$ function such that $|y_1-y_2|<\delta$ implies

$$|c(t,s,y_1)-c(t,s,y_2)|\le\varepsilon\gamma(t,s),\quad t,s\in\mathrm{R},$$

where $\vartheta=\sup_{t\in\mathrm{R}}\int_{-\infty}^{t}\gamma(t,s)\mathrm{d}s<\infty$.

(C2) $g:\mathrm{R}\times\mathrm{C}^n\times\mathrm{C}^n\to\mathrm{C}^n$ is a continuous function.

Invariance conditions:

(I1) For every $r>0$, $t,s\in\mathrm{R}$, $|y|\le r$, there exist $\lambda,\gamma:\mathrm{R}^2\to[0,\infty)$ functions and positive constants $c_1,\ \vartheta_1$ with $c_1<\mu^{-1}$ for which

$$|c(t,s,y)|\le\lambda(t,s)\,|y|+\gamma(t,s),$$

where $\sup_{t\in\mathrm{R}}\int_{-\infty}^{t}\lambda(t,s)\mathrm{d}s=c_1$, $\sup_{t\in\mathrm{R}}\int_{-\infty}^{t}\gamma(t,s)\mathrm{d}s=\vartheta_1$.

(I2) For every $r>0$, there exist positive constants $c_2,\ \vartheta_2$ with $2c_2<\mu^{-1}$ for which $|g(t,x,y)|\le c_2\left(|x|+|y|\right)+\vartheta_2$, for every $|x|,|y|\le r$, $r>0$, and $t\in\mathrm{R}$.

Consider the operator

$$\begin{aligned}(\Gamma y)(t) &= \int_{-\infty}^{\infty} G(t,s)F_1(s,y(s))\mathrm{d}s + \int_{-\infty}^{\infty} G(t,s)F_2(s,y(s))\mathrm{d}s \\ &=: (\Gamma_1 y)(t) + (\Gamma_2 y)(t),\end{aligned} \tag{20}$$

where $F = F_1 + F_2$ is defined by (20) and, for example, F_1 satisfies conditions *(D), (P), (L1)* and F_2 satisfies conditions (C2), (I2).

Using Krasnoselskii's Theorem A, we will prove:

Theorem 1. *Let assumptions* (D),(P),(L1),(C2),(I2) *hold. Assume that* $L_1 + 2c_2 < \mu^{-1}$. *Then system* (18) *has at least one T-periodic solution.*

By the symmetry of the conditions, we will obtain as Theorem 1:

Theorem 2. $2L_2 + c_1 < \mu^{-1}$ *and if* (D),(P),(L2),(C1) *and* (I1) *are fulfilled, then system* (18) *has at least one T-periodic solution.*

Let $\mathbb{B} = \{y : \mathrm{R} \to \mathrm{C}^n | y(t)$ is $T-$ periodic continuous function$\}$, the Banach space with the supremum norm $\|y\| = \sup_{t\in[0,T]} |y(t)|$.

We will prove Theorem 1 establishing some lemmas. Clearly, $\Gamma_1, \Gamma_2 : \mathbb{B} \to \mathbb{B}$.

Firstly, we have that Γ_1 is a contraction, where Γ_1 is given by (20).

Lemma 2. *Under conditions* (D),(P),(L1), $\Gamma_1 : \mathbb{B} \to \mathbb{B}$, *given by* (20), *is a contraction mapping.*

Proof. By conditions *(D)* and *(P)*, using Proposition 5, $y \in \mathbb{B}$ implies that $\Gamma_1 y \in \mathbb{B}$. We shall prove that Γ_1 is a contraction mapping in $\mathbb{B}$. For $y_1, y_2 \in \mathbb{B}$, by conditions *(L1)*

$$|F_1(t,y_1) - F_2(t,y_2)| \le \int_{-\infty}^{t} \lambda(t,s)\, |y_1(s) - y_2(s)|\, \mathrm{d}s$$

and

$$|\Gamma_1 y_1(t) - \Gamma_1 y_2(t)| \le L_1 \|y_1 - y_2\| \int_{-\infty}^{\infty} |G(t,s)|\, \mathrm{d}s \le \mu L_1 \|y_1 - y_2\|.$$

Then, as $\mu L_1 < 1$, Γ_1 is a contraction mapping. □

Similarly, Γ_2 given by (20), may be also a contraction operator.

Lemma 3. *Under conditions* (D),(P),(L2), $\Gamma_2 : \mathbb{B} \to \mathbb{B}$, *given by* (19), *is a contraction mapping.*

Let $B_N = B(0,N) \subset \mathbb{B}$ be the closed ball centered at $0 \in \mathbb{B}$ with radius N and also let $C_N = \{x \in \mathrm{C}^n |\, |x| \le N\}$. Now, we will demonstrate that $\Gamma_1 : B_N \to B_N$ is a compact operator for some $N \in \mathrm{N}$.

Lemma 4. *Condition* (I1) *implies that there is* $N \in \mathrm{N}$ *big enough such that* $\Gamma_1 : B_N \to B_N$.

