J. Evol. Equ. 14 (2014), 417–444
© 2014 Springer Basel
1424-3199/14/020417-28, *published online* February 21, 2014
DOI 10.1007/s00028-014-0221-y

Journal of Evolution Equations

Periodic solutions of a fractional neutral equation with finite delay

VERÓNICA POBLETE AND JUAN C. POZO

Abstract. In this paper, we prove the maximal regularity property of an abstract fractional differential equation with finite delay on periodic Besov and Triebel–Lizorkin spaces and use these results to guarantee the existence and uniqueness of periodic solution of a neutral fractional differential equation with finite delay. The main tool used to achieve our goal is an operator-valued version of Miklhin's Fourier multiplier theorem and fixed-point argument.

1. Introduction

The fractional calculus which allows us to consider integration and differentiation of any order, not necessarily integer, has been the object of an extensive study for analyzing not only anomalous diffusion on fractals (physical objects of fractional dimension, such as some amorphous semiconductors or strongly porous materials. See [2,34] and references therein), but also fractional phenomena in optimal control (see, e.g., [35–37]). As indicated in [14,33] and the related references given there, the advantages of fractional derivatives become apparent in modeling mechanical and electrical properties of real materials, as well as in the description of rheological properties of rocks, and in many other fields. One of the emerging branches of the study is the Cauchy problems for abstract differential equations involving fractional derivatives in time. In recent decades, there has been a lot of interest in this type of problems, its applications and various generalizations (cf. e.g., [5,11,18] and references therein). It is significant to study this class of problems, because, in this way, one is more realistic to describe the memory and hereditary properties of various materials and processes (cf. [21,28,35,36]).

In the same manner, several systems of great interest in science are modeled by partial neutral functional differential equations. The reader can see [1,17,38,39]. Many of these equations can be written as abstract neutral functional differential equations. Additionally, it is well known that one of the most interesting topics, both from a theoretical as practical point of view, of the qualitative theory of differential equations

Mathematics Subject Classification (2010): 34K40, 39A23, 34K37
Keywords: Maximal regularity, Fourier multipliers, Strong solutions, Fractional neutral equations.
The authors are partially financed by FONDECYT 1110090. The second author is financed by MECESUP PUC 0711.

and functional differential equations is the existence of periodic solutions. In particular, the existence of periodic solutions of abstract neutral functional differential equation has been considered in several works [16,19,20,22].

Let $0 < \beta < \alpha \leqslant 2$. This paper is devoted to the study of sufficient conditions that guarantee the existence and uniqueness of a periodic strong solution for the following fractional order abstract neutral differential equation with finite delay

$$D^{\alpha}\big(u(t) - Bu(t-r)\big) = Au(t) + Fu_t + GD^{\beta}u_t + f(t), \quad t \in [0, 2\pi], \tag{1.1}$$

where the fractional derivative is taken in sense of Liouville–Grünwald–Letnikov, the delay $r > 0$ is a fixed number, $A : D(A) \subseteq X \to X$ and $B : D(B) \subseteq X \to X$ are closed linear operators defined in a Banach space X such that $D(A) \subseteq D(B)$. The function u_t is given by $u_t(\theta) = u(t+\theta)$ for $\theta \in [-2\pi, 0]$, and denotes the history of the function $u(\cdot)$ at t and $D^{\beta}u_t(\cdot)$ is defined by $D^{\beta}u_t(\cdot) = \big(D^{\beta}u\big)_t(\cdot)$. The operators F and G are called delay operators, and they belong to appropriate spaces, which will be described later. The map f is a X-valued function which belongs to either periodic Besov spaces, or periodic Triebel–Lizorkin spaces.

We prove the maximal regularity property of an auxiliary equation, on periodic Besov spaces and periodic Triebel–Lizorkin spaces, and using this result together with fixed-point argument to show existence and uniqueness of periodic solution of Eq. (1.1). Here, the auxiliary equation is given by

$$D^{\alpha}u(t) = Au(t) + Fu_t + GD^{\beta}u_t + f(t), \quad t \in [0, 2\pi], \tag{1.2}$$

with boundary periodic conditions depending of the values of the numbers α and β. All terms in the Eq. (1.2) are defined in the same manner as in the Eq. (1.1).

Our main results involve, among other considerations, a boundedness condition for the family

$$\big\{(ik)^{\alpha}\big((ik)^{\alpha} - F_k - (ik)^{\beta}G_k - A\big)^{-1}\big\}_{k\in\mathbb{Z}},$$

and regularity properties for the families of bounded operators $\{F_k\}_{k\in\mathbb{Z}}$ and $\{G_k\}_{k\in\mathbb{Z}}$, defined by

$$F_k x = F(e_k x) \quad \text{and} \quad G_k x = G(e_k x), \text{ where } (e_k x)(t) = e^{ikt}x$$
$$\text{with } x \in X, \quad t \in [-2\pi, 0] \text{ and } k \in \mathbb{Z}.$$

In recent years, several particular cases of the Eq. (1.2) have been studied. If $\alpha = 1$ and $F \equiv G \equiv 0$, Arendt and Bu [3,4] have studied L^p-maximal regularity and $B^s_{p,q}$-maximal regularity, and Bu and Kim [8], have studied $F^s_{p,q}$-maximal regularity. On the other hand, Lizama [30] has obtained a characterization of the existence and uniqueness of strong L^p-solutions, and Lizama and Poblete [31] study C^s-maximal regularity of the corresponding equation on the real line. In the same manner, if $\alpha = 2$ and $\beta = 1$, Bu [6] characterizes C^s-maximal regularity on $\mathbb{R}$. Furthermore, if $\alpha = 2$ and $\beta = 1$, Bu and Fang [7] have studied this equation simultaneosly in periodic Lebesgue spaces, periodic Besov spaces and periodic Triebel–Lizorkin spaces. Moreover, if $1 < \alpha < 2$ and $G \equiv 0$, Lizama and Poblete [32] study L^p-maximal regularity for this equation in the periodic case.

This paper is organized as follows. In Sect. 2, we introduce some important notation and collect relevant theorems and concepts that are needed to establish our main results. In Sects. 3, 4, 5 and 6, we present our results of existence and uniqueness of a periodic solution of the Eqs. (1.1) and (1.2). Finally, in Sect. 7, we apply our abstract results to concrete situations.

2. Preliminaries

Most of the notation used throughout this work is standard. So, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ denote the set of natural, integers, real and complex numbers, respectively.

Further, X and Y always are complex Banach spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$; the subscript will be dropped when there is no danger of confusion. We denote the space of all bounded linear operators from X to Y by $\mathcal{L}(X; Y)$. In the case $X = Y$, we will write briefly $\mathcal{L}(X)$. Let A be an operator defined in X. We will denote its domain by $D(A)$, its domain endowed with the graph norm by $[D(A)]$, its resolvent set by $\rho(A)$, and its spectrum by $\sigma(A) = \mathbb{C} \setminus \rho(A)$.

Let $1 \leqslant p < \infty$, $J \subseteq \mathbb{R}$ an interval of real numbers, and X a Banach space. By $L^p(J; X)$, we denote the Banach space of all p-integrable functions (in sense of Bochner) endowed with the norm

$$\|f\|_{L^p(J,X)} = \left(\int_J \|f(t)\|_X^p\right)^{1/p}.$$

For the rest of the paper, we will identify $\mathbb{T}$ with the group defined as the quotient $\mathbb{R}/2\pi\mathbb{Z}$, and we shall identify the spaces of vector or operator-valued functions defined on $[0, 2\pi]$ to their periodic extensions to $\mathbb{R}$. Let $f \in L^1(\mathbb{T}; X)$. For $k \in \mathbb{Z}$, we denote the k-th Fourier coefficient of the function f by

$$\widehat{f}(k) = \frac{1}{2\pi}\int_0^{2\pi} e^{-ikt} f(t)\mathrm{d}t.$$

There exist several notions of fractional differentiation. In this paper, we use the fractional differentiation in sense of Liouville–Grünwald–Letnikov. This concept was introduced in [15,29] and has been widely studied by several authors. In these works, the fractional derivative is defined directly as a limit of a fractional difference quotient. In [10], the authors apply this approach based on fractional differences to study fractional differentiation of periodic scalar functions. This idea has been used to extend the definition of fractional differentiation to vector-valued functions, (see [26]). In the case of periodic functions, this concept enables one to set up a fractional calculus in the L^p setting with the usual rules, as well as provides a connection with the classical Weyl fractional derivative (see [37]).

Let $\alpha > 0$ and $f \in L^p(\mathbb{T}; X)$ for $1 \leqslant p < \infty$ the Riemann difference of f is defined by

$$\Delta_t^\alpha f(x) = \sum_{j=0}^{\infty} (-1)^j \binom{\alpha}{j} f(x - tj),$$

where $\binom{\alpha}{j} = \frac{\alpha(\alpha-1)\cdots(\alpha-j-1)}{j!}$ is the binomial coefficient. The Riemann difference of the function f exists almost everywhere, (see [10]). Moreover, $\sum_{j=0}^{\infty} |\binom{\alpha}{j}| < \infty$, and

$$\|\Delta_t^\alpha f\|_{L^p(\mathbb{T};X)} \leqslant \sum_{j=0}^{\infty} \left|\binom{\alpha}{j}\right| \|f\|_{L^p(\mathbb{T};X)}.$$

The following definition is a direct extension of [10, Definition 2.1] to the vector-valued case. See [26] for its connection with differential equations.

DEFINITION 2.1. Let X be a Banach space, $\alpha > 0$ and $1 \leqslant p < \infty$. Let $f \in L^p(\mathbb{T}; X)$. If there is $g \in L^p(\mathbb{T}; X)$ such that $\lim_{t \to 0^+} t^{-\alpha} \Delta_t^\alpha f = g$ in the $L^p(\mathbb{T}; X)$ norm, then the funcion g is called the αth-***Liouville–Grünwald–Letnikov derivative*** of f in the mean of order p.

We abbreviate this terminology by αth-derivative and we denote it by $D^\alpha f = g$. We also mention here a few properties of this fractional derivative. The proof of the following proposition follows the same steps as in the scalar case given in [10, Proposition 4.1].

PROPOSITION 2.2. *Let* $1 \leqslant p < \infty$ *and* $f \in L^p(\mathbb{T}; X)$. *For* $\alpha, \beta > 0$ *the following properties hold:*

- *If* $D^\alpha f \in L^p(\mathbb{T}; X)$, *then* $D^\beta f \in L^p(\mathbb{T}; X)$ *for all* $0 < \beta < \alpha$.
- $D^\alpha D^\beta f = D^{\alpha+\beta} f$ *whenever one of the two sides is well defined.*

REMARK 2.3. Let $f \in L^p(\mathbb{T}; X)$ and $\alpha > 0$. It has been proved by Butzer and Westphal [10] that $D^\alpha f \in L^p(\mathbb{T}; X)$ if and only if there exists $g \in L^p(\mathbb{T}; X)$ such that for all $k \in \mathbb{Z}$ it holds $(ik)^\alpha \widehat{f}(k) = \widehat{g}(k)$, where $(ik)^\alpha = |k|^\alpha e^{\frac{\pi i \alpha}{2} \operatorname{sgn}(k)}$. In this case $D^\alpha f = g$.

On the other hand, periodic Besov spaces and periodic Triebel–Lizorkin spaces form part of functions spaces which have a lot of interest in mathematics. They generalize many important functions spaces. For example, if $0 < s < 1$, the periodic Hölder continuous functions space of index s, is a particular case of periodic Besov spaces, see [4] for more details. However, the main reason for working in these spaces is that a certain form of Mikhlin's multiplier theorem holds for operator-valued symbols defined in arbitrary Banach spaces X. This is a dramatic contrast to Lebesgue spaces where the corresponding theorem merely holds for Hilbert spaces even when $p = 2$ (for more information [13]).

Let X be a Banach space. Let $\mathcal{S}(\mathbb{R})$ be the Schwartz space of all rapidly decreasing smooth functions on $\mathbb{R}$. Let $\mathcal{D}(\mathbb{T})$ be the space of all infinitely differentiable functions on $\mathbb{T}$ equipped with the topology given by the seminorms $\|f\|_n = \sup_{t\in\mathbb{T}} |f^{(n)}(t)|$, where $n \in \mathbb{N}\cup\{0\}$. Let $\mathcal{D}'(\mathbb{T}; X) = \mathcal{L}(\mathcal{D}(\mathbb{T}); X)$ be the space of all bounded linear operators from $\mathcal{D}(\mathbb{T})$ to X. The elements of $\mathcal{D}'(\mathbb{T}; X)$ are called X-valued distributions on $\mathbb{T}$. Let $k \in \mathbb{Z}$, denote by e_k the function $e_k(t) = \mathrm{e}^{ikt}$ for $t \in \mathbb{T}$. For $x \in X$, we denote by $(e_k \otimes x)$ the X-valued function given by $(e_k \otimes x)(t) = e_k(t)x$. Consequently we have that $(e_k \otimes x) \in \mathcal{D}'(\mathbb{T}; X)$.

In order to define the periodic X-valued Besov spaces, we denote by $\Phi(\mathbb{R})$ the set of all systems $\{\phi_j\}_{j\geqslant 0} \subseteq \mathcal{S}(\mathbb{R})$ such that $supp(\phi_0) \subseteq [-2, 2]$, and for all $j \in \mathbb{N}$

$$\operatorname{supp}(\phi_j) \subseteq \left[-2^{j+1}, -2^{j-1}\right] \cup \left[2^{j-1}, 2^{j+1}\right], \qquad \sum_{j\geqslant 0} \phi_j(t) = 1, \text{ for } t \in \mathbb{R}$$

and, for $\alpha \in \mathbb{N}\cup\{0\}$, there is a $C_\alpha > 0$ such that $\sup_{j\geqslant 0, x\in\mathbb{R}} 2^{\alpha j}\|\phi_j^{(\alpha)}(x)\| \leqslant C_\alpha$.

DEFINITION 2.4. [4] Let $1 \leqslant p, q \leqslant \infty$, $s \in \mathbb{R}$ and $\phi = \{\phi_j\}_{j\geqslant 0} \in \Phi(\mathbb{R})$. The ***X-valued periodic Besov space*** is defined by

$$B^{s,\phi}_{p,q}(\mathbb{T}; X) = \left\{ f \in \mathcal{D}'(\mathbb{T}; X) : \|f\|_{B^{s,\phi}_{p,q}} < \infty \right\},$$

where

$$\|f\|_{B^{s,\phi}_{p,q}} = \left(\sum_{j\geqslant 0} 2^{jsq} \left\| \sum_{k\in\mathbb{Z}} e_k \otimes \phi_j(k)\widehat{f}(k) \right\|_p^q \right)^{\frac{1}{q}},$$

with usual modifications when $q = \infty$. The space $B^{s,\phi}_{p,q}$ is independent of $\phi \in \Phi(\mathbb{R})$ and different choices of $\phi \in \Phi(\mathbb{R})$ generate equivalent norms. As consequence, we will denote $\|\cdot\|_{B^{s,\phi}_{p,q}}$ simply by $\|\cdot\|_{B^s_{p,q}}$.

