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Computing quaternion quotient graphs via representations of orders

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ABSTRACT

We give a method to describe the quotient of the local Bruhat–Tits tree T_P for $\mathrm{PGL}_2(K)$, where K is a global function field, by certain subgroups of $\mathrm{PGL}_2(K)$ of arithmetical significance. In particular, we can compute the quotient of T_P by an arithmetic subgroup $\mathrm{PGL}_2(A)$, where $A = A_P$ is the ring of functions that are regular outside P, recursively for a place P of any degree, when K is a rational function field. We achieve this by proving that the infinite matrices whose coordinates are the numbers of neighbors of a vertex in T_P corresponding to orders in a fixed isomorphism class commute for different places P, using tools from the theory of representations of orders. The latter result holds for every global function field K.

1. Introduction

In the late seventies, J.-P. Serre and H. Bass showed that the structure of a group Γ acting on a tree T can be recovered from the structure of the quotient graph $\Gamma\backslash T$ [17, Ch. I]. This theory, now known as Bass–Serre Theory, was used to find generators of certain arithmetic subgroups Γ of $\mathrm{PGL}_2(K)$. [17, Ch. II] is mostly concerned with the case $\Gamma = \Gamma_A = \mathrm{PGL}_2(A)$, for the ring $A = A_P$ of functions that are regular outside a single place P, of a smooth irreducible curve X with field of functions $K = K(X)$

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and structure sheaf $\mathcal{O}_X$. Using this method, Serre generalized Nagao's Theorem, which expresses $\mathrm{PGL}_2(\mathbb{F}[t])$, for any field $\mathbb{F}$, as a free product with amalgamation [17, Ch. II, Th. 6]. He gave the following structural result for these quotient graphs [17, Ch. II, Th. 9]:

Theorem S. *The graph $\Gamma_A \backslash T_P$, where T_P is the local Bruhat–Tits tree for the group* $\mathrm{PGL}_2(K)$ *at P, is obtained by attaching a finite number of cusps, or infinite half lines, to a certain finite graph Y. The set of such cusps is indexed by the elements in the Picard group* $\mathrm{Pic}(A) = \mathrm{Pic}(X)/\langle \bar{P} \rangle$.

Serre also determined the explicit structure of the quotient graph in some specific examples [17, §II.2.4]. The proof of Theorem **S** relies heavily on the fact that the vertices of T_P are in correspondence with certain equivalence classes of vector bundles. A.W. Mason has given a more elementary proof of these facts [8,9]. The latter author applied these graphs to the study of the lowest index non-congruence subgroup of Γ_A, in a series of joint works with A. Schweitzer [10,11]. A few additional quotient graphs are described in [12] and [16]. M. Papikian has studied the case where Γ_A is replaced by the group $\Gamma = \mathrm{PGL}_1(D)$, where D is a maximal A-order in a quaternion division algebra $\mathfrak{A}$ [13]. Note that $\Gamma = \Gamma_A$ when $\mathfrak{A} = \mathbb{M}_2(K)$ and $D = \mathbb{M}_2(A)$.

In this article we study family of quotient graphs that classify maximal X-orders on a quaternion K-algebra $\mathfrak{A}$ splitting at P. Since we use the theory of representation fields, we limit ourselves, in all that follows, to curves X defined over a finite field $\mathbb{F}$. These quotient graphs are closely related to the graph $\Gamma \backslash T_P$ studied by Serre, Mason, and Papikian. Let $G = G_P$ be the conjugation stabilizer $G = \mathrm{Stab}_{\mathfrak{A}^*}(D)$, for a maximal A-order D. Note that $\Gamma = D^* K^*/K^*$ is a normal subgroup of G, whence the group G/Γ acts on $\Gamma \backslash T_P$, and $G \backslash T_P$ is the quotient graph under this action (see Remark 1.6). We call $C_P(D) = G \backslash T_P$ the classifying graph, or C-graph of D at P, while $S_P(D) = \Gamma \backslash T_P$ is called the S-graph of D in this work. Note that $\Gamma = \mathrm{PGL}_2(A)$ when $D = \mathbb{M}_2(A)$. The S-graph can be recovered from the C-graph when X is the projective line (cf. Example 8.3).

Recall that an X-order $\mathfrak{D}$ on $\mathfrak{A}$ is a locally free sheaf of $\mathcal{O}_X$-algebras whose generic fiber is $\mathfrak{A}$ [2,6]. Such an order is completely determined by the completion $\mathfrak{D}_Q$ at every closed place $Q \in X$. Furthermore, the completion at any finite set of closed places can be modified to define a new order. In particular, an A-order can be extended to an X-order by choosing an arbitrary completion at P. An order is maximal if it is maximal at all places. It follows that the set of maximal orders $\mathfrak{D}$ with a fixed restriction $\mathfrak{D}(U) = D$ to the affine open subset $U = X \backslash \{P\}$ is in correspondence with the vertices of the local Bruhat–Tits tree T_P, and isomorphism classes of such orders are in correspondence with the vertices of $C_P(D)$. In what follows we write $C_P(\mathfrak{D})$ or $S_P(\mathfrak{D})$ instead of $C_P(D)$ or $S_P(D)$.

Recall that the set $\mathbb{O}$ of maximal X-orders in $\mathfrak{A}$ can be split into spinor genera (cf. Section 2). There exists an unramified abelian extension Σ/K of exponent 2, called the spinor class field, that classifies spinor genera via an explicit distance function

$$\rho : \mathbb{O} \times \mathbb{O} \to \mathrm{Gal}(\Sigma/K),$$

i.e., $\mathfrak{D}$ and $\mathfrak{D}'$ are in the same spinor genera if and only if $\rho(\mathfrak{D}, \mathfrak{D}') = \mathrm{Id}_\Sigma$. Spinor genera of A-orders are just isomorphism classes for any ring $A = A_S$ of S-integers, when the automorphism group of $\mathfrak{A}$ has strong approximation with respect to S. This is not the case for X-orders, where by convention $S = \emptyset$. However, spinor genera still plays an important role in the present setting. In fact, if $B \mapsto |[B, \Sigma/K]|$ denotes the Artin symbol on divisors, we have next result:

Theorem 1.1. *Let $\mathfrak{D}$ be a maximal X-order in a quaternion algebra $\mathfrak{A}$ and let P be a place splitting $\mathfrak{A}$. Let Σ be the spinor class field of maximal orders of $\mathfrak{A}$. Then, the set of vertices in the C-graph $C_P(\mathfrak{D})$ is in correspondence with the isomorphism classes in two spinor genera of maximal X-orders, if the Artin symbol $|[P, \Sigma/K]|$ is not trivial, and one spinor genera otherwise. In the former case, each C-graph is bipartite.*

It follows that the number of connected graphs that are needed to describe all isomorphism classes of maximal orders is either $[\Sigma : K]$ or $[\Sigma : K]/2$. We call the disjoint union of these graphs the full C-graph $C_P = C_P(\mathfrak{A})$. The full S-graph S_P is defined analogously.

When $\mathfrak{A} \cong \mathbb{M}_2(K)$, by a split maximal order we mean a conjugate of the sheaf:

$$\mathfrak{D}_B = \begin{pmatrix} \mathcal{O}_X & \mathfrak{L}^B \\ \mathfrak{L}^{-B} & \mathcal{O}_X \end{pmatrix},$$

where B is an arbitrary divisor on X and $\mathfrak{L}^B$ is the invertible sheaf defined by

$$\mathfrak{L}^B(U) = \{ f \in K \mid \mathrm{div}(f)|_U + B|_U \geqslant 0 \}.$$

The cusps in Serre's work are explicitly described in terms of the vector bundles corresponding to the orders $\mathfrak{D}_B$ for large enough values of $|\deg(B)|$ [17, Ch. II]. In this sense, next theorem is a partial refinement of Serre's result:

Theorem 1.2. *In the full C-graph of $\mathbb{M}_2(K)$ at P, the vertices corresponding to split maximal orders are located in a finite disjoint union of infinite lines or half-lines. The set of such lines is in correspondence with the pairs of the form $\{a, -a\}$ in the quotient group $\mathrm{Pic}(X)/\langle \bar{P} \rangle$, where $\bar{P} \in \mathrm{Pic}(X)$ denotes the class of P. A split order $\mathfrak{D}_B$ is in the line corresponding to $\{\bar{B} + \langle \bar{P} \rangle, -\bar{B} + \langle \bar{P} \rangle\}$. The half lines correspond to the elements of order 1 or 2.*

In the case of a matrix algebra, the spinor class field of maximal A-orders Σ_U is the maximal unramified exponent-2 abelian extension splitting P, and the Galois group $\mathrm{Gal}(\Sigma_U/K)$ is isomorphic to the maximal exponent-2 quotient of $\mathrm{Pic}(X)/\langle \bar{P} \rangle$ (cf. Section 2). The image of the cusp Δ_B in [17, §II.2.3] is part of the line containing the order

Fig. 1. Full C-graph for a degree 1 place in the projective line.

$\mathfrak{D}_{2B}$, so it is always in the trivial component of the C-graph (cf. Section 3). As follows from Theorem **S**, or more precisely from its natural extension to the full C-graph, the rest of the graph is finite. In Section 5 we give a general formula for the valencies of vertices in the S-graph, which allows us to compute valencies in the C-graph for all split vertices. When $X = \mathbb{P}_1$ is the projective line, the following result can be used to recover the S-graph from the C-graph (cf. Example 8.3):

Theorem 1.3. *If $X \cong \mathbb{P}_1$, $\mathfrak{A} \cong \mathbb{M}_2(K)$, and P is a place of odd degree, then C_P is isomorphic to S_P and connected. When P has even degree, there are two connected components in C_P and every vertex of C_P has exactly two pre-images in S_P.*

In particular, when $\deg(P) = 1$, then $C_P = S_P$ is as shown in Fig. 1 [17, Ex. II.2.4.1].

The multiplicity $M_P(\mathfrak{D}, \mathfrak{D}')$ of edges joining two particular vertices $\mathfrak{D}$ and $\mathfrak{D}'$ can be explicitly computed, at least for most split vertices, in terms of $N_P(\mathfrak{D}, \mathfrak{D}')$, the number of neighbors of $\mathfrak{D}$ in T_P that correspond to maximal orders isomorphic to $\mathfrak{D}'$. This is the case for all vertices when $X = \mathbb{P}_1$ and $\mathfrak{A} = \mathbb{M}_2(K)$ (cf. Section 6). In Section 7 we prove the following commutativity law:

Theorem 1.4. *Let X be a smooth curve over a finite field, and let $\mathfrak{A}$ be a quaternion algebra over $K = K(X)$. Then, for any pair of maximal orders $(\mathfrak{D}, \mathfrak{D}'')$ and any pair (P, Q) of prime divisors in X, we have*

$$\sum_{\mathfrak{D}'} N_P(\mathfrak{D}, \mathfrak{D}') N_Q(\mathfrak{D}', \mathfrak{D}'') = \sum_{\mathfrak{D}'} N_Q(\mathfrak{D}, \mathfrak{D}') N_P(\mathfrak{D}', \mathfrak{D}''),$$

where the sum extends over all isomorphism classes of maximal orders in $\mathfrak{A}$.

