

# On integral kernels for Dirichlet series associated to Jacobi forms

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## ABSTRACT

Every Jacobi cusp form of weight  $k$  and index  $m$  over  $\mathrm{SL}_2(\mathbb{Z}) \times \mathbb{Z}^2$  is in correspondence with  $2m$  Dirichlet series constructed with its Fourier coefficients. The standard way to get from one to the other is by a variation of the Mellin transform. In this paper, we introduce a set of integral kernels which yield the  $2m$  Dirichlet series via the Petersson inner product. We show that those kernels are Jacobi cusp forms and express them in terms of Jacobi Poincaré series. As an application, we give a new proof of the analytic continuation and functional equations satisfied by the Dirichlet series mentioned above.

## 1. Introduction

Let  $\mathfrak{S}_k$  be the space of cuspidal modular forms of integral weight  $k$  over the modular group  $\mathrm{SL}_2(\mathbb{Z})$ . This is a finite-dimensional  $\mathbb{C}$ -vector space equipped with a positive-definite Hermitian form  $\langle \cdot, \cdot \rangle$  called the Petersson inner product.

Any  $f$  in  $\mathfrak{S}_k$  is determined by its Fourier series representation

$$f(\tau) = \sum_{n=1}^{\infty} a_n \exp(2\pi i n \tau) \quad (\tau \in \mathbb{C}, \Im(\tau) > 0), \quad (1.1)$$

with complex coefficients  $a_n = O(n^{k/2})$ .

On the other hand, any sequence of complex numbers  $\{a_n\}_{n=1}^{\infty}$  with polynomial growth defines the Dirichlet series

$$L(f, s) := \sum_{n=1}^{\infty} a_n n^{-s} \quad (s \in \mathbb{C}, \Re(s) \gg 0),$$

and the completed Dirichlet series

$$\Lambda(f, s) := (2\pi)^{-s} \Gamma(s) L(f, s).$$

In 1936, Hecke [10] proved that the Fourier series (1.1) represents a cusp form  $f$  in  $\mathfrak{S}_k$  if and only if  $\Lambda(f, s)$  admits an analytic continuation to the whole complex plane, such a continuation is bounded on any vertical strip and satisfies the functional equation  $\Lambda(f, s) = i^k \Lambda(f, k - s)$ . This important result is known as Hecke's converse theorem.

One way to get  $\Lambda(f, s)$  from the Fourier series  $f$  is via the Mellin transform, for example,

$$\int_{y=0}^{\infty} f(iy) y^{s-1} dy = \Lambda(f, s).$$

Alternatively, one can use an integral kernel. Namely, if

$$\Omega_s^k(\tau) := \sum_{M \in \mathrm{SL}_2(\mathbb{Z})} \phi_s|_k[M](\tau), \quad (1.2)$$

where  $\phi_s(\tau) = \tau^{-s}$  and  $|_k$  denotes the usual slash operator used in the theory of modular forms, then  $\Omega_s^k(\tau) \in \mathfrak{S}_k$  for  $1 < \Re(s) < k - 1$ , and

$$\langle \Omega_s^k, f \rangle = \frac{\pi}{2^{k-2} e^{\pi i s/2}} \frac{\Gamma(k-1)}{\Gamma(s)\Gamma(k-s)} \Lambda(\bar{f}, k-s) \quad \text{for all } f \in \mathfrak{S}_k. \quad (1.3)$$

Here,  $\bar{f}(\tau) := \overline{f(-\bar{\tau})}$ , where the bar in the right-hand side denotes complex conjugation.

The study of kernel functions similar to (1.2) in the theory of modular forms was pioneered by Petersson [19]. The introduction of the particular series (1.2) is due to Good [7], who used it to get positive lower bounds for the sum  $\sum_j |\Lambda(f_j, s)|^2$ , where  $\{f_j\}_j$  is any orthonormal basis of  $\mathfrak{S}_k$  and  $k/2 \leq s \leq (k+1)/2$ , if  $k \gg 0$ . (Good's paper is quite general, as he deals with finitely generated Fuchsian groups of the first kind.) Another application of the cusp form (1.2) to a non-vanishing problem of  $L$ -functions is developed by Kohnen [14], and a generalization of it to the theory of Siegel modular forms is presented by Kohnen and Sengupta [15].

From the Eichler–Shimura isomorphism theorem, one could guess that the cusps forms  $\Omega_s^k(\tau)$  with  $s = 2, 3, \dots, k-2$  are particularly important. They were first considered by Cohen [2], and have been studied and used by several authors since then. (For example, Diamantis and O'Sullivan [3], Fukuhara [5], Fukuhara and Yang [6], Kohnen and Zagier [16].) For a recent, comprehensive look at the properties of (1.2), see [3].

The main purpose of this work is to show the existence of integral kernels (in fact, a set of  $2m$  kernels) analogous to (1.2) and identities similar to (1.3) in the case of Jacobi cusp forms of weight  $k$  and index  $m$  over the Jacobi group  $\mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$ .

*Jacobi forms.* A Jacobi form is a generalization of the classical concept of modular form. Important examples of the former are theta functions in two variables and Fourier–Jacobi coefficients of Siegel modular forms over  $\mathrm{Sp}_2(\mathbb{Z})$ . Even though some Jacobi forms have been around for a long time, the systematic study of these objects is more recent (see [4] for a good introduction to the subject).

Let  $k$  and  $m$  be positive integers, fixed from now on. As in the case of elliptic modular forms, the set  $J_{k,m}^{\mathrm{cusp}}$  of weight  $k$ , index  $m$  Jacobi cusp forms over the group  $\Gamma^J := \mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$  is a finite-dimensional  $\mathbb{C}$ -space with a Petersson inner product  $\langle \cdot, \cdot \rangle$ .

Every  $f \in J_{k,m}^{\mathrm{cusp}}$  has a Fourier series representation in integral powers of  $\exp(2\pi i\tau)$  and  $\exp(2\pi iz)$ , which can be written as a combination of  $2m$  Fourier series in the variable  $\tau$  and certain theta functions  $\Theta_{m,\mu}(\tau, z)$ ;

$$f(\tau, z) = \sum_{\mu=1}^{2m} f_\mu(\tau) \Theta_{m,\mu}(\tau, z) \quad \text{where } f_\mu(\tau) = \sum_{D=1}^{\infty} c_\mu(D) \exp\left(\pi i \frac{D}{2m} \tau\right). \quad (1.4)$$

In [1], Berndt introduced the  $2m$ -tuple of Dirichlet series  $L_\mu(f, s) = (4m)^s \sum_{D=1}^{\infty} c_\mu(D) D^{-s}$  ( $1 \leq \mu \leq 2m$ ) associated to the Jacobi cusp form  $f$ , and in [18] the author proved the following analogue of Hecke's converse theorem.

A combination of theta functions  $f$  as in (1.4) is in  $J_{k,m}^{\mathrm{cusp}}$  if and only if the  $2m$  Dirichlet series  $\Lambda_\mu(f, s) := (2\pi)^{-s} \Gamma(s) L_\mu(f, s)$  have analytic continuation to the whole complex plane, such a continuation is bounded on vertical strips, and they satisfy the system of  $2m$  functional equations

$$\Lambda_\alpha(f, s) = \frac{i^k}{\sqrt{2m}} \sum_{\beta=0}^{2m-1} \exp\left(\pi i \frac{\alpha\beta}{m}\right) \Lambda_\beta\left(k-s-\frac{1}{2}\right) \quad (1 \leq \alpha \leq 2m). \quad (1.5)$$

*Results.* To any  $f$  in  $J_{k,m}^{\mathrm{cusp}}$  with series representation (1.4), let us associate the function  $\bar{f}(\tau, z) := \overline{f(-\bar{\tau}, -\bar{z})}$  (the bars in the right-hand side denote complex conjugation). It is not difficult to see that  $\bar{f}$  is a Jacobi cusp form in  $J_{k,m}^{\mathrm{cusp}}$ , and that its series representation is (1.4) with coefficients  $\overline{c_\mu(D)}$  instead of  $c_\mu(D)$ .

Let  $\mathcal{H} := \{\tau \in \mathbb{C} \mid \text{Im}(\tau) > 0\}$  be the complex upper half-plane and  $\Omega_{t_o, s}^{k, m} : \mathcal{H} \times \mathbb{C} \rightarrow \mathbb{C}$  be the function given by the series

$$\Omega_{t_o, s}^{k, m}(\tau, z) := \sum_{h \in H^J \backslash \Gamma^J} \phi_{t_o, s|_{k, m}}[h](\tau, z),$$

where  $\phi_{t_o, s}(\tau, z) := \tau^{-s} \exp(-2\pi i m(z - t_o)^2/\tau)$ ,  $s$  is a complex number,  $t_o \in (2m)^{-1}\mathbb{Z}$ ,  $H^J$  is the subgroup of  $\Gamma^J$  given by  $H^J := \{\text{Id}\} \times \mathbb{Z} \times \{0\}$  and  $|_{k, m}$  is the slash operator used in the theory of Jacobi forms. Our main result is the following.

**THEOREM 1.1.** *Let  $k$  and  $m$  be positive integers with  $k > 6$  and  $t_o \in (2m)^{-1}\mathbb{Z}$ .*

*If  $s \in \mathbb{C}$  with  $1 < \Re(s) < k - 3$ , then the series  $\Omega_{t_o, s}^{k, m}(\tau, z)$  defines a Jacobi cusp form in  $J_{k, m}^{\text{cusp}}$ . Moreover,*

$$\langle \Omega_{t_o, s}^{k, m}, f \rangle = \frac{\pi}{2^{k-2}} \frac{\Gamma(k-3/2)}{e^{\pi i s/2} \Gamma(s-1/2) \Gamma(k-s)} \frac{1}{2m} \sum_{\mu=1}^{2m} \exp(-2\pi i \mu t_o) \Lambda_{\mu}(\bar{f}, k-s),$$

for all  $f \in J_{k, m}^{\text{cusp}}$  and all  $s \in \mathbb{C}$  with  $\frac{3}{2} < \Re(s) < \frac{k}{2} - 2$ .

The identity in this theorem is the analogue of (1.3) that we are after. It is clear from it that a linear combination of the functions  $\Omega_{t_o, s}^{k, m}$  with  $t_o \in \mathbb{Z}/2m\mathbb{Z}$  yield an integral expression for each  $\Lambda_{\mu}(\bar{f}, k-s)$ . Alternatively, one could use the functional equations (1.5) in order to get an integral representation for every  $\Lambda_{\mu}(\bar{f}, s - \frac{1}{2})$ .

The restriction  $\frac{3}{2} < \Re(s) < \frac{k}{2} - 2$  in the theorem can be replaced by the extended vertical strip  $\frac{3}{2} < \Re(s) < k - 3$  via the analytic continuation of both sides of the equation (see final remark of the paper).

Let us recall something else from the classical case. The representation of the cusp form  $\Omega_s^k$  in terms of Poincaré series is

$$\Omega_s^k(\tau) = \frac{(2\pi)^s}{e^{\pi i s/2} \Gamma(s)} \sum_{n=1}^{\infty} n^{s-1} P_{k, n}(\tau),$$

where the  $n$ th Poincaré series is defined as  $P_{k, n}(\tau) := \sum_{M \in \Gamma_{\infty} \backslash \text{SL}_2(\mathbb{Z})} \exp(2\pi i n \tau) |_{k, n}[M]$ , and  $\Gamma_{\infty}$  is the stabilizer of infinity in  $\text{SL}_2(\mathbb{Z})$ .

If we consider the corresponding collection of Poincaré series in the case of Jacobi forms, namely,

$$P_{k, m, (n, r)}(\tau, z) := \sum_{h \in \Gamma_{\infty}^J \backslash \Gamma^J} \exp(2\pi i (n\tau + rz)) |_{k, m}[h],$$

where  $\Gamma_{\infty}^J$  is the stabilizer of infinity in  $\Gamma^J$  and  $n, r \in \mathbb{Z}$  with  $4mn > r^2$ , we can prove a similar statement for the new kernel functions.

