J. Evol. Equ. 11 (2011), 57–70

1424-3199/11/010057-14, *published online* September 11, 2010
DOI 10.1007/s00028-010-0081-z

Journal of Evolution Equations

Periodic solutions of fractional differential equations with delay

CARLOS LIZAMA AND VERÓNICA POBLETE

Abstract. In this paper, we give a necessary and sufficient conditions for the existence and uniqueness of periodic solutions of inhomogeneous abstract fractional differential equations with delay. The conditions are obtained in terms of R-boundedness of operator-valued Fourier multipliers determined by the abstract model.

1. Introduction

Recent investigations into physics, engineering, biological sciences and other fields have demonstrated that the dynamics of many systems are described more accurately using fractional differential equations and that fractional differential equations with delay are often more realistic to describe natural phenomena than those without delay (see [3,19,25–27] and [21]).

The aim of this paper is to study the existence of periodic solutions for the equation

$$D_t^\alpha u(t) = Au(t) + Fu_t + f(t), \qquad t \in [0, 2\pi], \quad 1 \leq \alpha \leq 2, \tag{1.1}$$

where $(A, D(A))$ is a (unbounded) linear operator on a Banach space X, $u_t(\cdot) = u(t + \cdot)$ on $[-r, 0]$, $r > 0$, and the delay operator F is supposed to belong to $\mathcal{B}(L^p([-r, 0]; X), X)$ for some $1 \leq p < \infty$. The state space $L^p([-r, 0]; X)$ is a typical choice with regard to certain applications (e.g. to control theory, or to numerical methods, see [16]).

We observe that similar fractional differential equations on the positive real line have been studied by Clément and Prüss [11], Clément et al. [12] when $0 < \alpha < 1$. In case $\alpha = 1$, Eq. (1.1) with periodic boundary condition in the Lebesgue vector-valued spaces has been studied in the article [22] and, in scales of Besov and Triebel–Lizorkin spaces, by Bu and Fang [9]. The case $\alpha = 2$ has been recently treated in the article [8],

---

Mathematics Subject Classification (2000): 34G10, 34K13, 47D06
Keywords: Operator-valued Fourier multipliers, R-boundedness, Periodic vector-valued Lebesgue spaces, Delay equations.
The first author is partially supported by FONDECYT Grant 1100485.
The second author is partially financed by FONDECYT de Iniciación 11075046.

simultaneously in the scale of Lebesgue, Besov and Triebel–Lizorkin vector-valued spaces. Time fractional differential equations with periodic boundary conditions have recently been treated by Bu [7] and by Keyantuo and Lizama [18]. To the knowledge of the authors, time fractional evolution equations with periodic boundary conditions *and delay* have not been studied until now. One of the difficulties is to determine the right definition of fractional derivative to be used in this case. We consider here the framework of the so-called Liouville-Grünwald-Letnikov fractional derivative, studied in [6] (see also [15] and [20]) in the scalar case and used in [18] in the vector-valued case.

With the above definition, in this paper, we succeed to find necessary and sufficient conditions for the existence and uniqueness of periodic solution of (1.1) in the vector-valued Lebesgue space $L^p(0, 2\pi; X)$, $1 < p < \infty$ (see Theorem 3.5 below).

Considering the scalar case:

$$D_t^\alpha u(t) = \rho u(t) + u(t-\tau) + f(t), \quad t \in [0, 2\pi], \quad 1 \le \alpha \le 2, \tag{1.2}$$

where $\rho \in \mathbb{R}$, we show that if $\tau = 2\pi$ then the unique periodic solution is explicitly given by

$$u(t) = \int_{-\infty}^{t} (t-s)^{\alpha-1} E_{\alpha,\alpha}((1+\rho)(t-s)^\alpha) f(s) \mathrm{d}s \tag{1.3}$$

where $E_{\alpha,\alpha}$ denotes the Mittag-Leffler function. If $0 < \tau < 2\pi$, our characterization in the finite-dimensional case (Corollary 3.9) shows the interesting fact that the number of nonperiodic solutions of (1.2), except for those in the set $\{(-1, \tau)/\tau \in [0, 2\pi]\}$, is greater than 4 for $\alpha^* < \alpha < 2$, but is exactly 4 for all $1 < \alpha < \alpha^*$, where $\alpha^* \approx 1{,}8163$. This property reveals a distinguished behavior of fractional differential equations with delay, which is not present in the case without delay (cf. [18]).

This paper is organized as follows: Sect. 2 collects some results about the Liouville-Grünwald-Letnikov fractional derivative of a function $f \in L^p(0, 2\pi; X)$ and operator-valued Fourier multipliers in vector-valued Lebesgue spaces. Section 3 is devoted to our main abstract result (Theorem 3.5) and some important consequences that are new even in the scalar case (Corollary 3.9). After that, we discuss periodic solutions of the scalar Eq. (1.2) and then we establish an abstract criteria in case X is a UMD space (Theorem 3.14).

2. Preliminaries

Let X, Y be complex Banach spaces. We denote by $\mathcal{B}(X, Y)$ be the space of all bounded linear operators from X to Y. When $X = Y$, we write simply $\mathcal{B}(X)$. For a linear operator A on X, we denote its domain by $D(A)$ and its resolvent set by $\rho(A)$, and for $\lambda \in \rho(A)$, we write $R(\lambda, A) = (\lambda I - A)^{-1} = (\lambda - A)^{-1}$.

We shall identify the spaces of (vector or operator-valued) functions defined on $[0, 2\pi]$ to their periodic extensions to $\mathbb{R}$. Thus, throughout, we consider the space

$L^p(0, 2\pi; X)$, $1 \leq p \leq \infty$ of all 2π-periodic Bochner measurable X-valued functions f such that the restriction of f to $[0, 2\pi]$ is p-integrable (essentially bounded if $p = \infty$).

In the paper [6], Butzer and Westphal studied the fractional derivative directly as a limit of a fractional difference quotient. In the case of periodic functions, it enables one to set up a fractional calculus in the L^p setting with the usual rules, as well as the connection with the classical Weyl fractional derivative (see [23]).