Note that both sums in the theorem are actually finite. In particular, when $X = \mathbb{P}_1$ and $\mathfrak{A} = \mathbb{M}_2(K)$, then C_P can be completely determined from the infinite matrix $N_P = (N_P(\mathfrak{D}_{iQ}, \mathfrak{D}_{jQ}))_{i,j \in \mathbb{N}}$, where $\deg(Q) = 1$. When $P = Q$, this matrix is the following (cf. Section 6):

$$N_Q = N_1 := \begin{pmatrix} 0 & p & 0 & 0 & \cdots \\ p+1 & 0 & p & 0 & \cdots \\ 0 & 1 & 0 & p & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

In this context, all matrices N_P, when $X = \mathbb{P}_1$ and $\mathfrak{A} = \mathbb{M}_2(K)$, are described by Theorem 1.5 bellow. This allows us to compute C_P, and therefore S_P, for all P in this case.

(a) Loop with inversion.

(b) Loop without inversion.

Fig. 2. Two types of loops.

Theorem 1.5. *For any place P in $\mathbb{P}_1$, the matrix $N_P = N_{\deg(P)}$ depends only on the degree of P, and can be computed by the recurrence relation*

$$N_d = N_1^d - \sum_{i=1}^{[d/2]} \binom{d}{i} p^i N_{d-2i}. \tag{1}$$

Remark 1.6. Serre defined quotient graphs only for a group G acting without inversions on a graph T [17, §I.3.1], where an inversion is a pair (g, y) where $g \in G$ and y is an edge of T such that gy in the opposite edge $\bar{y}$. However, as was noted by Serre himself, on way around this problem is to replace the graph by its barycentric subdivision T^1. When this is done, inversions appear in the quotient graph $G\backslash T$ as shown in Fig. 2(a), where the symbol $*$ corresponds to a vertex of T^1 that is not a vertex of T. On the other hand, loops not corresponding to any inversion appear in this quotient graph as shown in Fig. 2(b). We denote all loops as in Fig. 4(b) in all that follows. Note, however that loops of multiplicity one are always as in Fig. 2(a).

2. Orders and spinor genera

Recall that an X-lattice or X-bundle in a K-vector space V is a locally free subsheaf of the constant sheaf V [6]. For any sheaf of groups Λ on X we let $\Lambda(U)$ denote the group of U-sections. In particular, $\Lambda(X)$ is the $\mathbb{F}$-vector space of global sections. An order $\mathfrak{D}$ in a K-algebra $\mathfrak{A}$ is an X-lattice in $\mathfrak{A}$ such that $\mathfrak{D}(U)$ is a ring for any open subset U. In particular, the structure sheaf $\mathcal{O}_X$ is an X-order in K. In all that follows, we assume that $\mathbb{F}$ is the whole constant field of K, i.e., $\mathcal{O}_X(X) = \mathbb{F}$, as otherwise $\mathbb{F}$ can be replaced with a larger field. We also let $\mathfrak{A}$ be a central simple K-algebra. In this section we review the basic facts about spinor genera and spinor class fields of orders. See [2] for details.

Let $|X|$ be the set of closed points in X. Let $\mathbb{A} = \mathbb{A}_X$ be the adele ring of X, i.e., the subring of $\prod_{P\in|X|} K_P$ of elements that are integral at almost all places. Let $\mathfrak{A}_\mathbb{A} = \mathfrak{A} \otimes_K \mathbb{A}$ be the adelization of $\mathfrak{A}$. Both $\mathbb{A}$ and $\mathfrak{A}_\mathbb{A}$ are given the adelic topology [18, §IV.1]. More generally, for any finite dimensional K-vector space V, we can define the adelization $V_\mathbb{A} = V \otimes_K \mathbb{A} \cong \mathbb{A}^{\dim_K V}$ endowed with the product topology. For any $\mathcal{O}_X$-lattice Λ, the adelization $\Lambda_\mathbb{A} = \prod_{P\in|X|} \Lambda_P$, is an open and compact subgroup of $V_\mathbb{A}$. In particular, the ring of integral ideles $\mathcal{O}_\mathbb{A} = (\mathcal{O}_X)_\mathbb{A}$ is open and compact in $\mathbb{A}$. Furthermore, every open and compact $\mathcal{O}_\mathbb{A}$-sub-modules of $V_\mathbb{A}$ is the adelization of an X-lattice. For any X-lattice Λ and any adelic element $a \in (\mathrm{End}_K(V))_\mathbb{A} \cong \mathrm{End}_\mathbb{A}(V_\mathbb{A})$, the lattice $L = a\Lambda$ is the X-lattice defined by $L_\mathbb{A} = a\Lambda_\mathbb{A}$.

Since any two maximal X-orders are locally conjugate at all places, if we fix a maximal X-order $\mathfrak{D}$, any other maximal X-order on $\mathfrak{A}$ has the form $\mathfrak{D}' = a\mathfrak{D}a^{-1}$ for some adelic element $a \in \mathfrak{A}_{\mathbb{A}}^*$. In a more general theory it is said that two maximal X-orders are always in the same genus [4]. Two maximal X-orders $\mathfrak{D}$ and $\mathfrak{D}'$ are in the same spinor genus if a can be chosen of the form $a = bc$, where $b \in \mathfrak{A}$ and $N(c) = 1_{\mathbb{A}}$, while $N : \mathfrak{A}_{\mathbb{A}}^* \to \mathbb{A}^* =: J_X$ is the reduced norm on adeles. When this holds we write $\mathfrak{D}' \in \mathrm{Spin}(\mathfrak{D})$. The spinor class field is defined as the class field corresponding to the group $K^*H(\mathfrak{D}) \subseteq J_X$, where

$$H(\mathfrak{D}) = \{N(a) \mid a \in \mathfrak{A}_{\mathbb{A}}^*,\ a\mathfrak{D}a^{-1} = \mathfrak{D}\}.$$

Let $t \mapsto [t, \Sigma/K]$ denote the Artin map on ideles. The distance between the maximal X-orders $\mathfrak{D}$ and $\mathfrak{D}'$ is the element $\rho(\mathfrak{D}, \mathfrak{D}') \in \mathrm{Gal}(\Sigma/K)$ defined by $\rho(\mathfrak{D}, \mathfrak{D}') = [N(a), \Sigma/K]$, for any adelic element $a \in \mathfrak{A}_{\mathbb{A}}^*$ satisfying $\mathfrak{D}' = a\mathfrak{D}a^{-1}$. Note that this implies $\rho(\mathfrak{D}, \mathfrak{D}'') = \rho(\mathfrak{D}, \mathfrak{D}')\rho(\mathfrak{D}', \mathfrak{D}'')$ for any triple $(\mathfrak{D}, \mathfrak{D}', \mathfrak{D}'')$ of maximal X-orders. Two such orders are in the same spinor genus if and only if their distance is trivial. The spinor class field can be defined also for maximal S-orders for any finite subset $S \subseteq |X|$. In fact, the spinor class field $\Sigma' = \Sigma^S$ of S-orders, or equivalently, the spinor $\Sigma' = \Sigma_U$ of $\mathcal{O}_X(U)$-orders, for the affine set $U = |X| \backslash S$, is the largest subfield of Σ splitting completely at every place in S.

One important property of spinor genera is that they coincide with conjugacy classes whenever strong approximation holds. In the context of X-orders, this implies that two maximal orders are in the same spinor genus if and only if they are isomorphic (as sheaves) on every affine subset U whose complement S has a place splitting $\mathfrak{A}$. More generally, for a given affine subset U satisfying this condition, two S-orders $\mathfrak{D}(U)$ and $\mathfrak{D}'(U)$ are isomorphic if and only if the distance $\rho(\mathfrak{D}, \mathfrak{D}')$ is in the group $\langle |[P, \Sigma/K]| \mid P \in S\rangle$ (see [2, §2] or [3, §2]). In all that follows, we assume $S = \{P\}$ for a fixed place P splitting $\mathfrak{A}$, the *place at infinity*.

Let $\mathfrak{H}$ be a suborder of a maximal X-order $\mathfrak{D}$, and let

$$H(\mathfrak{D}|\mathfrak{H}) = \{N(a) \mid a \in \mathfrak{A}_{\mathbb{A}}^*,\ a^{-1}\mathfrak{H}_{\mathbb{A}}a \subseteq \mathfrak{D}_{\mathbb{A}}\} \subseteq J_X.$$

When the set $K^*H(\mathfrak{D}|\mathfrak{H}) \subseteq J_X$ is a group, or equivalently, the set

$$\Phi = \{\rho(\mathfrak{D}, \mathfrak{D}') \mid \mathfrak{H} \subseteq \mathfrak{D}'\} \subseteq \mathrm{Gal}(\Sigma/K)$$

is a group, then the class field $F(\mathfrak{H}) = \Sigma^{\Phi}$, corresponding to $K^*H(\mathfrak{D}|\mathfrak{H})$, is called the representation field for $\mathfrak{H}$. The order $\mathfrak{H}$ embeds into some order in $\mathrm{Spin}(\mathfrak{D}')$ precisely when $\rho(\mathfrak{D}, \mathfrak{D}')$ is trivial on $F(\mathfrak{H})$. The representation field is not always defined for central simple algebras of arbitrary dimension, but this is indeed the case for quaternion algebras [1]. When $\mathfrak{A}$ is a quaternion algebra and $\mathfrak{H}$ is the maximal X-order in a maximal subfield L, then $F(\mathfrak{H}) = L \cap \Sigma$ [2, §5, Cor. 2].

Example 2.1. When $\mathfrak{A} \cong \mathbb{M}_2(K)$, then $H(\mathfrak{D}) = J_X \cap \prod_{P\in|X|} \mathcal{O}_P^* K_P^{*2}$, so that Σ is the largest unramified exponent-2 abelian extension of K. When $X = \mathbb{P}_1$, so that $K = \mathbb{F}(t)$, then $\Sigma = \mathbb{L}(t)$ for the unique quadratic extension $\mathbb{L}$ of $\mathbb{F}$. If $\mathfrak{H}$ is the maximal order of $\mathbb{L}(t)$, then $F(\mathfrak{H}) = \mathbb{L}(t)$.