**PROPOSITION 1.2.** *Let  $k$  and  $m$  be positive integers with  $k > 6$  and  $t_o \in (2m)^{-1}\mathbb{Z}$ .*

*If  $s \in \mathbb{C}$  with  $1 < \Re(s) < k - 3$ , then*

$$\begin{aligned} \Omega_{t_o, s}^{k, m}(\tau, z) &= \frac{(2\pi)^{s-1/2}}{e^{\pi i s/2} \Gamma(s-1/2)} \frac{1}{\sqrt{2m}} \sum_{\mu=1}^{2m} \exp(-2\pi i \mu t_o) \\ &\times \sum_{\substack{D=1 \\ 4m|D+\mu^2}}^{\infty} \left(\frac{D}{4m}\right)^{s-3/2} P_{k, m, ((D+\mu^2)/4m, \mu)}(\tau, z). \end{aligned}$$

Note that the previous formula yields all Fourier coefficients of  $\Omega_{t_o, s}^{k, m}$  in closed form, as the Fourier expansion of every  $P_{k, m, (n, r)}$  is known (see [9, p. 519]).

Note also that the same formula can be used to get a second proof of the identity given in Theorem 1.1, as the inner product  $\langle f, P_{k, m, (n, r)} \rangle$  is essentially equal to the coefficient of  $\exp(2\pi i n \tau) \exp(2\pi i r z)$  in the Fourier expansion of  $f$  (see [9, p. 519]). We do not pursue this line of reasoning here though.

As an application of our main result, we present a new proof of the analytic properties satisfied by the series  $\Lambda_\mu(f; s)$  ( $1 \leq \beta \leq 2m$ ). They were first established in [1] utilizing a variation of the Mellin transform (see also [18]). Here, we obtain those properties from Theorem 1.1.

**PROPOSITION 1.3.** *Let  $k$  and  $m$  be positive integers with  $k > 9$  and  $f \in J_{k, m}^{\text{cusp}}$ .*

*Then every completed Dirichlet series  $\Lambda_\mu(f, s)$ , with  $1 \leq \mu \leq 2m$ , admits an analytic continuation to the whole complex plane, and they satisfy the set of  $2m$  functional equations*

$$\Lambda_\beta(\bar{f}, s) = \frac{i^k}{\sqrt{2m}} \sum_{\mu=1}^{2m} \exp\left(\pi i \frac{\mu\beta}{m}\right) \Lambda_\mu\left(\bar{f}, k - s - \frac{1}{2}\right) \quad (1 \leq \beta \leq 2m).$$

This article is divided into five sections. In the next one, we recall the basic definitions and properties of Jacobi forms which are needed in the rest of the paper. In Section 3, we introduce the kernel functions via infinite series and prove their convergence on a vertical strip of the  $s$ -plane. In Section 4, we give a proof of Theorem 1.1. Finally, in Section 5, we give proofs of Propositions 1.2 and 1.3.

*Notation.* Throughout this article,  $\text{Id}$  denotes the 2 by 2 identity matrix, we abbreviate the exponential function as  $e(w) = \exp(2\pi i w)$  for any  $w \in \mathbb{C}$ , put  $e^m(w) = \exp(2\pi i m w)$  for  $m \in \mathbb{Z}$ , and we write  $\Re(w)$  (respectively,  $\Im(w)$ ) for the real (respectively, imaginary) part of the complex number  $w$ . Also, we always set  $w^s := \exp(s \log w)$  for any  $w \in \mathbb{C} - (-\infty, 0]$  and  $s \in \mathbb{C}$ , where  $\log w = \log |w| + i \arg w$  with  $-\pi < \arg w < \pi$ . Throughout Section 3, we denote by  $B(\tau, r)$  (respectively,  $D(\tau, r)$ ) the hyperbolic (respectively, euclidean) open ball of centre  $\tau$  and radius  $r$ .

## 2. Basic definitions

Let  $G^J$  be the real Jacobi group, namely, the set of triples  $h = [\gamma, Y, \zeta]$ , where  $\gamma \in \text{SL}_2(\mathbb{R})$ ,  $Y \in \mathbb{R}^2$  and  $\zeta \in S^1 := \{w \in \mathbb{C} \mid |w| = 1\}$ , together with the product

$$h_1 h_2 = \left[ \gamma_1 \gamma_2, Y_1 \gamma_2 + Y_2, \zeta_1 \zeta_2 \exp\left(2\pi i \det\begin{pmatrix} Y_1 \gamma_2 \\ Y_2 \end{pmatrix}\right) \right].$$

This group acts on  $\mathcal{H} \times \mathbb{C}$ . Namely, any  $h = [\gamma, Y, \zeta]$  in  $G^J$  with components

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{R}) \quad \text{and} \quad Y = (\lambda, \nu) \in \mathbb{R}^2,$$

send the pair  $(\tau, z) \in \mathcal{H} \times \mathbb{C}$  to

$$h \cdot (\tau, z) = h(\tau, z) := \left( \frac{a\tau + b}{c\tau + d}, \frac{z + \lambda\tau + \nu}{c\tau + d} \right).$$

For positive integers  $k$  and  $m$ , let  $j_{k, m} : G^J \times \mathcal{H} \times \mathbb{C} \rightarrow \mathbb{C}$  be the 1-cocycle

$$j_{k, m}(h, \tau, z) := \zeta^m (c\tau + d)^{-k} e^m \left( \frac{-c(z + \lambda\tau + \nu)^2}{c\tau + d} + \lambda^2 \tau + 2\lambda z + \lambda \nu \right).$$

This map is used in the definition of a  $G^J$ -action on the collection of holomorphic functions  $f : \mathcal{H} \times \mathbb{C} \rightarrow \mathbb{C}$ , namely

$$f|_{k,m}[h](\tau, z) := j_{k,m}(h, \tau, z)f(h(\tau, z)).$$

The semi-direct product  $\Gamma^J = \mathrm{SL}_2(\mathbb{Z}) \ltimes (\mathbb{Z}^2 \times \{1\})$  is a discrete subgroup of  $G^J$  which plays a central role in this theory. For convenience, we drop the 1 in the third component of its triples from now on.

**DEFINITION 2.1.** Let  $k$  and  $m$  be positive integers. A Jacobi form of weight  $k$  and index  $m$  over  $\Gamma^J$  is any holomorphic function  $f : \mathcal{H} \times \mathbb{C} \rightarrow \mathbb{C}$  which satisfies

$$f|_{k,m}[h] = f,$$

for every  $h \in \Gamma^J$ , and has a Fourier series representation

$$f|_{k,m}[\sigma^{-1}, 0, 0, 1](\tau, z) = \sum_{\substack{n,r \in \mathbb{Z} \\ 4mnt_\sigma \geq r^2}} c_\sigma(n, r) e\left(\frac{n}{t_\sigma}\tau\right) e\left(\frac{r}{t_\sigma}z\right), \quad (2.1)$$

for each  $\sigma \in \mathrm{SL}_2(\mathbb{Q})$ . Here,  $t_\sigma$  is a positive integer depending on  $\sigma^{-1}(\infty) \in \mathbb{Q} \cup \{\infty\}$ .

The set  $J_{k,m}$  of all these functions is a finite-dimensional  $\mathbb{C}$ -vector space. A very important subspace of it is  $J_{k,m}^{\mathrm{cusp}}$ , the space of Jacobi cusp forms. It is defined as the set of functions  $f$  in  $J_{k,m}$  such that all its series representations (2.1) are indexed by integers  $n, r$  such that  $4mnt_\sigma > r^2$ . In particular, any  $f$  in  $J_{k,m}^{\mathrm{cusp}}$  has a series representation

$$f(\tau, z) = \sum_{\substack{n,r \in \mathbb{Z} \\ 4mn > r^2}} c(n, r) e(n\tau) e(rz) \quad (c(n, r) \in \mathbb{C}). \quad (2.2)$$

Set  $c_r(D) := c(n, r)$  if  $D = 4mn - r^2$ . Some of the functional equations in Definition 2.1 imply that (2.2) can be written as

$$f(\tau, z) = \sum_{\mu=1}^{2m} f_\mu(\tau) \Theta_{m,\mu}(\tau, z) \quad \text{where } f_\mu(\tau) = \sum_{D=1}^{\infty} c_\mu(D) e\left(\frac{D}{4m}\tau\right), \quad (2.3)$$

and  $\Theta_{L,\mu}(\tau, z)$  denotes the theta function

$$\Theta_{L,\mu}(\tau, z) := \sum_{\substack{l \in \mathbb{Z} \\ l \equiv \mu(2L)}} e\left(\frac{l^2}{4L}\tau\right) e(lz) \quad (L, \mu \in \mathbb{Z}, L > 0).$$

The representation (2.3) is called the theta decomposition of  $f$  (at infinity). As in the case of elliptic cusp forms, any  $f$  in  $J_{k,m}^{\mathrm{cusp}}$  with series representation (2.2) determines a constant  $K \in \mathbb{R}$  such that

$$|c(n, r)| = |c_r(D)| \leq KD^{k/2} \quad \text{for all } n, r. \quad (2.4)$$

**DEFINITION 2.2.** To any  $f$  in  $J_{k,m}^{\mathrm{cusp}}$  with theta decomposition (2.3), we associate the  $2m$  Dirichlet series

$$L_\mu(f; s) := \sum_{D=1}^{\infty} c_\mu(D) \left(\frac{D}{4m}\right)^{-s} \quad \text{for } \mu = 1, 2, \dots, 2m.$$

Also, we set  $\Lambda_\mu(f; s) := (2\pi)^{-s} \Gamma(s) L_\mu(f; s)$ , where  $\Gamma(s)$  is Euler's gamma function.

By (2.4), each of these series is uniformly convergent on compact subsets of the complex half-plane  $\operatorname{Re}(s) > 1 + k/2$ .

In this work, we often use the following parametrization of  $\mathcal{H} \times \mathbb{C}$ ; a pair  $(\tau, z)$  in  $\mathcal{H} \times \mathbb{C}$  is identified with  $(x, y, p, q)$  in  $\mathbb{R}^4$  if and only if  $\tau = x + iy$  and  $z = p\tau + q$ . In terms of these variables, the  $G^J$ -invariant volume element in  $\mathcal{H} \times \mathbb{C}$  is given by  $dV = dV(\tau, z) := y^{-2} dx dy dp dq$ .

Consider next the function  $\mu_{k,m} : \mathcal{H} \times \mathbb{C} \rightarrow \mathbb{C}$  given as

$$\mu_{k,m}(\tau, z) := y^{k/2} e^m(p^2 iy) = \Im(\tau)^{k/2} e^m \left( \frac{\Im(z)^2}{\Im(\tau)} i \right).$$

It satisfies

$$\mu_{k,m}(\tau, z) = |c\tau + d|^k \left| e^m \left( \frac{c(z + \lambda\tau + \nu)^2}{c\tau + d} - \lambda^2\tau - 2\lambda z \right) \right| \mu_{k,m}(h(\tau, z)), \quad (2.5)$$

for any  $h = \left[ \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \lambda, \nu, \zeta \right]$  in  $G^J$  (see [20, p. 177]). From (2.5), one deduces that the function  $f(\tau, z)g(\tau, z)\mu_{k,m}(\tau, z)^2$  is  $\Gamma^J$ -invariant whenever  $f$  and  $g$  are in  $J_{k,m}^{\text{cusp}}$ .

DEFINITION 2.3. The Petersson inner product of two cusp forms  $f, g$  in  $J_{k,m}^{\text{cusp}}$  is

$$\langle f, g \rangle := \int_{\Gamma^J \backslash \mathcal{H} \times \mathbb{C}} f(\tau, z) \overline{g(\tau, z)} e^m(2p^2 iy) y^k dV.$$

### 3. The kernel function

The complex-valued map on  $\mathcal{H} \times \mathbb{C}$  given by

$$\phi_{t_o, s}(\tau, z) := \frac{1}{\tau^s} e^m \left( \frac{-(z - t_o)^2}{\tau} \right) \quad (3.1)$$

is well defined for any complex number  $s$  and any real parameter  $t_o$ . Straightforward computations yield the identity

$$\phi_{t_o, s}|_{k,m}[\text{Id}, \lambda, \nu, 1](\tau, z) = e^m(-\lambda\nu) e^m(2\lambda t_o) \phi_{-\nu+t_o, s}(\tau, z) \quad \text{for all } \lambda, \nu \in \mathbb{R}, \quad (3.2)$$

and show that  $H^J := \{[\text{Id}, \lambda, 0] \mid \lambda \in \mathbb{Z}\}$  is contained in the stabilizer of  $\phi_{t_o, s}(\tau, z)$  in  $\Gamma^J$ . In particular, one has  $\phi_{t_o, s}(\tau, z) = \phi_{0, s}|_{k,m}[\text{Id}, 0, -t_o, 1](\tau, z)$  for all  $t_o \in \mathbb{R}$ . For the purposes of this work, it suffices to consider  $t_o$  in  $(2m)^{-1}\mathbb{Z}$ .