Let $\alpha > 0$. Given $f \in L^p(0, 2\pi; X)$, $(1 \leq p < \infty)$ the Riemann difference

$$\Delta_t^\alpha f(x) := \sum_{j=0}^{\infty} (-1)^j \binom{\alpha}{j} f(x - tj) \tag{2.1}$$

(where $\binom{\alpha}{j} = \frac{\alpha(\alpha-1)\cdots(\alpha-j-1)}{j!}$ is the binomial coefficient) exists almost everywhere and

$$\|\Delta_t^\alpha f\|_{L^p(0,2\pi;X)} \leq \sum_{j=0}^{\infty} |\binom{\alpha}{j}| \|f\|_{L^p(0,2\pi;X)} = O(1) \tag{2.2}$$

since $\binom{\alpha}{j} = O(j^{-j-1})$ as $j \to \infty$.

The following definition is the direct extension of [6, Definition 2.1] to the vector-valued case. See also [18] for their connection with fractional differential equations.

DEFINITION 2.1. Let X be a complex Banach space, $\alpha > 0$ and $1 \leq p < \infty$. If for $f \in L^p(0, 2\pi; X)$ there exists $g \in L^p(0, 2\pi; X)$ such that $\lim_{t \to 0^+} t^{-\alpha} \Delta_t^\alpha f = g$ in the $L^p(0, 2\pi; X)$ norm, then g is called the αth Liouville-Grünwald-Letnikov derivative of f in the mean of order p. We use the notation $g = D^\alpha f$.

EXAMPLE 2.2. The αth fractional derivative of e^{iax} for any real a is given by $(ia)^\alpha e^{iax}$. In particular, $D^\alpha \sin x = \sin(x + \frac{\pi}{2}\alpha)$ and $D^\alpha \cos x = \cos(x + \frac{\pi}{2}\alpha)$.

We also have the following properties.

PROPOSITION 2.3. *For* $f \in L^p(0, 2\pi; X)$, $1 \leq p < \infty$, α, $\beta > 0$ *we have*

(i) *If* $D^\alpha f \in L^p(0, 2\pi; X)$, *then* $D^\beta f \in L^p(0, 2\pi; X)$ *for all* $0 < \beta < \alpha$,

(ii) $D^\alpha D^\beta f = D^{\alpha+\beta} f$ *whenever one of the two sides is well defined.*

Proof. The proof is the same as in the scalar case, which is given in [6, Proposition 4.1]. □

We recall that the Fourier series of $f \in L^p(0, 2\pi; X)(1 \leq p < \infty)$ is defined for $k \in \mathbb{Z}$ by

$$\hat{f}(k) = \frac{1}{2\pi} \int_0^{2\pi} e^{-ikt}(t) f(t) dt.$$

In what follows we denote by $H^{\alpha,p}(0, 2\pi; X)$ the vector-valued function space

$$H^{\alpha,p}(0, 2\pi; X) := \{u \in L^p(0, 2\pi; X) : \text{ there exists } v \in L^p(0, 2\pi; X) \text{ such that } \hat{v}(k) = (ik)^\alpha \hat{u}(k), \text{ for all } k \in \mathbb{Z}\}.$$

By a result of Butzer and Westphal [6, Theorem 4.1], we have that if $f \in L^p(0, 2\pi; X)$, then $D^\alpha f \in L^p(0, 2\pi; X)$ if and only if there exists $g \in L^p(0, 2\pi; X)$ such that $(ik)^\alpha \hat{f}(k) = \hat{g}(k)$, and in this case we have in fact $D^\alpha f = g$. In consequence,

$$H^{\alpha,p}(0, 2\pi; X) = \{u \in L^p(0, 2\pi; X) : D^\alpha u \in L^p(0, 2\pi; X)\}.$$

Let $1 < \alpha \leq 2,\ p > 1$ and $u \in H^{\alpha,p}(0, 2\pi; X)$. For any $\beta < \alpha$ it follows, from Proposition 2.3, that if $u \in H^{\alpha,p}(0, 2\pi; X)$ then $D^\beta u \in H^{\alpha-\beta,p}(0, 2\pi; X)$.

Given $0 < \gamma < 1$, denote by $C^\gamma_{per}(0, 2\pi; X)$ the space of all vector-valued γ-Hölder continuous functions u on $[0, 2\pi]$ satisfying $u(0) = u(2\pi)$. It was shown in [28] (see also [7]) that when $1/p < \gamma < 1 + 1/p$, then

$$H^{\gamma,p}(0, 2\pi; X) \subset C^{\gamma-1/p}_{per}(0, 2\pi; X). \tag{2.3}$$

Hence

$$H^{\alpha-\beta,p}(0, 2\pi; X) \subset C^{\alpha-\beta-1/p}_{per}(0, 2\pi; X),$$

whenever $1/p < \alpha - \beta < 1 + 1/p$.

Note that for $\beta := \alpha - 1$ we have that $1/p < 1 < 1 + 1/p$ and then

$$D^{\alpha-1}u \in H^{1,p}(0, 2\pi; X) \subset C^{1-1/p}_{per}(0, 2\pi; X).$$

It follows that $D^{\alpha-1}u(0) = D^{\alpha-1}u(2\pi)$. Note also that $H^{\alpha,p}(0, 2\pi; X) \subset H^{1,p}(0, 2\pi; X)$ and hence $u(0) = u(2\pi)$.

We remark that in the very recent paper [7], S.Q. Bu studied L^p-well posedness of Eq. (1.1) without delay, i.e. $F \equiv 0$, and periodic boundary conditions. Bu's paper consider the fractional derivative in the sense of Weyl. The connection of our fractional derivative with the classical Weyl derivative was established by Butzer and Westphal in [6, Proposition 6.1]. We also observe that Bu's paper states conditions on α in order to have Hölder regularity of the solutions of Eq. (1.1) without delay and periodic boundary conditions.