Proof of Theorem 1.1. Consider the maximal affine subset $U = X\backslash\{P\}$. The spinor class field Σ_U of maximal $\{P\}$-orders is the maximal subfield of Σ splitting completely at P. In particular, $\Sigma_U = \Sigma$ if and only if P splits completely in Σ/K. Otherwise, we have $[\Sigma : \Sigma_U] = 2$. If P in unramified for $\mathfrak{A}$, any two maximal order $\mathfrak{D}$ and $\mathfrak{D}'$ are isomorphic on U if and only if their distance $\rho(\mathfrak{D}, \mathfrak{D}')$ is trivial on Σ_U. If this is the case, replacing $\mathfrak{D}'$ by a (global) conjugate it can be assumed that $\mathfrak{D}(U) = \mathfrak{D}'(U) = D$. The set of maximal orders satisfying the last condition is in correspondence with the set of vertices of T_P. Two such orders, $\mathfrak{D}$ and $\mathfrak{D}'$, are conjugate if and only if $\mathfrak{D}' = g\mathfrak{D}g^{-1}$ for some $g \in G$.

Let e_P be an idele that is 1 outside of P and a uniformizing parameter π_P at P. Note that if $\mathfrak{D}$ and $\mathfrak{D}'$ are neighbors in T_P, their completions $\mathfrak{D}_P$ and $\mathfrak{D}'_P$ have, in some basis, the form

$$\mathfrak{D}_P = \begin{pmatrix} \mathcal{O}_P & \mathcal{O}_P \\ \mathcal{O}_P & \mathcal{O}_P \end{pmatrix}, \qquad \mathfrak{D}'_P = \begin{pmatrix} \mathcal{O}_P & \pi_P^{-1}\mathcal{O}_P \\ \pi_P\mathcal{O}_P & \mathcal{O}_P \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \pi_P \end{pmatrix} \mathfrak{D}_P \begin{pmatrix} 1 & 0 \\ 0 & \pi_P \end{pmatrix}^{-1}$$

(cf. [17, §II.1.1]). We conclude that $\rho(\mathfrak{D}, \mathfrak{D}') = [e_P, \Sigma/K] = |[P, \Sigma/K]|$. The result follows. In particular, when $|[P, \Sigma/K]| \neq \mathrm{id}_\Sigma$, neighboring vertices are in different spinor genera, so the graph is bipartite. □

3. Orders and vector bundles

In this section, notations are as in Section 2, except that we assume $\mathfrak{A} = \mathbb{M}_n(K)$. In this case, any maximal X-order on $\mathfrak{A}$ has the form $\mathfrak{D} = b\mathfrak{D}_0 b^{-1}$ where $b \in \mathfrak{A}_{\mathbb{A}}^*$ is a matrix with adelic coefficients and $\mathfrak{D}_0 \cong \mathbb{M}_n(\mathcal{O}_X)$. Note that the adelization is $\mathfrak{D}_{0\mathbb{A}} \cong \mathbb{M}_n(\mathcal{O}_{\mathbb{A}})$, where $\mathcal{O}_{\mathbb{A}} \cong \prod_{P\in|X|} \mathcal{O}_P$ is the ring of integral adeles (Section 2). In particular, $\mathfrak{D}_{0\mathbb{A}}^*$ is the group of adelic matrices c satisfying $c\mathcal{O}_X^n = \mathcal{O}_X^n$. It follows that $\mathfrak{D}_{\mathbb{A}}^*$ is the group of all adelic matrices c satisfying $c\Lambda = \Lambda$, where $\Lambda = b\mathcal{O}_X^n$. Since the stabilizer of any order $\mathfrak{D}_P$ in $\mathbb{M}_2(K_P)$ is $\mathfrak{D}_P^* K_P^*$, it follows that two X-lattices Λ_1 and Λ_2 correspond to the same maximal order, if and only if $\Lambda_1 = d\Lambda_2$ for some $d \in J_X$. Let $\mathrm{div}(d)$ denote the divisor generated by d, i.e., $\mathfrak{L}^{-\mathrm{div}(d)} = d\mathcal{O}_X$. Note that every divisor is generated by an idele in this sense, e.g., $\mathrm{div}(e_P) = P$. Next result follows:

Proposition 3.1. *There is a correspondence between conjugacy classes of maximal X-orders in $\mathbb{M}_n(K)$ and isomorphism classes of X-lattices on K^n up to tensor product with one dimensional X-lattices*[1] *in K.*

[1] In other words, isomorphism classes of maximal X-orders correspond to isomorphism classes of n-dimensional vector bundles up to tensor product with invertible bundles.

Let $\mathfrak{D}_E = \mathcal{E}\mathrm{nd}_{\mathcal{O}_X}(E)$ be the maximal order corresponding to the vector bundle E. A finite algebra $\mathbb{B}$ acts faithfully as a ring of global endomorphisms of a vector bundle E if and only if $\mathbb{B}$ embeds into the ring of global sections $\mathfrak{D}_E(X)$. Note that the split maximal order $\mathfrak{D}_B$ defined in the introduction is the order $\mathfrak{D}_{E_B}$ corresponding to the bundle $E_B = \mathfrak{L}^B \oplus \mathcal{O}_X$. More generally, the maximal order corresponding to the bundle $\mathfrak{L}^B \oplus \mathfrak{L}^A = \mathfrak{L}^A \otimes_{\mathcal{O}_X} (\mathfrak{L}^{B-A} \oplus \mathcal{O}_X)$ is $\mathfrak{D}_{B-A}$. Furthermore, a maximal X-order $\mathfrak{D} = \mathfrak{D}_E$ is split if and only if any of the following equivalent conditions is satisfied:

1. The algebra $\mathbb{F}^2 = \mathbb{F} \times \mathbb{F}$ acts globally on E,
2. $\mathbb{F}^2$ embeds into the ring of global sections $\mathfrak{D}(X)$.
3. The commutative order $\mathfrak{H} = \mathcal{O}_X \times \mathcal{O}_X$ embeds into $\mathfrak{D}$.

It follows from [2, Cor. 5.6] that every spinor genera of maximal orders contain split orders. In fact, if $B = \mathrm{div}(b)$, then $\mathfrak{D}_B = c\mathfrak{D}_0 c^{-1}$ where $c = \left(\begin{smallmatrix} 1 & 0 \\ 0 & b \end{smallmatrix}\right)$. In particular, the corresponding distance element is $\rho(\mathfrak{D}_0, \mathfrak{D}_B) = [b, \Sigma/K] \in \mathrm{Gal}(\Sigma/K)$, and therefore $\rho(\mathfrak{D}_A, \mathfrak{D}_B) = [a^{-1}b, \Sigma/K] = |[B - A, \Sigma/K]|$, if $A = \mathrm{div}(a)$. By Example 2.1, the spinor genera $\mathrm{Spin}(\mathfrak{D}_A)$ and $\mathrm{Spin}(\mathfrak{D}_B)$ coincide if and only if the class $\overline{B - A}$ belongs to $2\,\mathrm{Pic}(X)$.

In general, if $\mathbb{B} \subseteq \mathbb{M}_2(K)$ is a finite $\mathbb{F}$-algebra, the dimension $\dim_{\mathbb{F}}\mathbb{B}$ can be arbitrarily large. However, we have next result:

Proposition 3.2. *Assume $\mathbb{B} = \mathbb{B}' \oplus \mathbb{R}$ is a finite $\mathbb{F}$-algebra contained in $\mathbb{M}_n(K)$, where $\mathbb{B}'$ is semisimple and $\mathbb{R}$ is the radical of $\mathbb{B}$. Then $\dim_{\mathbb{F}}\mathbb{B}' = \dim_K(K\mathbb{B}')$, and the sum $K\mathbb{B}' + K\mathbb{R}$ is direct.*

Proof. If $\mathbb{B} = \bigoplus_{i=1}^n P_i\mathbb{B}$, where $P_1, \ldots, P_n$ are the minimal central idempotents of $\mathbb{B}$, then $K\mathbb{B} = \bigoplus_{i=1}^n P_iK\mathbb{B}$. Therefore, we can assume that $\mathbb{B}'$ is simple. If $\mathbb{B}' = \mathbb{M}_n(\mathbb{L})$ where $\mathbb{L}/\mathbb{F}$ is a finite extension, then $K\mathbb{B}'$ is a quotient of $K \otimes_{\mathbb{F}} \mathbb{B}' \cong \mathbb{M}_n(K \otimes_{\mathbb{F}} \mathbb{L})$. Since $\mathbb{F}$ is the full constant field of K, the tensor product $K \otimes_{\mathbb{F}} \mathbb{L}$ is a field. It follows that $K \otimes_{\mathbb{F}} \mathbb{B}'$ is simple and therefore equals $K\mathbb{B}'$. The last statement follows since the two sided ideal generated by an arbitrary non-invertible element u in $K\mathbb{B}'$ contains a non-trivial idempotent, and therefore u cannot belong to $K\mathbb{R}$. $\square$

Corollary 3.3. *For any maximal X-order $\mathfrak{D}$ in $\mathbb{M}_2(K)$, the semi-simple part of the ring $\mathfrak{D}(X)$ is isomorphic to an element in the set $\{\mathbb{F}, \mathbb{F} \times \mathbb{F}, \mathbb{L}, \mathbb{M}_2(\mathbb{F})\}$, where $\mathbb{L}$ is the unique quadratic extension of $\mathbb{F}$. Only in the first two cases a non-trivial radical $\mathbb{R}$ can exist, and in that case $\dim_K K\mathbb{R} = 1$.*

Example 3.4. The bundles E admitting an $\mathbb{L}$-vector space structure are those satisfying $\mathfrak{D}_E(X) \cong \mathbb{L}$ or $\mathfrak{D}_E(X) \cong \mathbb{M}_2(\mathbb{F})$. By the Matrix Units Theorem [14, p. 30] we have that $\mathfrak{D}(X) \cong \mathbb{M}_2(\mathbb{F})$ implies $\mathfrak{D} \cong \mathbb{M}_2(\mathcal{O}_X)$. Moreover, $\mathbb{L}$ embeds into $\mathfrak{D}_E(X)$ if and only if $\mathfrak{H}_{\mathbb{L}} = \mathbb{L} \otimes_{\mathbb{F}} \mathcal{O}_X$ embeds into $\mathfrak{D}_E$. Note that $\mathfrak{H}_{\mathbb{L}}$ is the maximal order of $L = K\mathbb{L}$, and also the push-forward sheaf $\mathfrak{H}_{\mathbb{L}} = f_*(\mathcal{O}_Y)$, where $Y = X \times_{\mathrm{Spec}(\mathbb{F})} \mathrm{Spec}(\mathbb{L})$ and $f : Y \to X$ is the projection on the first coordinate. The extension L/K is unramified, whence $L \subseteq \Sigma$ (cf. Example 2.1). Let σ be the generator of $\mathrm{Gal}(L/K)$. Then by the definition of the Artin map, $|[B, L/K]| = \sigma^{\deg(B)}$ for any divisor B. Since $F(\mathfrak{H}_{\mathbb{L}}) = \Sigma \cap L = L$ [2, §5, Cor. 2], the order $\mathfrak{H}_{\mathbb{L}}$ embeds in, precisely, the spinor genera Φ satisfying any of the following equivalent conditions:

1. For some (any) $\mathfrak{D} \in \Phi$, we have $\rho(\mathfrak{D}_0, \mathfrak{D})|_L = \mathrm{Id}_L$.
2. For some (any) $\mathfrak{D} = c\mathfrak{D}_0c^{-1} \in \Phi$, the integer $\deg(\mathrm{div}(N(c)))$ is even.
3. Φ contains an order of the form $\mathfrak{D}_B$, where B is a divisor of even degree.