The following series is the main object of study in this article and our proposed analogue of the kernel (1.2).

DEFINITION 3.1. For  $t_o \in (2m)^{-1}\mathbb{Z}$  and  $s \in \mathbb{C}$ , let

$$\Omega_{t_o, s}^{k,m}(\tau, z) := \sum_{h \in H^J \backslash \Gamma^J} \phi_{t_o, s}|_{k,m}[h](\tau, z).$$

Formally,  $\Omega_{t_o+\nu, s}^{k,m} = \Omega_{t_o, s}^{k,m}$  for all  $\nu \in \mathbb{Z}$  by (3.2), hence the collection  $\{\Omega_{t_o, s}^{k,m}\}_{t_o \in (2m)^{-1}\mathbb{Z}}$  has at most  $2m$  distinct series. The rest of this section is devoted to establish a region of convergence for all these series. To this end, we start with some simple remarks, prove a technical lemma, and then give a proof of Proposition 3.4, which exhibit such a region.

REMARK 3.2. (a) A complete, minimal collection of coset representatives for the elements in  $H^J \backslash \Gamma^J$  is

$$\{[\text{Id}, 0, \nu][M, 0, 0] \mid M \in \Gamma, \nu \in \mathbb{Z}\} \subseteq \Gamma^J. \quad (3.3)$$

(b) From equation (2.5), one gets

$$|\phi_{t_o, s}|_{k, m}[h](\tau, z) |\mu_{k, m}(\tau, z)| = |\phi_{t_o, s}(h(\tau, z))| |\mu_{k, m}(h(\tau, z))| \quad \text{for all } h \in \Gamma^J. \quad (3.4)$$

(c) Any hyperbolic ball is an euclidean ball. In particular, the hyperbolic ball  $B(\tau, \frac{1}{2})$  of centre  $\tau = x + iy \in \mathcal{H}$  and radius  $\frac{1}{2}$  is equal to the euclidean ball  $D(\tau_0, r_0)$ , of centre  $\tau_0 = x + iy \cosh(\frac{1}{2})$  and radius  $r_0 = y \sinh(\frac{1}{2})$ .

(d) The invariant height function of the group  $\Gamma = \text{SL}_2(\mathbb{Z})$  is the real-valued map on  $\mathcal{H}$  defined as

$$\Im(\tau)_\Gamma := \max_{M \in \Gamma} \Im(M(\tau)).$$

There exists a positive constant  $c_\Gamma > 1$  such that

$$\Im(\tau)_\Gamma \leq c_\Gamma \left( y + \frac{1}{y} \right) \quad \text{for any } \tau = x + iy \in \mathcal{H} \quad (3.5)$$

(see, for example, [12, Lemma A.1]).

LEMMA 3.3. For every  $(\tau, z) \in \mathcal{H} \times \mathbb{C}$ , there exists  $R = R_{\tau, z} > 0$  such that the image of  $B(\tau, \frac{1}{2}) \times D(z, \frac{1}{2}) \subseteq \mathcal{H} \times \mathbb{C}$  under any  $h = [M, 0, 0] \in \Gamma^J$  is contained in the cartesian product  $B(M(\tau), \frac{1}{2}) \times \bar{D}(0, R)$ .

*Proof.* Let  $(\tau', z')$  be any point in  $B(\tau, \frac{1}{2}) \times D(z, \frac{1}{2})$  and  $(\tau'', z'') = h(\tau', z')$ .

Then  $\tau'' = M(\tau') \in B(M(\tau), \frac{1}{2})$  since any  $M \in \Gamma$  is an isometry of the hyperbolic plane. On the other hand,  $z'' = z'(c\tau' + d)^{-1}$ , where  $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ . Thus,

$$|z''|^2 = \frac{|z'|^2}{y'} \Im(M(\tau')) \quad \text{where } \tau' = x' + iy' \text{ with } x', y' \in \mathbb{R}. \quad (3.6)$$

Let  $\rho(\tau, \tau')$  be the hyperbolic distance between  $\tau$  and  $\tau'$ . Then  $\tau' \in B(\tau, \frac{1}{2})$  if and only if  $\cosh \rho(\tau, \tau') < \cosh(\frac{1}{2})$ . This inequality can be written as

$$1 + \frac{|\tau' - \tau|^2}{2yy'} < \cosh\left(\frac{1}{2}\right)$$

(see, for example, [11, p. 7]), which in turn yields

$$\frac{y'}{2y} + \frac{y}{2y'} = 1 + \frac{(y' - y)^2}{2yy'} \leq 1 + \frac{(x' - x)^2 + (y' - y)^2}{2yy'} < \cosh\left(\frac{1}{2}\right).$$

Let  $C = 2 \cosh(\frac{1}{2})$ . The argument above shows that for any  $\tau'$  in  $B(\tau, \frac{1}{2})$ , one has  $y' < Cy$  and  $y < Cy'$ . From the last inequality, (3.6) and  $z' \in D(z, \frac{1}{2})$ , we get the relation

$$|z''|^2 \leq |z'|^2 \frac{C}{y} \Im(M(\tau')) \leq \left( |z| + \frac{1}{2} \right)^2 \frac{C}{y} \Im(M(\tau')).$$

As  $M(\tau') \in B(M(\tau), \frac{1}{2})$ , we have that  $\Im(M(\tau')) < C \Im(M(\tau))$ . Hence,  $\Im(M(\tau')) < C \Im(\tau)_\Gamma$ , where  $\Im(\tau)_\Gamma$  is the value at  $\tau$  of the invariant height function (item (d) in Remark 3.2). From these inequalities and (3.5), we obtain

$$|z''|^2 < \left( |z| + \frac{1}{2} \right)^2 \frac{C^2}{y} c_\Gamma \left( y + \frac{1}{y} \right).$$

Let  $R = C(|z| + \frac{1}{2})c_\Gamma\sqrt{1+1/y^2}$ . Then the last inequality yields  $(\tau'', z'') \in B(M(\tau), \frac{1}{2}) \times D(0, R)$ , and the lemma follows.  $\square$

**PROPOSITION 3.4.** *Let  $k > 6$ ,  $t_o \in (2m)^{-1}\mathbb{Z}$  and  $s \in \mathbb{C}$ .*

*If  $1 < \Re(s) < k - 3$ , then the series  $\Omega_{t_o, s}^{k, m}(\tau, z)$  is absolute and uniformly convergent on any compact subset of  $\mathcal{H} \times \mathbb{C}$ .*

*In particular,  $\Omega_{t_o, s}^{k, m}(\tau, z)$  defines a holomorphic function on  $\mathcal{H} \times \mathbb{C}$  for any such  $s$ .*

*Proof.* For any  $(\tau, z) \in \mathcal{H} \times \mathbb{C}$ , consider the hyperbolic ball  $B(\tau, \frac{1}{2})$  and the euclidean ball  $D(z, \frac{1}{2})$ . As mentioned in item (c) of Remark 3.2,  $B(\tau, \frac{1}{2}) = D(\tau_0, r_0)$  for certain  $\tau_0 \in \mathcal{H}$  and  $r_0 > 0$ .

Since the function  $\phi_{t_o, s|k, m}[h](\tau, z)$  is holomorphic on  $\mathcal{H} \times \mathbb{C}$  for every  $h \in \Gamma^J$ , Cauchy's estimate theorem for holomorphic functions on several variables yields

$$|\phi_{t_o, s|k, m}[h](\tau, z)| \leq \frac{4}{\pi^2 r_0^2} \int_{D(\tau_0, r_0) \times D(z, 1/2)} |\phi_{t_o, s|k, m}[h](\tau', z')| dx' dy' du' dv', \quad (3.7)$$

where  $\tau' = x' + iy'$  and  $z' = u' + iv'$  with  $x', y', u', v' \in \mathbb{R}$ .

Now we observe that the real-valued map  $(\tau', z') \mapsto \mu_{k, m}(\tau', z')y'^{-3}$  is continuous on  $\mathcal{H} \times \mathbb{C}$ , and therefore has a minimum value on the closure of  $D(\tau_0, r_0) \times D(z, \frac{1}{2})$ , say  $m_{\tau, z}$ . Then

$$1 \leq \frac{\mu_{k, m}(\tau', z')y'^{-3}}{m_{\tau, z}} \quad \text{for all } (\tau', z') \in D(\tau_0, r_0) \times D\left(z, \frac{1}{2}\right).$$

If we use the last inequality, then replace  $D(\tau_0, r_0)$  by  $B(\tau, \frac{1}{2})$  in the domain of integration of (3.7), and change the cartesian coordinates of  $z'$  by the real variables  $p', q'$  such that  $z' = p'\tau' + q'$ , we get from (3.7) the relation

$$|\phi_{t_o, s|k, m}[h](\tau, z)| \leq \frac{4}{\pi^2 r_0^2 m_{\tau, z}} \int_{B(\tau, 1/2) \times D(z, 1/2)} |\phi_{t_o, s|k, m}[h](\tau', z')| \mu_{k, m}(\tau', z') dV(\tau', z').$$

This expression, valid for all  $h \in \Gamma^J$ , together with (3.4) yield

$$\begin{aligned} & 2^{-2} \pi^2 r_0^2 m_{\tau, z} \sum_{h \in H^J \setminus \Gamma^J} |\phi_{t_o, s|k, m}[h](\tau, z)| \\ & \leq \sum_{h \in H^J \setminus \Gamma^J} \int_{B(\tau, 1/2) \times D(z, 1/2)} |\phi_{t_o, s}(h(\tau', z'))| \mu_{k, m}(h(\tau', z')) dV(\tau', z') \\ & = \sum_{h \in H^J \setminus \Gamma^J} \int_{h(B(\tau, 1/2) \times D(z, 1/2))} |\phi_{t_o, s}(\tau', z')| \mu_{k, m}(\tau', z') dV(\tau', z'). \end{aligned}$$

Next we consider the set (3.3) of coset representatives for the elements in  $H^J \setminus \Gamma^J$  and use the previous lemma in order to get

$$\begin{aligned} & 2^{-2} \pi^2 r_0^2 m_{\tau, z} \sum_{h \in H^J \setminus \Gamma^J} |\phi_{t_o, s|k, m}[h](\tau, z)| \\ & \leq \sum_{M \in \Gamma} \sum_{\nu \in \mathbb{Z}} \int_{[\text{Id}, 0, \nu][M, 0, 0] \cdot (B(\tau, 1/2) \times D(z, 1/2))} |\phi_{t_o, s}(\tau', z')| \mu_{k, m}(\tau', z') dV(\tau', z') \\ & \leq \sum_{M \in \Gamma} \sum_{\nu \in \mathbb{Z}} \int_{[\text{Id}, 0, \nu] \cdot (B(M(\tau), 1/2) \times D(0, R))} |\phi_{t_o, s}(\tau', z')| \mu_{k, m}(\tau', z') dV(\tau', z') \\ & = \sum_{M \in \Gamma} \sum_{\nu \in \mathbb{Z}} \int_{B(M(\tau), 1/2)} \int_{D(\nu, R)} |\phi_{t_o, s}(\tau', z')| \mu_{k, m}(\tau', z') dV(\tau', z'), \quad (3.8) \end{aligned}$$



for some positive real number  $R$  (which depends on  $(\tau, z)$ ).

