

Semi-Classical Limit for Radial Non-Linear Schrödinger Equation

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Abstract: We consider the nonlinear Schrödinger equation

$$\varepsilon^2 \Delta u - V(x)u + |u|^{p-1}u = 0, \quad x \in \mathbb{R}^N,$$

with superlinear and subcritical nonlinearity. Assuming that the potential is radially symmetric we find radial sign-changing solutions of the equation that concentrate in a ball, as the parameter ε goes to zero. We study the asymptotic profile of these highly oscillatory solutions, completely characterizing their behavior by means of an envelope function.

1. Introduction

In this article we are interested in the study of highly oscillatory *standing waves* for the nonlinear Schrödinger equation

$$i \hbar \psi_t = -(\hbar^2/2m) \Delta \psi + W(x)\psi - \gamma |\psi|^{p-1} \psi, \quad (1.1)$$

for a radial potential W and constants $m, \gamma > 0$, as the parameter \hbar approaches zero.

This celebrated equation has been used to describe numerous physical phenomena. Among them we mention fluid dynamics, plasma physics and dispersive phenomena in waves, in particular optical waves. In all these cases the complex function ψ represents a density, through $|\psi|^2$. Standing waves are obtained by considering in (1.1) the Ansatz

$$\psi(x, t) = \exp(-iEt/\hbar)u(x).$$

After proper scaling, we find that the amplitude u satisfies

$$\varepsilon^2 \Delta u - V(x)u + |u|^{p-1}u = 0, \quad x \in \mathbb{R}^N, \quad (1.2)$$

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for $\varepsilon > 0$ and $V(x) = W(x) - E$. It is the purpose of this article to analyze the asymptotic behavior of highly oscillatory sign-changing solutions of (1.2) in $H^1(\mathbb{R}^N)$, concentrating in a ball of finite radius around the origin, as the parameter $\varepsilon \rightarrow 0$. These solutions represent excited bound states of the system that keep the overall mass, that is the integral of u^2 , bounded away from zero along the limiting process.

The semi-linear elliptic problem (1.2) was first studied, in a pioneering work, by Floer and Weinstein [13], in the one dimensional case, for V positive and $p = 3$. They show that as $\varepsilon \rightarrow 0$, positive single peaked solutions exist near any non-degenerate critical point of V . Since then, numerous authors have extended this result in many directions. We mention the works by Oh [19], Rabinowitz [21], Wang [24], Ambrosetti, Badiale and Cingolani [1], del Pino and Felmer [7, 8], among many others. In all cases the potential is considered positive and concentration occurs at isolated points in \mathbb{R}^N . Concerning multiple concentration or clusters we have the contribution by Kang and Wei [14] in dimension N and by del Pino, Felmer and Tanaka [9] in dimension one.

More related to our work, the N -dimensional radial case, we find articles by Benci and D'Aprile [5], D'Aprile [6] and Ambrosetti, Malchiodi and Ni [3], where positive solutions are constructed, concentrating around a sphere centered at the origin. More recently, Ambrosetti, Malchiodi and Ni [4], and Malchiodi, Ni and Wei [18] have obtained clusters of positive solutions for (1.2), concentrating on a sphere whose radius is located at a positive maximum point of the effective potential (1.4).

In this paper we divert in two directions from previous works. On one hand we allow the potential V to take negative values near the origin. We observe that this situation may occur when we consider standing waves for (1.1) with high values of E , that is highly excited states. On the other hand we consider oscillatory sign-changing solutions that keep their L^2 norm away from zero as $\varepsilon \rightarrow 0$. The asymptotic behavior of our solutions is so that their frequencies increase as ε^{-1} , their amplitudes stay away from zero and the oscillations take place in a ball of finite radius, while away from that ball the solutions decay as $e^{-r/\varepsilon}$. In this way our solutions concentrate rather than in spheres, in a whole ball of finite radius.

Our analysis goes further, by identifying an envelope function that completely describes the asymptotic amplitude of the solutions. By means of this envelope we can also determine the asymptotic frequency at any given radius, and the mass and energy distribution in the concentration ball, see comments after Theorem 1.2.

Let us describe our results more precisely. Our first goal is to find solutions for (1.2) having high energies. We achieve this by using the variational formulation of the problem, taking advantage of the even character of the associated functional

$$J_\varepsilon(u) = \int_{\mathbb{R}^N} \frac{\varepsilon^2}{2} |\nabla u|^2 + \frac{1}{2} V(x) u^2 - \frac{1}{p+1} |u|^{p+1} dx. \quad (1.3)$$

For our existence theory we assume the potential V satisfy the following hypothesis:

(V₁) $V : [0, \infty) \rightarrow \mathbb{R}$ is of class C^1 and $\liminf_{r \rightarrow \infty} V(r) > 0$.

In the appendix we prove the following existence result

Theorem 1.1. *Assume that the potential V satisfies (V₁) and that $1 < p < (N+2)/(N-2)$ if $N \geq 3$ and $p > 1$ if $N = 2$. Then, for every $c > 0$ there is a sequence (ε_n, u_n) of radial solutions of (1.2), with ε_n converging to zero and such that $J_{\varepsilon_n}(u_n) = c$, for all $n \in \mathbb{N}$.*

In order to analyze the asymptotic behavior of the solutions of (1.2) we require extra hypotheses on the potential. First we need

(V₂) V is uniformly continuous.

Second, an hypothesis that is better presented in terms of the *effective potential*, defined as

$$U(r) = r^{\alpha(p-1)}V(r), \tag{1.4}$$

where $\alpha = 2(N - 1)/(p + 3)$. We assume

(U) There is $d > 0$ and $\eta > 0$ such that $U(r)(r - d) > 0$ if $r > 0, r \neq d$, and $U'(r) \geq \eta$ if $r \geq d$.

For positive potentials we slightly change the hypothesis on U :

(U⁺) $U(r) > 0$ and $U'(r) > 0$ if $r > 0$, and there exists $\eta > 0$ such that $U'(r) \geq \eta$ if $r \geq 1$.

Our goal is to study the asymptotic behavior of the solutions $\{u_n\}$ found in Theorem 1.1. The first result we get is the oscillatory character of the functions u_n . Thinking these functions as dependent on the radius r , this means that the zeroes of u_n become dense in an interval of the form $(0, R)$, as $\varepsilon_n \rightarrow 0$. In order to describe the asymptotic behavior of the sequence, we associate to each u_n an approximate envelope function e_n , obtained simply by joining through straight lines their maxima. This piece-wise linear function has the information on the amplitude of the oscillatory solution u_n . See the precise definition in Sect. 5. Our main theorem is the identification of the limit of the sequence $\{e_n\}$.

We consider the equation

$$\begin{cases} w''(y) - V(r)w(y) + |w|^{p-1}w(y) = 0, & y \in \mathbb{R}, \\ w(0) = s, \quad w'(0) = 0, \end{cases} \tag{1.5}$$

where $r, s > 0$ are parameters, $w = w(r, s; y)$. We denote by $T = T(r, s)$ a quarter of a period of w , if w is periodic with zeroes. When w is positive with exponential decay, we set $T = \infty$. Then we introduce the functions

$$Q(r, s) = \frac{1}{T} \int_0^T w^2 dy \quad \text{and} \quad R(r, s) = \frac{1}{T} \int_0^T |w|^{p+1} dy, \tag{1.6}$$

if $T < \infty$, and $Q(r, s) = R(r, s) = 0$ if $T = \infty$. We also define

$$H(r, s) = \frac{(V'(r) + \alpha(p - 1)V(r)/r)(s^2 - Q(r, s))}{2(s^p - V(r)s)} - \alpha \frac{s}{r}, \tag{1.7}$$

and the asymptotic energy functional

$$\bar{J}(e) = \frac{p - 1}{2(p + 1)} \int_0^\infty R(r, e(r))r^{N-1} dr, \tag{1.8}$$

for a function $e(r)$. Here is our main result.

Theorem 1.2. *We assume that V satisfies the hypotheses (V_1) – (V_2) , (U) or (U^+) , and that p satisfies $1 < p < \min\{5, (N + 2)/(N - 2)\}$. Let (ε_n, u_n) be a sequence of radial solutions of (1.2) such that $J_{\varepsilon_n}(u_n) = c > 0$. Then the sequence of approximate envelopes e_n converges locally uniformly in $\mathbb{R}^+ = \{r > 0\}$ to a function e , which is the unique solution of the differential equation*

$$e' = H(r, e) \quad r > 0, \tag{1.9}$$

subject to the condition

$$\bar{J}(e) = c. \tag{1.10}$$

We point out that the function H fails to be Lipschitz continuous over the graph of the function e_0 , defined in (5.1). Thus, condition (1.10) replaces the initial condition in order to obtain uniqueness of the solution.

The envelope function carries asymptotic information on the sequence $\{u_n\}$. In particular, the functions

$$\mathcal{E}(r) = \frac{p - 1}{2(p + 1)} R(r, e(r)) r^{N-1} \quad \text{and} \quad \rho(r) = Q(r, e(r)) r^{N-1}, \tag{1.11}$$

correspond to asymptotic energy and mass densities, respectively. The function $e(r)$ itself represents the asymptotic amplitude and $T^{-1}(r, e(r))$ the asymptotic frequency. In particular, the number of zeroes of u in an interval (r_0, r_1) is approximately $\varepsilon_n^{-1} \int_{r_0}^{r_1} T^{-1}(r, e(r)) dr$.

Our results can also be described using the effective potential U . If we define $v_n(r) = r^\alpha u_n(r)$ and the corresponding sequence of approximate envelopes for v_n , say \tilde{e}_n , we can prove that \tilde{e}_n converges locally uniformly in \mathbb{R}^+ to the function $\tilde{e}(r) = r^\alpha e(r)$ which is a solution of

$$\tilde{e}' = \frac{U'(r)(\tilde{e}^2 - \tilde{Q}(r, \tilde{e}(r)))}{2(\tilde{e}^p - U(r)\tilde{e})}, \tag{1.12}$$

where $\tilde{Q}(r, s) = s^{2\alpha} Q(r, r^{-\alpha}s)$.

As a consequence of Theorem 1.2 we can prove the following surprising result on the behavior of u_n near the origin.