4. Split maximal orders

Next we study in greater detail the order $\mathfrak{D}_B$ defined in the introduction. Note that if $B = D + \mathrm{div}(b)$, for any idele b, then $\mathfrak{D}_B = c\mathfrak{D}_Dc^{-1}$, where $c = \left(\begin{smallmatrix} 1 & 0 \\ 0 & b \end{smallmatrix}\right)$. It follows that, when B is a principal divisor, then $\mathfrak{D}_B \cong \mathfrak{D}_0 = \mathbb{M}_2(\mathcal{O}_X)$, and therefore, the ring of global sections $\mathfrak{D}_B(X)$ is isomorphic to the matrix algebra $\mathbb{M}_2(\mathbb{F})$. If B is not principal, then $\mathfrak{L}^B$ and $\mathfrak{L}^{-B}$ cannot have a global section simultaneously. In fact, if $\mathrm{div}(f) + B \geqslant 0$ and $\mathrm{div}(g) - B \geqslant 0$, then $B = \mathrm{div}(g) = \mathrm{div}(f^{-1})$. We conclude that $\mathfrak{D}_B(X) \cong (\mathbb{F} \times \mathbb{F}) \oplus \mathfrak{L}^{\pm B}u$, where $u^2 = 0$. Observe that, since $\mathfrak{D}_B \cong \mathfrak{D}_{-B}$, we can always assume $\mathfrak{L}^{-B}(X) = \{0\}$ when B is not principal.

Proposition 4.1. *Assume $\mathfrak{L}^{-B}(X) = \mathfrak{L}^{-D}(X) = \{0\}$. Any matrix U satisfying $\mathfrak{D}_B = U\mathfrak{D}_DU^{-1}$ has the form $\left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right)$, in which case B is linearly equivalent to D, or $\left(\begin{smallmatrix} 0 & a \\ c & 0 \end{smallmatrix}\right)$, in which case B is linearly equivalent to $-D$.*

Proof. Let U be as stated. If W_B and W_D denote the K-vector spaces spanned by $\mathfrak{D}_B(X)$ and $\mathfrak{D}_D(X)$, respectively, then $W_B = UW_DU^{-1}$. Let $\{E_{i,j}\}_{i,j}$ be the canonical basis of the matrix algebra $\mathbb{M}_2(K)$. There are two cases two be considered:

1. If $\mathfrak{L}^B(X) \neq \{0\}$, then $W_B = W_D = KE_{1,1} \oplus KE_{2,2} \oplus KE_{1,2}$.
2. If $\mathfrak{L}^B(X) = \{0\}$, then $W_B = W_D = KE_{1,1} \oplus KE_{2,2}$.

In the first case, U has the form $\left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right)$. In particular we have

$$\mathfrak{L}^{-B}E_{2,1} = E_{2,2}\mathfrak{D}_BE_{1,1} = E_{2,2}\bigl(U\mathfrak{D}_DU^{-1}\bigr)E_{1,1} = a^{-1}c\mathfrak{L}^{-D}E_{2,1}.$$

We conclude that $B = D + \mathrm{div}(ac^{-1})$, and therefore B and D are linearly equivalent. In the second case U has either the form $\left(\begin{smallmatrix} a & 0 \\ 0 & c \end{smallmatrix}\right)$, which is similar to the previous case, or the form $\left(\begin{smallmatrix} 0 & a \\ c & 0 \end{smallmatrix}\right)$, so that by a similar computation $B = -D + \mathrm{div}(ac^{-1})$, and B is linearly equivalent to $-D$. $\square$

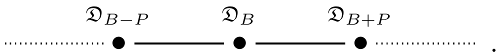

Fig. 3. Line (path) corresponding to $\{\bar{B}+\langle\bar{P}\rangle, -\bar{B}+\langle\bar{P}\rangle\} \subseteq \mathrm{Pic}(X)/\langle\bar{P}\rangle$.

(a) $2\bar{B}=0$

(b) $2\bar{B}=\bar{P}$

Fig. 4. Folded lines.

Proof of Theorem 1.2. Let $e \in \mathfrak{A}$ be a non-trivial idempotent and let $Z = Ke \oplus K(1-e)$ be the split semisimple commutative subalgebra generated by e. Let $\mathfrak{Z} = \mathcal{O}_X e \oplus \mathcal{O}_X(1-e)$ be the unique maximal order in Z. By identifying the vector space K^2 with Z, we see that the only local lattices that are invariant under $\mathfrak{Z}_P$ are the fractional ideals $(\pi_P^r e + \pi_P^s(1-e))\mathfrak{Z}_P$, whence the corresponding maximal orders, the ones containing $\mathfrak{Z}_P$, lie in a maximal path in the tree (or in the language of buildings, an apartment). We conclude that the maximal orders in that path are split, and moreover, this path has the form shown in Fig. 3. It follows from Proposition 4.1, that two orders in the path can be conjugate if and only if the divisor classes $\bar{B}, \bar{P} \in \mathrm{Pic}(X)$ satisfy $\bar{B}+N\bar{P} = \pm(\bar{B}+M\bar{P})$, for some integers N and M. Since P has positive degree, only the equation with a negative sign can have non-trivial solutions. In fact, this implies $2\bar{B} = (N+M)\bar{P}$. Replacing B by $B+kP$ for some integer k if needed, we can assume $(N+M) \in \{0,1\}$, whence either $2\bar{B} = 0$ and $\bar{B}$ is an element of order 2 in $\mathrm{Pic}_0(X)$, or $2\bar{B} = \bar{P}$, and in the latter case the place P has even degree. $\square$

Remark 4.2. Note that, when $2\bar{B} = 0$ or $2\bar{B} = \bar{P}$, the image of this line in the C-graph has, respectively, one of the forms shown in Fig. 4. In the sequel, they will be called folded lines of type (a) or (b), respectively.

5. Valencies in the S-graph

In all of this section, let $\mathbb{F} = \mathbb{F}_p$ with p a prime power, and let $d = \deg(P)$, so that $\mathbb{F}(P) = \mathbb{F}_{p^d}$. Let $\mathbb{R}$ be the radical of $\mathfrak{D}(X)$ and $\tilde{\mathbb{R}} \subseteq \mathbb{R}$ the kernel of the natural map $\mathfrak{D}(X) \twoheadrightarrow \Delta \subseteq \mathfrak{D}_P/\pi_P\mathfrak{D}_P$, for any local uniformizing parameter π_P at P. The conjugation-stabilizer in Γ of a vertex $\mathfrak{D}$ is the group invertible elements $\mathfrak{D}(X)^*$, and its action on the set of neighbors of $\mathfrak{D}$ can be realized identifying Δ^* with a subgroup of $\mathrm{GL}_2(\mathbb{F}(P))$, which acts naturally on the set of $\mathbb{F}(P)$-points of the projective line $\mathbb{P}_1$ [17, §II.1.1]. The number of orbits for all orders is given in Table 1. In this table, ϵ is 0 when P has odd degree and 1 otherwise. We let $r = \dim_{\mathbb{F}}(\mathbb{R}/\tilde{\mathbb{R}}) \leqslant d$. To prove these values we compute, in each case, the number of elements in Δ^* with any possible Jordan form. Consider, for example, an order of Type I. The number of elements with the Jordan form $B = \left(\begin{smallmatrix} b & 1 \\ 0 & b \end{smallmatrix}\right)$ for every fixed $b \in \mathbb{F}^*$ is

Table 1
Types of orders, and number of orbits in each case.

Type	$\mathfrak{D}(X)$	Number of orbits (valency)
I	$\mathbb{M}_2(\mathbb{F})$	$1+\frac{p^{d-1}-1}{p^2-1}+\frac{p}{p+1}\epsilon$
II	$\mathbb{F}+\mathbb{R}$	$p^{d-r}+1$
III	$(\mathbb{F}\times\mathbb{F})+\mathbb{R}$	$2+\frac{p^{d-r}-1}{p-1}$
IV	$\mathbb{L}$	$\frac{p^d+2p\epsilon+1}{p+1}$

Table 2
Number of elements in Δ^* with every Jordan form for different types of orders.

Jordan forms	$\begin{pmatrix} b & 1 \\ 0 & b \end{pmatrix}$	$\begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix}$	$\begin{pmatrix} b & 0 \\ 0 & c \end{pmatrix}$	NEV
Fixed points	1	p^d+1	2	2ϵ
I	$(p-1)^2(p+1)$	$p-1$	$\frac{1}{2}(p^2-1)(p-2)p$	$\frac{1}{2}(p-1)^2p^2$
II	$(p-1)(p^r-1)$	$p-1$	0	0
III	$(p-1)(p^r-1)$	$p-1$	$(p-1)(p-2)p^r$	0
IV	0	$p-1$	0	$(p-1)p$

$$\frac{|\Delta^*|}{|C_{\Delta^*}(B)|}=\frac{(p^2-1)(p^2-p)}{(p-1)p}=(p+1)(p-1),$$

where $C_{\Delta^*}(B)$ is the centralizer, while we have $p-1$ possible values of the eigenvalue b. The remaining values in Table 2 for Type I are computed analogously. Types II and III are simpler since their elements are already in triangular form. When a radical $\mathbb{R}$ is present, the off-diagonal element can be chosen among p^r possibilities. In this table, NEV stands for *no eigenvalues* on $\mathbb{F}$. Orders of Type IV have only non-scalar elements of this form. These elements have eigenvalues over the extension $\mathbb{F}(P)$ if and only if $d=\deg P$ is even. Now the result follows by a straightforward application of Burnside's Counting Lemma [15, §26.10], according two which the number of orbits in an action is the average number of fixed points for an element in the group.

Example 5.1. We can have vertices of valency 1 (or endpoints) only if $d=1$, and in this case they are exactly the maximal orders representing $\mathfrak{H}_{\mathbb{L}}$, as in Example 3.4 (compare to [13, §5]).