Let Φ_α be the function defined by

$$\Phi_\alpha(t) = \sum_{k \in \mathbb{Z}\setminus\{0\}} \frac{e^{ikt}}{(ik)^\alpha}, \quad t \in \mathbb{R}\setminus 2\pi\mathbb{Z}, \quad \alpha > 0 \tag{2.4}$$

where $(ik)^\alpha = |k|^\alpha e^{\frac{\pi i \alpha}{2}\mathrm{sgn}k}$. Note that $\Phi_\alpha \in L^1(0, 2\pi)$ (see [28] for more details) and hence for $u \in L^p(0, 2\pi; X)$, $1 \leq p < \infty$ and $\alpha > 0$, we can define

$$I^\alpha u(t) = \frac{1}{2\pi} \int_0^{2\pi} u(t-s)\Phi_\alpha(s)\mathrm{d}s. \tag{2.5}$$

The following lemma is contained in [6, Theorem 4.1].

LEMMA 2.4. *Let* $1 \leq p < \infty$ *and* $u \in L^p(0, 2\pi; X)$. *The following statements are equivalent:*

(i) *There exists* $w \in L^p(0, 2\pi; X)$ *and* $x \in X$ *such that*

$$u(t) = x + \frac{1}{2\pi} \int_0^{2\pi} w(t-s)\Phi_\alpha(s)ds \ a.e. \text{ on } [0, 2\pi], \tag{2.6}$$

and $\int_0^{2\pi} w(t)dt = 0$.

(ii) $u \in H^{\alpha,p}(0, 2\pi; X)$.

We will need the following definition of operator-valued Fourier multipliers.

DEFINITION 2.5. For $1 \leq p \leq \infty$, $\alpha \geq 0$ we say that a sequence $\{M_k\}_{k\in\mathbf{Z}} \subset \mathcal{B}(X, Y)$ is an $(L^p, H^{\alpha,p})$-multiplier, if for each $f \in L^p(0, 2\pi; X)$ there exists $u \in H^{\alpha,p}(0, 2\pi; Y)$ such that

$$\hat{u}(k) = M_k \hat{f}(k) \quad \text{for all } k \in \mathbb{Z}.$$

In particular, in case $\alpha = 0$ (therefore $H^{\alpha,p} = H^{0,p} = L^p$) the definition coincides with the one contained in [2, Proposition 1.1]. The proof of the following lemma is similar to that of [2, Lemma 2.2] taking into account Lemma 2.4 above.

LEMMA 2.6. *Let* $1 \leq p < \infty$, $\alpha > 0$ *and* $(M_k)_{k\in\mathbf{Z}} \subset \mathcal{B}(X)$. *The following assertions are equivalent*

(i) $(M_k)_{k\in\mathbb{Z}}$ *is an* $(L^p, H^{\alpha,p})$*-multiplier;*

(ii) $((ik)^\alpha M_k)_{k\in\mathbb{Z}}$ *is an* (L^p, L^p)*- multiplier.*

A Banach space X is said to be UMD, if the Hilbert transform is bounded on $L^p(\mathbb{R}; X)$ for some (and then all) $p \in (1, \infty)$. Here, the Hilbert transform H of a function $f \in \mathcal{S}(\mathbb{R}; X)$, the Schwartz space of rapidly decreasing X-valued functions, is defined by

$$Hf := \frac{1}{\pi} PV\left(\frac{1}{t}\right) * f.$$

These spaces are also called $\mathcal{HT}$ spaces. It is a well-known theorem that the set of Banach spaces of class $\mathcal{HT}$ coincides with the class of UMD spaces. This has been shown by Bourgain [4] and Burkholder [5].

DEFINITION 2.7. Let X and Y be Banach spaces. A family of operators $\mathcal{T} \subset \mathcal{B}(X, Y)$ is called R-bounded, if there is a constant $C > 0$ and $p \in [1, \infty)$ such that for each $N \in \mathbb{N}$, $T_j \in \mathcal{T}$, $x_j \in X$ and for all independent, symmetric, $\{-1, 1\}$-valued random variables r_j on a probability space $(\Omega, \mathcal{M}, \mu)$ the inequality

$$\| \sum_{j=1}^{N} r_j T_j x_j \|_{L^p(\Omega,Y)} \leq C \| \sum_{j=1}^{N} r_j x_j \|_{L^p(\Omega,X)} \tag{2.7}$$

is valid. The smallest such C is called R-bound of $\mathcal{T}$, we denote it by $R_p(\mathcal{T})$.

Several properties of R-bounded families can be founded in the recent monograph of Denk–Hieber–Prüss [13, Section 3].

We remark that large classes of classical operators are R-bounded (cf. [14] and references therein). Hence, this assumption is not too restrictive for the applications that we consider in this article.

The following theorem, due to Arendt and Bu [2, Theorem 1.3], is the discrete analog of the operator-valued version of Mikhlin's theorem due to Weis [24] and play an important role in the following sections.

THEOREM 2.8. *Let X, Y be UMD spaces and let $\{M_k\}_{k\in\mathbb{Z}} \subseteq \mathcal{B}(X,Y)$. If the sets $\{M_k\}_{k\in\mathbb{Z}}$ and $\{k(M_{k+1}-M_k)\}_{k\in\mathbb{Z}}$ are R-bounded, then $\{M_k\}_{k\in\mathbb{Z}}$ is an L^p-multiplier for $1 < p < \infty$.*

3. Periodic solutions

We consider in this section the equation

$$D^\alpha u(t) = Au(t) + Fu_t + f(t), \quad t \in [0, 2\pi], \ 1 < \alpha \leq 2, \tag{3.1}$$

where $A : D(A) \subseteq X \to X$ is a linear, closed operator; $f \in L^p(0, 2\pi; X)$, $p \geq 1$. Setting $r_{2\pi} := 2\pi N$, for some $N \in \mathbb{N}$, $F : L^p([-r_{2\pi}, 0]; X) \to X$ is a linear, bounded operator and u_t is an element of $L^p([-r_{2\pi}, 0]; X)$ which is defined as $u_t(\theta) = u(t+\theta)$ for $-r_{2\pi} \leq \theta \leq 0$.