Corollary 1.1. *There is a constant $C > 0$ such that*

$$|u_n(r)| \leq \frac{C}{r^\alpha}, \quad \text{for all } r > 0$$

and

$$\lim_{n \rightarrow \infty} \|u_n\|_\infty = \infty.$$

At this point we mention the earlier work by Felmer and Torres [12] where the one dimensional case of (1.2) is studied. In [12] the existence of an envelope equation like (1.12), is proved but where U is replaced by V . The fact that it is the effective potential what governs the concentration phenomena has been already observed in [5, 6, 3, 4], and [18] in the case when concentration of positive solutions occurs at spheres away from the origin. For recent results in related one dimensional problems see Felmer and Martínez [10] and Felmer, Martínez and Tanaka [11].

Remark 1.1. For the nonlinear Schrödinger equation in the radial case we have shown that a concentration phenomena of sign-changing solutions occurs in a set with non-empty interior. We conjecture that, if the effective potential has a local maximum at the origin, then there exist positive highly oscillatory solutions concentrating in a ball, with a singularity at the origin.

For concentration phenomena in a lower dimensional set, other than points, we should mention the recent results by Malchiodi and Montenegro [16] and [17] and Malchiodi [15] in the case of a related Neumann problem, in a bounded domain.

Remark 1.2. In this article we have considered that the effective potential U does not have critical points in (d, ∞) ; in this way concentration occurs only in a ball near the origin. If there are critical points in $(0, d)$, then we expect concentration of oscillating solutions in *fat spheres* around these points. We do not pursue this line of research, but we mention the work by Felmer, Martínez and Tanaka [11], where an analogous situation is considered in the unbalanced Allen-Cahn equation.

Remark 1.3. Our hypotheses on the potential imply control of the growth of V at infinity, that can be interpreted as a confinement condition. The strength of these hypotheses is used in obtaining a uniform estimate of the L^∞ norm of the sequence $\{u_n\}$, a fact that is proved in Sect. 3. This is perhaps the hardest part of the paper.

There is a wide class of potentials satisfying our hypotheses. They are satisfied, for example, by a potential behaving like mr for large r , $m > 0$. Another particularly interesting case is the constant potential $V \equiv 1$. Here, our Theorem 1.2 holds if

$$\frac{2N + 1}{2N - 3} \leq p < \frac{N + 2}{N - 2}.$$

This certainly exclude the case $N = 2$, where we require the extra assumption $p < 5$, see (3.5). We do not know if the constraint $p < 5$ can be removed.

Our work is organized in the following way. In Sect. 2 we prove some preliminary results. In Sect. 3 we prove that u_n is locally bounded in \mathbb{R}^+ and that v_n is uniformly bounded. In Sect. 4 we prove that the zeroes of u_n and v_n , are densely distributed in a bounded interval. This allows us to define the approximate envelopes e_n and \tilde{e}_n . In Sects. 5 and 6 we study the asymptotic behavior of e_n and \tilde{e}_n , and we characterize completely their limits through the solutions of the corresponding envelope equations.

2. Preliminary Properties of Solutions

In this section we introduce some elements in order to study the asymptotic behavior of the solutions (ε_n, u_n) given by Theorem 1.1. Let us first observe that, as a function of r a solution u of (1.2) satisfies the ordinary differential equation

$$\begin{cases} \varepsilon^2 \left(u'' + \frac{N-1}{r} u' \right) - V(r)u + |u|^{p-1}u = 0, & r > 0, \\ u'(0) = 0, \quad \lim_{r \rightarrow \infty} u(r) = \lim_{r \rightarrow \infty} u'(r) = 0. \end{cases} \tag{2.1}$$

We notice that the function $v = r^\alpha u$ satisfies equation

$$\varepsilon^2 r^{(p-1)\alpha} \left(v'' + \frac{(p-1)\alpha}{2} \frac{v'}{r} \right) - U_\varepsilon(r)v + |v|^{p-1}v = 0, \tag{2.2}$$

where

$$U_\varepsilon(r) = U(r) + \alpha \left(\frac{(p+1)\alpha}{2} - 1 \right) \varepsilon^2 r^{(p-1)\alpha-2},$$

with $U(r)$ and α as defined in the Introduction. We observe that the exponent $(p-1)\alpha-2$ is negative, so that the function U_ε has a singularity at the origin. If $N \geq 3$ then the coefficient $(p+1)\alpha/2 - 1$ is positive, while if $N = 2$ it is negative. In any case, U_ε converges to U in a C^1 uniform sense in any interval of the form (r_0, ∞) , with $r_0 > 0$.

In the next two lemmas we prove preliminary properties of u_n and v_n .

Lemma 2.1. *Given $\bar{r} > d$ there exists $\varepsilon_0 > 0$ such that if (ε, u) is a solution of (2.1) with $\varepsilon \in (0, \varepsilon_0)$, then u , and also $v(r) = r^\alpha u(r)$, do not possess positive minima nor negative maxima in $[\bar{r}, \infty)$.*

Proof. Multiplying (2.2) by v' we see that

$$\frac{d}{dr} \left(\varepsilon^2 r^{(p-1)\alpha} \frac{|v'|^2}{2} - U_\varepsilon(r) \frac{v^2}{2} + \frac{|v|^{p+1}}{p+1} \right) + U'_\varepsilon(r) \frac{v^2}{2} = 0. \tag{2.3}$$

By the positivity of the potential V at infinity we see that both u and v decay exponentially. This together with the uniform continuity of V implies that

$$\lim_{r \rightarrow \infty} \varepsilon^2 r^{(p-1)\alpha} \frac{v'(r)^2}{2} - U_\varepsilon(r) \frac{v(r)^2}{2} + \frac{|v(r)|^{p+1}}{p+1} = 0. \tag{2.4}$$

Consider $r_1 \geq \bar{r}$, a critical point of u with $m = u(r_1) > 0$. Integrating (2.3) between r_1 and infinity and using that $U'_\varepsilon(r) > 0$ in $[\bar{r}, \infty)$ for all $\varepsilon > 0$ small, we find that

$$\varepsilon^2 r_1^{(p-1)\alpha} \frac{v'(r_1)^2}{2} - U_\varepsilon(r_1) \frac{v(r_1)^2}{2} + \frac{|v(r_1)|^{p+1}}{p+1} \geq 0,$$

and since $v(r_1) = r_1^\alpha m$ and $v'(r_1) = \alpha r_1^{\alpha-1} m$ we obtain

$$c \frac{\varepsilon^2}{r_1^2} + \frac{2}{p+1} m^{p-1} \geq V(r_1), \tag{2.5}$$

for a certain constant c . If r_1 is a positive minimum point of u , from (2.1) we see $V(r_1) \geq m^{p-1}$, and combining with (2.5) we get

$$c \frac{\varepsilon^2}{r_1^2} \geq \left(\frac{p-1}{p+1} \right) V(r_1),$$

which is impossible if $\varepsilon > 0$ is small enough. Here we used that $V(r)$ is bounded away from zero in $[\bar{r}, \infty)$ as can be seen from (V_1) and (U) or (U^+) . This completes the proof in the case of u .

Now we consider v in the case when U changes sign (the case U positive is similar). Let d_ε be the point near d where U_ε changes sign. Let $r_1 \geq d_\varepsilon$ be the critical point of $v(r) = r^\alpha u(r)$. Since $U'_\varepsilon(r) > 0$ in $[d_\varepsilon, \infty)$, integrating (2.3) between r_1 and infinity we obtain

$$\frac{2}{p+1} m^{p-1} \geq U_\varepsilon(r_1), \tag{2.6}$$

where $m = v(r_1)$. Thus, if r_1 is a positive minimum point of v , from Eq. (2.2) we see that $m^{p-1} \leq U_\varepsilon(r_1)$, providing a contradiction. \square

Lemma 2.2. *Let (ε, u) be a solution of (2.1). If $0 < r_1 < r_2$ are two consecutive critical points of v then*

- i) $|v(r_1)| < |v(r_2)|$ if $U'_\varepsilon > 0$ in $[r_1, r_2]$, and
- ii) $|v(r_1)| > |v(r_2)|$ if $U'_\varepsilon < 0$ in $[r_1, r_2]$.

Here we can replace $<, >$ by \leq, \geq .

Proof. It is enough to prove the lemma in case $U'_\varepsilon(r) < 0$ in $[r_1, r_2]$. Defining $h_i = |u(r_i)|, i = 1, 2$ and considering the functions

$$F_i(s) = \frac{s^{p+1}}{p+1} - U_\varepsilon(r_i) \frac{s^2}{2}, \quad s > 0, \quad i = 1, 2,$$

after integrating (2.3) between r_1 and r_2 we find

$$F_2(h_2) - F_1(h_1) = - \int_{r_1}^{r_2} U'_\varepsilon(r) \frac{v^2}{2} dr.$$

Noticing that $F_1(h_2) - F_2(h_2) = (U_\varepsilon(r_2) - U_\varepsilon(r_1)) h_2^2/2$, we find

$$F_1(h_2) - F_1(h_1) = \int_{r_1}^{r_2} \frac{U'_\varepsilon(r)}{2} (h_2^2 - v^2) dr. \tag{2.7}$$

Now we assume for contradiction that $h_1 \leq h_2$. If $U_\varepsilon > 0$ in $[r_1, r_2]$, from the equation for v we see that $h_1 \geq (U_\varepsilon(r_1))^{1/(p-1)}$, and then F_1 is increasing in $[h_1, h_2]$, since

$$F'_1(s) = s^p - U_\varepsilon(r_1)s > 0$$

for $s > (U_\varepsilon(r_1))^{1/(p-1)}$. Thus we obtain that the left-hand side in (2.7) is positive, while the right-hand side is negative. If $U_\varepsilon < 0$ in $[r_1, r_2]$, the function F_1 is also increasing and we get the same contradiction. The remainder cases are treated similarly. \square

3. Uniform Bounds for the Solutions

In this section we consider the sequence (ε_n, u_n) of solutions of (2.1) with $J_{\varepsilon_n}(u_n) = c$ and $\varepsilon_n \rightarrow 0$. The goal is to obtain uniform estimates for u_n and $v_n = r^\alpha u_n$. This task is perhaps the hardest part in all our analysis.

It is not hard to check that the sequence u_n has an increasing number of zeroes and critical points, as $n \rightarrow \infty$. The contrary would lead to $J_{\varepsilon_n}(u_n) \rightarrow 0$. We can see this either by analyzing the min-max procedure or by an asymptotic study of u_n . Our first lemma says that critical points of u_n are not isolated.