Example 5.2. If $\mathfrak{A}$ is a division algebra, there are no radicals, so in particular $r=0$. Furthermore, every vertex is of Type II or Type IV. We conclude that a vertex has valency $\frac{p^d+2p\epsilon+1}{p+1}$ if the corresponding maximal order represents $\mathfrak{H}_{\mathbb{L}}$ and p^d+1 otherwise (compare to [13, §5]).

Example 5.3. When $\mathfrak{D}=\mathfrak{D}_B$ is split and $\deg(B)>0$, so that the elements of Δ are upper triangular matrices, the orbit of the element $0:=[0;1]\in\mathbb{P}_1(\mathbb{F}(P))$ corresponds to the order $\mathfrak{D}_{B-P}$, while $\mathfrak{D}_{B+P}$ corresponds to the orbit of the element $\infty:=[1;0]$. Furthermore, $\pi_P\mathfrak{L}_P^B=\mathfrak{L}_P^{B-P}$, whence r is the dimension of $\mathfrak{L}^B(X)/\mathfrak{L}^{B-P}(X)$, i.e., $r=l(B)-l(B-P)$ in the notations of [7, Ch. 8]. Note that when $r=N$, the corresponding

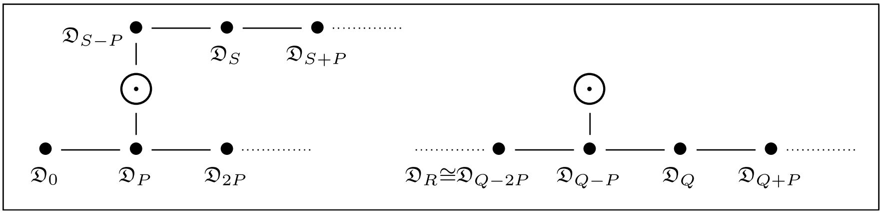

Fig. 5. Lines and folded lines for the quotient graph in Example 5.6.

vertex has valency 2. By Riemann–Roch's Theorem, this holds whenever $\deg(B) \geqslant 2g-2+d$ [17, §II.2.3, Lem. 6].

A vertex $\mathfrak{D}$ in the C-graph C_P is said to be unramified for the covering $C_P \twoheadrightarrow S_P$ if it has the same valency than one (and therefore, every) vertex in its pre-image. For unramified vertices, the valencies in the C-graph are the ones we have already computed.

Proposition 5.4. *A split maximal order $\mathfrak{D}_B$ in $\mathbb{M}_2(K)$ is unramified, unless the class of the divisor B has order exactly 2 in the Picard group $\mathrm{Pic}(X)$.*

Proof. Assume first that B is not principal and $\mathfrak{L}^{-B}(X) = \{0\}$. Let U be a global matrix satisfying $U\mathfrak{D}_B U^{-1} = \mathfrak{D}_B$. By Proposition 4.1, we conclude that $U = \left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right)$ or $U = \left(\begin{smallmatrix} 0 & a \\ c & 0 \end{smallmatrix}\right)$. In the first case $ac^{-1}\mathfrak{L}^{-B} = \mathfrak{L}^{-B}$, and therefore $ac^{-1} \in \mathbb{F}^*$. Replacing U by a scalar multiple if needed, we can assume $a, c \in \mathbb{F}^*$. Then comparing the upper-left coordinate of the identity $U\mathfrak{D}_B U^{-1} = \mathfrak{D}_B$, we obtain $\mathcal{O}_X + a^{-1}b\mathfrak{L}^{-B} = \mathcal{O}_X$, and therefore $b\mathfrak{L}^{-B} \subset \mathcal{O}_X$. It follows that $b\mathcal{O}_X \subseteq \mathfrak{L}^B$, whence $b \in \mathfrak{L}^B(X)$. We conclude that $U \in \mathfrak{D}(X)^*$. In the second case B is linearly equivalent to $-B$ by Proposition 4.1.

Assume now that B is principal, so $\mathfrak{D}_B(X) \cong \mathbb{M}_2(\mathbb{F})$. Then any Global matrix U satisfying $U\mathfrak{D}_B U^{-1} = \mathfrak{D}_B$ must, in particular, satisfy $U\mathfrak{D}_B(X)U^{-1} = \mathfrak{D}_B(X)$. Since $\mathfrak{D}_B(X)$ is simple, every automorphism of it is inner. It follows that $U \in K^*\mathfrak{D}_B(X)^*$. $\square$

Example 5.5. When $2B = \mathrm{div}(f)$, we have $U\mathfrak{D}_B U^{-1} = \mathfrak{D}_B$ for $U = \left(\begin{smallmatrix} 0 & f \\ 1 & 0 \end{smallmatrix}\right)$. In fact, by the proof of Proposition 4.1, any matrix $U' = \left(\begin{smallmatrix} 0 & a \\ c & 0 \end{smallmatrix}\right)$ satisfying this condition is in the coset $\left(\begin{smallmatrix} 0 & f \\ 1 & 0 \end{smallmatrix}\right) K^*\mathfrak{D}_B(X)^*$.

Example 5.6. Assume X is the projective curve defined by the equation $y^2z = x^3 + xz^2 + z^3$, and let $P = [0;1;0]$ be the point at infinity. Then $\mathrm{Pic}(X)/\langle \bar{P} \rangle \cong \mathrm{Pic}_0(X)$ is a cyclic group with 4 elements generated by the class of either $Q = [0;1;1]$ or $R = [0;-1;1]$, while $S = [1;0;1]$ has order 2. The C-graph has three lines corresponding to the sets $\{\bar{0}\}$, $\{\bar{S}\}$, and $\{\bar{Q}, \bar{R}\}$. The first two lines are in the trivial component, since $\bar{0}$ and $\bar{S}$ are squares. By valency considerations, we conclude that the C-graph has the shape shown in Fig. 5. In the figure, the symbol $\odot$ represents an unknown portion of the graph. The explicit description of $S_P(\mathfrak{D}_0)$, given in [16, p. 87], shows that $C_P(\mathfrak{D}_0)$ is as shown in Fig. 6. In the notations of the reference, $\mathfrak{O}$ and $\mathfrak{V}_1$ are the images of the vertices o and

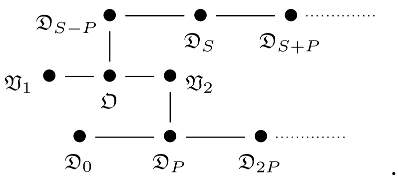

Fig. 6. Actual form of $C_P(\mathfrak{D}_0)$ in Example 5.6.

$v(-1)$, respectively, while $\mathfrak{V}_2$ is the image of both $v(1)$ and $v(\infty)$. The ramified vertices are $\mathfrak{D}$ and $\mathfrak{D}_{S-P}$, the latter being the image of $v(0)$. The order $\mathfrak{D}$ is of Type II with a trivial radical, since o has valency 4 in the S-graph.

Proof of Theorem 1.3. Assume first that $d = \deg(P)$ is odd. In particular $\Sigma_U = K$ (cf. Section 2), whence C_P is connected. We claim that $G = \Gamma$ in this case. Let M be such that $M\mathfrak{D}_0(U)M^{-1} = \mathfrak{D}_0(U)$. The determinant of M has even valuation at every place $Q \in U$, and therefore also on P, since principal divisors have degree 0. Since the Picard group in this case is $\mathrm{Pic}(\mathbb{P}_1) \cong \mathbb{Z}$, $\det(M) = g^2a$, for some $g \in K^*$ and $a \in \mathbb{F}^*$. We conclude that $g^{-1}M \in \mathfrak{D}_0(U)^*$, and the result follows.

Assume now that $d = \deg(P)$ is even. In this case $\Sigma_U = \Sigma$ is a quadratic extension (cf. Example 2.1), whence C_P has two connected components. Then P is linearly equivalent to some divisor of the form $2B$. Let M be as above. It follows as in the previous paragraph that $\det(M) \in hK^{*2}\mathbb{F}^* \cup K^{*2}\mathbb{F}^*$, for any $h \in K^*$ with $\mathrm{div}(h) = P - 2B$, whence we conclude that $|G/\Gamma| \leqslant 2$.

Strong approximation for the group $\mathrm{SL}_n(K)$ with respect to $S = \{P\}$ guarantees the existence of a matrix $M \in G$ with $\det(M) \in hK^{*2}\mathbb{F}^*$, so equality follows. In fact, choose any global matrix N with determinant in $hK^{*2}\mathbb{F}^*$. There exists, at every place $Q \neq P$ a local matrix $M_Q \in G_Q$ with $\det(M_Q) = \det(N)$. By strong approximation for $\mathrm{SL}_2(K)$ with respect to $\{P\}$, there exists a global matrix M' close enough to each NM_Q^{-1}, so that $M = N^{-1}M' \in G$. This proves the claim. Furthermore, each orbit has exactly two vertices by the proof of Proposition 5.4. □

Example 5.7. When $\mathfrak{A} = \mathbb{M}_2(K)$, $X \cong \mathbb{P}_1$, and $d = \deg P$ is odd, these graphs are described in [17, §II.2.4]. When $d = 1$, we have the graph described in Section 1 by valency considerations alone. In particular, no other vertices exists. This proves Grothendieck–Birkhoff Theorem [5, Th. 2.1] in the particular case of two dimensional vector bundles over a finite field.

Example 5.8. Let X and $\mathfrak{A}$ be as in the previous example. The C-graphs for $d = 2$ and $d = 4$ are shown in Fig. 7. The double lines are deduced by valency considerations, this cannot be done for all lines when $d \geqslant 6$. The line joining $\mathfrak{D}_{2Q}$ and $\mathfrak{D}_0$ for $d = 4$ must be there since both lines are in the same connected component. Note that existence of loops on every component implies $|G/\Gamma| = 2$, thus giving another proof of Theorem 1.3. For every even integer d, it can be proved using Theorem 1.5, and rather lengthy computations,

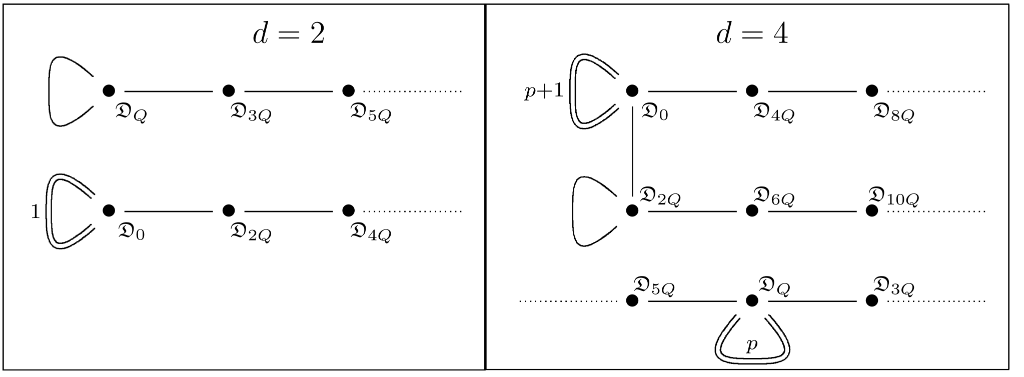

Fig. 7. Two examples of full C-graphs.

that such loops actually exist in either component. This is immediate, however, for the component containing a single-line loop, which corresponds to a folded line of type (b).