Next, we define the notion of strong solution of the fractional differential equation with delay (1.1) and the associated concept of well-posedness.

DEFINITION 3.1. Let $1 \leq p < \infty$. A function u is called a strong L^p-solution of (1.1) if $u \in H^{\alpha,p}(0, 2\pi; X) \cap L^p(0, 2\pi; D(A))$ and Eq. (1.1) holds for almost all $t \in [0, 2\pi]$.

DEFINITION 3.2. Let $1 \leq p < \infty$. We say that problem (1.1) is strongly L^p-well posed (or has maximal regularity) if for every $f \in L^p(0, 2\pi; X)$ there exists a unique strong L^p-solution of (1.1).

The concept of maximal regularity has received much attention in recent years. It is connected to the question of closedness of the sum of two closed operators. It has proven very efficient in the treatment of nonlinear problems in partial differential equations, especially semi-linear and quasi-linear ones (see for example [10]).

Denote by $e_\lambda(t) := e^{i\lambda t}$ for all $\lambda \in \mathbb{R}$, and define the operators $\{B_\lambda\}_{\lambda\in\mathbb{R}} \subseteq \mathcal{B}(X)$ by

$$B_\lambda x = F(e_\lambda x), \quad \text{for all } \lambda \in \mathbb{R} \quad \text{and} \quad x \in X. \tag{3.2}$$

Defining the *real spectrum* of (3.1) by

$$\sigma(\Delta) = \{s \in \mathbb{R} : (is)^{\alpha} I - B_s - A \in \mathcal{B}(D(A), X) \text{ is not invertible}\},$$

and denote $\rho(\Delta) = \mathbb{R} \backslash \sigma(\Delta)$. We prove the following result.

PROPOSITION 3.3. *Lets A be a closed linear operator defined on a UMD space X and $1 < \alpha \leq 2$. Suppose that $\mathbb{Z} \subset \rho(\Delta)$. Then the following assertions are equivalent.*

(i) $\{(ik)^{\alpha}((ik)^{\alpha} I - B_k - A)^{-1}\}_{k \in \mathbb{Z}}$ *is an (L^p, L^p)-multiplier for $1 < p < \infty$.*

(ii) $\{(ik)^{\alpha}((ik)^{\alpha} I - B_k - A)^{-1}\}_{k \in \mathbb{Z}}$ *is R-bounded.*

Proof. By [2, Proposition 1.11] it follows that (i) implies (ii). Conversely, define $M_k = (ik)^{\alpha}(N_k - A)^{-1}$, where $N_k := (ik)^{\alpha} I - B_k$. By Theorem 2.8 is sufficient to prove that the set $\{k(M_{k+1} - M_k)\}_{k \in \mathbb{Z}}$ is R-bounded. From the proof of [22, Proposition 3.2] we have that the set $\{B_k\}_{k \in \mathbb{Z}}$ is R-bounded.

Let $a_k = 1/(ik)^{\alpha}$, $k \neq 0$. Next we note the following identity

$$\begin{aligned} k[M_{k+1} - M_k] &= k a_{k+1} M_{k+1} B_{k+1} M_k - k a_k M_{k+1} B_k M_k \\ &\quad + k \frac{a_{k+1} - a_k}{a_k} M_{k+1}[M_k - I] - k(a_{k+1} - a_k) M_{k+1} B_k M_k. \end{aligned}$$

Observe that for $\gamma > 0$ we have that $|(i(k+1))^{\gamma} - (ik)^{\gamma}|$ can be estimated by $(ik)^{\gamma - 1}$ uniformly in k according to the definition of $|(ik)^{\gamma}|$ and the mean value theorem. This implies that $\{k(a_{k+1} - a_k)\}$ and $\{k\frac{a_{k+1} - a_k}{a_k}\}$ are bounded sequences. Since $\{k a_k\}$ also is bounded for $\alpha > 1$, taking into account that the products and sums of R-bounded sequences is R-bounded (see [13]), the proof is finished. $\square$

PROPOSITION 3.4. *Let X be a Banach space and let $A : D(A) \subset X \to X$ be a closed linear operator. Suppose that for every $f \in L^p(0, 2\pi; X)$, there exists a unique strong L^p-solution of (3.1) for $1 < p < \infty$. Then*

(i) $\mathbb{Z} \subset \rho(\Delta)$,

(ii) $\{(ik)^{\alpha}((ik)^{\alpha} I - B_k - A)^{-1}\}_{k \in \mathbb{Z}}$ *is R-bounded.*

Proof. Follows the same lines of [22, Proposition 3.3]. $\square$

Our main result in this paper, establish that the converse of Proposition 3.4 is true, provided X is an UMD space.

THEOREM 3.5. *Let X be a UMD space and let $A : D(A) \subset X \to X$ be a closed linear operator. Then, the following assertions are equivalent for $1 < p < \infty$ and $1 < \alpha \leq 2$.*

(i) *For every $f \in L^p(0, 2\pi; X)$, there exists a unique strong L^p-solution of (3.1);*

(ii) $\mathbb{Z} \subset \rho(\Delta)$ *and* $\{(ik)^{\alpha}((ik)^{\alpha} I - B_k - A)^{-1}\}_{k \in \mathbb{Z}}$ *is R-bounded.*

Proof. Let $f \in L^p(0, 2\pi; X)$. Define $N_k = ((ik)^\alpha I - B_k - A)^{-1}$. By Proposition 3.3, the family $\{M_k := (ik)^\alpha N_k\}_{k\in\mathbb{Z}}$ is an (L^p, L^p)-multiplier. By Lemma 2.6, it is equivalent to the fact that the family $\{N_k\}_{k\in\mathbb{Z}}$ is an $(L^p, H^{\alpha,p})$-multiplier, i.e. there exists $u \in H^{\alpha,p}(0, 2\pi; X)$ such that

$$\hat{u}(k) = N_k \hat{f}(k) = ((ik)^\alpha I - B_k - A)^{-1} \hat{f}(k). \tag{3.3}$$

In particular, $u \in L^p(0, 2\pi; X)$ and there exists $v \in L^p(0, 2\pi; X)$ such that

$$\hat{v}(k) = (ik)^\alpha \hat{u}(k), \tag{3.4}$$

Moreover, $D^\alpha u \in L^p(0, 2\pi; X)$ and $\widehat{D^\alpha u}(k) = \hat{v}(k)$.