Moreover, if $r \in \mathbb{T}$ is fixed, we say that a function $u : [r, r + 2\pi] \to X$ belongs $B^s_{p,q}([r, r+2\pi]; X)$ if and only if the periodic extension to $\mathbb{R}$ of the function u belongs to $B^s_{p,q}(\mathbb{T}; X)$.

We recall some important properties of these spaces. Let $1 \leqslant p, q \leqslant \infty$, $s \in \mathbb{R}$ be fixed.

- The X-valued periodic space $B^s_{p,q}(\mathbb{T}; X)$ is a Banach space.
- If $s > 0$, the natural injection from $B^s_{p,q}(\mathbb{T}; X)$ into $L^p(\mathbb{T}; X)$ is a continuous linear operator.
- For all $\varepsilon > 0$, we have that $B^{s+\varepsilon}_{p,q}(\mathbb{T}; X) \subseteq B^s_{p,q}(\mathbb{T}; X)$.
- (Lifting property) Let $f \in \mathcal{D}'(\mathbb{T}; X)$ and $\gamma \in \mathbb{R}$ then $f \in B^s_{p,q}(\mathbb{T}; X)$ if and only if $\sum_{k\neq 0} e_k \otimes (ik)^\gamma \widehat{f}(k) \in B^{s-\gamma}_{p,q}(\mathbb{T}; X)$.

To define the periodic X-valued Triebel–Lizorkin spaces, we use the same notation for $\mathcal{S}(\mathbb{R})$, $\mathcal{D}(\mathbb{T})$, $\mathcal{D}'(\mathbb{T}; X)$ and $\Phi(\mathbb{R})$ as those which we have used in the definition of X-valued periodic Besov spaces.

DEFINITION 2.5. [9] Let $\phi = \{\phi_j\}_{j\geqslant 0} \in \Phi(\mathbb{R})$ be fixed, for $1 \leqslant p, q \leqslant \infty$, and $s \in \mathbb{R}$. The ***X-valued periodic Triebel–Lizorkin space*** is defined by

$$F_{p,q}^{s,\phi}(\mathbb{T}; X) = \left\{ f \in \mathcal{D}'(\mathbb{T}; X) : \|f\|_{F_{p,q}^{s,\phi}} < \infty \right\},$$

where

$$\|f\|_{F_{p,q}^{s,\phi}} = \left\| \left(\sum_{j \geqslant 0} 2^{jsq} \left\| \sum_{k \in \mathbb{Z}} e_k \otimes \phi_j(k) \widehat{f}(k) \right\|_X^q \right)^{\frac{1}{q}} \right\|_p ,$$

with the usual modification when $q = \infty$. The space $F_{p,q}^{s,\phi}$ is independent of $\phi \in \Phi(\mathbb{R})$ and different choices of $\phi \in \Phi(\mathbb{R})$ generate equivalent norms. Consequently, we simply denote $\|\cdot\|_{F_{p,q}^{s,\phi}}$ by $\|\cdot\|_{F_{p,q}^{s}}$.

Moreover, if $r \in \mathbb{T}$ is fixed, we say that a function $u : [r, r + 2\pi] \to X$ belongs $F_{p,q}^{s}([r, r+2\pi]; X)$ if and only if the periodic extension to $\mathbb{R}$ of the function u belongs to $F_{p,q}^{s}(\mathbb{T}; X)$.

Note that X-valued periodic Triebel–Lizorkin spaces have similar properties to those of X-valued periodic Besov spaces, the reader can see [9]. The following list summarizes the most elementary properties of Triebel–Lizorkin spaces. Let $1 \leqslant p, q \leqslant \infty$, $s \in \mathbb{R}$ be fixed.

- The X-valued periodic space $F_{p,q}^{s}(\mathbb{T}; X)$ is a Banach space.
- If $s > 0$, then the natural injection from $F_{p,q}^{s}(\mathbb{T}; X)$ into $L^p(\mathbb{T}; X)$ is a continuous linear operator.
- For all $\varepsilon > 0$, we have that $F_{p,q}^{s+\varepsilon}(\mathbb{T}; X) \subseteq F_{p,q}^{s}(\mathbb{T}; X)$.
- (Lifting property) Let $f \in \mathcal{D}'(\mathbb{T}; X)$ and $\gamma \in \mathbb{R}$ then $f \in F_{p,q}^{s}(\mathbb{T}; X)$ if and only if $\sum_{k \neq 0} e_k \otimes (ik)^\gamma \widehat{f}(k) \in F_{p,q}^{s-\gamma}(\mathbb{T}; X)$.

REMARK 2.6. It is simple to verify from the definition that if $u \in B_{p,q}^{s}(\mathbb{T}; X)$ and $t_0 \in [0, 2\pi]$ is fixed, then the function u_{t_0} defined on $[-2\pi, 0]$ by the formula $u_{t_0}(\theta) = u(t_0 + \theta)$, is an element of the Besov space $B_{p,q}^{s}(\mathbb{T}; X)$, and $\|u_{t_0}\|_{B_{p,q}^{s}} = \|u\|_{B_{p,q}^{s}}$. For periodic Triebel–Lizorkin, we have a similar result.

Let X be a Banach space. In order to develop certain conditions which we will need for the rest of the paper, we establish the following notation. Let $\{L_k\}_{k \in \mathbb{Z}} \subset \mathcal{L}(X)$ be a bounded family of operators. Set

$$\Delta^0 L_k = L_k, \quad \Delta L_k = \Delta^1 L_k = L_{k+1} - L_k$$

and for $n = 2, 3, \ldots,$ set

$$\Delta^n L_k = \Delta \left(\Delta^{n-1} L_k \right).$$

DEFINITION 2.7. [25] We say that a family of operators $\{L_k\}_{k \in \mathbb{Z}} \subset \mathcal{L}(X)$ is a ***$\mathcal{M}$-bounded family of order n*** $(n \in \mathbb{N}_0)$ if

$$\sup_{0 \leqslant l \leqslant n} \sup_{k \in \mathbb{Z}} \|k^l \Delta^l L_k\| < \infty. \tag{2.1}$$

Note that, for $j \in \mathbb{Z}$ fixed, $\sup_{0\leqslant l\leqslant n}\sup_{k\in\mathbb{Z}}\|k^l\,\Delta^l L_k\| < \infty$, if and only if $\sup_{0\leqslant l\leqslant n}\sup_{k\in\mathbb{Z}}\|k^l\,\Delta^l L_{k+j}\| < \infty$. The statement follows directly from the binomial formula.

In the preceding definition when $n = 0$, the $\mathcal{M}$-boundedness of order n for $\{L_k\}_{k\in\mathbb{Z}}$ simply means that the family of operators $\{L_k\}_{k\in\mathbb{Z}}$ is bounded. When $n = 1$, this is equivalent to

$$\sup_{k\in\mathbb{Z}}\|L_k\| < \infty \quad \text{and} \quad \sup_{k\in\mathbb{Z}}\|k\,(L_{k+1} - L_k)\| < \infty. \tag{2.2}$$

If $n = 2$, in addition to (2.2), we must have

$$\sup_{k\in\mathbb{Z}}\|k^2\,(L_{k+2} - 2L_{k+1} + L_k)\| < \infty. \tag{2.3}$$

For $n = 3$, in addition to (2.2) and (2.3), we must have

$$\sup_{k\in\mathbb{Z}}\|k^3\,(L_{k+3} - 3L_{k+2} + 3L_{k+1} - L_k)\| < \infty. \tag{2.4}$$

In the scalar case, that is, $\{a_k\}_{k\in\mathbb{Z}} \subseteq \mathbb{C}$, we will write $\Delta^n a_k = \Delta(\Delta^{n-1}a_k)$.

DEFINITION 2.8. [23] A sequence $\{a_k\}_{k\in\mathbb{Z}} \subseteq \mathbb{C}$ is called

(a) **1-regular** if the sequence $\left\{k\,\dfrac{\Delta^1 a_k}{a_k}\right\}_{k\in\mathbb{Z}}$ is bounded;

(b) **2-regular** if it is 1-regular and the sequence $\left\{k^2\,\dfrac{\Delta^2 a_k}{a_k}\right\}_{k\in\mathbb{Z}}$ is bounded;

(c) **3-regular** if it is 2-regular and the sequence $\left\{k^3\,\dfrac{\Delta^3 a_k}{a_k}\right\}_{k\in\mathbb{Z}}$ is bounded.

For useful properties and further details about n-regularity, see [27].

REMARK 2.9. Note that if $\{a_k\}_{k\in\mathbb{Z}}$ is an 1-regular sequence then, for all $j \in \mathbb{Z}$ fixed, the sequence $\left\{k\dfrac{a_{k+j} - a_k}{a_{k+j}}\right\}_{k\in\mathbb{Z}}$ is bounded. In the cases $n = 2, 3$, analogous properties hold.

The following definitions will be used with Besov and Triebel–Lizorkin spaces. Let X be a Banach space. We denote the space consisting of all 2π-periodic, X-valued functions by $E(\mathbb{T}; X)$.

One of the most powerful methods for proving maximal regularity of evolution equations is the technique of Fourier multipliers. For this reason, we recall some operator-valued Fourier multipliers theorems.

DEFINITION 2.10. Let X be a Banach space. We say that the family of operators $\{L_k\}_{k\in\mathbb{Z}} \subseteq \mathcal{L}(X)$ is an E**-multiplier** if for each $f \in E(\mathbb{T}; X)$, there exists a function $u \in E(\mathbb{T}; X)$ such that

$$\widehat{u}(k) = L_k\widehat{f}(k), \quad \text{for all } k \in \mathbb{Z}.$$

The following theorem, proved by Arendt and Bu in [4], establishes a sufficient condition ensuring when a family of operators $\{L_k\}_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-multiplier. It is remarkable that this theorem is valid on an arbitrary Banach space X.

THEOREM 2.11. *Let $1 \leqslant p, q \leqslant \infty$, and $s \in \mathbb{R}$. Let X be a Banach space. If the family of operators $\{L_k\}_{k\in\mathbb{Z}} \subseteq \mathcal{L}(X)$ is $\mathcal{M}$-bounded of order 2, then $\{L_k\}_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-multiplier.*

Next theorem, proved by Bu and Kim in [9] , establishes a sufficient condition which guarantees when a family of operators $\{L_k\}_{k\in\mathbb{Z}}$ is a $F^s_{p,q}$-multiplier. We remark, as well as in theorem 2.11, this theorem is valid for arbitrary Banach space X, however, more conditions are imposed to the family $\{L_k\}_{k\in\mathbb{Z}}$.

THEOREM 2.12. *Let $1 \leqslant p, q \leqslant \infty$, and $s \in \mathbb{R}$. Let X be a Banach space. If the family of operators $\{L_k\}_{k\in\mathbb{Z}} \subseteq \mathcal{L}(X)$ is $\mathcal{M}$-bounded of order 3, then $\{L_k\}_{k\in\mathbb{Z}}$ is a $F^s_{p,q}$-multiplier.*

In order to abbreviate the text of this work, we introduce the following notation. Let $1 \leqslant p, q \leqslant \infty$ and $s > 0$ and $0 < \beta < \alpha \leqslant 2$. Assume that A is an operator defined in a Banach space X, and that $F \in \mathcal{L}(B^{s+\alpha}_{p,q}([-2\pi, 0]; X); X)$ and $G \in \mathcal{L}(B^{s+\alpha-\beta}_{p,q}([-2\pi, 0]; X); X)$ or $F \in \mathcal{L}(F^{s+\alpha}_{p,q}([-2\pi, 0]; X); X)$ and $G \in \mathcal{L}(F^{s+\alpha-\beta}_{p,q}([-2\pi, 0]; X); X)$ are linear, bounded operators. For $k \in \mathbb{Z}$, we will write

$$a_k = (ik)^{\alpha} \quad \text{and} \quad b_k = (ik)^{\beta}, \tag{2.5}$$

where $(ik)^{\gamma} = |k|^{\gamma} e^{\frac{\pi i \gamma}{2}\operatorname{sgn}(k)}$. Note that $\{a_k\}_{k\in\mathbb{Z}}$ and $\{b_k\}_{k\in\mathbb{Z}}$ are 2, 3−regular sequences.

Now, the bounded linear operators F_k and G_k are defined by $F_k x = F(e_k x)$ and $G_k x = G(e_k x)$, where $(e_k x)(t) = e^{ikt} x$ for all $t \in [-2\pi, 0]$ and $x \in X$.

For reference purposes, we introduce the following conditions for the families $\{F_k\}_{k\in\mathbb{Z}}$ and $\{G_k\}_{k\in\mathbb{Z}}$.

(F2) For $l = 0, 1, 2$, the family of operators $\left\{\frac{k^l}{a_k}\Delta^l F_k\right\}_{k\in\mathbb{Z}\setminus\{0\}}$ are bounded.

(F3) The family $\{F_k\}_{k\in\mathbb{Z}}$ satisfies ***(F2)*** and the family $\left\{\frac{k^3}{a_k}\Delta^3 F_k\right\}_{k\in\mathbb{Z}\setminus\{0\}}$ is bounded.

(G2) For $l = 0, 1, 2$, the families of operators $\left\{\frac{b_k}{a_k} k^l \Delta^l G_k\right\}_{k\in\mathbb{Z}\setminus\{0\}}$ are bounded.

(G3) The family $\{G_k\}_{k\in\mathbb{Z}}$ satisfies ***(G2)*** and the family $\left\{\frac{b_k}{a_k} k^3 \Delta^3 G_k\right\}_{k\in\mathbb{Z}\setminus\{0\}}$ is bounded.