Lemma 3.1. *Let (ε_n, u_n) be a sequence of solutions of (2.1) such that $\varepsilon_n \rightarrow 0$ and $J_{\varepsilon_n}(u_n) = c$, for all $n \in \mathbb{N}$. If $\bar{r} > d$, and $x_n < y_n$ are sequences of consecutive critical points of u_n so that $y_n \geq \bar{r}$, for all $n \in \mathbb{N}$. Then $y_n - x_n \rightarrow 0$ as $n \rightarrow \infty$.*

Proof. Before starting our proof, let us consider a generic situation we encounter several times later. Let ζ_n be a maximum point of u_n and let $m_n = u_n(\zeta_n)$. It will be convenient to consider the re-scaled function

$$w_n(z) = u_n(\zeta_n + \varepsilon_n m_n^{(1-p)/2} z) / m_n, \tag{3.1}$$

that satisfies the equation

$$P(\zeta_n) \begin{cases} w''(z) + \frac{N-1}{\varepsilon_n^{-1} m_n^{(p-1)/2} \zeta_n + z} w'(z) - V_n(z) w(z) + |w|^{p-1} w(z) = 0, \\ w(0) = 1, \quad w'(0) = 0, \end{cases}$$

with

$$V_n(z) = V(\zeta_n + \varepsilon_n m_n^{(1-p)/2} z) / m_n^{p-1}.$$

Now we start our proof. Assume, without loss of generality, that our points y_n are maximum points of u_n . Then we re-scale around y_n obtaining w_n that satisfies $P(y_n)$ and we can follow the proof of Lemma 2.1, to get as (2.5),

$$c \frac{\varepsilon_n^2}{y_n^2} + \frac{2}{p+1} m_n^{p-1} \geq V(y_n) \geq \bar{V} \quad \text{and} \quad \liminf_{n \rightarrow \infty} m_n^{p-1} \geq \frac{p+1}{2} \bar{V},$$

where $\bar{V} = \inf_{r \in [\bar{r}, \infty)} V(r) > 0$. By the uniform continuity of V we find that $V_n(z)$ converges, up to sub-sequence, locally uniformly to some constant $\gamma \in [0, 2/(p+1)]$. On the other hand, w_n and also V_n are locally bounded in \mathbb{R} so that from equation $P(y_n)$ we see that w_n converges, up to a sub-sequence, to the solution of

$$E(\gamma) \quad w'' - \gamma w + |w|^{p-1} w = 0, \quad w(0) = 1, \quad w'(0) = 0.$$

Now we consider a constant $C > 0$ such that $\bar{r} - 2C > d$ and we assume that $u_n(r) > 0$ in $[y_n - 2C, y_n]$, up to a sub-sequence. This implies that $\gamma = \frac{2}{p+1}$ and w is the positive homoclinic solution.

Thus $u_n(y_n - C) \rightarrow 0$, and consequently $u_n(r) \rightarrow 0$, for all $r \in [y_n - 2C, y_n - C]$. From here we can easily prove that there is $\bar{r}_n \in [y_n - 2C, y_n - C]$ such that

$$0 < u_n(\bar{r}_n), u'_n(\bar{r}_n) \leq c_0 \exp(-c_1/\varepsilon_n), \tag{3.2}$$

for certain positive constants c_0, c_1 . We just need a comparison argument for the function $\bar{w}_n(z) = u_n(y_n - 2C + \varepsilon_n z)$ with the solution of

$$u'' - \rho^2 u = 0, \quad u(0) = u(C/\varepsilon_n) = 1, \tag{3.3}$$

for an appropriate $\rho > 0$. Now we use (3.2) to obtain

$$\varepsilon_n^{2(p-1)\alpha} \frac{v'_n(\bar{r}_n)^2}{2} + \frac{|v_n(\bar{r}_n)|^{p+1}}{p+1} \leq c_2 \bar{r}_n^{-(p+1)\alpha} e^{-c_1/\varepsilon_n}, \tag{3.4}$$

for certain $c_2 > 0$. On the other hand, using (U) or (U^+) and the convergence of w_n to w , by a direct estimate we get

$$\varepsilon_n \bar{r}_n^{2\alpha} m_n^{(5-p)/2} \|w\|_2^2 \leq c_2 \int_{\bar{r}_n}^\infty U'_{\varepsilon_n}(r) v_n(r)^2 dr. \tag{3.5}$$

Next we integrate (2.3) for (ε_n, v_n) between \bar{r}_n and infinity and we use that $\liminf_{n \rightarrow \infty} m_n > 0$, $p < 5$, (3.4) and (3.5) to obtain

$$\varepsilon_n \leq c_2 \bar{r}_n^{(p-1)\alpha} e^{-c_1/\varepsilon_n}, \tag{3.6}$$

enlarging c_2 if necessary. But $\varepsilon_n \bar{r}_n^{N-1}$ is bounded as the inequality

$$\frac{1}{2} \varepsilon_n \bar{r}_n^{N-1} m_n^{(p+3)/2} \int_{\mathbb{R}} |w|^{p+1} dz \leq \int_{\bar{r}_n}^{\infty} |u_n|^{p+1} r^{N-1} dr$$

shows, for n large. Thus, from (3.6) it follows that $\varepsilon_n^\lambda \leq \bar{c}_2 e^{-c_1/\varepsilon_n}$, with $\lambda = 1 + \frac{p-1}{N-1}\alpha$ and a proper c_2 . This is impossible for n large.

Thus, we have proved that there is a sequence $b_n < y_n$, such that $u_n(b_n) = 0$ and $y_n - b_n$ converges to zero. To complete the proof of the lemma it is enough to show that $b_n - x_n \rightarrow 0$. In order to accomplish this we use again the argument just given. For that purpose it will be sufficient to assume that $u_n < 0$ and $u'_n > 0$ in $(b_n - 2C, b_n)$, and prove that $u_n(b_n - C) \rightarrow 0$. Then we go step by step as before to reach a contradiction.

Let us assume that $u_n(b_n - C) \rightarrow -\infty$ then $u_n(r) \rightarrow -\infty$ in $[b_n - 2C, b_n - C]$, which contradicts the boundedness of the integral

$$\int_0^\infty |u_n|^{p+1} r^{N-1} dr. \tag{3.7}$$

Let us assume now that $\liminf_{n \rightarrow \infty} u_n(b_n - C) < 0$ and finite. Then there exists $\bar{x}_n \in [b_n - 2C, b_n - 3C/2]$ such that $u'(\bar{x}_n)$ is bounded, since the contrary would imply again that (3.7) is unbounded. We let $m_n = u_n(\bar{x}_n)$ and we re-scale u_n around \bar{x}_n to obtain w_n as in (3.1), satisfying equation $P(\bar{x}_n)$.

We claim that V_n converges locally uniformly to a constant $\gamma \in [0, 2/(p+1)]$. In fact, integrating (2.3) between \bar{x}_n and infinity we find

$$\varepsilon_n^2 \bar{x}_n^{(p-1)\alpha} \frac{v'(\bar{x}_n)^2}{2} + \frac{|v(\bar{x}_n)|^{p+1}}{p+1} \geq U_{\varepsilon_n}(\bar{x}_n) \frac{v(\bar{x}_n)^2}{2},$$

and replacing $v(\bar{x}_n) = \bar{x}_n^\alpha m_n$ and $v'(\bar{x}_n) = \alpha \bar{x}_n^{\alpha-1} m_n + \bar{x}_n^\alpha u'_n(\bar{x}_n)$ we obtain

$$\varepsilon_n^2 \left(\frac{\alpha}{\bar{x}_n} + \frac{u'(\bar{x}_n)}{m_n} \right)^2 + |m_n|^{p-1} \geq \frac{p+1}{2} \left(V(\bar{x}_n) + C_1 \frac{\varepsilon_n^2}{\bar{x}_n^2} \right),$$

from where the claim follows, as $m_n \leq \liminf_{n \rightarrow \infty} u_n(b_n - C) < 0$ and $u'(\bar{x}_n)$ is bounded.

Since V_n and w_n are locally bounded to the right of 0, and since $w'_n(0) = \varepsilon_n m_n^{-(p+1)/2} u'_n(\bar{x}_n)$ converges to zero, the sequence w_n converges, up to a sub-sequence, to the solution of equation $E(\frac{2}{p+1})$. This implies, in particular, that $u_n(b_n - C)$ converges to zero, obtaining a contradiction. \square

The next proposition is crucial, allowing to obtain upper bound for u_n away from the origin.

Proposition 3.1. *Let $r_0 > 0$ and (ε_n, u_n) be a sequence of solutions of (2.1) such that $\varepsilon_n \rightarrow 0$ and $J_{\varepsilon_n}(u_n) = c$ for all $n \in \mathbb{N}$. Then $\|u_n\|_{L^\infty[r_0, \infty)}$ is bounded.*

Proof. Let us denote by $y_{n,1} > y_{n,2} > \dots > y_{n,s(n)}$ the zeroes of u_n and by $x_{n,k}$ a maximum point of $|u_n|$ in $[y_{n,k+1}, y_{n,k}]$, for $k = 1, \dots, s(n) - 1$. Let $x_{n,0}$ be a maximum point of $|u_n|$ in $[y_{n,1}, \infty)$ and $x_{n,s(n)}$ be a maximum point of $|u_n|$ in $[0, y_{n,s(n)}]$. We also define $m_{n,k} = |u_n(x_{n,k})|$, $k = 0, \dots, s(n)$, for all $n \in \mathbb{N}$.

Our first goal is to prove that the sequence $x_{n,0}$ is bounded. To do so we assume the contrary and we prove that $J_{\varepsilon_n}(u_n)$ is unbounded.

We can assume that $u_n(x_{n,0}) > 0$. From the proof of Lemma 3.1 we know that the sequence of functions

$$w_n(z) = u_n(x_{n,0} + \varepsilon_n m_{n,0}^{(1-p)/2} z) / m_{n,0} \tag{3.8}$$

converges locally uniformly to the solutions of $E(\frac{2}{p+1})$, since $w_n > 0$ to the right of $x_{n,0}$. From Lemma 3.1 we also see that $y_{n,1} - y_{n,k} \rightarrow 0$, for all $k \geq 2$.