6. Multiplicities of edges

In this section, we show how the number of edges $M = M(\mathfrak{D}, \mathfrak{D}')$ can be computed in terms of the number of neighbors $N = N(\mathfrak{D}, \mathfrak{D}')$ defined in Section 1. As before, we limit ourselves to split vertices $\mathfrak{D} = \mathfrak{D}_B$. In this case, the set of neighbors of $\mathfrak{D}$ corresponding to a given edge in $S_P(\mathfrak{D})$ is in correspondence with an orbit of $\mathfrak{D}(X)^*$ on the projective line $\mathbb{P}^1(\mathbb{F}(P))$ (Section 5). As usual we identify the point $[\alpha; 1]$ in the projective line with the finite element $\alpha \in \mathbb{F}(P)$, while we let $\infty = [1; 0]$. Note that we can always assume that P is not in the support of B. We do this in all that follows. The computation is divided into three cases:

Case A: $2\bar{B} \neq 0$. We can assume $\mathfrak{L}^{-B}(X) = \{0\}$, so that global sections are upper triangular matrices, or we can replace B by $-B$, which corresponds to replacing z by $1/z$ in the projective line. The orders in this case are unramified vertices for the cover $S_P \twoheadrightarrow C_P$ (Proposition 5.4). In particular, C_P is locally homeomorphic to S_P at these points, so it suffices to consider Δ^*-orbits as in Section 5. On $\mathbb{P}^1(\mathbb{F}(P))$, an element $\left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right) \in \Delta^*$, acts as $t \mapsto c^{-1}(at + b)$. Note that $a, c \in \mathbb{F}^*$, while $b \in V_{B,P} := \{f(P) \mid f \in \mathfrak{L}^B(X)\}$. Since $f(P) = 0$ if and only if $f \in \mathfrak{L}^{B-P}(X)$, $\dim_{\mathbb{F}}(V_{B,P}) = r$. The only possible finite solutions of $t = at + b$, with $a \in \mathbb{F}^*$, occur when $t \in V_{B,P}$, and $V_{B,P}$ is a single orbit, namely $[0]$. The point ∞ is fixed by the whole group. The class $[0]$ corresponds to $\mathfrak{D}_{B-P}$ and the class $[\infty]$ corresponds to $\mathfrak{D}_{B+P}$.

Case B: $2\bar{B} = 0$, but $\bar{B} \neq 0$. These vertices are similar, in the S-graph, to those in case **A**, but they are ramified (cf. Example 5.5). In C_P, they are endpoints of folded lines of type (a). In this case, the radical is $\mathbb{R} = \mathfrak{L}^B(X)E_{1,2} = 0$. Let f be as in Example 5.5. Conjugation by $F = \left(\begin{smallmatrix} 0 & f \\ 1 & 0 \end{smallmatrix}\right)$ induces the map $x \mapsto f(P)/x$ on $\mathbb{P}^1(\mathbb{F}(P))$. The orbits under

Δ^* have the form $[t] = \mathbb{F}^* t$. It follows that the F-invariant orbits $[\lambda]$ are given by the solutions of $f(P)/t = at$ for $a \in \mathbb{F}^*$, or $t^2 = a^{-1} f(P)$.

There are several sub-cases to be considered here:

- When the characteristic char($\mathbb{F}$) is 2 or deg(P) is odd, there is always a unique invariant orbit corresponding to an X-order $\mathfrak{D}_1$. In this case we can have $\mathfrak{D}_1 \ncong \mathfrak{D}_{B+P}$ (case **B1**), or $\mathfrak{D}_1 \cong \mathfrak{D}_{B+P}$ (case **B2**).
- When char($\mathbb{F}$) $\neq 2$, deg(P) is even, and $f(P)$ is not a square in $\mathbb{F}(P)$, there are no invariant orbits. This is case **B3**.
- When char($\mathbb{F}$) $\neq 2$, deg(P) is even, and $f(P)$ is a square in $\mathbb{F}(P)$, there are two invariant orbits corresponding to $\mathfrak{D}_1$ and $\mathfrak{D}_2$. There are 4 different subcases.
  - The X-orders $\mathfrak{D}_1$ and $\mathfrak{D}_2$ can be isomorphic, in which case they can be isomorphic to $\mathfrak{D}_{B+P}$ (case **B4**), or not (case **B5**).
  - The X-orders $\mathfrak{D}_1$ and $\mathfrak{D}_2$ can fail to be isomorphic, in which case there can be one isomorphic to $\mathfrak{D}_{B+P}$ (case **B6**), or none (case **B7**).

Case C: $\bar{B} = 0$. In this case we can assume $B = 0$. Then $\mathfrak{D}_B$ is an unramified vertex (Proposition 5.4). It suffices therefore to find the number of elements in each orbit of $\Delta^* \cong \mathrm{PGL}_2(\mathbb{F})$ on $\mathbb{P}^1(\mathbb{F}(P))$. Let $\mathbb{L}$ be the unique quadratic extension of $\mathbb{F}$. We know from the specific shape of the graphs for $d = 1$ and $d = 2$ (Section 1 and Section 5, respectively), that $\mathrm{PGL}_2(\mathbb{F})$ has one orbit on $\mathbb{P}^1(\mathbb{F})$ and two orbits on $\mathbb{P}^1(\mathbb{L})$. We conclude that $\mathbb{P}^1(\mathbb{L})\backslash\mathbb{P}^1(\mathbb{F})$ is an orbit. Let $\hat{\mathfrak{D}}$ be the maximal order corresponding to this orbit. Since any equation of the type $x = (ax + b)/(cx + d)$ has all its roots in $\mathbb{L}$, all elements outside $\mathbb{L}$ have trivial stabilizer. Let μ be such that $\mathbb{L} = \mathbb{F}(\mu)$. There are three subcases:

- If P has odd degree, μ is not an $\mathbb{F}(P)$-point of the projective plane. This is case **C1**.
- If P has even degree, the class $[\mu]$ corresponds to an order $\hat{\mathfrak{D}}$. We can have $\hat{\mathfrak{D}} \cong \mathfrak{D}_{B+P}$ (case **C2**) or not (case **C3**).

Table 3 covers the number of neighbors corresponding to each edge, or equivalently the number of elements in each orbit. Note that the number of elements in the orbit $[\infty]$ is specified only when this orbit is different from $[0]$. A similar warning applies to $[\lambda]$ and $[\mu]$. Table 4 allows us to compute the multiplicity M of an edge in terms of the number N of neighbors of that type, in each case, assuming that we can identify the exceptional orders $\mathfrak{D}_1$, $\mathfrak{D}_2$, $\hat{\mathfrak{D}}$, or $\mathfrak{D}_{B-P}$. The X-order $\mathfrak{D}_{B-P}$ is considered exceptional to simplify the table. Certainly, $\mathfrak{D}_{B-P} \cong \mathfrak{D}_{B+P}$ except in case **A**. Observe that case **B** is not needed when $X = \mathbb{P}_1$.

Note that, whenever $p > 2$, we can tell these formulas apart by congruence conditions on N, except for cases **B4** and **B5**, where the presence of two equal invariant orbits can be mistaken by a single orbit. In actual computations, it is preferable avoiding these vertices if at all possible.

Table 3
Number of elements in each orbit.

Case	[0]	[∞]	[λ]	[μ]	Other
A	p^r	1	–	–	$(p-1)p^r$
B	2	–	$(p-1)$	–	$2(p-1)$
C	$p+1$	–	–	$p(p-1)$	$p(p^2-1)$

Table 4
$M = M(\mathfrak{D}_B, *)$ as a function of $N = N(\mathfrak{D}_B, *)$.

Case	$\mathfrak{D}_{B+P}$	Exceptional if $\not\cong \mathfrak{D}_{B+P}$	Other
A	$1+\frac{N-1}{(p-1)p^r}$	$1+\frac{N-p^r}{(p-1)p^r}$	$\frac{N}{(p-1)p^r}$
B1	$2+\frac{N-(p-1)-2}{2(p-1)}$	–	$\frac{N}{2(p-1)}$
B2	$1+\frac{N-2}{2(p-1)}$	$1+\frac{N-(p-1)}{2(p-1)}$	$\frac{N}{2(p-1)}$
B3	$1+\frac{N-2}{2(p-1)}$	–	$\frac{N}{2(p-1)}$
B4	$2+\frac{N-2}{2(p-1)}$	–	$\frac{N}{2(p-1)}$
B5	$1+\frac{N-2}{2(p-1)}$	$1+\frac{N}{2(p-1)}$	$\frac{N}{2(p-1)}$
B6	$2+\frac{N-(p-1)-2}{2(p-1)}$	$1+\frac{N-(p-1)}{2(p-1)}$	$\frac{N}{2(p-1)}$
B7	$2+\frac{N-2}{2(p-1)}$	$1+\frac{N-(p-1)}{2(p-1)}$	$\frac{N}{2(p-1)}$
C1	$1+\frac{N-(p+1)}{p(p^2-1)}$	–	$\frac{N}{p(p^2-1)}$
C2	$1+\frac{N-(p+1)}{p(p^2-1)}$	$1+\frac{N-p(p-1)}{p(p^2-1)}$	$\frac{N}{p(p^2-1)}$
C3	$2+\frac{N-(p^2+1)}{p(p^2-1)}$	–	$\frac{N}{p(p^2-1)}$

7. Representations

In this section we show how C-graphs can be used in the study of representation of orders and conversely. Let $\mathfrak{H}$ be an X-suborder of the maximal X-order $\mathfrak{D}$. In particular, this implies that $\mathfrak{H}(U) \subseteq \mathfrak{D}(U)$ for any open subset $U \subseteq X$. On the other hand, if $\mathfrak{H}$ is an X-order satisfying $\mathfrak{H}(U) \subseteq \mathfrak{D}(U)$ for $U = X\backslash\{P\}$, then $\mathfrak{H} \subseteq \mathfrak{D}$ if and only if $\mathfrak{H}_P \subseteq \mathfrak{D}_P$. For any effective divisor B we define the X-order $\mathfrak{H}^{[B]} = \mathcal{O}_X + \mathfrak{L}^{-B}\mathfrak{H}$. Representations of orders relate to local Bruhat–Tits trees by the following fundamental result:

Proposition 7.1. *Let P be a prime divisor of a global function field $K = K(X)$. Let $\mathfrak{H}$ be an arbitrary X-order in $\mathfrak{A}$. Then $\mathfrak{H}^{[tP]}$ is contained in a maximal X-order $\mathfrak{D}$ if and only if there exists a maximal X-order $\mathfrak{D}'$ containing $\mathfrak{H}$ such that the natural distance δ_P in the local Bruhat–Tits tree T_P satisfies $\delta_P(\mathfrak{D}, \mathfrak{D}') \leqslant t$.*

Proof. By an easy induction, it suffices to prove the case $t = 1$. Since $\mathfrak{H}^{[P]}$ coincides with $\mathfrak{H}$ outside of P, the result follows from the remarks at the beginning of this section if we proof the following result: *Let $\pi = \pi_P$ be a uniformizing parameter at P. Then a local order $\mathfrak{D}_P$ contains $\pi\mathfrak{H}_P$ if and only if either $\mathfrak{D}_P$ contains $\mathfrak{H}_P$ or $\mathfrak{D}_P$ has a neighbor containing $\mathfrak{H}_P$*. This is proved in [4, Prop. 2.4]. Here we give an alternative proof:

Two local orders $\mathfrak{D}_P$ and $\mathfrak{D}'_P$ are neighbors if and only if, up to a change of basis, they have the form

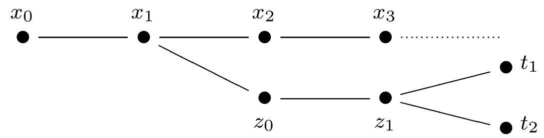

Fig. 8. $S_P(\mathfrak{D}_0)$ for the curve in Example 7.3 [17, §II.2.4.4].

$$\mathfrak{D}_P = \begin{pmatrix} \mathcal{O}_P & \mathcal{O}_P \\ \mathcal{O}_P & \mathcal{O}_P \end{pmatrix}, \qquad \mathfrak{D}'_P = \begin{pmatrix} \mathcal{O}_P & \pi^{-1}\mathcal{O}_P \\ \pi\mathcal{O}_P & \mathcal{O}_P \end{pmatrix}$$

(cf. [17, §II.1.1]) It follows that if $\mathfrak{H}_P$ is contained in $\mathfrak{D}_P$, then $\pi\mathfrak{H}_P$ is contained in every neighbor. Assume now that $\pi\mathfrak{H}_P$ is contained in $\mathfrak{D}_P$ as above, but $\mathfrak{H}_P$ is not contained in $\mathfrak{D}_P$. For any $h \in \mathfrak{H}_P$, let $\phi(h) \in \mathbb{M}_2(\mathbb{F}(P))$ be the reduction of πh. The image V of ϕ is a vector space over $\mathbb{F}(P)$ and, since the elements of $\mathfrak{H}_P$ are integral over $\mathcal{O}_P$, V contains only matrices with zero trace and determinant. Furthermore, some $h \in \mathfrak{H}_P$ is not in $\mathfrak{D}_P$, so V is not trivial. It must contain a non-zero element which, by a change of basis, it can be assumed to be $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. If V contains any other element of trace 0, say $\left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right)$ then it must contain $\left(\begin{smallmatrix} a & b+x \\ c & -a \end{smallmatrix}\right)$ for any $x \in \mathbb{F}(P)$, and therefore the determinant $-a^2 - (b+x)c$ must vanish identically, whence $c = a = 0$. We conclude that $\mathfrak{H}_P$ is contained in the lattice $\left(\begin{smallmatrix} \mathcal{O}_P & \pi^{-1}\mathcal{O}_P \\ \mathcal{O}_P & \mathcal{O}_P \end{smallmatrix}\right)$, and it contains an element h whose upper-right coordinate is not integer. Any such element with integral determinant is actually in the maximal order $\mathfrak{D}'_P = \left(\begin{smallmatrix} \mathcal{O}_P & \pi^{-1}\mathcal{O}_P \\ \pi\mathcal{O}_P & \mathcal{O}_P \end{smallmatrix}\right)$, which is a neighbor of $\mathfrak{D}_P$. If $h' \in \mathfrak{H}_P$ has an integral upper-right coordinate, the same conclusion follows by writing $h' = (h + h') - h$. $\square$

Example 7.2. Let $\mathfrak{H} = \mathfrak{H}_{\mathbb{L}}$ be as in Example 3.4 and Example 5.1, and let $\mathfrak{H}' = \mathfrak{H}^{[3Q+4R+S]}$, where Q, R, S are points of degrees $1, 2, 4$, respectively. We define the intermediate orders $\mathfrak{H}'' = \mathfrak{H}^{[3Q]}$, and $\mathfrak{H}''' = \mathfrak{H}^{[3Q+4R]}$. Note that the only maximal order containing a copy of $\mathfrak{H}$ is $\mathfrak{D}_0$ since it is the only vertex whose valency is 1 (cf. Example 5.1). Then the diagrams in Section 1 and Section 5 show that $\mathfrak{H}''$ is contained in $\mathfrak{D}_{tQ}$ for $t \leqslant 3$, while $\mathfrak{H}'''$ is contained in $\mathfrak{D}_{tQ}$ for $t \leqslant 11$, and finally $\mathfrak{H}'$ is contained in $\mathfrak{D}_{tQ}$ for $t \leqslant 15$.

Example 7.3. Let X be the curve over $\mathbb{F}_2$ defined by the projective equation $x^2z + xz^2 = y^3 + yz^2 + z^3$, and let $P = [1;0;0]$. The only non-trivial symmetry of $S_P(\mathfrak{D}_0)$ is the transposition interchanging t_1 and t_2 in Fig. 8. We conclude that either $C_P = S_P$ or C_P is obtained identifying these two vertices. The only vertices corresponding to split maximal orders are the x_i, whence they are the only orders representing $\mathfrak{H} \cong \mathcal{O}_X \times \mathcal{O}_X$. We conclude that the maximal order corresponding to t_1 contains a copy of $\mathfrak{H}^{[3P]}$, but not $\mathfrak{H}^{[2P]}$.

We say that $\mathfrak{H}$ is optimally contained in $\mathfrak{D}$, if $\mathfrak{H} \subseteq \mathfrak{D}$, but $\mathfrak{H}$ is not contained in $\mathfrak{D}^{[B]}$ for any effective divisor B. Let $\mathfrak{H}$ be an order of maximal rank, and let $\mathfrak{D}$ be a maximal order. The number of isomorphic copies of the order $\mathfrak{H}$ optimally contained into the maximal order $\mathfrak{D}$ is denoted $I(\mathfrak{D}|\mathfrak{H})$. The set of neighbors of $\mathfrak{D}$ in C_P is denoted $\mathcal{V}_P(\mathfrak{D})$. We let $N_P(\mathfrak{D},\mathfrak{D}')$ be as in Section 1.

Proposition 7.4. *Let $\mathfrak{H}$ be an order of maximal rank such that $\mathfrak{H}_P$ is maximal. Let $\mathfrak{D}$ be a maximal order. Then the number $I(\mathfrak{D}|\mathfrak{H}^{[P]})$ is given by the formula*:

$$I\big(\mathfrak{D}\big|\mathfrak{H}^{[P]}\big) = \sum_{\mathfrak{D}'\in\mathcal{V}_P(\mathfrak{D})} N_P\big(\mathfrak{D},\mathfrak{D}'\big)I\big(\mathfrak{D}'\big|\mathfrak{H}\big).$$

Proof. It is immediate from Proposition 7.1 that $\mathfrak{H}^{[P]}$ is optimally contained in a maximal order $\mathfrak{D}$ if and only if there exists a maximal order $\mathfrak{D}'$ optimally containing $\mathfrak{H}$ such that $\delta_P(\mathfrak{D},\mathfrak{D}') = 1$. Assume this is the case. Since $(\mathfrak{H}^{[P]})_Q = \mathfrak{H}_Q$ at every place $Q \neq P$, then $\mathfrak{H}$ is contained in the order $\mathfrak{D}'$ defined by the local conditions:

$$\mathfrak{D}'_Q = \left\{\begin{array}{ll}\mathfrak{D}_Q & \text{if } Q\neq P\\ \mathfrak{H}_Q & \text{if } Q=P\end{array}\right|.$$

This order is maximal and coincide with $\mathfrak{D}$ outside of P, whence it corresponds to a vertex in the Bruhat–Tits tree. On the other hand, $\mathfrak{D}'$ must be a neighbor of $\mathfrak{D}$, since $\mathfrak{D}'_P = \mathfrak{H}_P$ and $\mathfrak{D}_P$ contains $\mathfrak{H}_P^{[P]}$. Assume now that $\mathfrak{D}''$ is a second neighbor containing $\mathfrak{H}$. Then $\mathfrak{D}''$ coincide with $\mathfrak{D}$, and therefore also with $\mathfrak{D}'$, outside of P. Furthermore $\mathfrak{D}''_P \supseteq \mathfrak{H}_P = \mathfrak{D}'_P$, whence $\mathfrak{D}'' = \mathfrak{D}'$. We conclude that $\mathfrak{H}$ is contained in a unique P-neighbor of $\mathfrak{D}$ and the result follows. $\square$

Proof of Theorem 1.4. If $\mathfrak{H} = \mathfrak{D}''$ is a maximal order, then it follows from the preceding proposition that $I(\mathfrak{D}|\mathfrak{H}^{[P]}) = N_P(\mathfrak{D},\mathfrak{H})$. By a second application of the same result, for any pair of different places (P_1,P_2), we have

$$I\big(\mathfrak{D}\big|\mathfrak{H}^{[P_1+P_2]}\big) = \sum_{\mathfrak{D}'} N_{P_1}\big(\mathfrak{D},\mathfrak{D}'\big)N_{P_2}\big(\mathfrak{D}',\mathfrak{H}\big),$$

where the sum extends over all maximal orders, but only a finite number of terms are non-zero. As the left hand side of this equation is symmetric on P_1 and P_2, the result follows. $\square$

Proof of Theorem 1.5. Using the structure of cusps and case **A** of Table 3, we can see that, for m big enough, the m-th column C_m and the m-th row R_m of N_d have the forms

$$C_m = (\begin{array}{ccccccccc}0 & \cdots & 0 & p^d & 0 & \cdots & 0 & 1 & 0 & \cdots\end{array})^t,$$
$$R_m = (\begin{array}{ccccccccc}0 & \cdots & 0 & 1 & 0 & \cdots & 0 & p^d & 0 & \cdots\end{array}),$$

where the first non-zero entry, in each case, is in the position $m-d$ and the second one in the position $m+d$. It follows by an easy induction that the matrices

$$N_1^d, \quad \text{and} \quad \sum_{i=0}^{[d/2]} \binom{d}{i} p^i N_{d-2i},$$

coincide outside a finite set of coordinates. It follows that their difference B is finitely supported, and commutes with N_1. Eq. (1) is equivalent to $B = 0$. The result follows if we prove that the only finitely supported matrix that commutes with N_1 is the zero matrix.