3. Maximal regularity on periodic Besov spaces

Let $1 \leqslant p, q \leqslant \infty$, $s > 0$ and $0 < \beta < \alpha \leqslant 2$. The first objective of this section is the study of $B^s_{p,q}$-maximal regularity of the fractional neutral equation

$$D^{\alpha} u(t) = Au(t) + Fu_t + GD^{\beta} u_t + f(t), \qquad t \in [0, 2\pi], \tag{3.1}$$

where the fractional derivative is taken in sense of Liouville–Grünwald–Letnikov. The operator $A : D(A) \subseteq X \to X$ is a closed linear operator defined in a Banach space X. The function u_t is defined by $u_t(\theta) = u(t+\theta)$ for $\theta \in [-2\pi, 0]$, and denotes the history of the function $u(\cdot)$ at t. Further, $D^\beta u_t(\cdot)$ is defined by $D^\beta u_t(\cdot) = \left(D^\beta u\right)_t(\cdot)$. We suppose that $F \in \mathcal{L}(B^{s+\alpha}_{p,q}([-2\pi, 0]; X); X)$ and $G \in \mathcal{L}(B^{s+\alpha-\beta}_{p,q}([-2\pi, 0]; X); X)$. The mapping f is a X-valued function which belongs to the periodic Besov space $B^s_{p,q}(\mathbb{T}; X)$. Moreover, we assume that this equation has periodic boundary conditions depending of the numbers α and β,

$$\left.\begin{array}{ll} u(0) = u(2\pi) & \text{if } 0 < \beta < \alpha \leqslant 1, \\ u(0) = u(2\pi) \text{ and } D^{\alpha-1}u(0) = D^{\alpha-1}u(2\pi) & \text{if } 0 < \beta \leqslant 1 < \alpha \leqslant 2, \\ u(0) = u(2\pi)\,,\ D^{\alpha-1}u(0) = D^{\alpha-1}u(2\pi)\,, & \\ D^{\beta-1}u(0) = D^{\beta-1}u(2\pi) & \text{if } 1 < \beta < \alpha \leqslant 2, \end{array}\right\}$$

Let $\alpha > 0$. We establish a characterization of the periodic Besov space $B^{s+\alpha}_{p,q}(\mathbb{T}; X)$ in terms of the fractional derivative.

PROPOSITION 3.1. *Let X be a Banach space and $1 \leqslant p, q \leqslant \infty$ and $s > 0$. If $\alpha > 0$ then*

$$B^{s+\alpha}_{p,q}(\mathbb{T}; X) = \left\{u \in B^s_{p,q}(\mathbb{T}; X) : D^\alpha u \in B^s_{p,q}(\mathbb{T}; X)\right\}.$$

Proof. Suppose that $u \in B^s_{p,q}(\mathbb{T}; X)$ and $D^\alpha u \in B^s_{p,q}(\mathbb{T}; X)$. By the *lifting property* we have that

$$\sum_{k \neq 0} e_k \otimes \widehat{D^\alpha u}(k) \in B^s_{p,q}(\mathbb{T}; X).$$

Since $s > 0$, we have that $D^\alpha u \in L^p(\mathbb{T}; X)$ then $\widehat{D^\alpha u}(k) = (ik)^\alpha \hat{u}(k)$, for all $k \in \mathbb{Z}$, hence

$$\sum_{k \neq 0} e_k \otimes (ik)^\alpha \widehat{u}(k) \in B^s_{p,q}(\mathbb{T}; X).$$

Using again the *lifting property* we obtain that $u \in B^{s+\alpha}_{p,q}(\mathbb{T}; X)$.

Reciprocally, let $u \in B^{s+\alpha}_{p,q}(\mathbb{T}; X)$, it is clear $u \in B^s_{p,q}(\mathbb{T}; X)$. Furthermore,

$$\sum_{k \neq 0} e_k \otimes (ik)^\alpha \widehat{u}(k) \in B^s_{p,q}(\mathbb{T}; X) \subset L^p(\mathbb{T}; X). \tag{3.2}$$

It follows from [10, Theorem 4.1] that there exists $g \in L^p(\mathbb{T}; X)$ such that $\widehat{g}(k) = (ik)^\alpha \widehat{u}(k)$ for all $k \in \mathbb{Z}$. From (3.2) we have that $g \in B^s_{p,q}(\mathbb{T}; X)$. Therefore $D^\alpha u \in B^s_{p,q}(\mathbb{T}; X)$ and $\widehat{D^\alpha u}(k) = (ik)^\alpha \widehat{u}(k)$ for all $k \in \mathbb{Z}$. □

REMARK 3.2. Let $1 \leqslant p, q \leqslant \infty$ and $s > 0$. According to the preceding Proposition if $u \in B^{s+\alpha}_{p,q}(\mathbb{T}; X)$ and $0 < \beta < \alpha$ then $D^\beta u \in B^{s+\alpha-\beta}_{p,q}(\mathbb{T}; X)$.

Let $s > 0$, using the previous characterization we define the concept of $B^s_{p,q}$-maximal regularity of the Eq. (3.1).

DEFINITION 3.3. Let $1 \leqslant p, q \leqslant \infty$, $s > 0$ and let $f \in B^s_{p,q}(\mathbb{T}; X)$. A function u is called strong $B^s_{p,q}$-solution of the Eq. (3.1) if $u \in B^{s+\alpha}_{p,q}(\mathbb{T}; X) \cap B^s_{p,q}(\mathbb{T}; [D(A)])$ and u satisfies the Eq. (3.1) for almost $t \in [0, 2\pi]$ and the functions $t \mapsto Fu_t$, $t \mapsto GD^\beta u_t$ belong to $B^s_{p,q}(\mathbb{T}; X)$. We say that the Eq. (3.1) has $B^s_{p,q}$-maximal regularity if, for each $f \in B^s_{p,q}(\mathbb{T}; X)$ the Eq. (3.1) has unique strong $B^s_{p,q}$-solution.

One of the main results of this paper is the Theorem 3.9. Its proof depends of our next results related with some bounded families of operators.

LEMMA 3.4. *Let X be a Banach space. Consider $1 \leqslant p, q \leqslant \infty$, $s > 0$, and $0 < \beta < \alpha \leqslant 2$. Let $G \in \mathcal{L}(B^{s+\alpha-\beta}_{p,q}([-2\pi, 0]; X); X)$. If the family $\{G_k\}_{k\in\mathbb{Z}}$ satisfies the condition **(G2)** then*

$$\left\{\frac{k}{a_k}\Delta^1\big(b_k G_k\big)\right\}_{k\in\mathbb{Z}\setminus\{0\}} \quad \textit{and} \quad \left\{\frac{k^2}{a_k}\Delta^2\big(b_k G_k\big)\right\}_{k\in\mathbb{Z}\setminus\{0\}}$$

are bounded families of operators.

Proof. Is clear that $\Delta^1(b_k G_k) = \big(\Delta^1 b_k\big)G_{k+1} + b_k\,\Delta^1 G_k$, for all $k \in \mathbb{Z}$. Therefore,

$$\frac{k}{a_k}\,\Delta^1(b_k G_k) = k\,\frac{\Delta^1 b_k}{b_k}\,\frac{b_k}{a_k}G_{k+1} + \frac{b_k}{a_k}\,k\,\Delta^1 G_k, \text{ for all } k \in \mathbb{Z}\setminus\{0\}.$$

On the other hand, a direct computation shows that

$$\Delta^2(b_k G_k) = \Delta^1 b_{k+1}\left[\Delta^1 G_{k+1} + \Delta^1 G_k\right] + \big(\Delta^2 b_k\big)G_k + b_{k+1}\,\Delta^2 G_k, \quad \text{for all } k \in \mathbb{Z}.$$

Now, for all $k \in \mathbb{Z}\setminus\{0\}$, we have

$$\frac{k^2}{a_k}\Delta^2(b_k G_k) = k\,\frac{\Delta^1 b_{k+1}}{b_k}\,k\,\frac{b_k}{a_k}[\Delta^1 G_{k+1} + \Delta^1 G_k] + k^2\,\frac{\Delta^2 b_k}{b_k}\,\frac{b_k}{a_k}\,G_k + \frac{b_{k+1}}{a_k}\,k^2\,\Delta^2 G_k.$$

Since the sequence $\{b_k\}_{k\in\mathbb{Z}}$ is 2-regular and the family $\{G_k\}_{k\in\mathbb{Z}}$ satisfies the condition ***(G2)***, all terms included in the right-hand side of the preceding identities are uniformly bounded. Hence, the families of operators

$$\left\{\frac{k}{a_k}\Delta^1\big(b_k G_k\big)\right\}_{k\in\mathbb{Z}\setminus\{0\}} \quad \text{and} \quad \left\{\frac{k^2}{a_k}\Delta^2\big(b_k G_k\big)\right\}_{k\in\mathbb{Z}\setminus\{0\}} \quad \text{are bounded.}$$

□

If there exist the following inverses, we will denote

$$N_k = (a_k I - F_k - b_k G_k - A)^{-1}, \tag{3.3}$$

and

$$M_k = a_k\big(a_k I - b_k G_k - F_k - A\big)^{-1} = a_k N_k. \tag{3.4}$$

LEMMA 3.5. *Consider* $1 \leqslant p, q \leqslant \infty$, $s > 0$, *and* $0 < \beta < \alpha \leqslant 2$. *Let* A *be a closed linear operator defined in a Banach space* X. *Assume* $F \in \mathcal{L}(B^{s+\alpha}_{p,q}([-2\pi, 0]; X); X)$ *and* $G \in \mathcal{L}(B^{s+\alpha-\beta}_{p,q}([-2\pi, 0]; X); X)$ *and that the operators* $N_k \in \mathcal{L}(X)$, *for all* $k \in \mathbb{Z}$. *If the families* $\{F_k\}_{k\in\mathbb{Z}}$ *and* $\{G_k\}_{k\in\mathbb{Z}}$ *satisfy the condition* ***(F2)*** *and* ***(G2)*** *respectively, and the family of operators* $\{M_k\}_{k\in\mathbb{Z}}$ *is bounded, then*

$$\left\{ka_k\, \Delta^1 N_k\right\}_{k\in\mathbb{Z}} \quad \text{and} \quad \left\{k^2 a_k\, \Delta^2 N_k\right\}_{k\in\mathbb{Z}}$$

are bounded families of operators.

Proof. Observe that the equality

$$\begin{aligned}\Delta^1 N_k &= N_{k+1}\big(a_k - F_k - b_k G_k - a_{k+1} + F_{k+1} + b_{k+1}G_{k+1}\big)N_k \\ &= \big(-\Delta^1 a_k\big)N_{k+1}N_k + N_{k+1}\big(\Delta^1 F_k\big)N_k + N_{k+1}\,(\Delta^1 b_k G_k)\,N_k. \end{aligned} \tag{3.5}$$

holds for all $k \in \mathbb{Z}$. Therefore, for all $k \in \mathbb{Z} \setminus \{0\}$, we have

$$ka_k\, \Delta^1 N_k = -k\tfrac{\Delta^1 a_k}{a_k}\, a_k N_{k+1} M_k + a_k N_{k+1}\, \tfrac{k}{a_k}(\Delta^1 F_k)\, M_k + a_k N_{k+1} \frac{k}{a_k}\, \Delta^1(b_k G_k) M_k.$$

Is obvious that if $k = 0$ the operator $ka_k\, \Delta^1 N_k$ is bounded. Since the sequence $\{a_k\}_{k\in\mathbb{Z}}$ is 2-regular and the families of operators $\{F_k\}_{k\in\mathbb{Z}}$ and $\{G_k\}_{k\in\mathbb{Z}}$ verify ***(F2)*** and ***(G2)*** respectively, it follows from Lemma 3.4 that $\left\{ka_k\, \Delta^1 N_k\right\}_{k\in\mathbb{Z}}$ is bounded family of operators.

On the other hand, for all $k \in \mathbb{Z}$, we have

$$\begin{aligned}\Delta^2 N_k = &\big[\Delta^1 N_{k+1} + \Delta^1 N_k\big]\big[-\Delta^1 a_{k+1} + \Delta^1 F_{k+1} + \Delta^1(b_{k+1}G_{k+1})\big]N_{k+1} \\ &+ N_k\big[-\Delta^2 a_k + \Delta^2 F_k + \Delta^2(b_k G_k)\big]N_{k+1}. \end{aligned} \tag{3.6}$$

Therefore, for all $k \in \mathbb{Z} \setminus \{0\}$

$$\begin{aligned}k^2 a_k\, \Delta^2 N_k = ka_k\big[\Delta^1 N_{k+1} + \Delta^1 N_k\big]\, \frac{k}{a_k}\left[-\Delta^1 a_{k+1} + \Delta^1 F_{k+1} + \Delta^1(b_{k+1}G_{k+1})\right], \\ a_k N_{k+1} + M_k\left[-k^2\, \frac{\Delta^2 a_k}{a_k} + \frac{k^2}{a_k}\, \Delta^2 F_k + \frac{k^2}{a_k} \Delta^2(b_k G_k)\right] a_k N_{k+1}.\end{aligned}$$

Is clear that if $k = 0$ the operator $k^2 a_k\, \Delta^2 N_k$ is bounded. Since the sequence $\{a_k\}_{k\in\mathbb{Z}}$ is 2-regular, the families $\{F_k\}_{k\in\mathbb{Z}}$ and $\{G_k\}_{k\in\mathbb{Z}}$ satisfy the conditions ***(F2)*** and ***(G2)*** respectively, and the family $\{ka_k\, \Delta^1 N_k\}_{k\in\mathbb{Z}}$ is bounded, it follows from Lemma 3.4 that the family of operators $\left\{k^2 a_k\, \Delta^2 N_k\right\}_{k\in\mathbb{Z}}$ is bounded. □

LEMMA 3.6. *Consider* $1 \leqslant p, q \leqslant \infty$, $s > 0$, *and* $0 < \beta < \alpha \leqslant 2$. *Let* A *be a closed linear operator defined in a Banach space* X. *Assume* $F \in \mathcal{L}(B^{s+\alpha}_{p,q}([-2\pi, 0]; X); X)$ *and* $G \in \mathcal{L}(B^{s+\alpha-\beta}_{p,q}([-2\pi, 0]; X); X)$ *and that the operators* $N_k \in \mathcal{L}(X)$, *for all* $k \in \mathbb{Z}$. *If the families* $\{F_k\}_{k\in\mathbb{Z}}$ *and* $\{G_k\}_{k\in\mathbb{Z}}$ *satisfy the condition* ***(F2)*** *and* ***(G2)*** *respectively, and the family of operators* $\{M_k\}_{k\in\mathbb{Z}}$ *is bounded, then the family* $\{F_k N_k\}_{k\in\mathbb{Z}}$ *is a* $B^s_{p,q}$*-multiplier.*

Proof. According to Theorem 2.11, it suffices to show that the family of operators $\{F_k N_k\}_{k\in\mathbb{Z}}$ is a $\mathcal{M}$-bounded family of order 2. With this purpose, note that

$$\|F_k N_k\| = \|\frac{1}{a_k} F_k\, a_k N_k\|,$$

The family $\{F_k\}_{k\in\mathbb{Z}}$ satisfies ***(F2)*** and $\{M_k\}_{k\in\mathbb{Z}}$ is bounded, hence the family of operators $\left\{F_k N_k\right\}_{k\in\mathbb{Z}}$ is bounded.