Let us assume, for the moment, that $l_n \in \mathbb{N}$ is a sequence such that $y_{n,1} - y_{n,l_n+1} \rightarrow 0$ as $n \rightarrow \infty$. From the uniform continuity of V and Lemma 2.2 we obtain that

$$\lim_{n \rightarrow \infty} \frac{m_{n,k_n}}{m_{n,1}} = 1 \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{V(x_{n,k_n})}{m_{n,k_n}^{p-1}} = \frac{2}{p+1},$$

uniformly on the sequences $k_n \in \{1, 2, \dots, l_n\}$. This implies that the sequences of functions

$$w_{n,k_n} = |u_n(x_{n,k_n} + \varepsilon_n m_{n,k_n}^{(1-p)/2} z)| / m_{n,k_n}$$

converge to a solution w of equation $E(\frac{2}{p+1})$ and

$$\lim_{n \rightarrow \infty} \frac{1}{\varepsilon_n m_{n,k_n}^{(5-p)/2} x_{n,k_n}^{2\alpha}} \int_{y_{n,k_n+1}}^{y_{n,k_n}} v_n^2 dx = \int_{-\infty}^{\infty} w^2 dz,$$

uniformly in the sequence k_n . Integrating (2.3) between two consecutive zeroes of u_n we find

$$\begin{aligned} y_{n,k+1}^{(p-1)\alpha} v_n'(y_{n,k+1})^2 - y_{n,k}^{(p-1)\alpha} v_n'(y_{n,k})^2 &= \frac{1}{\varepsilon_n^2} \int_{y_{n,k+1}}^{y_{n,k}} U'_{\varepsilon_n}(r) v_n(r)^2 dr \\ &\geq \frac{\eta m_{n,k}^{(5-p)/2} x_{n,k}^{2\alpha} \|w\|_2^2}{\varepsilon_n 2}, \end{aligned}$$

and integrating between $y_{n,1}$ and infinity,

$$y_{n,1}^{(p-1)\alpha} v_n'(y_{n,1})^2 = \frac{1}{\varepsilon_n^2} \int_{y_{n,1}}^{\infty} U'_{\varepsilon_n}(r) v_n(r)^2 dr \geq \frac{\eta m_{n,0}^{(5-p)/2} x_{n,0}^{2\alpha} \|w\|_2^2}{\varepsilon_n 2},$$

from where

$$y_{n,k}^{(p-1)\alpha} v_n'(y_{n,k})^2 \geq c_0 k \varepsilon_n^{-1} m_{n,0}^{(5-p)/2} x_{n,0}^{2\alpha},$$

for some $c_0 > 0$, for all $k \in \{1, 2, \dots, l_n\}$. Since $v_n'(y_{n,k}) = y_{n,k}^\alpha u_n'(y_{n,k})$ and $y_{n,k}/x_{n,0} \rightarrow 1$, we find

$$u_n'(y_{n,k})^2 \geq \frac{c_0 k m_{n,0}^{(5-p)/2}}{\varepsilon_n x_{n,0}^{(p-1)\alpha}}. \tag{3.9}$$

Next we obtain an estimate for the distance between two zeroes of u_n . Let us assume that $r_n \rightarrow \infty$ is a sequence of maximum points of u_n and let $a_n < b_n$ be the consecutive zeroes of u_n so that $r_n \in (a_n, b_n)$. Let $m_n = u_n(r_n)$ and let us further assume that $w_n(z) = u_n(r_n + \varepsilon_n m_n^{(1-p)/2} z) / m_n$ converges to the solution w of $E(\frac{2}{p+1})$. We claim that

$$b_n - a_n \leq -\gamma_1 \frac{\varepsilon_n}{m_n^{(p-1)/2}} \log \left(\frac{\varepsilon_n^2 u_n'(b_n)^2}{m_n^{p+1}} \right). \tag{3.10}$$

Let us prove this claim. From (2.3) and for $r \in [a_n, b_n]$ we have

$$\varepsilon_n^2 b_n^{(p-1)\alpha} \frac{v_n'(r)^2}{2} - U_{\varepsilon_n}(a_n) \frac{v_n(r)^2}{2} + \frac{|v_n(r)|^{p+1}}{p+1} \geq \varepsilon_n^2 b_n^{(p-1)\alpha} \frac{v_n'(b_n)^2}{2}, \tag{3.11}$$

where we used that $U' > 0$. Let us consider $\mu_n = \left(\frac{p+1}{2} U_{\varepsilon_n}(a_n)\right)^{1/(p-1)}$ so that $U_{\varepsilon_n}(a_n) s^2 - \frac{2}{p+1} s^{p+1} \geq 0$ for all $s \in [0, \mu_n]$. Evaluating (3.11) at the maximum point of v_n in $[a_n, b_n]$ we see that $\mu_n \leq \max_{r \in [a_n, b_n]} v_n(r)$, and then there are two points $r_n^-, r_n^+ \in (a_n, b_n)$ with $r_n^- < r_n^+$ so that $v_n(r_n^-) = v_n(r_n^+) = \mu_n$. From (3.11) we also have that

$$(r_n^- - a_n) + (b_n - r_n^+) \leq 2 \int_0^{\mu_n} \frac{\varepsilon_n b_n^{(p-1)\alpha/2} ds}{\sqrt{\varepsilon_n^2 b_n^{(p-1)\alpha} v_n'(b_n)^2 + U_{\varepsilon_n}(a_n) s^2 - \frac{2}{p+1} s^{p+1}}},$$

and then, after changing the variable and taking into account that $v_n'(b_n) = b_n^\alpha u_n'(b_n)$, we find

$$(r_n^- - a_n) + (b_n - r_n^+) \leq \frac{2\varepsilon_n b_n^{(p-1)\alpha/2}}{\sqrt{U_{\varepsilon_n}(a_n)}} \int_0^1 \frac{dt}{\sqrt{\lambda_n \varepsilon_n^2 u_n'(b_n)^2 + t^2 - t^{p+1}}}, \tag{3.12}$$

where

$$\lambda_n = \left(\frac{2}{p+1}\right)^{2/(p-1)} \frac{b_n^{(p+1)\alpha}}{U_{\varepsilon_n}(a_n)^{(p+1)/(p-1)}}.$$

From the definition of U_{ε_n} , the uniform continuity of V and, since $V(r_n)/m_n^{p-1}$ approaches $2/(p+1)$, we obtain

$$\lim_{n \rightarrow \infty} m_n^{p+1} \lambda_n = \frac{p+1}{2} \quad \text{and} \quad \lim_{n \rightarrow \infty} m_n^{(p-1)/2} \frac{b_n^{(p-1)\alpha/2}}{\sqrt{U_{\varepsilon_n}(a_n)}} = \sqrt{\frac{p+1}{2}}. \tag{3.13}$$

On the other hand, it can be proved that there is a positive constant γ so that for all $\xi > 0$,

$$\int_0^1 \frac{ds}{\sqrt{\xi + s^2 - s^{p+1}}} \leq \gamma(1 - \log_-(\xi)),$$

where $\log_-(\xi) = \min\{0, \log(\xi)\}$. Then, combining (3.12) and (3.13) we find $\gamma_1 > 0$ such that

$$(r_n^- - a_n) + (b_n - r_n^+) \leq \gamma_1 \frac{\varepsilon_n}{m_n^{p-1/2}} \left(1 - \log_- \left(\frac{\varepsilon_n^2 u_n'(b_n)^2}{m_n^{p+1}} \right) \right). \tag{3.14}$$

But, since w_n converges to the solution of $E(\frac{2}{p+1})$, and since $V(r_n)/m_n^{p-1}$ approaches $2/(p + 1)$, we see that

$$\lim_{n \rightarrow \infty} \frac{u_n(r_n^-)}{m_n} = \lim_{n \rightarrow \infty} \frac{u_n(r_n^+)}{m_n} = 1,$$

and then $r_n^+ - r_n^- \leq C \varepsilon_n m_n^{(1-p)/2}$, for some $C > 0$. From here we finally conclude (3.10), proving our claim. We notice that the argument of \log_- in (3.14) converges to zero, since the distance between the corresponding zeroes of w_n is $\varepsilon_n^{-1} m_n^{(p-1)/2} (b_n - a_n)$, which diverges to infinity.

Next we apply (3.9) and (3.10) to obtain $\gamma_2 > 0$ so that for all $1 \leq k \leq l_n$,

$$y_{n,k} - y_{n,k+1} \leq -\gamma_2 \frac{\varepsilon_n}{m_{n,0}^{(p-1)/2}} \log \left(\frac{k \varepsilon_n}{x_{n,0}^{(p-1)\alpha} m_{n,0}^{3(p-1)/2}} \right).$$

Adding this inequality from $k = 1$ to l_n , and using that $M! \geq (\theta M)^M$, for some constant $\theta > 0$, and for all $M \in \mathbb{N}$, we obtain

$$y_{n,1} - y_{n,l_n+1} \leq -\gamma_2 \frac{\varepsilon_n l_n}{m_{n,0}^{(p-1)/2}} \log \left(\frac{\varepsilon_n l_n}{x_{n,0}^{(p-1)\alpha} m_{n,0}^{3(p-1)/2}} \right),$$

and then

$$T_n := y_{n,1} - y_{n,l_n+1} \leq \frac{\varepsilon_n l_n}{m_{n,0}^{(p-1)/2}} \left(\frac{\varepsilon_n l_n}{x_{n,0}^{(p-1)\alpha} m_{n,0}^{3(p-1)/2}} \right)^{-\rho}, \tag{3.15}$$

for a fixed $\rho \in (0, 1)$ and n sufficiently large.

We recall that w_{n,k_n} converge to w uniformly in the sequences k_n , with $k_n \in \{0, \dots, l_n\}$. Then, for large n ,

$$\int_{y_{n,k+1}}^{y_{n,k}} |u_n|^{p+1} r^{N-1} dr \geq \varepsilon_n x_{n,k}^{N-1} m_{n,k}^{(p+3)/2} \int_{\mathbb{R}} w^{p+1} dz,$$

for all $k \in \{0, \dots, l_n\}$. This and (1.10) imply that $\varepsilon_n l_n x_{n,0}^{N-1} m_{n,0}^{(p+3)/2}$ is bounded, which together with (3.15) lead us to a constant $c_1 > 0$ such that

$$T_n \leq c_1 \frac{\left(x_{n,0}^{(p-1)\alpha + N - 1} m_{n,0}^2 \right)^\rho}{x_{n,0}^{N-1} m_{n,0}^{p+1}}. \tag{3.16}$$

By choosing an appropriate $\rho > 0$, we see that the right-hand side in (3.16) converges to zero. But, on the other hand, we may choose l_n large enough so that T_n converges to zero at a lower rate, providing a contradiction.