In order to get an upper bound for the last double series in (3.8), we first look at

$$\sum_{\nu \in \mathbb{Z}} \int_{D(\nu, R)} |\phi_{t_o, s}(\tau', z')| \mu_{k, m}(\tau', z') dp' dq',$$

for any fixed  $M \in \Gamma$  and any fixed  $\tau' = x' + iy' \in B(M(\tau), \frac{1}{2})$ . If we write  $z'$  in terms of its cartesian coordinates  $z' = u' + iv'$ , and use the fact that the multiplicity of any complex number in the disjoint union  $\bigcup_{\nu \in \mathbb{Z}} D(\nu, R)$  is at most  $2R$ , then we obtain the inequalities

$$\begin{aligned} & \sum_{\nu \in \mathbb{Z}} \int_{D(\nu, R)} |\phi_{t_o, s}(\tau', z')| \mu_{k, m}(\tau', z') dp' dq' \\ & \leq 2R \int_{\bigcup_{\nu \in \mathbb{Z}} D(\nu, R)} |\phi_{t_o, s}(\tau', z')| \mu_{k, m}(\tau', z') y'^{-1} du' dv' \\ & \leq 2R \int_{u'=-\infty}^{\infty} \int_{v'=-R}^R |\phi_{t_o, s}(\tau', z')| \mu_{k, m}(\tau', z') y'^{-1} du' dv'. \end{aligned} \quad (3.9)$$

By (3.1) and the relation  $z' = p'\tau' + q'$ , we can write this double integral as

$$\begin{aligned} & \int_{u'=-\infty}^{\infty} \int_{v'=-R}^R \left| \frac{1}{\tau'^s} e^m \left( \frac{-(z' - t_o)^2}{\tau'} \right) \right| e^m (p'^2 iy') y'^{-1+k/2} du' dv' \\ & = \left| \frac{1}{\tau'^s} \right| y'^{-1+k/2} \int_{u'=-\infty}^{\infty} \int_{v'=-R}^R e^m \left( \frac{(q' - t_o)^2}{|\tau'|^2} iy' \right) du' dv'. \end{aligned} \quad (3.10)$$

Let us switch to the variables  $p', q'$  again. As  $p' = v'/y'$  and  $q' = u' - p'x'$ , we have

$$\begin{aligned} & \int_{u'=-\infty}^{\infty} \int_{v'=-R}^R e^m \left( \frac{(q' - t_o)^2}{|\tau'|^2} iy' \right) dv' du' \\ & = \int_{q'=-\infty}^{\infty} \int_{p'=-R/y'}^{R/y'} e^m \left( \frac{(q' - t_o)^2}{|\tau'|^2} iy' \right) y' dp' dq' \\ & = 2R \int_{q'=-\infty}^{\infty} e^m \left( \frac{(q' - t_o)^2}{|\tau'|^2} iy' \right) dq' = 2R \int_{\tilde{q}=-\infty}^{\infty} e^{-(2\pi m y' / |\tau'|^2) \tilde{q}^2} d\tilde{q}. \end{aligned}$$

The last expression is obtained after the change of variables  $\tilde{q} = q' - t_o$ . This integral is well known and equal to  $2R|\tau'|/\sqrt{2m}y'$  (see, for example, [8, p. 336]). Hence, from (3.9), (3.10) and the last remark, we may conclude

$$\sum_{\nu \in \mathbb{Z}} \int_{D(\nu, R)} |\phi_{t_o, s}(\tau', z')| \mu_{k, m}(\tau', z') dp' dq' \leq \sqrt{\frac{8}{m}} R^2 \left| \frac{1}{\tau'^{s-1}} \right| y'^{(k-3)/2}. \quad (3.11)$$

Consequently, from (3.8) and (3.11), we obtain the upper bound

$$\begin{aligned} & 2^{-2} \pi^2 r_0^2 m_{\tau, z} \sum_{h \in H^J \setminus \Gamma^J} |\phi_{t_o, s}|_{k, m}[h](\tau, z)| \\ & \leq \sqrt{\frac{8}{m}} R^2 \sum_{M \in \Gamma} \int_{B(M(\tau), 1/2)} \left| \frac{1}{\tau'^{s-1}} \right| y'^{(k-\tau)/2} dx' dy'. \end{aligned} \quad (3.12)$$

At this point, we observe it is possible to use part of the proof of [3, Proposition 5.1] in our case, and we do so. First, we recall from Remark 3.2(c) and the proof of Lemma 3.3 that

$$\begin{aligned} & \tau_0 = x + iy \cosh \left( \frac{1}{2} \right), \quad r_0 = y \sinh \left( \frac{1}{2} \right) \\ & \text{and } R = 2 \cosh \left( \frac{1}{2} \right) c_{\Gamma} \left( |z| + \frac{1}{2} \right) \frac{\sqrt{1+y^2}}{y}. \end{aligned} \quad (3.13)$$

Secondly, we obtain an explicit value for  $m_{\tau,z}$  assuming the condition  $k > 6$ .

If  $(\tau', z')$  is in  $B(\tau, \frac{1}{2}) \times D(z, \frac{1}{2}) = D(\tau_0, r_0) \times D(z, \frac{1}{2})$ , then

$$ye^{-1/2} < y' < ye^{1/2} \quad \text{and} \quad v - \frac{1}{2} < v' < v + \frac{1}{2}.$$

Hence, if we consider the square  $S = [ye^{-1/2}, ye^{1/2}] \times [v - \frac{1}{2}, v + \frac{1}{2}] \subseteq \mathbb{R}^2$ , then we have that  $m_{\tau,z}$  is the minimum value of the map  $\tilde{\mu}_{k,m} : S \rightarrow \mathbb{R}$ ,  $\tilde{\mu}_{k,m}(y', v') = y'^{-3+k/2} e^m (v'^2 i / y')$ . A straightforward computation yields that the partial derivative  $\partial \tilde{\mu}_{k,m} / \partial y'$  is not zero if  $k > 6$  (using that  $\partial \tilde{\mu}_{k,m} / \partial y' = 0$  is equivalent to  $-y'(k/2 - 3) = 2\pi m v'^2$  and  $y' = \Im(\tau')$  with  $\tau' \in B(\tau, \frac{1}{2}) \subseteq \mathcal{H}$ ). Thus,  $\tilde{\mu}_{k,m}$  attains its minimum at the boundary of  $S$ . A computation of such values (which we leave to the interested reader) allow us to conclude

$$m_{\tau,z} = \tilde{\mu}_{k,m}(ye^{-1/2}, v \pm \frac{1}{2}) = y^{-3+k/2} e^{3/2-k/4} e^{-2\pi m(v \pm 1/2)^2 / ye^{-1/2}} \quad \text{if } \pm v \geq 0. \quad (3.14)$$

Putting together (3.12)–(3.14), we are able to get the estimate

$$\sum_{h \in H^J \setminus \Gamma^J} |\phi_{t_o,s}|_{k,m}[h](\tau, z) \ll \frac{1+y^2}{y^{1+k/2}} e^{c_1/y} \sum_{M \in \Gamma} \int_{B(M(\tau), 1/2)} \left| \frac{1}{\tau'^{s-1}} \right| y'^{(k-7)/2} dx' dy', \quad (3.15)$$

where both, the positive real number  $c_1$  and the implied constant, depend on  $z$  but not on  $\tau$ .

On the other hand, we observe that  $\#\{M \in \Gamma \mid \rho(M(\tau), \tau) < 1\} \ll \Im(\tau)_\Gamma + 1$  for any  $\tau \in \mathcal{H}$  (see [11, p. 52]). This inequality and (3.5) yield

$$\#\{M \in \Gamma \mid \rho(M(\tau), \tau) < 1\} \ll \left(y + \frac{1}{y}\right),$$

where the implied constant is independent of  $\tau$ . From this argument, we conclude that the number of distinct  $M' \in \Gamma$  such that  $B(M'(\tau), \frac{1}{2})$  intersect any given ball  $B(M(\tau), \frac{1}{2})$  is at most  $y + 1/y$  times a constant. Hence, the multiplicity of any point of  $\mathcal{H}$  in the disjoint union  $\bigcup_{M \in \Gamma} B(M(\tau), \frac{1}{2})$  is at most  $y + 1/y$  times a constant independent of  $\tau$ . This fact and (3.15) yield

$$\begin{aligned} & \sum_{h \in H^J \setminus \Gamma^J} |\phi_{t_o,s}|_{k,m}[h](\tau, z) \\ & \ll y^{-k/2} \left(y + \frac{1}{y}\right)^2 e^{c_1/y} \int_{\bigcup_{M \in \Gamma} B(M(\tau), 1/2)} \left| \frac{1}{\tau'^{s-1}} \right| y'^{(k-7)/2} dx' dy', \end{aligned} \quad (3.16)$$

where the union in the domain of integration is not necessarily disjoint and the implied constant depends on  $z$ , but not on  $\tau$ . Any  $\tau' \in \bigcup_{M \in \Gamma} B(M(\tau), \frac{1}{2})$  satisfies

$$\frac{1}{|\tau'|^2} \ll \frac{y + 1/y}{y'}, \quad (3.17)$$

with the implied constant independent of  $\tau$  and  $\tau'$ . (A proof of this inequality is given in [3, Lemma 5.2, and the ensuing remark about equation (5.15)].) From the definition of  $\Im(\tau)_\Gamma$  and (3.5), we have

$$\Im(M(\tau)) \leq \Im(\tau)_\Gamma \leq c_\Gamma \left(y + \frac{1}{y}\right) \quad \text{for all } M \in \Gamma.$$

We also have  $y' < Cy$  for any  $\tau'$  in  $B(\tau, \frac{1}{2})$  with  $C = 2 \cosh(\frac{1}{2})$  as shown in the proof of Lemma 3.3. These two inequalities yield

$$y' = \Im(\tau') < T(\tau, \Gamma) := Cc_\Gamma \left(y + \frac{1}{y}\right) \quad \text{for every } \tau' \in \bigcup_{M \in \Gamma} B\left(M(\tau), \frac{1}{2}\right). \quad (3.18)$$

By the previous remarks, we conclude

$$\bigcup_{M \in \Gamma} B\left(M(\tau), \frac{1}{2}\right) \subseteq B' := \{\tau' \in \mathcal{H} \mid y' < T(\tau, \Gamma) \text{ and } \tau' \text{ satisfies (3.17)}\}. \quad (3.19)$$

Since  $1 < \sigma := \Re(s)$  by hypothesis, we pick any  $r \in \mathbb{R}$  such that  $1 < r < \sigma$ , and deduce from (3.16) and (3.19) the estimate

$$\sum_{h \in H^J \setminus \Gamma^J} |\phi_{t_o, s|k, m}[h](\tau, z)| \ll y^{-k/2} \left(y + \frac{1}{y}\right)^2 e^{c_1/y} \int_{B'} \frac{y'^{(k-7)/2}}{|\tau'|^r |\tau'|^{\sigma-r-1}} dx' dy'.$$

Using now inequality (3.17) in the last integral, one gets

$$\begin{aligned} & \sum_{h \in H^J \setminus \Gamma^J} |\phi_{t_o, s|k, m}[h](\tau, z)| \\ & \ll y^{-k/2} \left(y + \frac{1}{y}\right)^2 e^{c_1/y} \int_{y'=0}^{T(\tau, \Gamma)} \int_{x'=-\infty}^{\infty} \frac{y'^{(k-7)/2}}{(x'^2 + y'^2)^{r/2}} \left(\frac{y + 1/y}{y'}\right)^{(\sigma-r-1)/2} dx' dy' \\ & = y^{-k/2} \left(y + \frac{1}{y}\right)^{(\sigma-r+3)/2} e^{c_1/y} \int_{y'=0}^{T(\tau, \Gamma)} \int_{x'=-\infty}^{\infty} \frac{y'^{(k-\sigma+r-6)/2}}{(x'^2 + y'^2)^{r/2}} dx' dy'. \end{aligned} \quad (3.20)$$

At this point, we recall the identity

$$\int_{x'=-\infty}^{\infty} \frac{dx'}{|\tau'|^l} = \sqrt{\pi} \frac{\Gamma((l-1)/2)}{\Gamma(l/2)} \frac{1}{y'^{l-1}},$$

valid for  $l > 1$ , and conclude from (3.20)

$$\begin{aligned} & \sum_{h \in H^J \setminus \Gamma^J} |\phi_{t_o, s|k, m}[h](\tau, z)| \\ & \ll y^{-k/2} \left(y + \frac{1}{y}\right)^{(\sigma-r+3)/2} e^{c_1/y} \frac{\Gamma((r-1)/2)}{\Gamma(r/2)} \int_{y'=0}^{T(\tau, \Gamma)} y'^{(k-\sigma-r-4)/2} dy'. \end{aligned}$$