We claim that the family $\{B_k N_k\}_{k\in\mathbb{Z}}$ is an (L^p, L^p)-multiplier. In fact, it is clear that $\{B_k N_k\}_{k\in\mathbb{Z}}$ is R-bounded. On the other hand, since $\{B_k\}_{k\in\mathbb{Z}}$ is R-bounded (see the proof of [22, Proposition 3.2]) the identity

$$k(B_{k+1}N_{k+1} - B_k N_k) = ka_{k+1}B_{k+1}M_{k+1} - ka_k B_k M_k$$

shows that $\{k(B_{k+1}N_{k+1} - B_k N_k)\}_{k\in\mathbb{Z}}$ is also R-bounded. Then, the claim follows from Theorem 2.8.

By Fejer's theorem (see [17]) one has in $L^p([-r_{2\pi}, 0]; X)$

$$u_t(\theta) = u(t+\theta) = \lim_{n\to\infty} \frac{1}{n+1} \sum_{m=0}^{n} \sum_{k=-m}^{m} e^{ikt} e^{ik\theta} \hat{u}(k).$$

Hence in $L^p(0, 2\pi; X)$ we obtain

$$u_t = \lim_{n\to\infty} \frac{1}{n+1} \sum_{m=0}^{n} \sum_{k=-m}^{m} e^{ikt} e_k \hat{u}(k).$$

Then, since F is linear and bounded

$$Fu_t = \lim_{n\to\infty} \frac{1}{n+1} \sum_{m=0}^{n} \sum_{k=-m}^{m} e^{ikt} F(e_k \hat{u}(k)) = \lim_{n\to\infty} \frac{1}{n+1} \sum_{m=0}^{n} \sum_{k=-m}^{m} e^{ikt} B_k \hat{u}(k)$$

By (3.3) and (3.4) we have

$$\widehat{D^\alpha u}(k) = (ik)^\alpha \hat{u}(k) = A\hat{u}(k) + B_k \hat{u}(k) + \hat{f}(k)$$

for all $k \in \mathbb{Z}$. Then using that A is closed, we conclude that $u(t) \in D(A)$ (cf. [2, Lemma 3.1]) and, from the uniqueness theorem of Fourier coefficients, that (3.1) is valid for a.a. $t \in [0, 2\pi]$.

To show uniqueness, let $u \in L^p(0, 2\pi; D(A)) \cap H^{\alpha,p}(0, 2\pi; X)$ be such that $D^\alpha u(t) = Au(t) + Fu_t, \quad t \in [0, 2\pi]$, then $\hat{u}(k) \in D(A)$ and $(ik)^\alpha \hat{u}(k) = A\hat{u}(k) + B_k \hat{u}(k)$. Since $\mathbb{Z} \cap \sigma(\Delta) = \emptyset$ this implies that $\hat{u}(k) = 0$ for all $k \in \mathbb{Z}$ and thus $u = 0$. $\square$

The solution $u(\cdot)$ given in Theorem 3.5 actually satisfies the following maximal regularity property.

COROLLARY 3.6. *In the context of Theorem* 3.5*, if condition* (ii) *is fulfilled, we have* $D^\alpha u, Au, Fu_{(\cdot)} \in L^p(0, 2\pi; X)$. *Moreover, there exists a constant* $C > 0$ *independent of* $f \in L^p(0, 2\pi; X)$ *such that*

$$||D^\alpha u||_{L^p(0,2\pi;X)} + ||Au||_{L^p(0,2\pi;X)} + ||Fu_{(\cdot)}||_{L^p(0,2\pi;X)} \leq C||f||_{L^p(0,2\pi;X)}. \tag{3.5}$$

REMARK 3.7. From the inequality (3.5) we deduce that the operator L defined by:

$$(Lu)(t) = D^\alpha(t) - Au(t) - Fu_t \text{ with domain}$$
$$D(L) = H^{\alpha,p}(0, 2\pi; X) \cap L^p(0, 2\pi; D(A)),$$

is an isomorphism onto. Indeed, since A is closed, the space $H^{\alpha,p}(0, 2\pi; X) \cap L^p(0, 2\pi; D(A))$ becomes a Banach space under the norm

$$|||u||| := ||u||_p + ||D^\alpha u||_p + ||Au||_p.$$

We remark that such isomorphisms are crucial for the handling of nonlinear evolution equations (see [1]).

In the case of a Hilbert space, Theorem 3.5 takes a particularly simple form.

COROLLARY 3.8. *Let* H *be Hilbert space and let* $A : D(A) \subset H \to H$ *be a closed linear operator. Then, the following assertions are equivalent for* $1 < p < \infty$ *and* $1 < \alpha \leq 2$.

(i) *For every* $f \in L^p(0, 2\pi; H)$*, there exists a unique strong* L^p*-solution of (3.1);*

(ii) $\mathbb{Z} \subset \rho(\Delta)$ *and*

$$\sup_{k\in\mathbb{Z}} ||(ik)^\alpha((ik)^\alpha I - B_k - A)^{-1}|| < \infty. \tag{3.6}$$

Proof. This is a consequence of Plancherel's Theorem. □

For future reference, we state separately the finite dimensional case, i.e. $H = \mathbb{C}^n$.