Thus $x_{n,0}$ is bounded and we are ready to show that u_n is uniformly bounded in $[r_0, \infty)$. We first notice that if $\bar{r} > \limsup_{n \rightarrow \infty} x_{n,0}$ then $u_n(\bar{r})$ is bounded, since on the contrary the integral (3.7) would be unbounded. Now we see that the functions $|u_n|$ and $|v_n|$ decay exponentially, in a uniform way in the interval $[\bar{r}, \infty)$. Next, let r_n be the maximum point of $|u_n|$ in $[r_0, \infty)$. Integrating (2.3) for (ε_n, v_n) we obtain

$$\varepsilon_n^2 r_n^{(p-1)\alpha} \frac{v'(r_n)^2}{2} - U_{\varepsilon_n}(r_n) \frac{v(r_n)^2}{2} + \frac{|v_n(r_n)|^{p+1}}{p+1} = \int_{r_n}^{\infty} U'_{\varepsilon_n}(r) \frac{v_n(r)^2}{2} dr. \quad (3.17)$$

Since the functions U'_{ε_n} have polynomial growth and the v_n decay exponentially, the right-hand side in (3.17) is bounded. From here we see that $u_n(r_n)$ and also $v_n(r_n)$ are bounded. \square

Proposition 3.2. *Let (ε_n, u_n) be a sequence of solutions of (2.1) such that $\varepsilon_n \rightarrow 0$ and $J_{\varepsilon_n}(u_n) = c$ for all $n \in \mathbb{N}$. Then the functions $v_n(r) = r^\alpha u_n(r)$ are uniformly bounded in \mathbb{R}^+ .*

Proof. First we consider the case $N \geq 3$ and V negative near the origin. Let r_n be the maximum point of $|v_n(r)|$ and assume, for contradiction, that $|v_n(r_n)| \rightarrow \infty$, as $n \rightarrow \infty$. From Proposition 3.1 we see that $r_n \rightarrow 0$, and from Lemma 2.2 we see that $v_n(r) \neq 0$ in $(0, r_n)$, since the existence of a critical point to the left of r_n , would imply that $|v_n(r_n)|$ is not the maximum value of $|v_n|$. Let us assume $m_n = u_n(0) > 0$, then since V is negative near the origin, u_n has a local maximum point in zero and is decreasing in $(0, r_n)$, and since $v_n(r_n) \leq r_n^\alpha m_n$, we see that $m_n \rightarrow \infty$. Let us re-scale u_n defining

$$w_n(z) = u_n \left(\varepsilon_n m_n^{(1-p)/2} z \right) / m_n.$$

Then w_n satisfies equation $P(0)$, see (3.1) and the following equations, and w_n converges to the solution of

$$w''(z) + \frac{(N-1)}{z} w'(z) + |w|^{p-1} w(z) = 0, \quad w(0) = 1, w'(0) = 0. \quad (3.18)$$

It is well known (using Emden-Fowler transformation, for example) that this equation has infinitely many solutions. Let z_0 be the first zero of w and $\bar{y}_n = y_{n,s(n)}$ be the first zero of u_n , then

$$\lim_{n \rightarrow \infty} \frac{\varepsilon_n^2 \bar{y}_n^{-(p-1)\alpha} v_n'^2(\bar{y}_n)}{(\bar{y}_n^\alpha m_n)^{p+1}} = w'(z_0)^2.$$

Since $v_n(r_n) \leq \bar{y}_n^\alpha m_n$, we obtain that $\varepsilon_n^2 \bar{y}_n^{-(p-1)\alpha} v_n'^2(\bar{y}_n)$ converges to infinity. Let $a_0 > 0$, be such that $U'_\varepsilon(r) < 0$ in $(0, a_0)$. Then, integrating (2.3) between \bar{y}_n and a_0 we find that

$$\varepsilon_n^2 a_0^{(p-1)\alpha} \frac{v'_n(a_0)^2}{2} - U_{\varepsilon_n}(a_0) \frac{v_n(a_0)^2}{2} + \frac{|v_n(a_0)|^{p+1}}{p+1} \geq \varepsilon_n^2 \bar{y}_n^{-(p-1)\alpha} v_n'^2(\bar{y}_n), \quad (3.19)$$

which is impossible in view of Proposition 3.1.

When V is positive, we consider a_n as the point where U_{ε_n} has its global minimum. Following the last part of the proof of Proposition 3.1, we see that $v_n(a_n)$ is bounded, since U'_{ε_n} is bounded in $[a_n, r_0]$, for any given r_0 . Let r_n be the maximum point of $|v_n(r)|$ and assume, for contradiction, that $|v_n(r_n)| \rightarrow \infty$, as $n \rightarrow \infty$. As before we see that

$r_n \rightarrow 0$ and that $v_n(r) \neq 0$ in $(0, r_n)$. If t_n is the maximum point of u_n in $[0, r_n]$, which exists since $u'_n(r_n) < 0$, then $m_n = u_n(t_n)$ satisfies $v_n(r_n) \leq r_n^\alpha m_n$ and then $m_n \rightarrow \infty$.

Now we re-scale u_n around t_n defining

$$w_n(z) = u_n \left(t_n + \varepsilon_n m_n^{(1-p)/2} z \right) / m_n.$$

Then w_n satisfies equation $P(t_n)$ and it converges to the solution of

$$w''(z) + \frac{(N-1)}{z + \bar{t}} w'(z) + |w|^{p-1} w(z) = 0, \quad w(0) = 1, w'(0) = 0, \quad (3.20)$$

where $\bar{t} = \lim_{n \rightarrow \infty} \varepsilon_n^{-1} m_n^{(p-1)/2} t_n$; here we allow $\bar{t} = \infty$. In any case, this equation has also infinitely many zeroes, and then we can repeat the argument given above, just changing a_0 by a_n in (3.19). This finishes the proof in the case $N \geq 3$.

Now we consider the case $N = 2$ and we assume first that V is negative near the origin. Let $s_n > 0$ be so that $U'_{\varepsilon_n}(s_n) = 0$ and $s_n \rightarrow 0$ as $n \rightarrow 0$. We have in this case that $U'_{\varepsilon_n}(r) > 0$, for all $r \in (0, s_n)$, if n is large.

We start our argument assuming that v_n is bounded in $[s_n, \infty)$ and unbounded in $(0, s_n]$. Noticing that $v_n(0) = 0$, if v_n does not have critical points then v_n is bounded in $[0, s_n]$. Thus we can assume that v_n has critical points in $(0, s_n]$. Let $b_n \in (0, s_n)$ so that $v'_n \neq 0$ in (b_n, s_n) , then using that $U'_{\varepsilon_n} > 0$ in $(0, s_n)$ and Lemma 2.2 we have that $v_n(b_n) \rightarrow \infty$, as $n \rightarrow \infty$.

Let us assume that $v'_n(s_n) < 0$ and $v_n(s_n) > 0$, and denote by z_n the first critical point of v_n to the right of s_n . Integrating (2.3) from b_n to z_n we get

$$\begin{aligned} -U_{\varepsilon_n}(z_n) \frac{v_n(z_n)^2}{2} + \frac{|v_n(z_n)|^{p+1}}{p+1} + U_{\varepsilon_n}(b_n) \frac{v_n(b_n)^2}{2} - \frac{|v_n(b_n)|^{p+1}}{p+1} = \\ - \int_{b_n}^{s_n} U'_{\varepsilon_n}(r) \frac{v_n^2(r)}{2} dr - \int_{s_n}^{z_n} U'_{\varepsilon_n}(r) \frac{v_n^2(r)}{2} dr. \end{aligned}$$

Since the right-hand side here is bounded below, we see that our assumption implies that $|v_n(z_n)| \rightarrow \infty$, which is a contradiction.

If we have $v'_n(s_n) > 0$, we repeat the same argument. Our conclusion is that v_n is unbounded in $[s_n, \infty)$. Let $\bar{r} > 0$ so that $U'(r) < 0$, for all $r \in (0, \bar{r})$, then $U'_{\varepsilon_n}(r) < 0$ in (s_n, \bar{r}) , if n is large enough. Let z_n be the first critical point of v_n to the right of s_n , then integrating 2.3 between z_n and \bar{r} we get

$$\begin{aligned} -U_{\varepsilon_n}(\bar{r}) \frac{v_n(\bar{r})^2}{2} + \frac{|v_n(\bar{r})|^{p+1}}{p+1} + U_{\varepsilon_n}(z_n) \frac{v_n(z_n)^2}{2} - \frac{|v_n(z_n)|^{p+1}}{p+1} = \\ - \int_{z_n}^{\bar{r}} U'_{\varepsilon_n}(r) \frac{v_n^2(r)}{2} dr. \end{aligned}$$

By Proposition 3.1, $v_n(\bar{r})$ is bounded and we see that the right-hand side is bounded below. We conclude that $v_n(z_n)$ is bounded. But then v_n is bounded in (s_n, \bar{r}) , using Lemma 2.2, completing the proof.

We are left with the case V positive, which is direct from Lemma 2.2 since U_{ε_n} is increasing. \square

Remark 3.1. From this proposition there exists $C > 0$ such that

$$|u_n(r)| \leq \frac{C}{r^\alpha}, \quad \text{for all } r > 0,$$

proving the first part of Corollary 1.1.

4. Zeroes and Critical Points are Dense

In this section we study the behavior of zeroes and critical points of the sequence u_n , as n goes to infinity. Let us consider now the number $\bar{d} = \liminf_{n \rightarrow \infty} y_{n,1}$, where $y_{n,1}$ is the rightmost zero of u_n .