On the other hand, for all $k \in \mathbb{Z}$ we have

$$k\,\Delta^1(F_k N_k) = \frac{k}{a_k}\big(\Delta^1 F_k\big) a_k N_{k+1} + \frac{1}{a_k} F_k\, k\, a_k\, \Delta^1 N_k,$$

and

$$k^2\,\Delta^2(F_k N_k) = \frac{1}{a_k} F_{k+1}\, k^2\, a_k\, \Delta^2 N_k + \frac{k^2}{a_k}\big(\Delta^2 F_k\big) M_k + \frac{k}{a_k}\big(\Delta^1 F_{k+1}\big) k a_k \big[\Delta^1 N_{k+1} + \Delta^1 N_k\big].$$

It follows from Lemmas 3.4 and 3.5 that $\{F_k N_k\}_{k\in\mathbb{Z}}$ is a $\mathcal{M}$-bounded family of order 2. □

LEMMA 3.7. *Consider* $1 \leqslant p, q \leqslant \infty, s > 0$, *and* $0 < \beta < \alpha \leqslant 2$. *Let* A *be a closed linear operator defined in a Banach space* X. *Assume further that* $F \in \mathcal{L}(B^{s+\alpha}_{p,q}([-2\pi, 0]; X); X)$ *and* $G \in \mathcal{L}(B^{s+\alpha-\beta}_{p,q}([-2\pi, 0]; X); X)$. *Suppose that the operators* $N_k \in \mathcal{L}(X)$, *for all* $k \in \mathbb{Z}$. *If the family* $\{F_k\}_{k\in\mathbb{Z}}$ *satisfies the condition* ***(F2)***, $\{G_k\}_{k\in\mathbb{Z}}$ *satisfies the condition* ***(G2)***, *and the family of operators* $\{M_k\}_{k\in\mathbb{Z}}$ *is bounded, then the family* $\{b_k G_k N_k\}_{k\in\mathbb{Z}}$ *is a* $B^s_{p,q}$*-multiplier.*

Proof. According to the Theorem 2.11 it suffices to show that the family $\{b_k G_k N_k\}_{k\in\mathbb{Z}}$ is a $\mathcal{M}$-bounded family of operators of order 2.

For this, note that $\|b_k G_k N_k\| = \|\frac{b_k}{a_k} G_k a_k N_k\|$. The family $\{G_k\}_{k\in\mathbb{Z}}$ satisfies ***(G2)*** and $\{M_k\}_{k\in\mathbb{Z}}$ is bounded, hence the family of operators $\left\{b_k G_k N_k\right\}_{k\in\mathbb{Z}}$ is bounded.

On the other hand, for all $k \in \mathbb{Z} \setminus \{0\}$ we have

$$k\big(\Delta^1 b_k G_k N_k\big) = \frac{k}{a_k}\big(\Delta^1 b_k G_k\big) a_k N_{k+1} + \frac{b_k}{a_k} G_k k a_k \big(\Delta^1 N_k\big)$$

and

$$k^2\,\Delta^2(b_k G_k N_k) = \frac{k}{a_k}\,\Delta^1(b_{k+1} G_{k+1})\, k a_k\big[\Delta^1 N_{k+1} + \Delta^1 N_k\big] + \frac{k^2}{a_k}\Delta^2(b_k G_k)\, M_k + \frac{b_{k+1}}{a_k} G_{k+1}\, k^2 a_k \big(\Delta^2 N_k\big).$$

Writing in this manner the preceding families, it follows from Lemmas 3.4 and 3.5 that the family $\{b_k G_k N_k\}_{k\in\mathbb{Z}}$ is a $\mathcal{M}$-bounded family of order 2. □

LEMMA 3.8. *Consider* $1 \leqslant p, q \leqslant \infty$, $s > 0$, *and* $0 < \beta < \alpha \leqslant 2$. *Let* A *be a closed linear operator defined in a Banach space* X. *Assume* $F \in \mathcal{L}(B^{s+\alpha}_{p,q}([-2\pi, 0]; X); X)$ *and* $G \in \mathcal{L}(B^{s+\alpha-\beta}_{p,q}([-2\pi, 0]; X); X)$ *and that the operators* $N_k \in \mathcal{L}(X)$, *for all* $k \in \mathbb{Z}$. *If the families* $\{F_k\}_{k\in\mathbb{Z}}$ *and* $\{G_k\}_{k\in\mathbb{Z}}$ *satisfy the condition* ***(F2)*** *and* ***(G2)*** *respectively, then the following assertions are equivalent.*

(i) *The family of operators* $\{M_k\}_{k\in\mathbb{Z}}$ *is bounded.*
(ii) *The family of operators* $\{M_k\}_{k\in\mathbb{Z}}$ *is a* $B^s_{p,q}$*-multiplier.*

Proof. $(i) \Rightarrow (ii)$. According to Theorem 2.11, it suffices to show that $\{M_k\}_{k\in\mathbb{Z}}$ is a $\mathcal{M}$-bounded family of order 2. From the hypotheses we already know that $\sup\limits_{k\in\mathbb{Z}} \|M_k\| < \infty$. Moreover, for all $k \in \mathbb{Z} \setminus \{0\}$ we have the identity

$$k\,\Delta^1 M_k = k\,\frac{\Delta^1 a_k}{a_k}\,a_k N_{k+1} + k a_k\,\Delta^1 N_k.$$

On the other hand, we have

$$k^2\,\Delta^2 M_k = k\,\frac{\Delta^1 a_{k+1}}{a_k}\,k a_k\big[\Delta^1 N_{k+1} + \Delta^1 N_k\big] + k^2\,\frac{\Delta^2 a_k}{a_k} M_k + k^2 a_{k+1}\,\Delta^2 N_k.$$

Since the sequence $\{a_k\}_{k\in\mathbb{Z}}$ is 2-regular, it follows from Lemmas 3.4 and 3.5 that $\{M_k\}_{k\in\mathbb{Z}}$ is a $\mathcal{M}$-bounded family of order 2.

$(ii) \Rightarrow (i)$. It follows from closed graph theorem that there exists $C \geqslant 0$ (independent of f) such that for $f \in B^s_{p,q}(\mathbb{T}; X)$ we have,

$$\Big\|\sum_{k\in\mathbb{Z}} e_k \otimes M_k \widehat{f}(k)\Big\|_{B^s_{p,q}} \leqslant C\|f\|_{B^s_{p,q}}.$$

Let $x \in X$ and define $f(t) = \mathrm{e}^{ikt}x$ for $k \in \mathbb{Z}$ fixed. Then the above inequality implies

$$\|e_k\|_{B^s_{p,q}}\|M_k x\|_{B^s_{p,q}} = \|e_k M_k x\|_{B^s_{p,q}} \leqslant C\|e_k\|_{B^s_{p,q}}\|x\|_{B^s_{p,q}}.$$

Hence for all $k \in \mathbb{Z}$ we have $\|M_k\| \leqslant C$. Thus $\sup_{k\in\mathbb{Z}} \|M_k\| < \infty$. □

The next theorem establishes a characterization of $B^s_{p,q}$-maximal regularity for the Eq. (3.1).

THEOREM 3.9. *Consider* $1 \leqslant p, q \leqslant \infty$, $s > 0$, *and* $0 < \beta < \alpha \leqslant 2$. *Let* A *be a closed linear operator defined in a Banach space* X. *If the families* $\{F_k\}_{k\in\mathbb{Z}}$ *and* $\{G_k\}_{k\in\mathbb{Z}}$ *satisfy the conditions* ***(F2)*** *and* ***(G2)*** *respectively, then the following assertions are equivalent.*

(i) *The Eq.* (3.1) *has* $B^s_{p,q}$*-maximal regularity.*
(ii) *The families* $\{N_k\}_{k\in\mathbb{Z}}$ *and* $\{M_k\}_{k\in\mathbb{Z}}$ *are bounded.*

Proof. $(i) \Rightarrow (ii)$. We show that for $k \in \mathbb{Z}$ the operators $((ik)^\alpha I - (ik)^\beta G_k - F_k - A)$ are invertible. For this, let $k \in \mathbb{Z}$ and $x \in X$, and define $h(t) = \mathrm{e}^{ikt}x$. By the assertion

(i) there exists $u \in B^{s+\alpha}_{p,q}(\mathbb{T}; X) \cap B^s_{p,q}(\mathbb{T}; [D(A)])$ such that the functions $t \mapsto Fu_t$ and $t \mapsto GD^\beta u_t$ belong to $B^s_{p,q}(\mathbb{T}; X)$ and the function u satisfies the equation

$$D^\alpha u(t) = Au(t) + Fu_t + GD^\beta u_t + h(t). \tag{3.7}$$

Since the function $Fu_. \in B^s_{p,q}(\mathbb{T}; X)$ and $s > 0$, we have that $Fu_. \in L^p(\mathbb{T}; X)$. Hence, by Fejér's Theorem (see [24]), we have $\widehat{Fu_.}(k) = F_k\widehat{u}(k)$ for all $k \in \mathbb{Z}$. By using Remark 3.2, in similar manner, we have that $\widehat{GD^\beta u_.}(k) = G_k\widehat{D^\beta u}(k)$ for all $k \in \mathbb{Z}$. It follows from $\alpha > \beta$ and $u \in B^{s+\alpha}_{p,q}(\mathbb{T}; X)$, that $u \in B^{s+\beta}_{p,q}(\mathbb{T}; X)$. Therefore $\widehat{D^\beta u}(k) = (ik)^\beta\widehat{u}(k)$ for all $k \in \mathbb{Z}$. Consequently, $\widehat{GD^\beta u_.}(k) = (ik)^\beta G_k\widehat{u}(k)$ for all $k \in \mathbb{Z}$.

Applying the Fourier transform on both sides of the Eq. (3.7), we obtain

$$\big((ik)^\alpha - F_k - (ik)^\beta G_k - A\big)\widehat{u}(k) = \widehat{h}(k) = x,$$

since x is arbitrary, we have that for $k \in \mathbb{Z}$ the operators $\big((ik)^\alpha - F_k - (ik)^\beta G_k - A\big)$ are surjective.

On the other hand, let $z \in D(A)$, and assume that $((ik)^\alpha - F_k - (ik)^\beta G_k - A)z = 0$. Substituting $u(t) = \mathrm{e}^{ikt}z$ in Eq. (3.1), we see that u is a periodic solution of this equation when $f \equiv 0$. The uniqueness of solution implies that $z = 0$.

Since for all $k \in \mathbb{Z}$ the linear operators N_k are closed defined in whole space X, it follows from closed graph theorem that $N_k \in \mathcal{L}(X)$. Thus $\{N_k\}_{k\in\mathbb{Z}} \subseteq \mathcal{L}(X)$.

Let $f \in B^s_{p,q}(\mathbb{T}; X)$. By (i), there exists a function $u \in B^{s+\alpha}_{p,q}(\mathbb{T}; X) \cap B^s_{p,q}(\mathbb{T}; [D(A)])$ such that the functions $t \mapsto Fu_t$ and $t \mapsto GD^\beta u_t$ belong to $B^s_{p,q}(\mathbb{T}; X)$ and u is the unique strong solution of the equation

$$D^\alpha u(t) = Au(t) + Fu_t + GD^\beta u_t + f(t), \quad t \in [0, 2\pi].$$

Applying Fourier transform on the both sides of the preceding equation, we have

$$\big((ik)^\alpha - F_k - (ik)^\beta G_k - A\big)\widehat{u}(k) = \widehat{f}(k), \quad \text{for all } k \in \mathbb{Z}.$$

Since for all $k \in \mathbb{Z}$ the operators$\big((ik)^\alpha - F_k - (ik)^\beta G_k - A\big)$ are invertible, we have

$$\widehat{u}(k) = \big((ik)^\alpha - F_k - (ik)^\beta G_k - A\big)^{-1}\widehat{f}(k), \quad \text{for all } k \in \mathbb{Z}.$$

Hence, $(ik)^\alpha\widehat{u}(k) = \widehat{D^\alpha u}(k) = (ik)^\alpha N_k\widehat{f}(k) = M_k\widehat{f}(k)$ for all $k \in \mathbb{Z}$.

Since $u \in B^{s+\alpha}_{p,q}(\mathbb{T}; X)$, it follows from Proposition 3.1 that $D^\alpha u \in B^s_{p,q}(\mathbb{T}; X)$. Therefore, by definition the family $\{M_k\}_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-multiplier. It follows from Lemma 3.8 that $\{M_k\}_{k\in\mathbb{Z}}$ is a bounded family of operators.

$(ii) \Rightarrow (i)$. We are assuming that the hypothesis and (ii) condition of Lemma 3.8 are satisfied. Therefore, $\{M_k\}_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-multiplier. Define the family of operator $\{I_k\}_{k\in\mathbb{Z}}$, by $I_k = \frac{1}{(ik)^\alpha}I$ when $k \neq 0$ and $I_0 = I$. It follows from Theorem 2.11 that $\{I_k\}_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-multiplier. Since $N_k = I_kM_k$ for all $k \in \mathbb{Z}\setminus\{0\}$ we have $\{N_k\}_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-multiplier. For an arbitrary function $f \in B^s_{p,q}(\mathbb{T}; X)$ there are two functions $u, w \in B^s_{p,q}(\mathbb{T}, X)$ such that

$$\widehat{u}(k) = N_k \widehat{f}(k) \quad \text{and} \quad \widehat{w}(k) = (ik)^\alpha N_k \widehat{f}(k) \quad \text{for all } k \in \mathbb{Z}. \tag{3.8}$$

Therefore, $\widehat{w}(k) = (ik)^\alpha \widehat{u}(k) = \widehat{D^\alpha u}(k)$ for all $k \in \mathbb{Z}$. By the uniqueness of the Fourier coefficients, $D^\alpha u = w$. This implies that $D^\alpha u \in B^s_{p,q}(\mathbb{T}; X)$. It follows from Proposition 3.1 that $u \in B^{s+\alpha}_{p,q}(\mathbb{T}; X)$ and $D^\beta u \in B^{s+\alpha-\beta}_{p,q}(\mathbb{T}; X)$.

On the other hand, it follows from Lemma 3.6 that $\{F_k N_k\}_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-multiplier. Consequently, there exists a function $g \in B^s_{p,q}(\mathbb{T}; X)$ such that

$$\widehat{g}(k) = F_k N_k \widehat{f}(k) \quad \text{for all } k \in \mathbb{Z}.$$

By equality in (3.8) we have $\widehat{g}(k) = F_k \widehat{u}(k)$ for all $k \in \mathbb{Z}$.