COROLLARY 3.9. *Let* $1 < p < \infty$, $1 < \alpha \leq 2$ *and* $f \in L^p(0, 2\pi; \mathbb{C}^n)$. *A necessary and sufficient condition for the existence of a unique strong* L^p*-solution of* (3.1) *is that*

$$det((ik)^\alpha I - B_k - A) \neq 0, \quad \textit{for all } k \in \mathbb{Z}$$

where A *is a* $n \times n$ *matrix and* $(B_k)_k$ *is a sequence of* $n \times n$ *matrices.*

EXAMPLE 3.10. Set $X = \mathbb{C}$ and $1 < \alpha < 2$. For $\rho \in \mathbb{R}\backslash\{-1\}$ consider the fractional differential equation with delay

$$D_t^\alpha x(t) = \rho x(t) + x(t - 2\pi) + f(t), \quad t \in [0, 2\pi]. \tag{3.7}$$

Defining $Fx := x(-2\pi)$ we obtain the abstract form (1.1) with $A = \rho I$. Note that $B_k = e^{2\pi ik} = 1$ for all $k \in \mathbb{Z}$, and hence we only have to examine the sequence $s_k := (ik)^\alpha - 1 - \rho$. Since $\rho \in \mathbb{R}\backslash\{-1\}$, we observe that $s_k \neq 0$ for all $k \in \mathbb{Z}$ and hence Corollary 3.9 implies the existence of a unique solution $x \in L^p(0, 2\pi)$. In this case, we can show explicitly that solution, noting that the Laplace transform of the function

$$M(t) := t^{\alpha-1} E_{\alpha,\alpha}((1 + \rho)t^\alpha),$$

where $E_{\alpha,\alpha}$ denotes the Mittag-Leffler function, can be extended to the imaginary axis and is given by

$$\mathcal{L}(M)(ik) = \frac{1}{(ik)^\alpha - (1 + \rho)}, \qquad k \in \mathbb{Z}.$$

It follows that the explicit form of the periodic solution is

$$x(t) = \int_{-\infty}^{t} M(t - s) f(s) \mathrm{d}s. \tag{3.8}$$

Indeed, note that the Fourier transform of (3.8) is given by the product of the Laplace transform of M (evaluated in the imaginary axis) and the Fourier transform of f. Then, a straightforward calculation shows that it coincides with the Fourier transform of the given Eq. (3.9). The claim then follows from the uniqueness of Fourier coefficients.

EXAMPLE 3.11. Again, set $X = \mathbb{C}$ and $1 < \alpha < 2$. We consider the fractional differential equation with delay

$$D_t^\alpha x(t) = ax(t) + x(t - \tau) + f(t), \quad t \in [0, 2\pi], \tag{3.9}$$

where $\tau \in [0, 2\pi]$ and $a \in \mathbb{R}$. Defining $Fx := x(-\tau)$ we obtain the abstract form (1.1) with $A = aI$. Note that $B_k = e^{i\tau k}$ for all $k \in \mathbb{Z}$, and hence we have to examine the zeroes of the sequence $f_\alpha(m) := (im)^\alpha - e^{i\tau m} - a, \quad m \in \mathbb{Z}$. Define the set

$$\mathcal{M}_\alpha = \{(a, \tau) \in \mathbb{R} \times [0, 2\pi] / \ f_\alpha(m) = 0 \quad \text{for some} \ \ m \in \mathbb{Z}\}. \tag{3.10}$$

Note that $(-1, \tau) \in \mathcal{M}_\alpha$ for all $\tau \in [0, 2\pi]$, since in such case $f_\alpha(0) = 0$.

For $s > 0$, $\lfloor s \rfloor$ denotes the largest integer less than or equal to s. Fixed $n := \left\lfloor \frac{1}{\sin^{1/\alpha}(\alpha\pi/2)} \right\rfloor \in \mathbb{N}$, observe that there exists numbers $\tau_1 = \tau_1(\alpha), \ldots, \tau_{4n} = \tau_{4n}(\alpha) \in [0, 2\pi]$ such that for each $|m| \leq n$, $m \neq 0$, we have

$$\sin(m\tau_j) = |m|^\alpha \sin(\alpha\pi/2).$$

Now define $a_j = n^\alpha \cos(\alpha\pi/2) - \cos(\tau_j n)$; $j = 1, \ldots 4n$. Then $(a_j, \tau_j) \in \mathcal{M}_\alpha \backslash \{(-1, \tau)/\tau \in [0, 2\pi]\}$. It is then easy to prove that

$$\mathcal{M}_\alpha = \{(-1, \tau)/\tau \in [0, 2\pi]\} \cup \{(a_j, \tau_j)\}_{j=1,\ldots,4n}$$

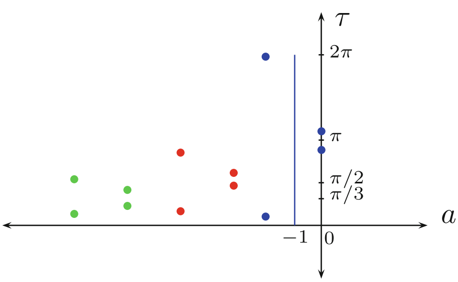
is the set of zeroes of the function f_α in $\mathbb{Z}$. Some concrete examples are:

$$\begin{aligned}
&\mathcal{M}_{3/2} = \{(-1, \tau)/\tau \in [0, 2\pi]\} \cup \{(0, 3\pi/4), (0, 5\pi/4), (-\sqrt{2}, \pi/4), (-\sqrt{2}, 7\pi/4)\},\\
&\mathcal{M}_{5/4} = \{(-1, \tau)/\tau \in [0, 2\pi]\}\\
&\qquad \cup\{(0, 5\pi/8), (0, 11\pi/8), (-2\cos(3\pi/8), 3\pi/8), (-2\cos(3\pi/8), 13\pi/8)\},\\
&\mathcal{M}_{7/4} = \{(-1, \tau)/\tau \in [0, 2\pi]\}\\
&\qquad \cup\{(0, 7\pi/8), (0, 9\pi/8), (-2\cos(\pi/8), \pi/8), (-2\cos(\pi/8), 15\pi/8)\},\\
&\mathcal{M}_{9/5} = \{(-1, \tau)/\tau \in [0, 2\pi]\}\\
&\qquad \cup\{(0, 9\pi/10), (0, 11\pi/10), (-2\cos(\pi/10), \pi/10), (-2\cos(\pi/10), 19\pi/10)\},\\
&\mathcal{M}_{19/10} = \{(-1, \tau)/\tau \in [0, 2\pi]\}\\
&\qquad \cup\{(0, 19\pi/20), (0, 21\pi/20), (-2\cos(\pi/20), \pi/20), (-2\cos(\pi/20), 39\pi/20),\\
&\qquad (a_2^+, \frac{\pi}{2} - \frac{\beta_2}{2}), (a_2^+, \frac{\pi}{2} + \frac{\beta_2}{2}), (a_2^-, \pi - \frac{\beta_2}{2}), (a_2^-, \frac{\beta_2}{2})\}\\
&\mathcal{M}_{195/10} = \{(-1, \tau)/\tau \in [0, 2\pi]\}\\
&\qquad \cup\{(0, 195\pi/200), (0, 205\pi/200), (-2\cos(5\pi/200), 5\pi/200),\\
&\qquad (-2\cos(5\pi/200), 395\pi/200), (a_2^+, \frac{\pi}{2} - \frac{\beta_2}{2}), (a_2^+, \frac{\pi}{2} + \frac{\beta_2}{2}),\\
&\qquad (a_2^-, \pi - \frac{\beta_2}{2}), (a_2^-, \frac{\beta_2}{2}), (a_3^+, \frac{\pi}{3} - \frac{\beta_3}{3}), (a_3^+, \frac{\pi}{3} + \frac{\beta_3}{3}),\\
&\qquad (a_3^-, \frac{2\pi}{3} - \frac{\beta_3}{3}), (a_3^-, \frac{\beta_3}{3})\}
\end{aligned}$$

where $\beta_j = \arcsin(j^\alpha \sin(\alpha\pi/2))$ and $a_j^\pm = j^\alpha \cos(\alpha\pi/2) \pm \cos(\beta_j)$, $j = 2, 3$.

We then conclude from the above and Corollary 3.9 that for all $(a, \tau) \notin \mathcal{M}_\alpha$ there exists a unique periodic solution of Eq. (3.9).

REMARK 3.12. It is remarkable that the number of points in the set $\mathcal{M}^* = \mathcal{M}_\alpha \backslash \{(-1, \tau)/\tau \in [0, 2\pi]\}$ is exactly the same (=4) until the value approximate $\alpha^* \approx 1.816373004$ corresponding to the unique root of $2^\alpha \sin(\alpha\pi/2) - 1 = 0$ in the open interval $1 < \alpha < 2$, and increases as α approach to 2. It reflects the surprising fact that the probability, in some sense, to find periodic solutions of the Eq. (3.9) decreases for α $(> \alpha^*)$ near to 2 but, however, is the same for $\alpha \in (1, \alpha^*)$. In the following figure shows the pairs (a, τ) in the case $\alpha = 1.95$.



EXAMPLE 3.13. Let A be a closed linear operator defined on a Hilbert space H and suppose that $\{(ik)^\alpha\}_{k\in\mathbb{Z}} \subset \rho(A)$ and $\sup_k ||A((ik)^\alpha - A)^{-1}|| =: M < \infty$. From the identity

$$(ik)^\alpha I - A - B_k = (I - B_k((ik)^\alpha - A)^{-1})((ik)^\alpha - A)$$

it follows that $(ik)^\alpha I - A - B_k$ is invertible whenever $||B_k((ik)^\alpha - A)^{-1}|| < 1$.

Next observe that $||B_k|| \le {r_{2\pi}}^{1/p}||F||$. Hence

$$||B_k((ik)^\alpha - A)^{-1}|| = ||B_k A^{-1}A((ik)^\alpha - A)^{-1}|| \le {r_{2\pi}}^{1/p}||F||||A^{-1}||M =: \xi$$

Therefore, under the condition

$$||F|| < \frac{1}{||A^{-1}||M{r_{2\pi}}^{1/p}} \tag{3.11}$$

we obtain that $\mathbb{Z} \cap \sigma(\Delta) = \emptyset$, and the identity

$$\begin{aligned}((ik)^\alpha I - A - B_k)^{-1} &= ((ik)^\alpha - A)^{-1}(I - B_k((ik)^\alpha - A)^{-1}) \\ &= ((ik)^\alpha - A)^{-1}\sum_{n=0}^{\infty}[B_k((ik)^\alpha - A)^{-1}]^n. \qquad (3.12)\end{aligned}$$

It follows that

$$\begin{aligned}||(ik)^\alpha((ik)^\alpha I - A - B_k)^{-1}|| &\le ||(ik)^\alpha((ik)^\alpha - A)^{-1}||\sum_{n=0}^{\infty}||B_k((ik)^\alpha - A)^{-1}||^n \\ &\le \frac{1+M}{1-\xi},\end{aligned}$$

and hence condition (ii) in Corollary 3.8 is satisfied.

The above example can be adapted to obtain the following criterion in case of UMD spaces.

THEOREM 3.14. *Let X be a UMD space and let $A : D(A) \subset X \to X$ be a closed linear operator such that $\{(ik)^\alpha\}_{k\in\mathbb{Z}} \subset \rho(A)$ and $R_p(\{A((ik)^\alpha - A)^{-1}\}_{k\in\mathbb{Z}}) =: M < \infty$. Suppose that*

$$||F|| < \frac{1}{(2r_{2\pi})^{1/p}||A^{-1}||M}. \tag{3.13}$$

Then for every $f \in L^p(0, 2\pi; X)$, there exists a unique strong L^p-solution of (3.1).

Proof. Follows the same lines of [22, Theorem 3.9]. □

To close this paper, and as an application, we want to compare the periodic problem

$$D^\alpha u(t) = Au(t) + f(t), \quad t \in [0, 2\pi) \tag{3.14}$$

with the delay Eq. (3.1). As a direct consequence of Theorem 3.14 and [18, Theorem 3.1] we have the following result.