Proposition 4.1. *For every interval $(a, b) \subset (0, \bar{d})$, with $a < b$, there exists $n_0 \in \mathbb{N}$ such that (a, b) contains at least one zero of u_n , for all $n \geq n_0$.*

Proof. We first prove the proposition in case $(a, b) \subset (0, d)$. Let us assume the result is not true. We can assume that $u_n(r) > 0$ in (a, b) . We first analyze the case when, up to a sub-sequence, u_n does not have a critical point in $[a, b]$. Let us consider the case $u'_n(a) < 0$ for all $n \in \mathbb{N}$. Then, since

$$\varepsilon_n^2 \frac{d}{dr} \left(r^{N-1} u'_n \right) = r^{N-1} \left(V(r) - |u_n|^{p-1} \right) u_n$$

and V is negative in (a, b) , we see that $r^{N-1} u'_n(r) < a^{N-1} u'_n(a)$ for all $r \in (a, b)$. And then

$$u_n(a) = u_n(b) + \int_a^b -u'_n dr \geq a^{N-1} |u'_n(a)| \int_a^b r^{1-N} dr,$$

which implies $u'_n(a)/u_n(a)$ is bounded. Let us define $m_n = u_n(a)$ and $w_n(z) = u_n(a + \varepsilon_n z)/m_n$, where we assume that m_n converges up to a sequence to $m \geq 0$. The functions w_n satisfy

$$\frac{d}{dz} \left((a + \varepsilon_n z)^{N-1} w'_n \right) = (a + \varepsilon_n z)^{N-1} \left(V(a + \varepsilon_n z) - m_n^{p-1} |w_n|^{p-1} \right) w_n. \quad (4.1)$$

Since w_n is uniformly bounded to the right of 0 and $w'_n(0) = \varepsilon_n u'_n(a)/u_n(a)$ converges to zero, integrating (4.1) between zero and $z > 0$ we see that the functions $(\varepsilon_n^{-1} a + z)^{-1} w'_n(z)$ are locally uniformly bounded. Then we can prove that w_n converges, up to a sub-sequence, to the solution of

$$w'' - V(a)w + m^{p-1} |w|^{p-1} w = 0, \quad \text{and} \quad w(0) = 1, w'(0) = 0, \quad (4.2)$$

which is periodic with zeroes. This is impossible.

On the other hand, if for some sub-sequence we have $u'_n(a) > 0$, then from the equation we see that $u''_n < 0$ in (a, b) and then $u'_n(r) > u'_n(b)$ for all $r \in (a, b)$. Thus

$$u_n(b) = u_n(a) + \int_a^b u'_n dr \geq (b - a) u'_n(b),$$

and then $u'_n(b)/u_n(b)$ is bounded. Re-scaling u_n as before, but around b , we reach again a contradiction.

Finally, if there is a sequence $x_n \in [a, b]$ with $u'_n(x_n) = 0$, then x_n is a maximum of u_n and if we define $m_n = u_n(x_n)$ and

$$w_n(z) = u_n(x_n + \varepsilon_n z) / m_n,$$

we can prove, using the argument as before, that there exists $\bar{x} \in [a, b]$ and $m \geq 0$ such that w_n converges, up to a sub-sequence, to the solution of (4.2), but with a replaced by \bar{x} . This is impossible again.

To end we consider the case $(a, b) \subset (d, \bar{d})$. We observe that Lemma 3.1 implies that for any sequence of two consecutive zeroes $a_n < b_n$ of u_n and $\liminf_{n \rightarrow \infty} b_n \geq b$, we have $b_n - a_n \rightarrow 0$. We may assume that $y_{n,1} > b$ and take b_n as the first zero of u_n to the right of b , we see then that $a_n \in (a, b)$, for n large enough. \square

5. The Envelope Function

In this section we construct the envelope function associated to the sequence of solution (ε_n, u_n) under study. We obtain this function as the limit of piece-wise linear functions joining the peaks of the functions u_n .

We start with some qualitative results that we need next. It will be convenient to consider the *trivial envelope*, which is given by

$$e_0(r) = \left(\frac{p+1}{2} V(r) \right)^{\frac{1}{p-1}}, \tag{5.1}$$

for $r \geq d$ and $e_0(r) = 0$ for $r < d$. We can easily check that this function satisfies (1.9) for $r > d$. In the next two lemmas we analyze the behavior u_n in relation to e_0 .

Lemma 5.1. *Let x_n be a point of maximum for $|u_n|$ for $n \in \mathbb{N}$, and assume that $x_n \rightarrow \bar{x}$, then $\liminf_{n \rightarrow \infty} |u_n(x_n)| \geq e_0(\bar{x})$.*

Proof. If $\bar{x} > d$ then the result is a consequence of (2.5), which implies

$$C_2 \frac{\varepsilon_n^2}{x_n^2} + \frac{2}{p+1} |u_n(x_n)|^{p-1} \geq V(x_n). \quad \square$$

In what follows we assume, taking a sub-sequence if necessary, that $x_{n,1}$ converges to \bar{d} . We have

Lemma 5.2. *If $\bar{d} > 0$ then*

$$\lim_{n \rightarrow \infty} |u_n(x_{n,1})| = e_0(\bar{d}).$$

Proof. Without loss of generality, we may assume that $u_n(x_{n,1}) > 0$. From the proof of Proposition 4.1 we know that $\bar{d} \geq d$. If $\bar{d} > d$, from the proof of Lemma 3.1 we have that the sequence

$$w_n(z) = \frac{u_n(x_{n,1} + \varepsilon_n m_n^{(1-p)/2} z)}{m_n},$$

with $m_n = u_n(x_{n,1})$, converges to the solution of $E(\frac{2}{p+1})$. This implies that $V_n(0) = V(x_{n,1})/m_n^{p-1}$ converges to $\frac{2}{p+1}$, and then the result follows.

If $\bar{d} = d$ and if, up to a sub-sequence, we have that $\lim_{n \rightarrow \infty} u_n(x_{n,1}) > 0$, then the sequence $w_n(z) = u_n(x_{n,1} + \varepsilon_n z)$, converges to the solution of

$$w'' + |w|^{p-1}w = 0, \quad \text{and} \quad w(0) = \lim_{n \rightarrow \infty} u_n(x_{n,1}), \quad w'(0) = 0.$$

Since this solution is periodic with zeroes, we reach a contradiction. Thus, we conclude that $\lim_{n \rightarrow \infty} u_n(x_{n,1}) = e_0(d) = 0$. \square

Next we study the behavior of the critical points of u_n in $(0, d)$. It will be useful to consider the functions v_n .

Lemma 5.3. *Assuming that V is negative near the origin. Given $r_0 \in (0, d)$, let $x_n \geq r_0$ be a critical point of v_n , $n \in \mathbb{N}$. If $v_n(x_n) \rightarrow 0$ then $v_n(z_n) \rightarrow 0$, for any sequence z_n of critical points of v_n such that $z_n \geq r_0$ and $\limsup_{n \rightarrow \infty} z_n \leq d$.*

Proof. Let $b_0 < d$ so that $U'_{\varepsilon_n} > 0$ in $[b_0, d]$ for n sufficiently large. Let z_n as in the lemma and assume that $x_n \in [b_0, d]$ and $v_n(x_n) \rightarrow 0$. We claim that $v_n(z_n) \rightarrow 0$. If $z_n \leq x_n$, from Lemma 2.2 we have $|v_n(z_n)| \leq |v_n(x_n)|$ and if $x_n \leq z_n$ then from (2.3) and, since $U'_{\varepsilon_n} > 0$ in (x_n, z_n) , we find

$$\frac{|v_n(z_n)|^{p+1}}{p+1} - U_{\varepsilon_n}(z_n) \frac{v_n(z_n)^2}{2} \leq \frac{|v_n(x_n)|^{p+1}}{p+1} - U_{\varepsilon_n}(x_n) \frac{v_n(x_n)^2}{2}.$$

In both cases it follows that $v_n(z_n) \rightarrow 0$, proving the claim.

Next we show the result when $x_n, z_n \in [r_0, b_0]$. We observe that there exist constants $m, M > 0$ so that $-U_{\varepsilon_n}(r)/2 \geq m$ and $|U'_{\varepsilon_n}(r)/2| \leq M$ in $[r_0, b_0]$, for all n large. Since $s \in [r_0, b_0]$, from (2.3) we have

$$\begin{aligned} \varepsilon_n^2 \frac{v_n'(s)^2}{2} - U_{\varepsilon_n}(s) \frac{v_n(s)^2}{2} + \frac{|v_n(s)|^{p+1}}{p+1} &= -U_{\varepsilon_n}(x_n) \frac{v_n(x_n)^2}{2} + \frac{|v_n(x_n)|^{p+1}}{p+1} \\ &\quad - \int_{x_n}^s U'_{\varepsilon_n} \frac{v_n^2}{2} dr, \end{aligned}$$

from where we obtain that

$$m \cdot v_n(s)^2 \leq -U_{\varepsilon_n}(x_n) \frac{v_n(x_n)^2}{2} + \frac{|v_n(x_n)|^{p+1}}{p+1} + M \left| \int_{x_n}^s v_n^2 dr \right|.$$

Using Gronwall's inequality we find a constant $C > 0$ such that

$$v_n(s)^2 \leq C \left(-U_{\varepsilon_n}(x_n) \frac{v_n(x_n)^2}{2} + \frac{|v_n(x_n)|^{p+1}}{p+1} \right),$$

for all $s \in [r_0, b_0]$. From here it follows that $v_n(r) \rightarrow 0$ uniformly in $[r_0, b_0]$.

The conclusion in the general case follows from the fact that the critical points of v_n are densely distributed in $[0, d]$. \square

Corollary 5.1. *In case V is negative near the origin, assume that x_n is a sequence of critical points of u_n such that $x_n \rightarrow \bar{x} \in (r_0, d)$ and*

$$\liminf_{n \rightarrow \infty} |u_n(x_n)| > 0.$$

Then there exists a constant $C > 0$ such that $|u_n(z_n)| > C$ for any sequence z_n of critical points of v_n such that $z_n \geq r_0$ and $\limsup_{n \rightarrow \infty} z_n \leq d$. Moreover, u_n possesses a zero between any pair of consecutive critical points of u_n , for all $n \in \mathbb{N}$ sufficiently large.