As we have shown, $\widehat{Fu_.}(k) = F_k \widehat{u}(k)$ for all $k \in \mathbb{Z}$. By the uniqueness of the Fourier coefficients, $Fu_. = g$. This implies that that $Fu_. \in B^s_{p,q}(\mathbb{T}; X)$. Hence, the function $t \mapsto Fu_t$ belongs to $B^s_{p,q}(\mathbb{T}; X)$.

In the same manner, it follows from Lemma 3.7 that $\{(ik)^\beta G_k N_k\}_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-multiplier. Hence there exists a function $h \in B^s_{p,q}(\mathbb{T}; X)$ such that

$$\widehat{h}(k) = (ik)^\beta G_k N_k \widehat{f}(k) \quad \text{for all } k \in \mathbb{Z}.$$

Using again the equality (3.8) we have

$$\widehat{h}(k) = (ik)^\beta G_k \widehat{u}(k) \quad \text{for all } k \in \mathbb{Z}.$$

Since $(ik)^\beta G_k \widehat{u}(k) = \widehat{GD^\beta u_.}(k)$ for all $k \in \mathbb{Z}$. By the uniqueness of the Fourier coefficients we have that $GD^\beta u_. = h$. This implies that that $GD^\beta u_. \in B^s_{p,q}(\mathbb{T}; X)$, and the function $t \mapsto GD^\beta u_t$ belongs to $B^s_{p,q}(\mathbb{T}; X)$. It follows from equality (3.8) that

$$\widehat{u}(k) = \big((ik)^\alpha - F_k - (ik)^\beta G_k - A\big)^{-1} \widehat{f}(k).$$

Thus,

$$\big((ik)^\alpha - F_k - (ik)^\beta G_k - A\big)\widehat{u}(k) = \widehat{f}(k)$$

for all $k \in \mathbb{Z}$. Using the fact that A is a closed operator, from the fact that $B^s_{p,q}(\mathbb{T}; X)$ is continuously embedded into $L^p(\mathbb{T}; X)$ and [3, Lemma 3.1] it follows that $u(t) \in D(A)$ for almost $t \in [0, 2\pi]$. Moreover, by uniqueness of Fourier coefficients we have

$$D^\alpha_t u(t) = Au(t) + Fu_t + GD^\beta u_t + f(t)$$

for almost $t \in [0, 2\pi]$. Since f, $Fu_.$, $GD^\beta u_.$ and $D^\alpha u \in B^s_{p,q}(\mathbb{T}; X)$, we conclude that $Au \in B^s_{p,q}(\mathbb{T}; X)$. This implies that $u \in B^s_{p,q}(\mathbb{T}; [D(A)])$. Therefore, u is a strong $B^s_{p,q}$-solution of Eq. (3.1).

Since $\big((ik)^\alpha I - (ik)^\beta G_k - F_k - A\big)^{-1}$ is invertible for all $k \in \mathbb{Z}$, this strong $B^s_{p,q}$-solution is unique. Therefore the Eq. (3.1) has $B^s_{p,q}$-maximal regularity. □

When the operators A, F and G satisfy some additional conditions, our next corollary provides a simple criterion to verify that the family $\{N_k\}_{k\in\mathbb{Z}}$ is bounded. Let $\alpha > 0$, for $k \in \mathbb{Z}$, we define the operators $S_k = ((ik)^\alpha - A)^{-1}$.

COROLLARY 3.10. *Let $1 \leqslant p, q \leqslant \infty$, $s > 0$ and $0 < \beta < \alpha \leqslant 2$. Let X be a Banach space. Assume further that the sequence $\{(ik)^\alpha\}_{k\in\mathbb{Z}} \subseteq \rho(A)$ and the families $\{F_k\}_{k\in\mathbb{Z}}$ and $\{G_k\}_{k\in\mathbb{Z}}$ satisfy the conditions* ***(F2)*** *and* ***(G2)*** *respectively. If the family of operators $\big\{(ik)^\alpha\big((ik)^\alpha - A\big)^{-1}\}_{k\in\mathbb{Z}}$ is bounded, and $\sup_{k\in\mathbb{Z}}\Big\|\big((ik)^\beta G_k + F_k\big)\big((ik)^\alpha - A\big)^{-1}\Big\| < 1$, then the Eq. (3.1) has $B^s_{p,q}$-maximal regularity.*

Proof. Since $\sup_{k\in\mathbb{Z}}\big\|\big((ik)^\beta G_k + F_k\big)\big((ik)^\alpha - A\big)^{-1}\big\| < 1$, we have that the family

$$\Big\{\big(I - \big((ik)^\beta G_k + F_k\big)S_k\big)^{-1}\Big\}_{k\in\mathbb{Z}}$$

is bounded. In addition

$$\begin{aligned} N_k &= \Big[\big((ik)^\alpha - A\big)\big(I - \big((ik)^\beta G_k + F_k\big)S_k\big)\Big]^{-1} \\ &= \big(I - \big((ik)^\beta G_k + F_k\big)S_k\big)^{-1}\big((ik)^\alpha - A\big)^{-1}. \end{aligned}$$

Therefore the family $\{(ik)^\alpha N_k\}_{k\in\mathbb{Z}}$ is bounded. Since the families $\{F_k\}_{k\in\mathbb{Z}}$ and $\{G_k\}_{k\in\mathbb{Z}}$ satisfy the conditions ***(F2)*** and ***(G2)*** respectively, it follows from Theorem 3.9 that the Eq. (3.1) has $B^s_{p,q}$-maximal regularity. □

4. Existence and uniqueness of periodic strong solution of a neutral equation in Besov spaces

Let $1 \leqslant p, q \leqslant \infty$, $s > 0$, and $0 < \beta < \alpha \leqslant 2$, and $0 < r < 2\pi$. Consider $A : D(A) \subseteq X \to X$ and $B : D(B) \subseteq X \to X$ linear closed operators such that $D(A) \subseteq D(B)$, and the operators $F \in \mathcal{L}(B^{s+\alpha}_{p,q}([-2\pi, 0]; X); X)$ and $G \in \mathcal{L}(B^{s+\alpha-\beta}_{p,q}([-2\pi, 0]; X); X)$. In this section we use the results about $B^s_{p,q}$-maximal regularity of the Eq. (3.1) to prove that the abstract fractional neutral differential equation

$$D^\alpha\big(u(t) - Bu(t-r)\big) = Au(t) + Fu_t + GD^\beta u_t + f(t), \quad t \in [0, 2\pi], \quad (4.1)$$

has a unique periodic strong $B^s_{p,q}$-solution, provided that $f \in B^s_{p,q}(\mathbb{T}; X)$.

Let $1 \leqslant p, q \leqslant \infty$ and $s > 0$. Suppose that the Eq. (3.1) have $B^s_{p,q}$-maximal regularity, hence for each $g \in B^s_{p,q}(\mathbb{T}; X)$ there exists a unique strong $B^s_{p,q}$-solution v of the equation

$$D^\alpha v = Av + Fv_t + GD^\beta v_t + g(t). \quad (4.2)$$

Denote by Ψ the operator $\Psi : B^s_{p,q}(\mathbb{T}; X) \to B^s_{p,q}(\mathbb{T}; X)$ defined by the formula $\Psi(g) = D^\alpha v$, where v is the unique strong $B^s_{p,q}$-solution of the Eq. (4.2). This linear operator is well defined. Moreover, by the closed graph theorem there exists a constant $M \geqslant 0$ such that for all $f \in B^s_{p,q}(\mathbb{T}; X)$ we have

$$\|D^\alpha u\|_{B^s_{p,q}} + \|Au\|_{B^s_{p,q}} + \|Fu_\cdot\|_{B^s_{p,q}} + \|GD^\beta u_\cdot\|_{B^s_{p,q}} \leqslant M\|f\|_{B^s_{p,q}}.$$

LEMMA 4.1. *Let $1 \leqslant p, q \leqslant \infty$, $s > 0$, and $0 < \beta < \alpha \leqslant 2$. Let be X a Banach space. Assume that B is a bounded linear operator such that $\|B\|\ \|\Psi\| < 1$ and $N_k \in \mathcal{L}(X)$, for all $k \in \mathbb{Z}$. Suppose further that the families $\{F_k\}_{k\in\mathbb{Z}}$ and $\{G_k\}_{k\in\mathbb{Z}}$ satisfy the conditions **(F2)** and **(G2)** respectively. If $\{(ik)^\alpha N_k\}_{k\in\mathbb{Z}}$ is a bounded family of operators, such that $\sup_{k\in\mathbb{Z}} |k|^\alpha \|B\|\|N_k\| < 1$, then the family $\{(I - e^{-ikr}(ik)^\alpha BN_k)^{-1}\}_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-multiplier.*

Proof. Denote $R_k = (I - \mathrm{e}^{-ikr}(ik)^\alpha BN_k)^{-1}$ for all $k \in \mathbb{Z}$. Since $\sup_{k\in\mathbb{Z}} |k|^\alpha \|B\|\|N_k\| < 1$, the family of operators $\{R_k\}_{k\in\mathbb{Z}} \subseteq \mathcal{L}(X)$. Let $f \in B^s_{p,q}(\mathbb{T}; X)$ fixed. Define the map $\mathcal{P} : B^s_{p,q}(\mathbb{T}; X) \to B^s_{p,q}(\mathbb{T}; X)$ by

$$\mathcal{P}\varphi(t) = B\Psi(\varphi)(t - r) + f(t).$$

By Theorem 3.9 the map $\mathcal{P}$ is well defined. Moreover, this mapping is a contraction, thus there exists a function $g \in B^s_{p,q}(\mathbb{T}; X)$ such that

$$g(t) = B\Psi(g)(t - r) + f(t) = BD^\alpha u(t - r) + f(t), \tag{4.3}$$

where u is the unique strong $B^s_{p,q}$-solution of the equation

$$D^\alpha u(t) = Au(t) + Fu_t + GD^\beta u_t + g(t), \qquad t \in [0, 2\pi], \quad 0 < \beta < \alpha \leqslant 2. \tag{4.4}$$

Applying the Fourier transform to the both sides of Eq. (4.3) we have

$$\widehat{g}(k) = \mathrm{e}^{-ikr}(ik)^\alpha B\widehat{u}(k) + \widehat{f}(k), \qquad \text{for all } k \in \mathbb{Z}. \tag{4.5}$$

On the other hand, applying the Fourier transform to the both sides of Eq. (4.4) we have

$$\widehat{u}(k) = N_k\widehat{g}(k), \qquad \text{for all } k \in \mathbb{Z}. \tag{4.6}$$

Therefore, $\widehat{g}(k) = \mathrm{e}^{-ikr}(ik)^\alpha BN_k\widehat{g}(k) + \widehat{f}(k)$, for all $k \in \mathbb{Z}$. This implies that $\widehat{g}(k) = R_k\widehat{f}(k)$ for all $k \in \mathbb{Z}$. Hence, the family of operators $\{(I - \mathrm{e}^{-ikr}(ik)^\alpha BN_k)^{-1}\}_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-multiplier. □

The following theorem establishes the existence and uniqueness of a strong $B^s_{p,q}$-solution for the Eq. (4.1). We use the same notations introduced in the preceding lemma.

THEOREM 4.2. *Let $1 \leqslant p, q \leqslant \infty$, $s > 0$, and $0 < \beta < \alpha \leqslant 2$. Let be X a Banach space. Assume that B is a bounded linear operator such that $\|B\|\ \|\Psi\| < 1$ and $N_k \in \mathcal{L}(X)$, for all $k \in \mathbb{Z}$. Suppose further that the families $\{F_k\}_{k\in\mathbb{Z}}$ and $\{G_k\}_{k\in\mathbb{Z}}$ satisfy the conditions **(F2)** and **(G2)** respectively. If $\{(ik)^\alpha N_k\}_{k\in\mathbb{Z}}$ is a bounded family of operators, such that $\sup_{k\in\mathbb{Z}} |k|^\alpha \|B\|\|N_k\| < 1$, then for each $f \in B^s_{p,q}(\mathbb{T}; X)$ there exists an unique strong $B^s_{p,q}$-solution of Eq. (4.1).*

Proof. It follows from Lemma 4.1 that the family of operators $\{(I - e^{-ikr}(ik)^{\alpha} BN_k)^{-1}\}_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-multiplier. Denote $R_k = (I - e^{-ikr}(ik)^{\alpha} BN_k)^{-1}$. Let $f \in B^s_{p,q}(\mathbb{T}; X)$. Since $\{R_k\}_{k\in\mathbb{Z}}$ is $B^s_{p,q}$-multiplier, there exists $g \in B^s_{p,q}(\mathbb{T}; X)$ such that

$$\widehat{g}(k) = R_k \widehat{f}(k) \quad \text{for all } k \in \mathbb{Z}. \tag{4.7}$$

On the other hand, by Theorem 3.9, there exists a function $u \in B^s_{p,q}(\mathbb{T}; X)$ such that u is the unique strong $B^s_{p,q}$-solution of equation

$$D^{\alpha}u(t) = Au(t) + Fu_t + GD^{\beta}u_t + g(t), \qquad t \in [0, 2\pi], \quad 0 < \beta < \alpha \leqslant 2. \tag{4.8}$$

Applying the Fourier transform to the both sides of the preceding equality we have $\widehat{u}(k) = N_k\widehat{g}(k)$ for all $k \in \mathbb{Z}$.

It follows from equality (4.7) that $\widehat{u}(k) = N_k R_k \widehat{f}(k)$ for all $k \in \mathbb{Z}$. Note that

$$N_k R_k = \big((ik)^{\alpha} - e^{-ikr}(ik)^{\alpha} B - (ik)^{\beta} G_k - F_k - A\big)^{-1} \text{ for all } k \in \mathbb{Z}.$$

Thus, $\big((ik)^{\alpha} - e^{-ikr}(ik)^{\alpha} B - (ik)^{\beta} G_k - F_k - A\big)\widehat{u}(k) = \widehat{f}(k)$ for all $k \in \mathbb{Z}$.