COROLLARY 3.15. *Assume that X is a UMD space. Let $1 < p < \infty$. If for each $f \in L^p(0, 2\pi; X)$ there is a unique strong L^p-solution of Eq.* (3.14) *and condition* (3.13) *is satisfied, then for all $f \in L^p(0, 2\pi; X)$ there is a unique strong L^p-solution of Eq.* (3.1).

Acknowledgments

The authors are grateful to the referee for useful remarks and to call our attention to the recently published paper [7].

REFERENCES

[1] AMANN H., *Linear and Quasilinear Parabolic Problems,* Volume I: Abstract Linear Theory. Monographs in Mathematics, vol 89., Birkhäuser, Basel-Boston-Berlin, 1995.

[2] ARENDT W. AND BU S.Q., *The operator-valued Marcinkiewicz multiplier theorem and maximal regularity,* Math. Z. **240** (2002), 311–343.

[3] BENCHOHRA M., HENDERSON J., NTOUYAS S.K. AND OUAHAB A., *Existence results for fractional order functional differential equations with infinite delay,* J. Math. Anal. Appl., (2008), 1340–1350.

[4] BOURGAIN J., *Some remarks on Banach spaces in which martingale differences sequences are unconditional,* Arkiv Math. **21**(1983), 163–168.

[5] BURKHÖLDER D.L., *A geometrical condition that implies the existence of certain singular integrals on Banach-space-valued functions,* In: Conference on Harmonic Analysis in Honour of Antoni Zygmund, Chicago 1981, W. Becker, A.P. Calderón, R. Fefferman, P.W. Jones (eds), Belmont, Cal. Wadsworth (1983), 270–286.

[6] BUTZER P.L. AND WESTPHAL U., *An access to fractional differentiation via fractional difference quotients,* 116–145, Lecture Notes in Math. **457**, Springer, Berlin, 1975.

[7] BU S.Q., *Well-posedness of equations with fractional derivative,* Acta Math. Sinica, English Series **26** (7) (2010), 1223–1232.

[8] BU S.Q. AND FANG Y., *Maximal regularity of second order delay equations in Banach spaces,* Sci. China Math. **53** (2010), no. 1, 51–62.

[9] BU S.Q. AND FANG Y., *Periodic solutions of delay equations in Besov spaces and Triebel–Lizorkin spaces,* Taiwanese J. Math. **13** (2009), no. 3, 1063–1076.

[10] CHILL R. AND SRIVASTAVA S., *L^p maximal regularity for second order Cauchy problems,* Math. Z. **251** (4) (2005), 751–781.

[11] CLÉMENT PH. AND PRÜSS J., *An operator-valued transference principle and maximal regularity on vector-valued L_p-spaces,* In: Evolution Equations and Their Applications in Physics and Life Sciences. Lumer, Weis eds., Marcel Dekker (2000), 67–87.

[12] CLÉMENT PH., GRIPENBERG G. AND LONDEN S.- O., *Schauder estimates for equations with fractional derivatives, Trans. Amer. Math. Soc.* **352** (2000), 2239–2260.

[13] DENK R., HIEBER M. AND PRÜSS J., *R-boundedness, Fourier Multipliers and Problems of Elliptic and Parabolic Type,* Mem. Amer. Math. Soc. **166** (788) (2003).

[14] GIRARDI M. AND WEIS L., *Criteria for R-boundedness of operator families,* Lecture Notes in Pure and Appl. Math., **234** Dekker, New York, 2003, 203–221.

[15] GRÜNWALD A.K., *Über begrenzte Derivationen und deren Anwendung,* Z. Angew. Math. Phys.,**12** (1867), 441–480.

[16] KAPPEL F., *Semigroups and Delay Equations,* in: Semigroups, Theory and Applications, vol II., (H. Brezis, M.G. Crandall, F. Kappel, eds.) Pitman Research Notes in Mathematics **152**, Longman, 1986, pp. 136–176.

[17] KATZNELSON Y., *An Introduction to Harmonic Analysis,* Wiley, New York, 1968.

[18] KEYANTUO V. AND LIZAMA C., *A characterization of periodic solutions for time-fractional differential equations in UMD spaces and applications,* Math. Nach., to appear.

[19] LAKSHMIKANTHAM V., *Theory of fractional functional differential equations,* Nonlinear Analysis, Theory Methods and Appl. **69** (2008), 3337–3343.

[20] LETNIKOV A.V., *Theory and differentiation of fractional order,* Mat. Sb. **3** (1868), 1–66.

[21] LIAO C. AND YE H., *Existence of positive solutions of nonlinear fractional delay differential equations,* Positivity, **13** (2009), 601–609.

[22] LIZAMA C., *Fourier Multipliers and Periodic Solutions of Delay Equations in Banach Spaces,* J. Math. Anal. Appl., **324**(2) (2006), 921–933.

[23] SAMKO S.G., KILBAS A.A. AND MARICHEV O.I., *Fractional Integrals and Derivatives,* Theory and Applications, Gordon and Breach, Yverdon, 1993.

[24] WEIS L., *Operator-valued Fourier multiplier theorems and maximal L_p-regularity,* Math. Ann. **319** (2001), 735–758.

[25] YE H., DING Y. AND GAO J., *The existence of a positive solution of $D^\alpha[x(t)x(0)] = x(t)f(t, xt)$,* Positivity, **11** (2007), 341–350.

[26] YU C. AND GAO G., *Some results on a class of fractional functional differential equations,* Commun. Appl. Nonlinear Anal., **11** (3) (2004) 67–75.

[27] ZHANG X., *Some results of linear fractional order time-delay system,* Appl. Math. Comput. **197** (2008), 407–411.

[28] ZYGMUND A., *Trigonometrical Series,* Cambridge University Press, 1959.

C. Lizama
Departamento de Matemática,
Facultad de Ciencias,
Universidad de Santiago de Chile,
Casilla 307-Correo 2, Santiago, Chile
E-mail: carlos.lizama@usach.cl

V. Poblete
Facultad de Ciencias,
Universidad de Chile,
Las Palmeras 3425, Santiago, Chile
E-mail: vpoblete@uchile.cl