In this expression, the implied constant is still independent of  $\tau$  and  $s$ . Finally, if  $\sigma < k - r - 2$ , then this estimate can be written as

$$\begin{aligned} & \sum_{h \in H^J \setminus \Gamma^J} |\phi_{t_o, s|k, m}[h](\tau, z)| \\ & \ll y^{-k/2} \left(y + \frac{1}{y}\right)^{(\sigma-r+3)/2} e^{c_1/y} \frac{\Gamma((r-1)/2)}{\Gamma(r/2)} T(\tau, \Gamma)^{(k-\sigma-r-2)/2}. \end{aligned} \quad (3.21)$$

Since the right-hand side of (3.21) is finite and  $r > 1$  is arbitrary, we conclude that the series  $\Omega_{t_o, s}^{k, m}(\tau, z)$  is absolutely convergent whenever the parameter  $\sigma = \Re(s)$  satisfies  $1 < \sigma < k - 3$ . Furthermore, as the constants  $r_0$ ,  $m_{\tau, z}$  and  $R_{\tau, z}$  depend continuously of the pair  $(\tau, z)$ , a straightforward variation of the previous argument yields the absolute uniform convergence of  $\Omega_{t_o, s}^{k, m}(\tau, z)$  on compact subsets of  $\mathcal{H} \times \mathbb{C}$ . This completes the proof of Proposition 3.4.  $\square$

REMARK 3.5. The implied constant in the estimate (3.21) is independent of  $\tau$ , and  $r$  is arbitrary such that  $1 < r < \sigma$ . Hence, from (3.18) and (3.21), we get

$$\begin{aligned} \sum_{h \in H^J \setminus \Gamma^J} |\phi_{t_o, s|k, m}[h](\tau, z)| & \ll y^{-k/2} \left(y + \frac{1}{y}\right)^{-\sigma+\epsilon+(k+1)/2} e^{c_1/y} \\ & = (y^{-\sigma+1/2+\epsilon} + y^{\sigma-1/2-\epsilon-k}) e^{c_1/y}, \end{aligned} \quad (3.22)$$

for any  $\epsilon > 0$ .

REMARK 3.6. In the proof of Proposition 3.4, we assume that the complex number  $s$  is fixed. However, the same argument works if we let  $s$  vary within any compact subset of the vertical strip  $1 < \Re(s) < k - 3$ . In particular, (3.21) holds in this more general situation, and

we can deduce that  $\Omega_{t_o, s}^{k, m}(\tau, z)$  is an absolute and uniformly convergent series of functions on any compact subset of the complex domain  $1 < \Re(s) < k - 3$ . As these functions are  $\phi_{t_o, s}|_{k, m}[h](\tau, z)$  with  $h \in \Gamma^J$ , they are clearly holomorphic functions of  $s$ . Consequently, the series  $\Omega_{t_o, s}^{k, m}(\tau, z)$  defines a holomorphic function of  $s$  on the region  $1 < \Re(s) < k - 3$ .

#### 4. Proof of the main result

In this section, we establish first sufficient conditions for the series  $\Omega_{t_o, s}^{k, m}(\tau, z)$  to be a Jacobi cusp form. Then we give the proof of Theorem 1.1.

PROPOSITION 4.1. *If  $k$  and  $m$  are positive integers with  $k > 6$  and  $t_o \in (2m)^{-1}\mathbb{Z}$ , then*

$$\Omega_{t_o, s}^{k, m} \in J_{k, m}^{\text{cusp}},$$

whenever  $1 < \Re(s) < k - 3$ .

*Proof.* By Proposition 3.4, we know that the series  $\Omega_{t_o, s}^{k, m}(\tau, z)$  is absolutely convergent and defines a holomorphic function on  $\mathcal{H} \times \mathbb{C}$  for any such  $s$ . From the absolute convergence and Definition 3.1, one gets the invariance property

$$\Omega_{t_o, s}^{k, m}|_{k, m}[h](\tau, z) = \Omega_{t_o, s}^{k, m}(\tau, z) \quad \text{for all } h \in \Gamma^J.$$

Now we can use some of these identities to imply the series representation

$$\Omega_{t_o, s}^{k, m}(\tau, z) = \sum_{\mu=1}^{2m} g_{\mu}(\tau) \Theta_{m, \mu}(\tau, z) \quad \text{with } g_{\mu}(\tau) = \sum_{D \in \mathbb{Z}} c_{\mu}(D) e\left(\frac{D}{4m} \tau\right),$$

where the coefficients  $c_{\mu}(D)$  are zero if  $4m$  does not divide  $D - \mu^2$ . Even though this is only the Fourier expansion of  $\Omega_{t_o, s}^{k, m}$  at infinity, it is easy to see that the proposition follows if we just show  $c_{\mu}(D) = 0$  whenever  $D < 0$ . A standard argument yields

$$\frac{1}{\sqrt{4mY}} c_{\mu}(D) = \int_{\tau=iY}^{1+iY} \int_{z \in \mathbb{C}/\mathbb{Z}\tau + \mathbb{Z}} \Omega_{t_o, s}^{k, m}(\tau, z) \overline{\Theta_{m, \mu}(\tau, z)} e^m(2p^2iy) e\left(-\frac{D}{4m} \tau\right) d\tau dz, \quad (4.1)$$

for any real number  $Y > 0$ . This expression shows that an estimate on the size of the right-hand side of (4.1) gives an estimate on the size of  $c_{\mu}(D)/\sqrt{Y}$ . Using the definition of  $\Theta_{m, \mu}(\tau, z)$ , the inequality  $|e^m(p^2iy)| = e^{-2\pi m p^2 y} \leq 1$ , and the relation  $z = p\tau + q$ , it is easy to deduce

$$|\overline{\Theta_{m, \mu}(\tau, z)}| e^m(2p^2iy) \leq |\overline{\Theta_{m, \mu}(\tau, z)}| e^m(p^2iy) \leq \sum_{l \in \mathbb{Z}} e^{-2\pi m y (p+l+\mu/2m)^2}.$$

From these inequalities and (3.22), one gets, for any  $Y > 1$ , the estimate

$$\begin{aligned} & \left| \int_{\tau=iY}^{1+iY} \int_{z \in \mathbb{C}/\mathbb{Z}\tau + \mathbb{Z}} \Omega_{t_o, s}^{k, m}(\tau, z) \overline{\Theta_{m, \mu}(\tau, z)} e^m(2p^2iy) e\left(-\frac{D}{4m} \tau\right) d\tau dz \right| \\ & \ll \int_{\tau=iY}^{1+iY} (y^{-\sigma+1/2+\epsilon} + y^{\sigma-1/2-\epsilon-k}) e^{c_1/y} \left| e\left(-\frac{D}{4m} \tau\right) \right| y \int_{p=0}^1 \sum_{l \in \mathbb{Z}} e^{-2\pi m y (p+l+\mu/2m)^2} dp d\tau. \end{aligned}$$

In the last step, we have used  $dz = \tau dp dq$  and  $|\tau| \ll y$  with the implied constant independent of  $y$ , whenever  $\tau$  is in the straight line segment from  $iY$  to  $1 + iY$  (this is why we ask the condition  $Y > 1$ ).

Now we observe that the inner integral in the right-hand side of the previous estimate is equal to

$$\int_{-\infty}^{\infty} e^{-2\pi m y p^2} dp = \frac{1}{\sqrt{2my}}$$

(see, for example, [8, p. 336]). Hence,

$$\begin{aligned} & \left| \int_{\tau=iY}^{1+iY} \int_{z \in \mathbb{C}/\mathbb{Z}\tau + \mathbb{Z}} \Omega_{t_o, s}^{k, m}(\tau, z) \overline{\Theta_{m, \mu}(\tau, z)} e^m(2p^2 iy) e\left(-\frac{D}{4m}\tau\right) d\tau dz \right| \\ & \ll \frac{1}{\sqrt{2m}} \int_{\tau=iY}^{1+iY} (y^{-\sigma+1/2+\epsilon} + y^{\sigma-1/2-\epsilon-k}) e^{c_1/y} \left| e\left(-\frac{D}{4m}\tau\right) \right| y^{1/2} d\tau \\ & = \frac{1}{\sqrt{2m}} (Y^{-\sigma+1+\epsilon} + Y^{\sigma-\epsilon-k}) e^{c_1/Y} e^{2\pi DY/4m}, \end{aligned}$$

whenever  $Y > 1$ . This estimate and (4.1) allow us to write

$$|c_\mu(D)| \ll (Y^{-\sigma+3/2+\epsilon} + Y^{-k+\sigma+1/2-\epsilon}) e^{c_1/Y} e^{2\pi DY/4m}, \quad (4.2)$$

for any  $Y > 1$ . Clearly, the right-hand side of (4.2) goes to zero if  $Y$  goes to infinity and  $D \leq 0$ . Consequently,  $c_\mu(D) = 0$  for every  $D \leq 0$ , as required.  $\square$

Let us show, prior to the proof of Theorem 1.1, that a particular series representation for the reproducing kernel obtained by Skoruppa and Zagier [20, p. 184, Equation (15)] has an analogue in our case. The latter is instrumental in the proofs of our main result and Propositions 1.2 and 1.3.

By Remark 3.2(a), we have

$$\Omega_{t_o, s}^{k, m}(\tau, z) = \sum_{M \in \Gamma} \sum_{\nu \in \mathbb{Z}} \phi_{t_o, s|k, m}[\text{Id}, 0, \nu]_{k, m}[M, 0, 0](\tau, z).$$

If  $t_o \in (2m)^{-1}\mathbb{Z}$ , say  $t_o = \beta/2m$  for some  $\beta \in \mathbb{Z}$ , then the previous expression and the identity  $\phi_{t_o, s}(\tau, z) = \phi_{0, s|k, m}[\text{Id}, 0, -\beta/2m, 1](\tau, z)$  yield

$$\Omega_{t_o, s}^{k, m}(\tau, z) = \sum_{M \in \Gamma} \sum_{\nu \in \mathbb{Z}} \phi_{0, s|k, m}[\text{Id}, 0, \nu]_{k, m} \left[ \text{Id}, 0, \frac{-\beta}{2m}, 1 \right]_{k, m} [M, 0, 0](\tau, z). \quad (4.3)$$

On the other hand, the inversion formula for the classical theta function is

$$\sum_{l \in \mathbb{Z}} e\left(\frac{-(z+l)^2}{\tau}\right) = \left(\frac{\tau}{2i}\right)^{1/2} \sum_{r \in \mathbb{Z}} e\left(\frac{r^2}{4}\tau + rz\right),$$

for all  $\tau \in \mathcal{H}$ ,  $z \in \mathbb{C}$ . This equation (with  $\tau/m$  instead of  $\tau$ ) together with (3.2) imply

$$\sum_{\nu \in \mathbb{Z}} \phi_{0, s|k, m}[\text{Id}, 0, \nu](\tau, z) = \frac{1}{\sqrt{2mi}} \frac{1}{\tau^{s-1/2}} \sum_{r_0=1}^{2m} \Theta_{m, r_0}(\tau, z). \quad (4.4)$$

Next we use (4.3), (4.4) and

$$\Theta_{m, r_0}(\tau, z - \beta/2m) = e(-r_0\beta/2m) \Theta_{m, r_0}(\tau, z),$$

in order to get

$$\Omega_{\beta/2m, s}^{k, m}(\tau, z) = \frac{1}{\sqrt{2mi}} \sum_{M \in \Gamma} \left( \frac{1}{\tau^{s-1/2}} \sum_{r_0=1}^{2m} e\left(\frac{-r_0\beta}{2m}\right) \Theta_{m, r_0}(\tau, z) \right) \Big|_{k, m} [M, 0, 0]. \quad (4.5)$$

This is the desired analogue of [20, p. 184, Equation (15)].