Proof. In view of Lemma 5.3, we only need to prove that u_n does not have positive minima nor negative maxima. Since $V \leq 0$ in $[0, d]$ and in view of Lemma 2.1, we just need to rule out the possibility of a sequence $y_n \rightarrow d$ of positive minima of $|u_n|$. Let $a_n < b_n$ be consecutive zeroes of u_n such that $y_n \in [a_n, b_n]$ and x_n is the point where $|u_n|$ reaches its maximum in (a_n, b_n) . Considering the sequence $w_n(z) = u_n(x_n + \varepsilon_n z)$, which converges, up to a sub-sequence, to the solution of

$$w'' + |w|^{p-1}w = 0, \quad \text{and} \quad w(0) = \lim_{n \rightarrow \infty} u_n(x_n) \neq 0, w'(0) = 0,$$

which is periodic with zeroes and does not have positive minima, nor negative maxima, we conclude the proof. \square

Now we are prepared to define the approximate envelope in a precise way. Let us assume for the moment that the hypotheses of Corollary 5.1 hold and let us define the function e_n as

$$e_n(r) = |u_n(x_{n,k+1})| + \frac{|u_n(x_{n,k})| - |u_n(x_{n,k+1})|}{x_{n,k} - x_{n,k+1}}(r - x_{n,k+1}), \quad r \in [x_{n,k+1}, x_{n,k}], \tag{5.2}$$

where $x_{n,1} > \dots > x_{n,s(n)}$ are the critical points of u_n . To extend e_n to $[0, \infty)$, we notice that e_0 is of class C^1 in $[d, \infty)$, $x_{n,1} \rightarrow \bar{d}$ and $|u_n(x_{n,1})| \rightarrow e_0(\bar{d})$, thus we can find a sequence $x_{n,0}$ such that $x_{n,0} > x_{n,1}$, $x_{n,0} - x_{n,1} \rightarrow 0$ and

$$\frac{e_0(x_{n,0}) - |u_n(x_{n,1})|}{x_{n,0} - x_{n,1}} \tag{5.3}$$

is bounded. We extend e_n to the right of $x_{n,1}$ as

$$e_n(r) = |u_n(x_{n,1})| + \frac{e_0(x_{n,0}) - |u_n(x_{n,1})|}{x_{n,0} - x_{n,1}}(r - x_{n,1}),$$

in $[x_{n,1}, x_{n,0}]$ and as e_0 in $[x_{n,0}, \infty)$. Now an important conclusion

Theorem 5.1. *Under the hypotheses of Theorem 1.2, the sequence e_n converges, up to a sub-sequence, locally uniformly in \mathbb{R}^+ to a function e which is a solution to the envelope equation (1.9).*

Proof. Let us assume first that there is a constant $C > 0$ such that $|u(x_{n,k})| \geq C$ for all n, k and let $r_0 > 0$. Multiplying (2.1) by u' we find

$$\frac{d}{dr} \left(\varepsilon^2 \frac{|u'|^2}{2} - V(r) \frac{u^2}{2} + \frac{|u|^{p+1}}{p+1} \right) = -\varepsilon^2 \frac{N-1}{r} |u'|^2 - V'(r) \frac{u^2}{2}. \tag{5.4}$$

Let $x_{n,k}$ and $x_{n,k+1}$ be two consecutive critical points of u_n . Integrating (5.4) for (ε_n, u_n) between $x_{n,k+1}$ and $x_{n,k}$ we obtain

$$\frac{h_2^{p+1}}{p+1} - \frac{h_1^{p+1}}{p+1} - V(x_{n,k}) \frac{h_2^2}{2} + V(x_{n,k+1}) \frac{h_1^2}{2} = - \int_{x_{n,k+1}}^{x_{n,k}} \varepsilon_n^2 \frac{N-1}{r} |u'_n|^2 + V'(r) \frac{u_n^2}{2} dr,$$

where $h_1 = |u_n(x_{n,k+1})|$ and $h_2 = |u_n(x_{n,k})|$. By the Mean Value Theorem we find $\xi_{n,k} \in (h_1, h_2)$ such that $h_2^{p+1} - h_1^{p+1} = (p + 1)\xi_{n,k}^p(h_2 - h_1)$, and then

$$\frac{h_2 - h_1}{x_{n,k} - x_{n,k+1}} = \frac{N_n}{D_n}, \tag{5.5}$$

where

$$N_n = \frac{h_1^2(V(x_{n,k}) - V(x_{n,k+1}))}{2(x_{n,k} - x_{n,k+1})} - \frac{1}{x_{n,k} - x_{n,k+1}} \int_{x_{n,k+1}}^{x_{n,k}} \varepsilon_n^2 \frac{N - 1}{r} |u'_n|^2 + V'(r) \frac{u_n^2}{2} dr \tag{5.6}$$

and

$$D_n = \xi_{n,k}^p - V(x_{n,k}) \frac{h_1 + h_2}{2}.$$

It is clear that for all $x_{n,k+1} \geq r_0$, both N_n and D_n are bounded. On the other hand, from Lemma 5.1 and under our assumption on the local maximum values of u_n , the denominator D_n is bounded away from zero uniformly for $0 \leq k \leq s(n)$. By the election we made for $x_{n,0}$, it is also clear that the right-hand side of (5.5) is bounded for $k = 1$. Thus, the sequence e_n is uniformly bounded and it is equicontinuous over $[r_0, \infty)$. The application of the Arzelà-Ascoli Theorem gives that e_n converges, up to a sub-sequence. Since r_0 is arbitrary, e_n converges locally uniformly in \mathbb{R}^+ to a function e .

We define the functions $f_n : \mathbb{R}^+ \rightarrow \mathbb{R}$ as the right-hand side of (5.5) for $r \in [x_{n,k+1}, x_{n,k}]$, $k = 0, \dots, s(n) - 1$, as (5.3) if $r \in [x_{n,1}, x_{n,0})$ and simply as $H(r, e_0(r))$ if $r \in [x_{n,0}, \infty)$. In what follows we prove that f_n converges point-wise to $H(r, e(r))$ in \mathbb{R}^+ .

Given $r \in (0, \bar{d})$, we let $x_n^- = x_{n,k(n)+1} \leq r$ and $x_n^+ = x_{n,k(n)} \geq r$ be the extreme points of u_n closest to r . By Proposition 4.1 we see that $x_n^-, x_n^+ \rightarrow r$ and $e_n(x_n^-), e_n(x_n^+) \rightarrow e(r)$. Then we have

$$\lim_{n \rightarrow \infty} h_{n,1}^2 \left(\frac{V(x_n^+) - V(x_n^-)}{x_n^+ - x_n^-} \right) = e(r)^2 V'(r)$$

and

$$\lim_{n \rightarrow \infty} \xi_n^p - V(x_n^+) \frac{h_{n,1} + h_{n,2}}{2} = e(r)^p - V(r)e(r),$$

where $h_{n,1} = |u_n(x_n^-)|$, $h_{n,2} = |u_n(x_n^+)|$ y $\xi_n = \xi_{n,k(n)}$.

Next we consider the integral term in (5.6). We let $w_n(y) = u_n(x_n^- + \varepsilon_n y)$ and we assume that x_n^- is a maximum point of u_n . Then w_n converges in to $w(y) = w(r, e(r); y)$ defined as the solution of (1.5).

Now we have to distinguish two cases. First, if $r \in (0, d]$, then $V(r) \leq 0$, w is periodic with zeroes and $(x_n^+ - x_n^-)/\varepsilon_n$ converges to $2T(r, e(r))$. Then, re-scaling we get

$$\lim_{n \rightarrow \infty} \frac{1}{x_n^+ - x_n^-} \int_{x_n^-}^{x_n^+} \frac{\varepsilon_n^2}{r} |u'_n|^2 dr = \frac{1}{T(r, e(r))} \int_0^{T(r, e(r))} \frac{|w'|^2}{r} dy$$

and

$$\lim_{n \rightarrow \infty} \frac{1}{x_n^+ - x_n^-} \int_{x_n^-}^{x_n^+} V'(r) u_n^2 dr = \frac{1}{T(r, e(r))} \int_0^{T(r, e(r))} V'(r) w^2 dy.$$

Second, if $r \in (d, \bar{d}]$, by Lemma 5.1 we have that $e(r) \geq e_0(r)$. If $e(r) > e_0(r)$ then the situation is as before. If $e(r) = e_0(r)$ then w is positive and decays exponentially. This implies that

$$\lim_{n \rightarrow \infty} \frac{1}{x_n^+ - x_n^-} \int_{x_n^-}^{x_n^+} V'(r) u_n^2 dr = \lim_{n \rightarrow \infty} \frac{1}{x_n^+ - x_n^-} \int_{x_n^-}^{x_n^+} \frac{\varepsilon_n^2}{r} |u_n'|^2 dr = 0.$$

Thus we have that for $r \in (0, \bar{d}]$,

$$\lim_{n \rightarrow \infty} f_n(r) = \frac{\frac{V'(r)}{2} (e(r)^2 - Q(r, e(r))) - \frac{N-1}{T(r, e(r))} \int_0^{T(r, e(r))} \frac{|w'|^2}{r} dy}{e(r)^p - V(r)e(r)}, \tag{5.7}$$

where $w(\cdot) = w(r, e(r); \cdot)$. We see that the right-hand side corresponds exactly to $H(r, e(r))$. In fact, multiplying (1.5) by w' and by w , after some computations we obtain

$$\frac{1}{T(r, s)} \int_0^{T(r, s)} |w'|^2 dy = V(r) (Q(r, s) - s^2) - \frac{2}{p+1} (R(r, s) - s^{p+1})$$

and

$$\frac{1}{T(r, s)} \int_0^{T(r, s)} |w'|^2 dy = -V(r)Q(r, s) + R(r, s),$$

respectively, from where

$$\frac{1}{T(r, s)} \int_0^{T(r, s)} |w'|^2 dy = \frac{1}{p+3} \left((p-1)V(r)Q(r, s) - (p+1)V(r)s^2 + 2s^{p+1} \right).$$

Replacing this in (5.7) we conclude. For $r > \bar{d}$ it is direct from the definition of e_0 .

Next, testing against a compactly supported smooth function, we can show that e is a weak solution of (1.9), which is C^1 since H is a continuous function in $\{(r, s)/r, s \in \mathbb{R}^+, s \geq e_0(r)\}$, as can be easily checked.

We have concluded the proof in case $|u(x_{n,k})| \geq C > 0$ for all n, k . If this is not the case, we know by Corollary 5.1 that u_n converges locally uniformly to zero in $(0, d)$, which implies e_n converges to the trivial envelope e_0 . Here we remark that in the definition of e_n , we may take as $x_{n,k}$ a maximum point of u_n in $[y_{n,k+1}, y_{n,k}]$, which may not be unique. In any case e_n converges to e_0 . \square

6. Characterizing the Envelope

In this section we complete the proof of Theorem 1.2. We already have a limiting envelope, but we do not know its uniqueness. We show in what follows that e can be characterized by means of an asymptotic energy involving the function $R(r, s)$.