Since A is a closed linear operator, it follows from uniqueness of Fourier coefficients that u satisfies the equation

$$D^{\alpha}\big(u(t) - Bu(t - r)\big) = Au(t) + Fu_t + GD^{\beta}u_t + f(t) \quad \text{for almost } t \in [0, 2\pi].$$

Hence u is a strong $B^s_{p,q}$-solution of Eq. (4.1). It only remains to show that the strong $B^s_{p,q}$-solution is unique. Indeed, let $f \in B^s_{p,q}(\mathbb{T}; X)$. Suppose that the Eq. (4.1) has two strong $B^s_{p,q}$-solutions, u_1 and u_2. A direct computation shows that

$$\big((ik)^{\alpha} - e^{-ikr}(ik)^{\alpha} B - (ik)^{\beta} G_k - F_k - A\big)(\widehat{u_1}(k) - \widehat{u_2}(k)) = 0$$

for all $k \in \mathbb{Z}$. Since $\big((ik)^{\alpha} - e^{-ikr}(ik)^{\alpha} B - (ik)^{\beta} G_k - F_k - A\big)$ is invertible, for all $k \in \mathbb{Z}$ we have that $\widehat{u_1}(k) = \widehat{u_2}(k)$. By the uniqueness of the Fourier coefficients we conclude that $u_1 \equiv u_2$. □

5. Maximal regularity on periodic Triebel–Lizorkin spaces

Let $1 \leqslant p, q \leqslant \infty$, $s > 0$ and $0 < \beta < \alpha \leqslant 2$. In this section, we study $F^s_{p,q}$-maximal regularity of the equation

$$D^{\alpha}u(t) = Au(t) + Fu_t + GD^{\beta}u_t + f(t), \qquad t \in [0, 2\pi], \tag{5.1}$$

where the mapping f is a X-valued function belonging to the periodic Triebel–Lizorkin space $F^s_{p,q}(\mathbb{T}; X)$ and the delay operators $F \in \mathcal{L}(F^{s+\alpha}_{p,q}([-2\pi, 0]); X)$ and $G \in \mathcal{L}(F^{s+\alpha-\beta}_{p,q}([-2\pi, 0]); X)$. The rest of the terms of this equation are defined as those of the Eq. (3.1). For this reason, we present a characterization of the periodic X-valued Triebel–Lizorkin $F^{s+\alpha}_{p,q}(\mathbb{T}; X)$ using the fractional derivative of Liouville–Grünwald–Letnikov.

PROPOSITION 5.1. *Let X be a Banach space, $1 \leqslant p, q \leqslant \infty$, and $s > 0$. If $\alpha > 0$ then*

$$F^{s+\alpha}_{p,q}(\mathbb{T}; X) = \{u \in F^s_{p,q}(\mathbb{T}; X) : D^\alpha u \in F^s_{p,q}(\mathbb{T}; X)\}.$$

Proof. The proof follows the same lines as those made in the proof Proposition 3.1. □

REMARK 5.2. Let $1 \leqslant p, q \leqslant \infty$ and $s > 0$. According to the preceding Proposition if $u \in F^{s+\alpha}_{p,q}(\mathbb{T}; X)$ and $0 < \beta < \alpha$ then $D^\beta u \in F^{s+\alpha-\beta}_{p,q}(\mathbb{T}; X)$.

Using this characterization, we define the $F^s_{p,q}$-maximal regularity for the solutions of Eq. (5.1) in the particular case $s > 0$.

DEFINITION 5.3. Let $1 \leqslant p, q \leqslant \infty, s > 0$ and let $f \in F^s_{p,q}(\mathbb{T}; X)$. A function u is called strong $F^s_{p,q}$-solution of Eq. (5.1) if $u \in F^{s+\alpha}_{p,q}(\mathbb{T}; X) \cap F^s_{p,q}(\mathbb{T}; [D(A)])$ and u satisfies the Eq. (5.1) for almost $t \in [0, 2\pi]$ and the functions $t \mapsto Fu_t,\ t \mapsto GD^\beta u_t$ belongs to $F^s_{p,q}(\mathbb{T}; X)$. We say that the Eq. (5.1) has $F^s_{p,q}$-maximal regularity if, for each $f \in F^s_{p,q}(\mathbb{T}; X)$ the Eq. (5.1) has unique strong $F^s_{p,q}$-solution.

One of the most important results of this chapter is the theorem 5.9. To prove it, we need the following results which are related with bounded families of operators.

LEMMA 5.4. *Let X be a Banach space. Consider $1 \leqslant p, q \leqslant \infty, s > 0$ and $0 < \beta < \alpha \leqslant 2$. Assume further $G \in \mathcal{L}(F^{s+\alpha-\beta}_{p,q}([-2\pi, 0]; X); X)$. If the family $\{G_k\}_{k\in\mathbb{Z}}$ satisfies the condition **(G3)**, then*

$$\left\{\frac{k^3}{a_k}\,\Delta^3(b_k G_k)\right\}_{k\in\mathbb{Z}\setminus\{0\}}$$

is a bounded family of operators.

Proof. For all $k \in \mathbb{Z}$, we obtain

$$\Delta^3(b_k G_k) = b_k\,\Delta^3 G_k + (b_{k+3} - b_k)\,\Delta^2 G_{k+1} + (\Delta^2 b_{k+1})(\Delta^1 G_{k+1}) \\ +(\Delta^3 b_k)G_{k+2} - 2(\Delta^2 b_k)(\Delta^1 G_{k+1}).$$

Now, for all $k \in \mathbb{Z} \setminus \{0\}$ we have the identity

$$\frac{k^3}{a_k}\Delta^3(b_k G_k) = k^3\,\frac{b_k}{a_k}\,\Delta^3 G_k + k\,\frac{b_{k+3} - b_k}{b_k}\,k^2\,\frac{b_k}{a_k}\,\Delta^2 G_{k+1} + k^2\,\frac{\Delta^2 b_{k+1}}{b_k}\,k\,\frac{b_k}{a_k}\Delta^1 G_{k+1} \\ +k^3\,\frac{\Delta^3 b_k}{b_k}\,\frac{b_k}{a_k}G_{k+2} - 2k^2\,\frac{\Delta^2 b_k}{b_k}\,k\,\frac{b_k}{a_k}\,\Delta^1 G_{k+1}.$$

Since the sequence $\{b_k\}_{k\in\mathbb{Z}}$ is 3-regular and $\{G_k\}_{k\in\mathbb{Z}}$ is a family satisfying condition ***(G3)***, it follows from Lemma 3.4 that

$$\left\{\frac{k^3}{a_k}\,\Delta^3(b_k G_k)\right\}_{k\in\mathbb{Z}\setminus\{0\}}$$

is a bounded family of operators. □

LEMMA 5.5. *Consider* $1 \leqslant p, q \leqslant \infty, s > 0$, *and* $0 < \beta < \alpha \leqslant 2$. *Let* A *be a closed linear operator defined in a Banach space* X. *Assume further that* $F \in \mathcal{L}(F^{s+\alpha}_{p,q}([-2\pi, 0]); X)$ *and* $G \in \mathcal{L}(F^{s+\alpha-\beta}_{p,q}([-2\pi, 0]); X)$. *Suppose that the operators* $N_k \in \mathcal{L}(X)$, *for all* $k \in \mathbb{Z}$, *and the families* $\{F_k\}_{k\in\mathbb{Z}}$ *and* $\{G_k\}_{k\in\mathbb{Z}}$ *satisfy the conditions* ***(F3)*** *and* ***(G3)*** *respectively. If the family of operators* $\{M_k\}_{k\in\mathbb{Z}}$ *is bounded, then*

$$\left\{k^3 a_k\, \Delta^3 N_k\right\}_{k\in\mathbb{Z}}$$

is a bounded family of operators.

Proof. Note that, for all $k \in \mathbb{Z}$, we have

$$\begin{aligned}
\Delta^3 N_k = &\left[\Delta^2 N_{k+1} + \Delta^2 N_k\right]\left[-\Delta^1 a_{k+2} + \Delta^1 F_{k+2} + \Delta^1(b_{k+2}G_{k+2})\right] N_{k+1}\\
&+\left[\Delta^1 N_{k+1} + \Delta^1 N_k\right]\left[-\Delta^2 a_{k+1} + \Delta^2 F_k + \Delta^2(b_{k+1}G_{k+1})\right] N_{k+1}\\
&+\left[\Delta^1 N_{k+1} + \Delta^1 N_k\right]\left[-\Delta^1 a_{k+1} + \Delta^1 F_k + \Delta^1(b_{k+1}G_{k+1})\right] \Delta^1 N_k\\
&+\Delta^1 N_k\left[-\Delta^2 a_{k+1} + \Delta^2 F_{k+1} + \Delta^2(b_{k+1}G_{k+1})\right] N_{k+2}\\
&+N_k\left[-\Delta^3 a_k + \Delta^3 F_k + \Delta^3(b_k G_k)\right] N_{k+2}\\
&+N_k\left[-\Delta^2 a_k + \Delta^2 F_k + \Delta^2(b_k G_k)\right] \Delta^1 N_{k+1}.
\end{aligned}$$

From the preceding identity, we conclude

$$\begin{aligned}
k^3 a_k\, \Delta^3 N_k = &\, k^2 a_k\left[\Delta^2 N_{k+1} + \Delta^2 N_k\right]\frac{k}{a_k}\left[-\Delta^1 a_{k+2} + \Delta^1 F_{k+2} + \Delta^1(b_{k+2}G_{k+2})\right] a_k N_{k+1}\\
&+k a_k\left[\Delta^1 N_{k+1} + \Delta^1 N_k\right]\frac{k^2}{a_k}\left[-\Delta^2 a_{k+1} + \Delta^2 F_k + \Delta^2(b_{k+1}G_{k+1})\right] a_k N_{k+1}\\
&+k a_k\left[\Delta^1 N_{k+1} + \Delta^1 N_k\right]\frac{k}{a_k}\left[-\Delta^1 a_{k+1} + \Delta^1 F_k + \Delta^1(b_{k+1}G_{k+1})\right] k a_k\, \Delta^1 N_k\\
&+k a_k\,(\Delta^1 N_k)\,\frac{k^2}{a_k}\left[-\Delta^2 a_{k+1} + \Delta^2 F_{k+1} + \Delta^2 b_{k+1}G_{k+1}\right] a_k N_{k+2}\\
&+M_k\,\frac{k^3}{a_k}\left[-\Delta^3 a_k + \Delta^3 F_k + \Delta^3(b_k G_k)\right] a_k N_{k+2}\\
&+M_k\frac{k^2}{a_k}\left[-\Delta^2 a_k + \Delta^2 F_k + \Delta^2(b_k G_k)\right] k a_k\, \Delta^1 N_{k+1},
\end{aligned}$$

for all $k \in \mathbb{Z} \setminus \{0\}$. Since the sequence $\{a_k\}_{k\in\mathbb{Z}}$ is a 3-regular sequence, it follows from Lemmas 3.4 and 3.5 that all the terms in the right hand of the preceding equality are uniformly bounded. Moreover, for $k = 0$ is clear that $k^3 a_k\, \Delta^3 N_k$ is a bounded operator. Therefore, $\left\{k^3 a_k\, \Delta^3 N_k\right\}_{k\in\mathbb{Z}}$ is a bounded family of operators. □

LEMMA 5.6. *Consider* $1 \leqslant p, q \leqslant \infty, s > 0$, *and* $0 < \beta < \alpha \leqslant 2$. *Let* A *be a closed linear operator defined in a Banach space* X. *Assume further that* $F \in \mathcal{L}(F^{s+\alpha}_{p,q}([-2\pi, 0]); X)$ *and* $G \in \mathcal{L}(F^{s+\alpha-\beta}_{p,q}([-2\pi, 0]); X)$. *Suppose that the operators* $N_k \in \mathcal{L}(X)$, *for all* $k \in \mathbb{Z}$. *If the families* $\{F_k\}_{k\in\mathbb{Z}}$ *and* $\{G_k\}_{k\in\mathbb{Z}}$ *satisfy the conditions* ***(F3)*** *and* ***(G3)*** *respectively, and the family of operators* $\{M_k\}_{k\in\mathbb{Z}}$ *is bounded, then the family* $\{F_k N_k\}_{k\in\mathbb{Z}}$ *is a* $F^s_{p,q}$*-multiplier.*

Proof. According to the Theorem 2.12, it suffices to show that the family of operators $\{F_k N_k\}_{k\in\mathbb{Z}}$ is a $\mathcal{M}$-bounded family of order 3. It follows from Lemma 3.6 that $\{F_k N_k\}_{k\in\mathbb{Z}}$ is a $\mathcal{M}$-bounded family of order 2. It remains to show that $\{k^3\,\Delta^3(F_k N_k)\}_{k\in\mathbb{Z}}$ is bounded. To prove this we first observe that for all $k\in\mathbb{Z}$ we have

$$\begin{aligned}\Delta^3(F_k N_k) &= F_k\,\Delta^3 N_k + (F_{k+3}-F_k)\,\Delta^2 N_{k+1} + (\Delta^2 F_{k+1})(\Delta^1 N_{k+1})\\ &\quad+(\Delta^3 F_k)\,N_{k+2} - 2(\Delta^2 F_k)(\Delta^1 N_{k+1}).\end{aligned}$$

Thus, for all $k\in\mathbb{Z}\setminus\{0\}$, it holds

$$\begin{aligned}k^3\,\Delta^3(F_k N_k) &= \frac{1}{a_k}F_k\,k^3 a_k\,\Delta^3 N_k + \frac{k}{a_k}\,(F_{k+3}-F_k)\,k^2 a_k\,\Delta^2 N_{k+1}\\ &\quad+\frac{k^2}{a_k}\,(\Delta^2 F_{k+1})\,k a_k\,\Delta^1 N_{k+1} + \frac{k^3}{a_k}\,(\Delta^3 F_k)\,a_k N_{k+2}\\ &\quad-2\frac{k^2}{a_k}(\Delta^2 F_k)k a_k\,\Delta^1 N_{k+1}.\end{aligned}$$

Since the family $\{F_k\}_{k\in\mathbb{Z}}$ satisfies the condition ***(F3)***, and clearly, when $k=0$ the operator $k^3\Delta^3(F_k N_k)$ is bounded, the family $\{F_k N_k\}_{k\in\mathbb{Z}}$ is a $\mathcal{M}$-bounded family of order 3. □

LEMMA 5.7. *Consider $1\leqslant p,q\leqslant\infty, s>0$ and $0<\beta<\alpha\leqslant 2$. Let A be a closed linear operator defined in a Banach space X. Assume further that $F\in\mathcal{L}(F^{s+\alpha}_{p,q}([-2\pi,0]);X)$ and $G\in\mathcal{L}(F^{s+\alpha-\beta}_{p,q}([-2\pi,0]);X)$. Suppose that the operators $N_k\in\mathcal{L}(X)$, for all $k\in\mathbb{Z}$. If the families $\{F_k\}_{k\in\mathbb{Z}}$ and $\{G_k\}_{k\in\mathbb{Z}}$ satisfy the conditions **(F3)** and **(G3)** respectively, and the family of operators $\{M_k\}_{k\in\mathbb{Z}}$ is bounded, then the family $\{b_k G_k N_k\}_{k\in\mathbb{Z}}$ is a $F^s_{p,q}$-multiplier.*