Let $B = \left(\begin{smallmatrix} A & O_1 \\ O_2 & O_3 \end{smallmatrix}\right)$, where A denotes a minimal finite square block, and each O_i is an infinite block of 0's. Then, if $N_1 = \left(\begin{smallmatrix} C & D \\ E & F \end{smallmatrix}\right)$ is the analogous decomposition for N_1, the condition $BN_1 = N_1B$ implies $AD = 0$ and $EA = 0$. Looking at the first column of AD and the first row of EA we obtain the equations

$$A\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & \cdots & 0 & q \end{pmatrix} A = \begin{pmatrix} 0 & \cdots & 0 & 0 \end{pmatrix},$$

where q is either p or $p+1$. This implies that the last row and the last column of A are 0 and this contradicts the minimality of A. The result follows. $\square$

Example 7.5. The first six of the matrices N_i are N_1 and

$$N_2 = N_1^2 - 2pI, \qquad N_3 = N_1^3 - 3pN_1, \qquad N_4 = N_1^4 - 4pN_1^2 + 2p^2I,$$
$$N_5 = N_1^5 - 5pN_1^3 + 5p^2N_1, \qquad N_6 = N_1^6 - 6pN_1^4 + 9p^2N_1^2 - 2p^3I,$$

where I denotes the identity matrix.

8. Examples

Example 8.1. This example is a straightforward application of Proposition 7.1. Assume $X \cong \mathbb{P}_1$ and $d = \deg P = 5$. By valency considerations, and recalling that the C-graph is bipartite in this case, it must be as shown in Fig. 9. The valencies in Table 1 give us the following system:

$$c + e = 1, \qquad c + a = p^2 + 1, \qquad a + b = p^2 + p + 1, \qquad b + e = p + 1.$$

It follows that either $(a, b, c, e) = (p^2, p + 1, 1, 0)$ or $(a, b, c, e) = (p^2 + 1, p, 0, 1)$. Assume the second solution. Let $\mathfrak{H} = \mathfrak{D}_{7Q}^{[P+R]}$, where R has degree 4. Define also $\mathfrak{H}' = \mathfrak{D}_{7Q}^{[R]}$ and $\mathfrak{H}'' = \mathfrak{D}_{7Q}^{[P]}$. One application of Proposition 7.1 shows that the set of maximal X-orders containing $\mathfrak{H}'$ is $\{\mathfrak{D}_{7Q}, \mathfrak{D}_{3Q}, \mathfrak{D}_{11Q}\}$. Now, a second application of Proposition 7.1 shows that the set of maximal X-orders containing $\mathfrak{H}$ is

$$A = \{\mathfrak{D}_{7Q}, \mathfrak{D}_{3Q}, \mathfrak{D}_{11Q}, \mathfrak{D}_{2Q}, \mathfrak{D}_{12Q}, \mathfrak{D}_{8Q}, \mathfrak{D}_{6Q}, \mathfrak{D}_{16Q}\}.$$

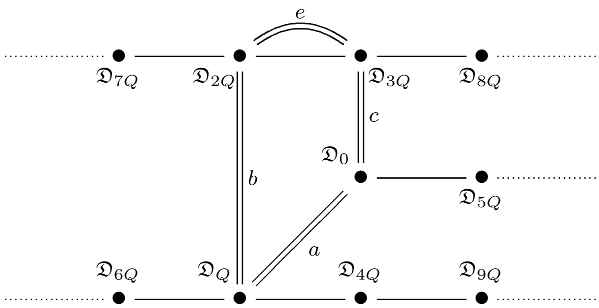

Fig. 9. The Full C-graph (or S-graph) when $\deg(P) = 5$ and $X = \mathbb{P}_1$.

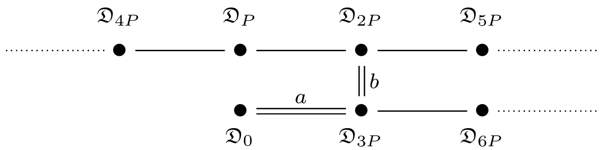

Fig. 10. Configuration of lines in Example 8.2

If we use $\mathfrak{H}''$ instead, we obtain the set $A \cup \{\mathfrak{D}_0\}$. As these sets are different, we conclude that the second solution is inconsistent, and therefore $(a, b, c, e) = (p^2, p + 1, 1, 0)$.

Example 8.2. Let X and P be as in Example 5.6, and let Z be the degree-3 prime divisor corresponding to the point $[\alpha; \alpha; 1] \in X_{\mathbb{F}_{27}}$, where $\alpha^3 - \alpha^2 + \alpha + 1 = 0$. Then $\mathrm{div}(x - y) = Z - 3P$, so Z is linearly equivalent to $3P$. Two lines in the local graph at Z, together with two edges with unknown multiplicities a and b, look as in Fig. 10. First we compute a. We use the equation

$$N_P(\mathfrak{D}_0, \mathfrak{D}_P)N_Z(\mathfrak{D}_P, \mathfrak{D}_{4P}) = N_Z(\mathfrak{D}_0, \mathfrak{D}_{3P})N_P(\mathfrak{D}_{3P}, \mathfrak{D}_{4P}),$$

corresponding to the pair $(\mathfrak{D}_0, \mathfrak{D}_{4P})$. Note that all other terms are 0, since $\mathfrak{D}_{5P}$ has valency 2 in T_Z, while neither $\mathfrak{D}_0$ nor $\mathfrak{D}_{4P}$ has other neighbors at P. This equation gives $N_Z(\mathfrak{D}_0, \mathfrak{D}_{3P}) = 4$, since all other values are known. For instance, $N_Z(\mathfrak{D}_P, \mathfrak{D}_{4P}) = 1$ since $\mathfrak{D}_{4P}$ corresponds to the orbit $[\infty]$ at $\mathfrak{D}_P$. By Table 3, case **C**, the class $[0] = [\infty]$ at $\mathfrak{D}_0$ has 4 elements, so $a = M_Z(\mathfrak{D}_0, \mathfrak{D}_{3P}) = 1$.

Next we show that $b = 0$. Assume $\mathfrak{D}_{2P}$ and $\mathfrak{D}_{3P}$ are neighbors at Z. Then the equation corresponding to the pair $(\mathfrak{D}_{3P}, \mathfrak{D}_{3P})$ reduces to

$$N_P(\mathfrak{D}_{3P}, \mathfrak{D}_{2P})N_Z(\mathfrak{D}_{2P}, \mathfrak{D}_{3P}) = N_Z(\mathfrak{D}_{3P}, \mathfrak{D}_{2P})N_P(\mathfrak{D}_{2P}, \mathfrak{D}_{3P}),$$

since $\mathfrak{D}_{4P}$ and $\mathfrak{D}_{3P}$ are not neighbors at Z. The extra edge for either $\mathfrak{D}_{2P}$ or $\mathfrak{D}_{3P}$ at Z corresponds to an orbit of size 18, whence the equation gives $3 \times 18 = 18 \times 1$. The contradiction yields the conclusion.

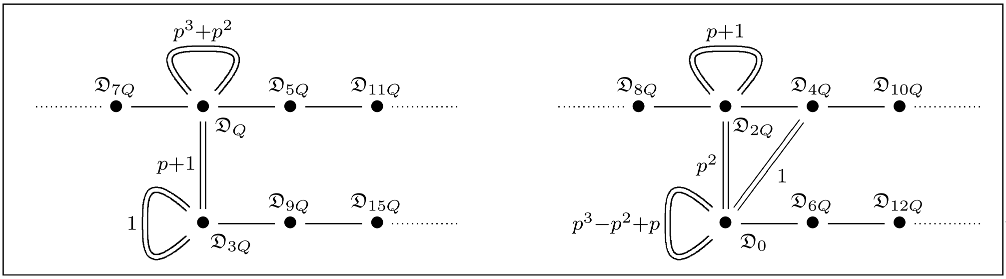

Fig. 11. The Full C-graph when $\deg(P) = 6$ and $X = \mathbb{P}_1$.

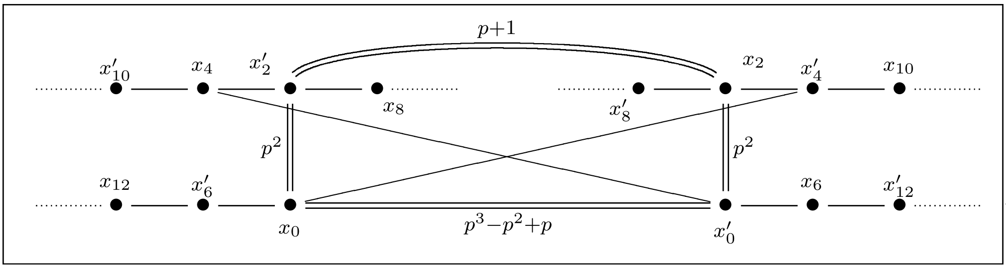

Fig. 12. The S-graph of $\mathfrak{D}_0$ when $\deg(P) = 6$ and $X = \mathbb{P}_1$.

Example 8.3. When $X = \mathbb{P}_1$ and $\deg(P) = 6$, the multiplicities (Fig. 11) can be computed one by one, as in the preceding example, or alternatively, we can use the matrix

$$N_6 = \begin{pmatrix} p^5(p-1) & 0 & p^5(p-1) & 0 & p^5(p-1) & 0 & p^6 & 0 & \cdots \\ 0 & p^4(p^2-1) & 0 & p^4(p^2-1) & 0 & p^6 & 0 & p^6 & \cdots \\ p^3(p^2-1) & 0 & p^3(p^2-1) & 0 & p^5 & 0 & 0 & 0 & \cdots \\ 0 & p^2(p^2-1) & 0 & p^4 & 0 & 0 & 0 & 0 & \cdots \\ p(p^2-1) & 0 & p^3 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & p^2 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ p+1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Note that $N_P(\mathfrak{D}_0, \mathfrak{D}_0) \equiv p(p-1) \pmod{p(p^2-1)}$, whence the exceptional edge for $\mathfrak{D}_0$ is in the loop. Recall that the S-graph is always bipartite, since Γ is contained in the group $\mathrm{GL}(V)^+$ in the notations of [17, §II.1.3]. Note that the distance between the two pre-images, x_i and x'_i, of every vertex $\mathfrak{D}_{iQ}$, is odd. In fact, the valuation of the element $M \in G$, constructed at the end of the proof of Theorem 1.3, is odd, so it must satisfy $M(x_i) = x'_i$ for every i. We conclude that the S-graph can be recovered as the unique bipartite 2-to-1 cover of the C-graph (Fig. 12).

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