*Proof of Theorem 1.1.* Let  $k$  and  $m$  be as in the statement of the theorem and  $t_o = \beta/2m$  with  $\beta \in \mathbb{Z}$ . By Proposition 4.1, we know that  $\Omega_{\beta/2m,s}^{k,m} \in J_{k,m}^{\text{cusp}}$  if  $1 < \Re(s) < k - 3$ .

Using next the definition of the Petersson inner product, equation (4.5) and the invariance of  $f$  under the action of  $\Gamma^J$ , we deduce

$$\begin{aligned} \langle \Omega_{t_o,s}^{k,m}, f \rangle &= \int_{\Gamma^J \backslash \mathcal{H} \times \mathbb{C}} \Omega_{t_o,s}^{k,m}(\tau, z) \overline{f(\tau, z)} \mu_{k,m}(\tau, z)^2 dV \\ &= \frac{1}{\sqrt{2mi}} \int_{\Gamma^J \backslash \mathcal{H} \times \mathbb{C}} \sum_{M \in \Gamma} \left( \frac{1}{\tau^{s-1/2}} \sum_{r_0=1}^{2m} e\left(\frac{-r_0\beta}{2m}\right) \Theta_{m,r_0}(\tau, z) \right) \Big|_{k,m} [M, 0, 0] \\ &\quad \times \overline{f(\tau, z)}|_{k,m} [M, 0, 0] \mu_{k,m}(\tau, z)^2 dV. \end{aligned}$$

We can simplify the previous integral if we consider the transformation formula for  $\mu_{k,m}$  given in (2.5). Namely, if we put  $\tau_M := (a\tau + b)(c\tau + d)^{-1}$  and  $z_M := z(c\tau + d)^{-1}$  for every  $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ , then

$$\begin{aligned} \sqrt{2mi} \langle \Omega_{t_o,s}^{k,m}, f \rangle &= \int_{\Gamma^J \backslash \mathcal{H} \times \mathbb{C}} \sum_{M \in \Gamma} \overline{f(\tau_M, z_M)} \left( \frac{1}{(\tau_M)^{s-1/2}} \sum_{r_0=1}^{2m} e\left(\frac{-r_0\beta}{2m}\right) \Theta_{m,r_0}(\tau_M, z_M) \right) \mu_{k,m}(\tau_M, z_M)^2 dV \\ &= \int_{\mathcal{H}} \int_{(\mathbb{Z}\tau + \mathbb{Z}) \backslash \mathbb{C}} \left( \frac{1}{\tau^{s-1/2}} \sum_{r_0=1}^{2m} e\left(\frac{-r_0\beta}{2m}\right) \Theta_{m,r_0}(\tau, z) \right) \overline{f(\tau, z)} \mu_{k,m}(\tau, z)^2 dV. \end{aligned}$$

The theta decomposition (2.3) of the cusp form  $f$  plus the relation

$$\int_{(\mathbb{Z}\tau + \mathbb{Z}) \backslash \mathbb{C}} \Theta_{m,\mu}(\tau, z) \overline{\Theta_{m,r_0}(\tau, z)} e^m(2p^2iy) dp dq = \begin{cases} \sqrt{\frac{1}{4my}} & \text{if } r_0 = \mu, \\ 0 & \text{otherwise} \end{cases} \quad (4.6)$$

yield

$$\sqrt{2mi} \langle \Omega_{t_o,s}^{k,m}, f \rangle = \frac{1}{2\sqrt{m}} \sum_{\mu=1}^{2m} e\left(\frac{-\mu\beta}{2m}\right) \int_{\mathcal{H}} \frac{1}{\tau^{s-1/2}} \overline{f_\mu(\tau)} y^{k-5/2} dx dy. \quad (4.7)$$

Now we compute the integral over  $\mathcal{H}$  in (4.7). Since  $f_\mu(\tau + 1) = e(-\mu^2/4m)f_\mu(\tau)$  for all  $\mu$ , we have

$$\begin{aligned} &\int_{\mathcal{H}} \frac{1}{\tau^{s-1/2}} \overline{f_\mu(\tau)} y^{k-5/2} dx dy \\ &= \int_{y=0}^{\infty} \int_{x=0}^1 \sum_{n \in \mathbb{Z}} \frac{1}{(\tau + n)^{s-1/2}} e\left(\frac{n\mu^2}{4m}\right) \overline{f_\mu(\tau)} y^{k-5/2} dx dy \\ &= \sum_{n_0(4m)} e\left(\frac{n_0\mu^2}{4m}\right) \int_{y=0}^{\infty} \int_{x=0}^1 \zeta_{4m\mathbb{Z}}\left(\tau + n_0, s - \frac{1}{2}\right) \overline{f_\mu(\tau)} y^{k-5/2} dx dy, \end{aligned} \quad (4.8)$$

where  $\zeta_{4m\mathbb{Z}}(\tau, s) := \sum_{l \in \mathbb{Z}} (\tau + 4ml)^{-s}$ . This series is a variation of the symmetrized Hurwitz zeta function  $\zeta_{\mathbb{Z}}(\tau, s) := \sum_{l \in \mathbb{Z}} (\tau + l)^{-s}$ , defined for any complex numbers  $\tau, s$  such that  $\tau \in \mathcal{H}$  and  $\Re(s) > 1$ . Indeed,

$$\zeta_{4m\mathbb{Z}}(\tau, s) = \frac{1}{(4m)^s} \zeta_{\mathbb{Z}}\left(\frac{\tau}{4m}, s\right). \quad (4.9)$$

Using Lipschitz summation, one can show

$$\zeta_{\mathbb{Z}}(\tau, s) = \frac{(2\pi)^s}{e^{\pi is/2} \Gamma(s)} \sum_{n=1}^{\infty} n^{s-1} e(n\tau) \quad (4.10)$$

(see, for example, [13, p. 1916]). This Fourier series representation yields that  $\zeta_{\mathbb{Z}}(\tau, s)$ , and therefore  $\zeta_{4m\mathbb{Z}}(\tau, s)$ , extends to an analytic function to the whole complex  $s$ -plane and

$$\zeta_{4m\mathbb{Z}}\left(\tau + n_0, s - \frac{1}{2}\right) = \frac{(2\pi)^{s-1/2} e^{\pi i/4}}{4m e^{\pi i s/2} \Gamma(s-1/2)} \sum_{l=1}^{\infty} \left(\frac{l}{4m}\right)^{s-3/2} e\left(\frac{l(\tau + n_0)}{4m}\right). \quad (4.11)$$

We recall that our goal is to compute the integral in (4.7), or equivalently, the double integral in the right-hand side of (4.8). We do so using the series representations (2.3) and (4.11), but first we need to establish a strip on the  $s$ -plane where the interchange of integration and summation is valid.

Let  $\sigma = \Re(s)$ . Using the estimate (2.4) for the Fourier coefficients  $c_{\mu}(D)$ , we have

$$\begin{aligned} & \sum_{l=1}^{\infty} \sum_{D=1}^{\infty} \int_{y=0}^{\infty} \int_{x=0}^1 \left| \left(\frac{l}{4m}\right)^{s-3/2} e\left(\frac{l(\tau + n_0)}{4m}\right) \overline{c_{\mu}(D)} e\left(\frac{D}{4m}\tau\right) \right| y^{k-5/2} dx dy \\ &= \frac{1}{(4m)^{\sigma-3/2}} \sum_{l=1}^{\infty} \sum_{D=1}^{\infty} l^{\sigma-3/2} \int_{y=0}^{\infty} \int_{x=0}^1 e^{-\pi l y/2m} |c_{\mu}(D)| e^{-\pi D y/2m} y^{k-5/2} dx dy \\ &\ll \frac{1}{(4m)^{\sigma-3/2}} \sum_{l=1}^{\infty} \sum_{D=1}^{\infty} l^{\sigma-3/2} D^{k/2} \int_{y=0}^{\infty} e^{-\pi(l+D)y/2m} y^{k-5/2} dy. \end{aligned}$$

The last integral is essentially Euler's gamma function, thus the expression above is equal to

$$\begin{aligned} & \frac{\Gamma(k-3/2)}{(4m)^{\sigma-3/2}} \left(\frac{2m}{\pi}\right)^{k-3/2} \sum_{l=1}^{\infty} \sum_{D=1}^{\infty} l^{\sigma-3/2} D^{k/2} (l+D)^{-k+3/2} \\ &= \frac{\Gamma(k-3/2)}{(4m)^{\sigma-3/2}} \left(\frac{2m}{\pi}\right)^{k-3/2} \sum_{n=2}^{\infty} n^{-k+3/2} \sum_{l=1}^{n-1} l^{\sigma-3/2} (n-l)^{k/2}. \end{aligned}$$

A straightforward comparison of the last double sum with Riemann's zeta function yields that the former is finite whenever  $\frac{3}{2} < \sigma < \frac{k}{2} - 2$ . This fact and Lebesgue's dominated convergence theorem allow us to interchange integration and summation in the expression below. Indeed, if we use the series representations (2.3) and (4.11) in the double integral (4.8), then we get

$$\begin{aligned} \int_{x=0}^1 \zeta_{4m\mathbb{Z}}\left(\tau + n_0, s - \frac{1}{2}\right) \overline{f_{\mu}(\tau)} dx &= \frac{(2\pi)^{s-1/2} e^{\pi i/4}}{4m e^{\pi i s/2} \Gamma(s-1/2)} \sum_{l=1}^{\infty} \left(\frac{l}{4m}\right)^{s-3/2} \\ &\quad \times e\left(\frac{ln_0}{4m}\right) \sum_{D=1}^{\infty} \overline{c_{\mu}(D)} \int_{x=0}^1 e\left(\frac{l\tau - D\bar{\tau}}{4m}\right) dx \\ &= \frac{(2\pi)^{s-1/2} e^{\pi i/4}}{4m e^{\pi i s/2} \Gamma(s-1/2)} \sum_{D=1}^{\infty} \overline{c_{\mu}(D)} \left(\frac{D}{4m}\right)^{s-3/2} \\ &\quad \times e\left(\frac{Dn_0}{4m}\right) e\left(\frac{D}{2m}iy\right). \end{aligned}$$

This equation and (4.8) yield

$$\begin{aligned}
& \int_{\mathcal{H}} \frac{1}{\tau^{s-1/2}} \overline{f_{\mu}(\tau)} y^{k-5/2} dx dy \\
&= \frac{(2\pi)^{s-1/2} e^{\pi i/4}}{4m e^{\pi i s/2} \Gamma(s-1/2)} \sum_{n_0(4m)} e\left(\frac{n_0 \mu^2}{4m}\right) \\
&\quad \times \sum_{D=1}^{\infty} \overline{c_{\mu}(D)} \left(\frac{D}{4m}\right)^{s-3/2} e\left(\frac{D n_0}{4m}\right) \int_{y=0}^{\infty} e\left(\frac{D}{2m} i y\right) y^{k-5/2} dy \\
&= \frac{(2\pi)^{s-1/2} e^{\pi i/4}}{4m e^{\pi i s/2} \Gamma(s-1/2)} (4\pi)^{-k+3/2} \Gamma\left(k - \frac{3}{2}\right) \\
&\quad \times \sum_{n_0(4m)} \sum_{D=1}^{\infty} \overline{c_{\mu}(D)} e\left(\frac{D + \mu^2}{4m} n_0\right) \left(\frac{D}{4m}\right)^{s-k}.
\end{aligned}$$

Since  $c_{\mu}(D) = 0$  whenever  $D + \mu^2$  is not divisible by  $4m$ , we obtain

$$\int_{\mathcal{H}} \frac{1}{\tau^{s-1/2}} \overline{f_{\mu}(\tau)} y^{k-5/2} dx dy = \frac{(2\pi)^{s-k+1} e^{\pi i/4} \Gamma(k-3/2)}{2^{k-3/2} e^{\pi i s/2} \Gamma(s-1/2)} \sum_{D=1}^{\infty} \overline{c_{\mu}(D)} \left(\frac{D}{4m}\right)^{s-k}. \quad (4.12)$$

Consequently, from equations (4.7) and (4.12), one gets

$$2m \langle \Omega_{t_o, s}^{k, m}, f \rangle = \frac{(2\pi)^{s-k+1} \Gamma(k-3/2)}{2^{k-1} e^{\pi i s/2} \Gamma(s-1/2)} \sum_{\mu=1}^{2m} e\left(\frac{-\mu\beta}{2m}\right) \sum_{D=1}^{\infty} \overline{c_{\mu}(D)} \left(\frac{D}{4m}\right)^{s-k},$$

for all  $s \in \mathbb{C}$  with  $\frac{3}{2} < \Re(s) < \frac{k}{2} - 2$ .