Proof. According to Theorem 2.12, it suffices to show that the family of operators $\{b_k G_k N_k\}_{k\in\mathbb{Z}}$ is $\mathcal{M}$-bounded of order 3. It follows from Lemma 3.7 that $\{b_k G_k N_k\}_{k\in\mathbb{Z}}$ is $\mathcal{M}$-bounded of order 2. It remains to show that $\{k^3\,\Delta^3(b_k G_k N_k)\}_{k\in\mathbb{Z}}$ is bounded. Note that, for all $k\in\mathbb{Z}$,

$$\begin{aligned}\Delta^3(b_k G_k N_k) &= b_k G_k\,\Delta^3 N_k + (b_{k+3}G_{k+3}-b_k G_k)\,\Delta^2 N_{k+1} + \Delta^2(b_{k+1}G_{k+1})\,\Delta^1 N_{k+1}\\ &\quad+(\Delta^3 b_k G_k)\,N_{k+2} - 2\Delta^2(b_k G_k)\,\Delta^1 N_{k+1}.\end{aligned}$$

Therefore, for all $k\in\mathbb{Z}\setminus\{0\}$, we have

$$\begin{aligned}k^3\,\Delta^3(b_k G_k N_k) &= \frac{b_k}{a_k}G_k\,k^3 a_k\,\Delta^3 N_k + \frac{k}{a_k}\,(b_{k+3}G_{k+3}-b_k G_k)\,k^2 a_k\,\Delta^2 N_{k+1}\\ &\quad+\frac{k^2}{a_k}\,\Delta^2(b_{k+1}G_{k+1})\,k a_k\,\Delta^1 N_{k+1} + \frac{k^3}{a_k}\,\Delta^3(b_k G_k)\,a_k N_{k+2}\\ &\quad-2\frac{k^2}{a_k}\,\Delta^2(b_k G_k)\,k a_k\,\Delta^1 N_{k+1}.\end{aligned}$$

Since $\{G_k\}_{k\in\mathbb{Z}}$ satisfies the condition ***(G3)***, it follows from Lemmas 3.4, 3.5, 5.4 and 5.5 that all the terms in the right hand of the preceding identity are uniformly bounded. In addition, $k^3\Delta^3(b_kG_kN_k)$ is a bounded operator when $k = 0$. Consequently, the family $\{b_kG_kN_k\}_{k\in\mathbb{Z}}$ is a $\mathcal{M}$-bounded family of order 3. □

LEMMA 5.8. *Let $1 \leqslant p, q \leqslant \infty$, $s > 0$ and $0 < \beta < \alpha \leqslant 2$. Let A be a closed linear operator defined in a Banach space X. Suppose that $F \in \mathcal{L}(F^{s+\alpha}_{p,q}([-2\pi, 0]); X)$ and $G \in \mathcal{L}(F^{s+\alpha-\beta}_{p,q}([-2\pi, 0]); X)$. Assume that the operators $N_k \in \mathcal{L}(X)$, for all $k \in \mathbb{Z}$. If the families $\{F_k\}_{k\in\mathbb{Z}}$ and $\{G_k\}_{k\in\mathbb{Z}}$ satisfy the conditions **(F3)** and **(G3)** respectively, then the following assertions are equivalent.*

(i) The family of operators $\{M_k\}_{k\in\mathbb{Z}}$ is bounded.

(ii) The family of operators $\{M_k\}_{k\in\mathbb{Z}}$ is a $F^s_{p,q}$-multiplier.

Proof. $(i) \Rightarrow (ii)$. According to Theorem 2.12 it suffices to show that $\{M_k\}_{k\in\mathbb{Z}}$ is $\mathcal{M}$-bounded of order 3. It follows from Lemma 3.8 that $\{M_k\}_{k\in\mathbb{Z}}$ is a family of operators $\mathcal{M}$-bounded of order 2. It remains to show that $\{k^3\,\Delta^3 M_k\}_{k\in\mathbb{Z}}$ is a bounded family of operators. For this we note

$$\begin{aligned}\Delta^3 M_k = {} & a_k\,\Delta^3 N_k + (a_{k+3} - a_k)\,\Delta^2 N_{k+1} + (\Delta^2 a_{k+1})(\Delta^1 N_{k+1})\\ & +(\Delta^3 a_k)N_{k+2} - 2\Delta^2 a_k\,\Delta^1 N_{k+1}.\end{aligned}$$

Therefore, for all $k \in \mathbb{Z}\setminus\{0\}$,

$$\begin{aligned}k^3\,\Delta^3 M_k = {} & k^3 a_k\,\Delta^3 N_k + k\frac{a_{k+3} - a_k}{a_k}\,k a_k\,\Delta^2 N_{k+1} + k^2\frac{\Delta^2 a_{k+1}}{a_k}\,k a_k\,\Delta^1 N_{k+1}\\ & +k^3\frac{\Delta^3 a_k}{a_k}\,a_k N_{k+2} - 2k^2\frac{\Delta^2 a_k}{a_k}\,k a_k\,\Delta^1 N_{k+1}.\end{aligned}$$

Since the sequence $\{a_k\}_{k\in\mathbb{Z}}$ is 3-regular, and all hypotheses of the Lemmas 3.5 and 5.4 are fulfilled, we conclude that all the operators included in the right-hand side of the equality above are uniformly bounded. Additionally, when $k = 0$ the operator $k^3\,\Delta^3 M_k$ is bounded. In consequence, the family of operators $\{M_k\}_{k\in\mathbb{Z}}$ is a $\mathcal{M}$-bounded family of order 3.

$(ii) \Rightarrow (i)$ This proof is analogous to the proof of the implication $(ii) \Rightarrow (i)$ of the Lemma 3.8, so we omit it. □

We are now ready to prove the main results of this section. We omit their proof because are analogous to the proof of the Theorem 3.9 and Corollary 3.10, respectively.

THEOREM 5.9. *Let $1 \leqslant p, q \leqslant \infty$, $s > 0$. Let be X a Banach space. If the families $\{F_k\}_{k\in\mathbb{Z}}$ and $\{G_k\}_{k\in\mathbb{Z}}$ satisfy the conditions **(F3)** and **(G3)** respectively, then the following assertions are equivalent.*

(i) The Eq. (5.1) has $F^s_{p,q}$-maximal regularity.

(ii) The families $\{N_k\}_{k\in\mathbb{Z}} \subseteq \mathcal{L}(X)$ and $\{M_k\}_{k\in\mathbb{Z}}$ are bounded.

Our next objective is to give other conditions on the operators A, F and G that imply the hypotheses of Theorem 5.9 and are easier to verify in applications. With this purpose, for $k \in \mathbb{Z}$, we define the operators $S_k = ((ik)^\alpha - A)^{-1}$.

COROLLARY 5.10. *Let* $1 \leqslant p, q \leqslant \infty$, $s > 0$. *Let be* X *a Banach space. Assume that* $\{(ik)^\alpha\}_{k\in\mathbb{Z}} \subseteq \rho(A)$ *and the families* $\{F_k\}_{k\in\mathbb{Z}}$ *and* $\{G_k\}_{k\in\mathbb{Z}}$ *satisfy the conditions* ***(F3)*** *and* ***(G3)*** *respectively. If the family of operators* $\big\{(ik)^\alpha S_k\}_{k\in\mathbb{Z}}$ *is bounded, and* $\sup_{k\in\mathbb{Z}}\Big\|\big((ik)^\beta G_k + F_k\big)S_k\Big\| < 1$, *then the solution of Eq.* (5.1) *has* $F^s_{p,q}$*-maximal regularity.*

6. Existence and uniqueness of periodic strong solution of neutral equation in Triebel–Lizorkin spaces

Let $1 \leqslant p, q \leqslant \infty$, $s > 0$ and $0 < \beta < \alpha \leqslant 2$. Let $A : D(A) \subseteq X \to X$ and $B : D(B) \subseteq X \to X$ linear closed operators such that $D(A) \subseteq D(B)$. By using the results about $F^s_{p,q}$-maximal regularity of the Eq. (5.1) obtained in Sect. 5, we prove that the fractional neutral differential equation

$$D^\alpha\big(u(t) - Bu(t-r)\big) = Au(t) + Fu_t + GD^\beta u_t + f(t), \quad t \in [0, 2\pi], \tag{6.1}$$

has a unique periodic strong $F^s_{p,q}$-solution. Suppose that the Eq. (5.1) have $F^s_{p,q}$-maximal regularity, then for each $g \in F^s_{p,q}(\mathbb{T}; X)$ there exists a unique strong $F^s_{p,q}$-solution v of the equation

$$D^\alpha v = Av + Fv_\cdot + GD^\beta v_\cdot + g(t). \tag{6.2}$$

Denote by Ψ the operator $\Psi : F^s_{p,q}(\mathbb{T}; X) \to F^s_{p,q}(\mathbb{T}; X)$ defined by the formula $\Psi(g) = D^\alpha v$, where v is the unique strong $F^s_{p,q}$-solution of the Eq. (6.2). This linear operator is well defined. Moreover, by the closed graph theorem there exists a constant $M \geqslant 0$ such that for all $f \in F^s_{p,q}(\mathbb{T}; X)$ we have

$$\|D^\alpha u\|_{F^s_{p,q}} + \|Au\|_{F^s_{p,q}} + \|Fu_\cdot\|_{F^s_{p,q}} + \|GD^\beta u_\cdot\|_{F^s_{p,q}} \leqslant M\|f\|_{F^s_{p,q}}.$$

With the following two results, we study the existence and uniqueness of a strong $F^s_{p,q}$-solution for the Eq. (6.1). We omit the details of their proofs because the are analogous to Lemma 4.1 and Theorem 4.2 respectively.

LEMMA 6.1. *Let* $1 \leqslant p, q \leqslant \infty$, $s > 0$, *and* $0 < \beta < \alpha \leqslant 2$. *Let be* X *a Banach space. Assume that* B *is a bounded linear operator such that* $\|B\|\ \|\Psi\| < 1$ *and* $N_k \in \mathcal{L}(X)$, *for all* $k \in \mathbb{Z}$. *Suppose further that the families* $\{F_k\}_{k\in\mathbb{Z}}$ *and* $\{G_k\}_{k\in\mathbb{Z}}$ *satisfy the conditions* ***(F3)*** *and* ***(G3)*** *respectively. If* $\{(ik)^\alpha N_k\}_{k\in\mathbb{Z}}$ *is a bounded family of operators, such that* $\sup_{k\in\mathbb{Z}} |k|^\alpha \|B\|\|N_k\| < 1$, *then the family* $\{(I - e^{-ikr}(ik)^\alpha BN_k)^{-1}\}_{k\in\mathbb{Z}}$ *is a* $F^s_{p,q}$*-multiplier.*

THEOREM 6.2. *Let* $1 \leqslant p, q \leqslant \infty$, $s > 0$, *and* $0 < \beta < \alpha \leqslant 2$. *Let be* X *a Banach space. Assume that* B *is a bounded linear operator such that* $\|B\|\ \|\Psi\| < 1$ *and* $N_k \in \mathcal{L}(X)$, *for all* $k \in \mathbb{Z}$. *Assume further that the families* $\{F_k\}_{k\in\mathbb{Z}}$ *and* $\{G_k\}_{k\in\mathbb{Z}}$ *satisfy the conditions* ***(F3)*** *and* ***(G3)*** *respectively. If* $\{(ik)^\alpha N_k\}_{k\in\mathbb{Z}}$ *is a bounded family of operators, such that* $\sup_{k\in\mathbb{Z}} |k|^\alpha \|B\|\|N_k\| < 1$, *then for each* $f \in F^s_{p,q}(\mathbb{T}; X)$ *there exists an unique strong* $F^s_{p,q}$*-solution of Eq.* (6.1).

7. Applications

In this last section, we present an application of our results to partial neutral functional differential equations. As we have already mentioned, equations of type (1.1) and (1.2), have been studied by several authors to model important physical systems. Next, we consider an integro-differential perturbation of the equation studied in [1,12].

EXAMPLE 7.1. Let $1 \leqslant p, q \leqslant \infty$, $s > 0$ and $1 < \beta < \alpha < 2$ and $0 < r < 2\pi$. Consider the following neutral fractional differential equation with finite delay

$$\left.\begin{aligned} \frac{\partial^\alpha}{\partial t^\alpha}\big[w(t,\xi) - bw(t-r,\xi)\big] = {}& \frac{\partial^2}{\partial \xi^2} w(t,\xi) + \int_{-2\pi}^0 q_1\gamma(s) w(t+s,\xi)\mathrm{d}s \\ & + \int_{-2\pi}^0 q_2 \gamma(s) \frac{\partial^\beta}{\partial t^\beta} w(t+s,\xi)\mathrm{d}s \\ & + \widetilde{f}(t,\xi),\ t \in \mathbb{R},\ \xi \in [0,\pi], \\ w(t,\xi) - bw(t-r,\xi) = {}& 0, \quad \xi = 0,\pi, \quad t \in \mathbb{R}. \end{aligned}\right\} \tag{7.1}$$

In order to rewrite the Eq. (7.1) in the abstract form of the Eq. (1.1), we consider X as the space $L^2([0,\pi];\mathbb{R})$. The operators A and B are defined by

$$\begin{aligned} A\varphi &= \frac{\partial^2 \varphi(\xi)}{\partial \xi^2} \quad \text{with domain} \\ D(A) &= \big\{\varphi \in L^2([0,\pi];\mathbb{R}) : \varphi'' \in L^2([0,\pi];\mathbb{R}),\ \varphi(0) = \varphi(\pi) = 0\big\}, \\ B\varphi &= b\varphi, \quad \text{where the constant } b \text{ is a positive number.} \end{aligned}$$

We assume that the function $\gamma : [-2\pi, 0] \to \mathbb{R}$ is a function of class C^2, and the operators $F : B_{p,q}^{s+\alpha}([-2\pi,0]; L^2([0,\pi];\mathbb{R})) \to L^2([0,\pi];\mathbb{R})$ and $G : B_{p,q}^{s+\alpha-\beta}([-2\pi,0]; L^2([0,\pi];\mathbb{R})) \to L^2([0,\pi];\mathbb{R})$ are described by the formula

$$(F\psi)(\xi) = \int_{-2\pi}^0 q_1\gamma(s)\psi(s)(\xi)\mathrm{d}s \quad \text{and} \quad (G\psi)(\xi) = \int_{-2\pi}^0 q_2\gamma(s)\psi(s)(\xi)\mathrm{d}s.$$

It follows from Cauchy–Schwartz inequality that $F\psi$ and $G\psi$ are elements of $L^2([0,\pi];\mathbb{R})$. Moreover, since $B_{p,q}^{s+\alpha}([-2\pi,0]; L^2([0,\pi];\mathbb{R}))$ is continuously embedded in $C([-2\pi,0]; L^2([0,\pi];\mathbb{R}))$, the maps F and G define bounded linear operators from $B_{p,q}^{s+\alpha}([-2\pi,0]; L^2([0,\pi];\mathbb{R}))$ and $B_{p,q}^{s+\alpha-\beta}([-2\pi,0]; L^2([0,\pi];\mathbb{R}))$ respectively to $L^2([0,\pi];\mathbb{R})$.