Proof. Suppose that for any $n \in \mathrm{N}$, there exists $y_n \in B_n$ such that $\|\Gamma_1(y_n)\| > n$. By condition (I1) there exists $N \in \mathrm{N}$ sufficiently large such that if $n \ge N$, then $\frac{|F_1(t,y_n)|}{n} \le c_1 + \frac{\vartheta_1}{n} < \mu^{-1}$ and

$$\frac{|\Gamma_1 y_n(t)|}{n} \le \frac{1}{n}\int_{-\infty}^{\infty} |G(t,s)|\, |F_1(s,y_n)|\, \mathrm{d}s < \mu\left(c_1 + \frac{\vartheta_1}{n}\right) < 1.$$

So, $\lim_{n\to\infty} \sup \frac{|\Gamma_1 y_n(t)|}{n} < 1$, contradicting $\|\Gamma_1 y_n\| > n$. Thus, there exists $N \in \mathrm{N}$ such that $\Gamma_1 : B_N \to B_N$. □

Similarly, for Γ_2 we have:

Lemma 5. *Condition* (I2) *implies that there exists* $N \in \mathrm{N}$ *big enough such that* $\Gamma_2 : B_N \to B_N$.

Lemma 6. *Under condition* (I1), $\Gamma_1 B_N$ *is a relatively compact set of* $\mathbb{B}$.

Proof. Since $\Gamma_1 B_N \subset B_N$, $\{\Gamma_1 y | y \in B_N\}$ is bounded in $\mathbb{B}$. Moreover, $A(t)$ and, by (I1), $F_1(t,y)$ are bounded respectively on $[0,T]$ and on $[0,T] \times B_N$.

Then $\frac{\mathrm{d}\Gamma_1 y(t)}{\mathrm{d}t}$ is bounded on $[0,T] \times B_N$, since $\frac{\mathrm{d}\Gamma_1 y(t)}{\mathrm{d}t} = A(t)\Gamma_1 y(t) + F_1(t,y)$.

Therefore $\{\Gamma_1 y | y \in B_N\}$ is equicontinuous. Thus, the conclusion follows from the Ascoli theorem. □

In a similar way, for Γ_2 we obtain.

Lemma 7. *Under condition* (I2), *$\Gamma_2 B_N$ is a relatively compact set of* B.

Finally, we prove the continuity of the operators.

Lemma 8. *Condition* (C1) *implies that $\Gamma_1 : B_N \to B_N$ is continuous.*

Proof. The function $c(t, s, y)$ is uniformly continuous on $[0, T] \times [0, T] \times C_N$ and by the periodicity in t, c is uniformly continuous on R $\times$ C_N. Thus, for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that $y_i \in B_N$, $\|y_1 - y_2\| \leq \delta$ implies $|F_1(t, y_1) - F_1(t, y_2)| \leq \varepsilon_1 = \frac{\varepsilon}{\mu\vartheta}$ for $t \in$ R. Then $\|\Gamma_1 y_1 - \Gamma_1 y_2\| \leq \varepsilon$. In fact, by (C1), $|c(t, s, y_1) - c(t, s, y_2)| \leq \varepsilon_1 \gamma(t, s)$ if $|y_1 - y_2| < \delta$ and then

$$|\Gamma_1(t, y_1) - \Gamma_1(t, y_2)| \leq \varepsilon_1 \mu \int_{-\infty}^{t} \gamma(t, s) \mathrm{d}s \leq \varepsilon_1 \mu \vartheta = \varepsilon. \quad \square$$

In a similar way:

Lemma 9. *The condition* (C2) *implies that $\Gamma_2 : B_N \to B_N$ is a continuous map.*

Finally, we prove Theorem 1:

Let N be big enough such that $\mu(L_1 + 2c_2) + (\vartheta_1 + \vartheta_2)N^{-1} < 1$ and let $S = B_N$ the ball, which may be obtained from Lemma 4. We have for $x, y \in S$, $\Gamma_1 x + \Gamma_2 y \in S$. By Lemma 2, Γ_1 is a contraction mapping. By Lemmas 6 and 8, Γ_2 is completely continuous. Using Theorem A, the proof of Theorem 1 is complete.

The proof of Theorem 2 is absolutely analogous.

As a direct consequence of the method, the contraction principle of Banach and Schauder's theorem imply respectively:

Theorem 3. $\mu(L_1 + 2L_2) < 1$ *and* (D),(P),(L1) *and* (L2) *are fulfilled, then there exists a unique T-periodic solution of system* (20).

Theorem 4. *If* $\mu(c_1 + 2c_2) < 1$ *and* (D),(P),(Ci) *and* (Ii) $i = 1, 2$ *hold, then there exist at least a T-periodic solution of system* (20).

4. Conclusions

The general results obtained are based on three general points: (1) a general type of dichotomy, namely, the integrable dichotomies satisfying that $\Phi(t)P\Phi^{-1}(t)$ is bounded. Any integrable (h, k)-dichotomy, and hence an exponential dichotomy, belongs to this important class of dichotomies. (2) The general conditions of the functional terms which allow an easy verification, applications to many cases and several extensions. General systems (9) may be studied. (3) The general fixed point theorem is used. Krasnoselskii's Theorem A includes Banach and Schauder's fixed point theorems, implying very natural and important results. So, Theorems 1–4 represent tangible situations showing the feasibility of our results.

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