This identity and the definition of  $\Lambda_{\mu}(f; s)$  yield the formula in Theorem 1.1.  $\square$

## 5. Proofs of Propositions 1.2 and 1.3

*Proof of Proposition 1.2.* Let  $t_o = \beta/2m$  with  $\beta \in \mathbb{Z}$  and  $\Gamma_{\infty}$  be the stabilizer of infinity in  $\Gamma$ . Then  $\Gamma_{\infty} := \{\pm \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix} \mid l \in \mathbb{Z}\}$ , and (4.5) can be written as

$$\begin{aligned}
& \sqrt{2mi} \Omega_{\beta/2m, s}^{k, m}(\tau, z) \\
&= \sum_{M \in \Gamma_{\infty} \setminus \Gamma} \left( \sum_{M' \in \Gamma_{\infty}} \left( \frac{1}{\tau^{s-1/2}} \sum_{\mu=1}^{2m} e\left(\frac{-\mu\beta}{2m}\right) \Theta_{m, \mu}(\tau, z) \right) \Big|_{k, m} [M', 0, 0] \right) \Big|_{k, m} [M, 0, 0].
\end{aligned}$$

Now

$$\begin{aligned}
& \sum_{M' \in \Gamma_{\infty}} \left( \frac{1}{\tau^{s-1/2}} \sum_{\mu=1}^{2m} e\left(\frac{-\mu\beta}{2m}\right) \Theta_{m, \mu}(\tau, z) \right) \Big|_{k, m} [M', 0, 0] \\
&= \sum_{l \in \mathbb{Z}} \frac{1}{(\tau+l)^{s-1/2}} \sum_{\mu=1}^{2m} \left( e\left(\frac{-\mu\beta}{2m}\right) + (-1)^k e\left(\frac{\mu\beta}{2m}\right) \right) \Theta_{m, \mu}(\tau+l, z),
\end{aligned}$$



since  $-\text{Id} \in \Gamma_\infty$  and  $\Theta_{m,\mu}(\tau, -z) = \Theta_{m,-\mu}(\tau, z)$ . Thus,

$$\begin{aligned} & \sum_{M' \in \Gamma_\infty} \left( \frac{1}{\tau^{s-1/2}} \sum_{\mu=1}^{2m} e\left(\frac{-\mu\beta}{2m}\right) \Theta_{m,\mu}(\tau, z) \right) \Big|_{k,m} [M', 0, 0] \\ &= \sum_{\mu=1}^{2m} \left( e\left(\frac{-\mu\beta}{2m}\right) + (-1)^k e\left(\frac{\mu\beta}{2m}\right) \right) \Theta_{m,\mu}(\tau, z) \sum_{l_0=1}^{4m} \sum_{l \in \mathbb{Z}} \frac{e(\mu^2 l_0 / 4m)}{(\tau + l_0 + 4ml)^{s-1/2}} \\ &= \sum_{l_0=1}^{4m} \sum_{\mu=1}^{2m} \left( e\left(\frac{\mu^2 l_0 - 2\mu\beta}{4m}\right) + (-1)^k e\left(\frac{\mu^2 l_0 + 2\mu\beta}{4m}\right) \right) \Theta_{m,\mu}(\tau, z) \zeta_{4m\mathbb{Z}}\left(\tau + l_0, s - \frac{1}{2}\right). \end{aligned}$$

Recalling the Fourier expansion of  $\zeta_{4m\mathbb{Z}}(\tau + l_0, s - \frac{1}{2})$  given in (4.11), we obtain

$$\begin{aligned} & \sqrt{2mi} \Omega_{\beta/2m,s}^{k,m}(\tau, z) \\ &= \frac{(2\pi)^{s-1/2} e^{\pi i/4}}{4m e^{\pi i s/2} \Gamma(s-1/2)} \sum_{l_0=1}^{4m} \sum_{\mu=1}^{2m} \left( e\left(\frac{\mu^2 l_0 - 2\mu\beta}{4m}\right) + (-1)^k e\left(\frac{\mu^2 l_0 + 2\mu\beta}{4m}\right) \right) \\ & \quad \times \sum_{D=1}^{\infty} \sum_{M \in \Gamma_\infty \setminus \Gamma} \left( \left(\frac{D}{4m}\right)^{s-3/2} \Theta_{m,\mu}(\tau, z) e\left(\frac{D(\tau + l_0)}{4m}\right) \right) \Big|_{k,m} [M, 0, 0]. \end{aligned}$$

Hence,

$$\begin{aligned} & \sqrt{2mi} \Omega_{\beta/2m,s}^{k,m}(\tau, z) \\ &= \frac{(2\pi)^{s-1/2} e^{\pi i/4}}{4m e^{\pi i s/2} \Gamma(s-1/2)} \sum_{l_0=1}^{4m} \sum_{\mu=1}^{2m} \left( e\left(\frac{\mu^2 l_0 - 2\mu\beta}{4m}\right) + (-1)^k e\left(\frac{\mu^2 l_0 + 2\mu\beta}{4m}\right) \right) \\ & \quad \times \sum_{D=1}^{\infty} \left(\frac{D}{4m}\right)^{s-3/2} e\left(\frac{Dl_0}{4m}\right) \sum_{\substack{M \in \Gamma_\infty \setminus \Gamma \\ r \in \mathbb{Z} \\ r \equiv \mu(2m)}} e\left(\frac{(r^2 + D)\tau + rz}{4m}\right) \Big|_{k,m} [M, 0, 0]. \end{aligned}$$

Now, for each  $D \geq 1$ ,  $1 \leq l_0 \leq 4m$  and  $1 \leq \mu \leq 2m$ , one has

$$\begin{aligned} & \sum_{\substack{M \in \Gamma_\infty \setminus \Gamma \\ r \in \mathbb{Z} \\ r \equiv \mu(2m)}} e\left(\frac{(r^2 + D)\tau + rz}{4m}\right) \Big|_{k,m} [M, 0, 0] \\ &= \frac{1}{2} \sum_{\substack{c,d,r \in \mathbb{Z} \\ (c,d)=1 \\ r \equiv \mu(2m)}} (c\tau + d)^{-k} e^m \left(\frac{-cz^2}{c\tau + d}\right) e\left(\frac{(r^2 + D)a\tau + b}{4m} \frac{a\tau + b}{c\tau + d} + r \frac{z}{c\tau + d}\right), \end{aligned}$$

where  $a, b$  are integers chosen such that  $ad - bc = 1$  in each term. Consequently,

$$\begin{aligned} & \sum_{\substack{M \in \Gamma_\infty \setminus \Gamma \\ r \in \mathbb{Z} \\ r \equiv \mu(2m)}} e\left(\frac{(r^2 + D)\tau + rz}{4m}\right) \Big|_{k,m} [M, 0, 0] = \frac{1}{2} \sum_{\substack{c,d,r \in \mathbb{Z} \\ (c,d)=1}} (c\tau + d)^{-k} \\ & \quad \times e^m \left(\frac{-cz^2}{c\tau + d} + r^2 \frac{a\tau + b}{c\tau + d} + 2r \frac{z}{c\tau + d}\right) e\left(\frac{\mu^2 + D}{4m} \frac{a\tau + b}{c\tau + d}\right) \\ & \quad \times e^\mu \left(\frac{z}{c\tau + d} + r \frac{a\tau + b}{c\tau + d}\right), \end{aligned} \tag{5.1}$$

after changing the variable  $r$  by  $\mu + 2mr$ . Using next that the Poincaré series

$$P_{k,m,(n,r)}(\tau, z) := \sum_{h \in \Gamma_\infty^J \setminus \Gamma^J} e(n\tau + rz)|_{k,m}[h]$$

is exactly the series above with the parameter  $n = (D + \mu^2)/4m$  (see, for example, [9, p. 520]), we conclude

$$\sum_{\substack{M \in \Gamma_\infty \setminus \Gamma \\ r \in \mathbb{Z} \\ r \equiv \mu(2m)}} e\left(\frac{(r^2 + D)}{4m}\tau + rz\right)\Big|_{k,m} [M, 0, 0] = \frac{1}{2} P_{k,m,((D+\mu^2)/4m,\mu)}(\tau, z).$$

Hence, the previous computations yield

$$\begin{aligned} \sqrt{2mi} \Omega_{\beta/2m,s}^{k,m}(\tau, z) &= \frac{(2\pi)^{s-1/2} e^{\pi i/4}}{8m e^{\pi i s/2} \Gamma(s-1/2)} \sum_{D=1}^{\infty} \left(\frac{D}{4m}\right)^{s-3/2} \sum_{l_0=1}^{4m} \sum_{\mu=1}^{2m} e\left(\frac{D + \mu^2}{4m} l_0\right) \\ &\quad \times \left( e\left(\frac{-\mu\beta}{2m}\right) + (-1)^k e\left(\frac{\mu\beta}{2m}\right) \right) P_{k,m,((D+\mu^2)/4m,\mu)}(\tau, z). \end{aligned}$$

From this equation and

$$\sum_{l_0=1}^{4m} e\left(\frac{D + \mu^2}{4m} l_0\right) = \begin{cases} 4m & \text{if } \frac{D + \mu^2}{4m} \in \mathbb{Z}, \\ 0 & \text{otherwise,} \end{cases}$$

one gets

$$\begin{aligned} \sqrt{2mi} \Omega_{\beta/2m,s}^{k,m}(\tau, z) &= \frac{(2\pi)^{s-1/2} e^{\pi i/4}}{2 e^{\pi i s/2} \Gamma(s-1/2)} \sum_{\mu=1}^{2m} \left( e\left(\frac{-\mu\beta}{2m}\right) + (-1)^k e\left(\frac{\mu\beta}{2m}\right) \right) \\ &\quad \times \sum_{D=1}^{\infty} \left(\frac{D}{4m}\right)^{s-3/2} P_{k,m,((D+\mu^2)/4m,\mu)}(\tau, z). \end{aligned}$$

All Poincaré series in the expression above are Jacobi forms in  $J_{k,m}^{\text{cusp}}$ , hence they satisfy the relation  $(-1)^k P_{k,m,(n,\mu)}(\tau, z) = P_{k,m,(n,\mu)}(\tau, -z)$ . On the other hand,  $P_{k,m,(n,\mu)}(\tau, -z) = P_{k,m,(n,-\mu)}(\tau, z)$  as we can see from the right-hand side of (5.1). These two changes in the last identity involving  $\Omega_{\beta/2m,s}^{k,m}(\tau, z)$  complete the proof.  $\square$

One application of our main result is Proposition 1.3, which we prove after we establish the following lemma.

LEMMA 5.1. *Let  $k > 1$  and  $f$  be a Jacobi form in  $J_{k,m}^{\text{cusp}}$ . Then  $\langle \Omega_{t_0,s}^{k,m}, f \rangle$  is a holomorphic function of  $s$  on the vertical strip  $\frac{3}{2} < \Re(s) < k - 3$ .*

*Proof.* Halfway in the proof of Theorem 1.1, we obtained the integral representation (4.7) of  $\langle \Omega_{t_0,s}^{k,m}, f \rangle$ . Such an expression is written as a finite sum of integrals in (4.8). Hence, it suffices to show that, for any integer  $n_0$ , the integral

$$\int_{y=0}^{\infty} \int_{x=0}^1 \zeta_{4m\mathbb{Z}}\left(\tau + n_0, s - \frac{1}{2}\right) \overline{f_\mu(\tau)} y^{k-5/2} dx dy \quad (5.2)$$

defines a holomorphic function of  $s$  on the region of the plane specified in the lemma.

In order to see this, we recall three simple facts.