Let identify $f(t) = \widetilde{f}(t,\cdot)$, and assume that $\widetilde{f}(t,\xi)$ is 2π-periodic at the variable t.

With all these considerations, the Eq. (7.1) takes the abstract form of the Eq. (1.2).

We will show that there exists $b > 0$ sufficiently small such that there exists a unique strong $B_{p,q}^s$-solution of Eq. (7.1), whenever $f \in B_{p,q}^s(\mathbb{T}; L^2([0,\pi]))$. For this purpose, we assume that q_1 and q_2 are positive numbers such that

$$\left|q_1 + q_2 \cos\left(\frac{\beta\pi}{2}\right)\right| \leqslant \left|q_2 \cos\left(\frac{\beta\pi}{2}\right)\right|$$

and

$$q_2 K < \sin\left(\frac{\alpha\pi}{2}\right),$$

where K is a constant satisfying $\|F\| \leqslant K$ and $\|G\| \leqslant K$.

Note that for $k \in \mathbb{Z}$ the operators F_k and G_k take the form

$$F_k\varphi = \int_{-2\pi}^{0} q_1\gamma(s)(e_k\varphi)(s)\mathrm{d}s \quad \text{and} \quad G_k\varphi = \int_{-2\pi}^{0} q_2\gamma(s)(e_k\varphi)(s)\mathrm{d}s.$$

By using the Cauchy–Schwartz inequality, we conclude that $F_k \in \mathcal{L}(L^2([0, \pi]; \mathbb{R}))$ and $G_k \in \mathcal{L}(L^2([0, \pi]; \mathbb{R}))$ for all $k \in \mathbb{Z}$. Integrating by parts twice, we obtain the following representation for the operator F_k and G_k.

$$F_k\varphi = \frac{iq_1[\gamma(-2\pi) - \gamma(0)]\varphi}{k} + \frac{q_1[\gamma'(0) - \gamma'(-2\pi)]\varphi}{k^2} - \frac{q_1}{k^2}\int_{-2\pi}^{0} \gamma''(s)\mathrm{e}^{iks}\varphi\mathrm{d}s,$$

and

$$G_k\varphi = \frac{iq_2[\gamma(-2\pi) - \gamma(0)]\varphi}{k} + \frac{q_1[\gamma'(0) - \gamma'(-2\pi)]\varphi}{k^2} - \frac{q_2}{k^2}\int_{-2\pi}^{0} \gamma''(s)\mathrm{e}^{iks}\varphi\mathrm{d}s.$$

With this representation, by a direct computation, it follows that the families $\{F_k\}_{k\in\mathbb{Z}}$ and $\{G_k\}_{k\in\mathbb{Z}}$ satisfy the conditions ***(F2)*** and ***(G2)*** respectively.

On another hand, the spectrum of A consists of eigenvalues $-n^2$, for $n \in \mathbb{N}$. Their associated eigenvectors are given by

$$x_n(\xi) = \sqrt{\frac{2}{\pi}}\sin(n\xi).$$

Moreover, the set $\{x_n : n \in \mathbb{N}\}$ is an orthonormal basis of $L^2([0, \pi]; \mathbb{R})$. In particular

$$A\varphi = \sum_{n=1}^{\infty} -n^2\langle\varphi, x_n\rangle x_n, \qquad \text{for all } \varphi \in D(A). \tag{7.2}$$

Therefore $\{(ik)^\alpha\}_{k\in\mathbb{Z}} \subseteq \rho(A)$ and

$$\left((ik)^\alpha I - A\right)^{-1}\varphi = \sum_{n\in\mathbb{N}} \frac{1}{(ik)^\alpha + n^2}\langle\varphi, x_n\rangle x_n. \tag{7.3}$$

Since $1 < \alpha < 2$ we have that $Re(ik)^\alpha < 0$ for $k \neq 0$. Thus, for $k \in \mathbb{Z} \setminus \{0\}$ and $n \in \mathbb{N}$ we have

$$|(ik)^\alpha + n^2| \geqslant \left|Im\left((ik)^\alpha\right)\right| = |k|^\alpha \sin\left(\frac{\alpha\pi}{2}\right).$$

Hence, for $k \neq 0$ we have the following estimative

$$\left\| \left((ik)^{\alpha} I - A \right)^{-1} \right\| \leqslant \frac{1}{|k|^{\alpha} \sin(\frac{\alpha\pi}{2})}. \tag{7.4}$$

It is clear from equality (7.3) that $\left\| \left((ik)^{\alpha} I - A \right)^{-1} \right\| < \infty$, in the case $k = 0$. On the other hand, for all $k \neq 0$ we have that

$$\left\| (ik)^{\beta} G_k + F_k \right\| \leqslant |(ik)^{\beta} q_2 + q_1| K \leqslant q_2 |(ik)^{\beta}| K = q_2 |k|^{\beta} K. \tag{7.5}$$

Hence, we have that

$$\sup_{k \in \mathbb{Z}} \| (ik)^{\alpha} \left((ik)^{\alpha} I - A \right)^{-1} \| < \infty,$$

and

$$\left\| \left((ik)^{\beta} G_k + F_k \right) \left((ik)^{\alpha} I - A \right)^{-1} \right\| \leqslant \frac{q_2 \, |k|^{\beta} \, K}{|k|^{\alpha} \sin(\frac{\alpha\pi}{2})}.$$

Since $q_2 K < \sin(\frac{\alpha\pi}{2})$ we have $\sup_{k \in \mathbb{Z}} \left\| \left((ik)^{\beta} G_k + F_k \right) S_k \right\| < 1$. From Corollary 3.10, it follows that fractional delay equation

$$D^{\alpha} u(t) = A u(t) + F u_t + G D^{\beta} u_t + f(t), \qquad t \in [0, 2\pi], \tag{7.6}$$

where the operators A, F and G are described as above, has $B^s_{p,q}$-maximal regularity. Thus, the mapping $\Psi : B^s_{p,q}(\mathbb{T}; X) \to B^s_{p,q}(\mathbb{T}; X)$, defined by $\Psi(f) = D^{\alpha} u$ where u is the unique strong $B^s_{p,q}$-solution of Eq. (7.6), is a bounded linear operator. Therefore, there exists $C_2 \geqslant 0$ such that $\|\Psi\| \leqslant C_2$.

Moreover, there exists $C_1 \geqslant 0$ such that $\sup_{k \in \mathbb{Z}} |k|^{\alpha} \|N_k\| \leqslant C_1$. If the constant $b > 0$ satisfies the condition $b < \min \left\{ \frac{1}{C_1}, \frac{1}{C_2} \right\}$ we have

$$\sup_{k \in \mathbb{Z}} b |k|^{\alpha} \|N_k\| < 1 \quad \text{and} \quad b \|\Psi\| < 1.$$

It follows from Theorem 3.9 that Eq. (7.1) has a unique strong $B^s_{p,q}$-solution.

Acknowledgments

We would like to thank the referee for several useful remarks.

REFERENCES

[1] M. ADIMY, A. ELAZZOUZI, AND K. EZZINBI, *Bohr-Neugebauer type theorem for some partial neutral functional differential equations*, Nonlinear Anal. **66** (2007), no. 5, 1145–1160.

[2] V. V. ANH AND N. N. LEONENKO, *Spectral analysis of fractional kinetic equations with random data*, J. Statist. Phys. **104** (2001), no. 5–6, 1349–1387.

[3] W. ARENDT AND S. BU, *The operator-valued Marcinkiewicz multiplier theorem and maximal regularity*, Math. Z. **240** (2002), no. 2, 311–343.
[4] W. ARENDT AND S. BU, *Operator-valued Fourier multipliers on periodic Besov spaces and applications*, Proc. Edinb. Math. Soc. (2) **47** (2004), no. 1, 15–33.
[5] E. G. BAZHLEKOVA, *Fractional Evolution Equations in Banach Spaces*, Eindhoven University of Technology, Eindhoven, 2001. Dissertation, Technische Universiteit Eindhoven, Eindhoven, 2001.
[6] S. Q. BU, *Maximal regularity of second order delay equations in Banach spaces*, Acta Math. Sin. (Engl. Ser.) **25** (2009), no. 1, 21–28.
[7] S. BU AND Y. FANG, *Maximal regularity of second order delay equations in Banach spaces*, Sci. China Math. **53** (2010), no. 1, 51–62.
[8] S. Q. BU AND J. M. KIM, *Operator-valued Fourier multipliers on periodic Triebel spaces*, Acta Math. Sin. (Engl. Ser.) **21** (2005), no. 5, 1049–1056.
[9] S. BU AND J.- M. KIM, *Operator-valued Fourier multiplier theorems on Triebel spaces*, Acta Math. Sci. Ser. B Engl. Ed. **25** (2005), no. 4, 599–609.
[10] P. L. BUTZER AND U. WESTPHAL, *An access to fractional differentiation via fractional difference quotients*, Fractional calculus and its applications (Proc. Internat. Conf., Univ. New Haven, West Haven, Conn., 1974), 1975, pp. 116–145. Lecture Notes in Math., Vol. 457.
[11] C. CUEVAS AND J. C. DE SOUZA, *S-asymptotically ω-periodic solutions of semilinear fractional integro-differential equations*, Appl. Math. Lett. **22** (2009), no. 6, 865–870.
[12] K. EZZINBI, S. FATAJOU, AND G. M. NGUÉRÉKATA, *Pseudo almost automorphic solutions for some partial functional differential equations with infinite delay*, Appl. Anal.**87** (2008), no. 5, 591–605.
[13] M. GIRARDI AND L. WEIS, *Criteria for R-boundedness of operator families*, Evolution equations, 2003, pp. 203–221.
[14] R. GORENFLO AND F. MAINARDI, *Fractional calculus: integral and differential equations of fractional order*, Fractals and fractional calculus in continuum mechanics (Udine, 1996), 1997, pp. 223–276.
[15] A. GRÜNWALD, *Uber begrenzte derivationen und deren anwendung*, Math. Phys. **12** (1867), 441–480.
[16] L. X. GUO, S. P. LU, B. DU, AND F. LIANG, *Existence of periodic solutions to a second-order neutral functional differential equation with deviating arguments*, J. Math. (Wuhan) **30** (2010), no. 5, 839–847.
[17] J. K. HALE, *Coupled oscillators on a circle*, Resenhas 1 (1994), no. **4**, 441–457. Dynamical phase transitions (São Paulo, 1994).
[18] E. HERNÁNDEZ, D. OREGAN, AND K. BALACHANDRAN, *On recent developments in the theory of abstract differential equations with fractional derivatives*, Nonlinear Anal. **73** (2010), no. 10, 3462–3471.
[19] H. R. HENRÍQUEZ, M. PIERRI, AND A. PROKOPCZYK, *Periodic solutions of abstract neutral functional differential equations*, J. Math. Anal. Appl. **385** (2012), no. 2, 608–621.
[20] H. R. Henríquez and V. Poblete, *Periodic solutions of neutral fractional differential equations*, Submitted.
[21] R. HILFER, *Applications of Fractional Calculus in Physics (R. Hilfer, ed.)*, World Scientific Publishing Co. Inc., River Edge, NJ, 2000.
[22] M. JIA, X. P. LIU, W. G. GE, AND Y. P. GUO, *Periodic solutions to a neutral integro-differential equation with three delays*, J. Systems Sci. Math. Sci. **27** (2007), no. 4, 615–623.
[23] N. J. KALTON AND L. WEIS, *The H^∞-calculus and sums of closed operators*, Math. Ann. **321** (2001), no. 2, 319–345.
[24] Y. KATZNELSON, *An Introduction to Harmonic Analysis*, Third, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 2004.
[25] V. KEYANTUO AND C. LIZAMA, *Periodic solutions of second order differential equations in Banach spaces*, Math. Z. 253 (2006), no. 3, 489514.
[26] V. KEYANTUO AND C. LIZAMA, *A characterization of periodic solutions for time-fractional differential equations in UMD spaces and applications*, Math. Nachr. **284** (2011), no. 4, 494506.
[27] V. KEYANTUO, C. LIZAMA, AND V. POBLETE, *Periodic solutions of integro-differential equations in vector-valued function spaces*, J. Differential Equations **246** (2009), no. 3, 1007–1037.

[28] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, vol. 204, Elsevier Science B.V., Amsterdam, 2006.

[29] A. V. Letnikov, *Theory and differentiation of fractional order*, Mat. Sb. **3**, 1–68.

[30] C. Lizama, *Fourier multipliers and periodic solutions of delay equations in Banach spaces*, J. Math. Anal. Appl. **324** (2006), no. 2, 921–933.

[31] C. Lizama and V. Poblete, *Maximal regularity of delay equations in Banach spaces*, Studia Math. **175** (2006), no. 1, 91–102.

[32] C. Lizama and V. Poblete, *Periodic solutions of fractional differential equations with delay*, J. Evol. Equ. **11** (2011), no. 1, 57–70.

[33] R. Metzler and J. Klafter, *The random walks guide to anomalous diffusion: a fractional dynamics approach*, Phys. Rep. **339** (2000), no. 1, 77.

[34] R. Metzler and J. Klafter, *Boundary value problems for fractional diffusion equations*, Phys. A **278** (2000), no. 1-2, 107–125.

[35] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, A Wiley-Interscience Publication, John Wiley & Sons Inc., New York, 1993.

[36] I. Podlubny, I. Petráš, B. M. Vinagre, P. OLeary, and L. Dorcák, *Analogue realizations of fractional-order controllers*, Nonlinear Dynam. **29** (2002), no. 1–4, 281–296. Fractional order calculus and its applications.

[37] S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional Integrals and Derivatives*, Gordon and Breach Science Publishers, Yverdon, 1993. Theory and applications, Edited and with a foreword by S. M. Nikolskiĭ, Translated from the 1987 Russian original, Revised by the authors.

[38] P. Weng, *Oscillation in periodic neutral parabolic differential system*, Bull. Inst. Math. Acad. Sinica **24** (1996), no. 1, 33–47.

[39] J. Wu and H. Xia, *Rotating waves in neutral partial functional-differential equations*, J. Dynam. Differential Equations **11** (1999), no. 2, 209–238.

Verónica Poblete
Facultad de Ciencias,
Universidad de Chile
Las Palmeras 3425, Santiago, Chile
E-mail: vpoblete@uchile.cl

Juan C. Pozo
Universidad de Chile,
Facultad de Ciencias,
Las Palmeras 3425, Santiago, Chile
E-mail: jpozo@ug.uchile.cl