(i) The symmetrized Hurwitz zeta function  $\zeta_{\mathbb{Z}}(\tau, s)$ , and therefore  $\zeta_{4m\mathbb{Z}}(\tau, s)$ , is holomorphic on the half-plane  $\Re(s) > 1$ .

(ii) One has  $f_\mu(\tau) = O(e^{-\pi y/2m})$  as  $y \rightarrow \infty$ , uniformly on  $x$  (as one can deduce from the Fourier expansion (2.3) of  $f_\mu$  and the polynomial growth (2.4) of its coefficients).

(iii) If  $\sigma = \Re(s) > 1$ , then

$$\zeta_{\mathbb{Z}}(\tau, s) \ll e^{-2\pi y}(1 + y^{-\sigma}),$$

where the implied constant depends continuously on  $s$ . (This is something one can obtain from the Fourier representation (4.10).) Consequently, this estimate and (4.9) yield

$$\begin{aligned} \zeta_{4m\mathbb{Z}}\left(\tau + n_0, s - \frac{1}{2}\right) &\ll e^{-\pi y/2m} \left(1 + \left(\frac{y}{4m}\right)^{-\sigma+1/2}\right) \\ &\ll \frac{e^{-\pi y/2m}}{(4m)^{-\sigma+1/2}} (1 + y^{-\sigma+1/2}) \end{aligned}$$

provide that  $\sigma = \Re(s) > \frac{3}{2}$ .

Now we use (ii) and (iii) in order to get the inequalities

$$\begin{aligned} &\int_{y=0}^{\infty} \int_{x=0}^1 \left| \zeta_{4m\mathbb{Z}}\left(\tau + n_0, s - \frac{1}{2}\right) \overline{f_\mu(\tau)} y^{k-5/2} \right| dx dy \\ &\ll \int_{y=0}^{\infty} e^{-\pi y/m} (y^{k-5/2} + y^{k-2-\sigma}) dy \\ &\ll \left(\frac{\pi}{m}\right)^{-k+3/2} \Gamma\left(k - \frac{3}{2}\right) + \left(\frac{\pi}{m}\right)^{-k+1+\sigma} \Gamma(k-1-\sigma). \end{aligned}$$

Note that the implied constants above depend continuously on  $s$ , and that the hypotheses  $k > 1$  and  $\Re(s) < k-3$  are used in the last step. From this relation, one deduces that the integral (5.2) is, as a function of  $s$ , absolute and uniformly convergent on any compact subset of the vertical strip  $\frac{3}{2} < \Re(s) < k-3$ .

At this point, we recall a standard result in complex analysis (see [17, p. 392], for example) in order to conclude from the previous statement and the holomorphicity of  $\zeta_{4m\mathbb{Z}}(\tau + n_0, s - \frac{1}{2})$  on  $\Re(s) > \frac{3}{2}$  that (5.2) defines a holomorphic function of  $s$ .  $\square$

*Proof of Proposition 1.3.* Let  $t_o = \beta/2m$  with  $\beta \in \mathbb{Z}$ . From Proposition 4.1, we know that  $\Omega_{t_o, s}^{k, m}$  is a Jacobi cusp form in  $J_{k, m}^{\text{cusp}}$  for  $1 < \Re(s) < k-3$ . This fact and equation (4.5) yield

$$\begin{aligned} &\sqrt{2mi} \Omega_{t_o, s}^{k, m}(\tau, z) \\ &= \sqrt{2mi} \Omega_{t_o, s}^{k, m} \Big|_{k, m} \left[ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, 0, 0 \right] (\tau, z) \\ &= \sum_{M \in \Gamma} \left( \frac{1}{\tau^{s-1/2}} \sum_{r_0=1}^{2m} e\left(\frac{-r_0\beta}{2m}\right) \Theta_{m, r_0}(\tau, z) \right) \Big|_{k, m} \left[ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, 0, 0 \right] [M, 0, 0]. \end{aligned}$$

Using the transformation formula of  $\Theta_{m, r_0}(\tau, z)$  under the action of  $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$  (see [4, p. 59], for example), we get

$$\begin{aligned} &\left( \frac{1}{\tau^{s-1/2}} \sum_{r_0=1}^{2m} e\left(\frac{-r_0\beta}{2m}\right) \Theta_{m, r_0}(\tau, z) \right) \Big|_{k, m} \left[ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, 0, 0 \right] \\ &= \frac{\tau^{-k}}{(-1/\tau)^{s-1/2}} \sum_{r_0=1}^{2m} e\left(\frac{-r_0\beta}{2m}\right) \sqrt{\frac{\tau}{2mi}} \sum_{l_0=1}^{2m} e\left(\frac{-r_0 l_0}{2m}\right) \Theta_{m, l_0}(\tau, z). \end{aligned} \quad (5.3)$$

This expression can be simplified a bit. Since

$$\sqrt{\frac{\tau}{2mi}} = \sqrt{\frac{-i\tau}{2m}} = e^{-\pi i/4} \frac{\sqrt{\tau}}{\sqrt{2m}}$$

and

$$\log\left(\frac{-1}{\tau}\right) = \log\left(\frac{1}{|\tau|}\right) + \log|\tau| + i\pi - \log(\tau),$$

for any  $\tau \in \mathcal{H}$  (so that  $(-1/\tau)^{s-1/2} = -i e^{\pi i s} \tau^{-s+1/2}$ ), one can write (5.3) as

$$\begin{aligned} & \frac{i e^{-\pi i s} e^{-\pi i/4}}{\sqrt{2m} \tau^{k-s}} \sum_{r_0=1}^{2m} \sum_{l_0=1}^{2m} e\left(\frac{-r_0(\beta + l_0)}{2m}\right) \Theta_{m, l_0}(\tau, z) \\ &= \frac{i e^{-\pi i s} e^{-\pi i/4}}{\sqrt{2m} \tau^{k-s}} 2m \Theta_{m, -\beta}(\tau, z) = \sqrt{2mi} e^{-\pi i s} \frac{1}{\tau^{k-s}} \Theta_{m, -\beta}(\tau, z). \end{aligned}$$

This last set of identities yields

$$\Omega_{\beta/2m, s}^{k, m}(\tau, z) = e^{-\pi i s} \sum_{M \in \Gamma} \left( \frac{1}{\tau^{k-s}} \Theta_{m, -\beta}(\tau, z) \right) \Big|_{k, m} [M, 0, 0]. \quad (5.4)$$

The right-hand side of (5.4) is a sub-series of  $\sqrt{2mi} \Omega_{0/2m, k-s+1/2}^{k, m}(\tau, z)$  by (4.5). Hence, by Proposition 3.4, it is an absolute and uniformly convergent series on any compact subset of  $\mathcal{H} \times \mathbb{C}$ , provided that  $1 < k - \Re(s) + \frac{1}{2} < k - 3$  (or equivalently,  $\frac{7}{2} < \Re(s) < k - \frac{1}{2}$ ). Consequently, (5.4) is an identity of analytic functions on  $\mathcal{H} \times \mathbb{C}$  whenever  $\frac{7}{2} < \Re(s) < k - 3$ .

Next we compute  $\langle \Omega_{t_o, s}^{k, m}, f \rangle$  in the case  $k > 6$  and  $\frac{7}{2} < \Re(s) < k - 3$ .

By (5.4), we have

$$\begin{aligned} \langle \Omega_{t_o, s}^{k, m}, f \rangle &= \int_{\Gamma^J \setminus \mathcal{H} \times \mathbb{C}} \Omega_{t_o, s}^{k, m}(\tau, z) \overline{f(\tau, z)} \mu_{k, m}(\tau, z)^2 dV \\ &= \int_{\Gamma^J \setminus \mathcal{H} \times \mathbb{C}} \sum_{M \in \Gamma} \left( \frac{e^{-\pi i s}}{\tau^{k-s}} \Theta_{m, -\beta}(\tau, z) \right) \Big|_{k, m} [M, 0, 0] \overline{f(\tau, z)} \Big|_{k, m} [M, 0, 0] \mu_{k, m}(\tau, z)^2 dV. \end{aligned}$$

Now we apply (2.5) as in the proof of Theorem 1.1 and, using the same notation introduced there, get

$$\begin{aligned} \langle \Omega_{t_o, s}^{k, m}, f \rangle &= \int_{\Gamma^J \setminus \mathcal{H} \times \mathbb{C}} \sum_{M \in \Gamma} \left( \frac{e^{-\pi i s}}{\tau_M^{k-s}} \Theta_{m, -\beta}(\tau_M, z_M) \right) \overline{f(\tau_M, z_M)} \mu_{k, m}(\tau_M, z_M)^2 dV \\ &= \int_{\mathcal{H}} \int_{(\mathbb{Z}\tau + \mathbb{Z}) \setminus \mathbb{C}} \left( \frac{e^{-\pi i s}}{\tau^{k-s}} \Theta_{m, -\beta}(\tau, z) \right) \overline{f(\tau, z)} \mu_{k, m}(\tau, z)^2 dV. \end{aligned}$$

We can simplify this expression if we recall the theta decomposition (2.3) of  $f$  and use the orthogonality relation (4.6). Namely,

$$\langle \Omega_{t_o, s}^{k, m}, f \rangle = \frac{e^{-\pi i s}}{2\sqrt{m}} \int_{\mathcal{H}} \frac{1}{\tau^{k-s}} \overline{f_{-\beta}(\tau)} y^{k-5/2} dx dy. \quad (5.5)$$

The last integral, with  $s - \frac{1}{2}$  instead of  $k - s$ , was already computed in the proof of Theorem 1.1. Therefore, from (4.12) and (5.5), we obtain

$$\begin{aligned} \langle \Omega_{t_o, s}^{k, m}, f \rangle &= \frac{1}{\sqrt{m}} \frac{(2\pi)^{-s+3/2} \Gamma(k-3/2)}{2^{k-1/2} e^{\pi i k/2} e^{\pi i s/2} \Gamma(k-s)} \sum_{D=1}^{\infty} \overline{c_{-\beta}(D)} \left( \frac{D}{4m} \right)^{-s+1/2} \\ &= \frac{\pi e^{-\pi i k/2}}{2^{k-3/2} \sqrt{m} e^{\pi i s/2}} \frac{\Gamma(k-3/2)}{\Gamma(k-s) \Gamma(s-1/2)} \Lambda_{-\beta} \left( \bar{f}, s - \frac{1}{2} \right), \end{aligned} \quad (5.6)$$

for all  $s \in \mathbb{C}$  with  $\frac{3}{2} < \Re(k - s - \frac{1}{2}) < \frac{k}{2} - 2$ , or equivalently  $(k+3)/2 < \Re(s) < k - 1$ .

By (2.4), we know that  $\Lambda_{-\beta}(\bar{f}, s - \frac{1}{2})$  defines a holomorphic function on the half-plane  $\Re(s) > (k+3)/2$ . By the same reason  $\Lambda_{\mu}(\bar{f}, k-s)$ , for  $1 \leq \mu \leq 2m$ , defines a holomorphic function on the half-plane  $k/2 - 1 > \Re(s)$ .

Then, the identities in Theorem 1.1 and (5.6) plus the holomorphicity of  $\langle \Omega_{t\sigma, s}^{k, m}, f \rangle$  proved in Lemma 5.1, yield the analytic continuation of  $\Lambda_{\mu}(\bar{f}, s)$  ( $1 \leq \mu \leq 2m$ ) and the functional equations

$$\Lambda_{\beta} \left( \bar{f}, s - \frac{1}{2} \right) = \frac{i^k}{\sqrt{2m}} \sum_{\mu=1}^{2m} e \left( \frac{\mu\beta}{2m} \right) \Lambda_{\mu}(\bar{f}, k-s) \quad (1 \leq \beta \leq 2m)$$

provide that

$$\frac{3}{2} < \frac{k}{2} - 2 \quad \text{and} \quad \frac{k+3}{2} < k-3.$$

This is the case when  $k > 9$ . □

REMARK 5.2. The analytic continuation of the series  $\Lambda_{\mu}(f, s)$  established in Proposition 1.3 together with Lemma 5.1 and Theorem 1.1, allow us to replace the condition  $\frac{3}{2} < \Re(s) < \frac{k}{2} - 2$  in our theorem by the less restrictive set of inequalities  $\frac{3}{2} < \Re(s) < k-3$ .

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