Proposition 6.1. *Let (ε_n, u_n) be a sequence of solutions of (2.1) with $\varepsilon_n \rightarrow 0$ and $J_{\varepsilon_n}(u_n) = c$. If e is the limiting envelope found in Sect. 5 then*

$$\lim_{n \rightarrow \infty} \int_a^b |u_n|^{p+1} r^{N-1} dr = \int_a^b R(r, e(r)) r^{N-1} dr,$$

for all $a, b \in \mathbb{R}^+$.

Proof. We first observe that $\sigma(r) := R(r, e(r))r^{N-1}$ is uniformly continuous in $[a, b]$, that is, given $\varepsilon > 0$ there exists $\delta > 0$ such that $x, y \in [a, b]$ and $|x - y| < \delta$ implies $|\sigma(x) - \sigma(y)| < \varepsilon$.

Let x_n^-, x_n^+ be two consecutive extreme points of u_n converging to \bar{r} , then we have

$$\lim_{n \rightarrow \infty} \frac{1}{x_n^+ - x_n^-} \int_{x_n^-}^{x_n^+} |u_n|^{p+1} r^{N-1} dr = \sigma(\bar{r}). \tag{6.1}$$

Consider a partition I_1, \dots, I_k of $[a, b]$ such that $|I_k| < \varepsilon$ and let r_i be the mid-point in I_i for all $i = 1, \dots, k$. Then, by uniform continuity of σ we have

$$\left| \frac{1}{x_n^+ - x_n^-} \int_{x_n^-}^{x_n^+} |u_n|^{p+1} r^{N-1} dr - \sigma(r_i) \right| < \varepsilon, \tag{6.2}$$

for all pair of extreme points x_n^-, x_n^+ of u_n in $I_i, i = 1, \dots, k$ and n large enough. This implies that

$$\left| \sum_{i=1}^k \sigma(r_i) |I_i| - \int_a^b |u_n|^{p+1} r^{N-1} dr \right| \leq \varepsilon(b - a) + o(1),$$

where $o(1) \rightarrow 0$ when $n \rightarrow \infty$. Since ε is arbitrary and σ is continuous, we conclude the proof. \square

To complete our arguments we need the monotonicity of $R(r, s)$. We have

Proposition 6.2. *$R(r, s)$ is strictly increasing as a function of s .*

Proof. By conservation of energy in Eq. (1.5) we have

$$\int_0^{T(r,s)} |w(y)|^{p+1} dy = \sqrt{\frac{p+1}{s^{p-1}}} s^{p+1} \int_0^1 G(t, \lambda) t^{p+1} dt,$$

and then

$$R(r, s) = s^{p+1} \int_0^1 G(t, \lambda) t^{p+1} dt / \int_0^1 G(t, \lambda) dt,$$

where

$$G(t, \lambda) = 1/\sqrt{1 - t^{p+1} - \lambda(1 - t^2)} \quad \text{and} \quad \lambda = (p + 1)V/(2s^{p-1}).$$

If $V(r) = 0$ then R is increasing in s since

$$\frac{\partial}{\partial s} R(r, s) = \frac{p + 1}{s} R(r, s) > 0.$$

In case $V(r) \neq 0$, differentiating we get

$$\frac{\partial}{\partial s} R(r, s) = \frac{p + 1}{s} R(r, s) + \left\{ s^{p+1} \frac{\int_0^1 G'(t, \lambda) t^{p+1} dt}{\int_0^1 G(t, \lambda) dt} - R(r, s) \frac{\int_0^1 G'(t, \lambda) dt}{\int_0^1 G(t, \lambda) dt} \right\} \frac{d\lambda}{ds},$$

where G' is the partial derivative of G with respect to λ . If $V(r) < 0$, then $\lambda < 0$ and $d\lambda/ds = -(p - 1)\lambda/s > 0$. Thus, since $G' > 0$, we just need to prove that

$$D(\lambda) = \frac{p + 1}{s} - \frac{d\lambda}{ds} \frac{\int_0^1 G'(t, \lambda) dt}{\int_0^1 G(t, \lambda) dt} > 0.$$

To do so, we notice that

$$\frac{G'(t, \lambda)}{G(t, \lambda)} = \frac{1}{2((1 - t^{p+1})/(1 - t^2) - \lambda)} < -\frac{1}{2\lambda},$$

and then

$$D(\lambda) > \frac{p + 1}{s} - \frac{(p - 1)\lambda}{s} \frac{1}{2\lambda} = \frac{p + 3}{2s} > 0.$$

If $V(r) > 0$, then we have $\lambda \in (0, 1)$ and $d\lambda/ds < 0$, and then we just need to prove that

$$E(\lambda) = \int_0^1 G'(t, \lambda) t^{p+1} dt \int_0^1 G(t, \lambda) dt - \int_0^1 G(t, \lambda) t^{p+1} dt \int_0^1 G'(t, \lambda) dt$$

is negative. To show this we define

$$g(t, \lambda) = \frac{G'(t, \lambda)}{G(t, \lambda)} = \frac{1}{2((1 - t^{p+1})/(1 - t^2) - \lambda)},$$

and we rewrite $E(\lambda)$ as

$$E(\lambda) = \frac{1}{2} \int_0^1 \int_0^1 G(t, \lambda) G(\tau, \lambda) (g(t, \lambda) - g(\tau, \lambda)) (t^{p+1} - \tau^{p+1}) dt d\tau.$$

Since $g(t, \lambda)$ is decreasing with respect to t , we conclude. \square

With the following corollary, whose proof is a direct consequence of Proposition 6.1 and Proposition 6.2, we conclude the proof of Theorem 1.2.

Corollary 6.1. *The sequence e_n converges to the unique solution e of Eq. (1.9) satisfying the energy condition (1.10).*

Remark 6.1. If $\tilde{e} = r^\alpha e$, it is not hard to see that \tilde{e} is positive at the origin. In fact, since $\tilde{J}(e) = c > 0$ then e and \tilde{e} are not trivial near the origin. This fact implies that e is not bounded at zero. Actually, for a certain constant C we have $e(r) \geq Cr^{-\alpha}$. This in turn implies that u_n is not bounded, since its critical points approach the origin. This proves the second part of Corollary 1.1.

Remark 6.2. Once we have identified the envelope e we can define the asymptotic energy and mass densities \mathcal{E} and ρ , as in (1.11). Then, from Proposition 6.1, we see that for every $0 \leq a < b \leq \infty$,

$$\lim_{n \rightarrow \infty} \int_a^b \left(\frac{\varepsilon_n^2}{2} |u'_n|^2 + \frac{1}{2} V(r) u_n^2 - \frac{1}{p+1} |u_n|^{p+1} \right) r^{N-1} dr = \int_a^b \mathcal{E}(r) dr,$$

and similarly

$$\lim_{n \rightarrow \infty} \int_a^b u_n^2 r^{N-1} dr = \int_a^b \rho(r) dr.$$

7. Appendix

In this Appendix we prove the existence of solutions for (1.2) using the variational method, taking advantage of the fact that the corresponding functional is even. Our proof, written in the radial case, can be directly extended to the general N dimensional case, considering some extra growth assumption for the potential at infinity.

We consider the Sobolev space

$$H = \left\{ u \in H^1(\mathbb{R}^N) / \int_{\mathbb{R}^N} V_+(x) u^2 dx < \infty \text{ and } u \text{ is radial} \right\},$$

where $V_+(x) = \max\{0, V(x)\}$, endowed with the inner product

$$\langle u, v \rangle = \int_{\mathbb{R}^N} \nabla u \cdot \nabla v + (1 + V_+(x)) uv dx.$$

We denote by $\|\cdot\|$ the norm in H associated with $\langle \cdot, \cdot \rangle$ and by $\|\cdot\|_q$ the usual norm of $L^q(\mathbb{R}^N)$. For functions u in H we define the quadratic functional Q_ε as

$$Q_\varepsilon(u) = \frac{1}{2} \int_0^\infty \left(\varepsilon^2 |u'|^2 + V(r) u^2 \right) r^{N-1} dr. \tag{7.1}$$

We will find critical points of Q_ε on the sphere $S = \{u \in H / \|u\|_{p+1} = 1\}$ using standard min-max theory for even functionals. Denoting by $\gamma(A)$ the Krasnoselski genus of the closed symmetric set $A \subset S$, we define

$$\mathcal{A}_k = \{A \subset S / A \text{ is closed and symmetric, } \gamma(A) \geq k\}$$

and, given $k \in \mathbb{N}$, we consider the min-max value

$$b_k(\varepsilon) = \inf_{A \in \mathcal{A}_k} \sup_{u \in A} Q_\varepsilon(u).$$

Since $N \geq 2$, the Strauss Lemma guarantees the compact embedding of H in $L^q(\mathbb{R}^N)$, for $1 \leq q < 2N/(N-2)$ if $N \geq 3$, and for $q \geq 1$ if $N = 2$, see [22]. Thus we can

apply Theorem 8.17 in [20] to obtain that each $b_k(\varepsilon)$ is a critical value of Q_ε on S and that

$$\lim_{k \rightarrow \infty} b_k(\varepsilon) = \infty. \tag{7.2}$$

If $v_k^\varepsilon \in H$ is a critical point associated to $b_k(\varepsilon)$, then $u_k^\varepsilon = (2b_k(\varepsilon))^{1/(p+1)}v_k^\varepsilon$ is a solution of (1.2) with $c_k(\varepsilon) \equiv J_\varepsilon(u_k^\varepsilon) = \frac{(p-1)}{(p+1)}b_k(\varepsilon)$. These values satisfy the following properties: 1) $c_k(\varepsilon)$ is a continuous function of ε , 2) If $k \leq \ell$ then $c_k(\varepsilon) \leq c_\ell(\varepsilon)$, and 3) If $\varepsilon \leq \varepsilon'$ then $c_k(\varepsilon) \leq c_k(\varepsilon')$.

These properties and the following lemma complete the proof of Theorem 1.1.

Lemma 7.1. *The critical values $c_k(\varepsilon)$ satisfy:*

- 1) $\lim_{k \rightarrow \infty} c_k(\varepsilon) = \infty$ and
- 2) Given $\alpha > 0$ and $k \in \mathbb{N}$, there exists ε_k such that $c_k(\varepsilon_k) < \alpha$.

Proof. The proof of 1) is direct from (7.2). To prove 2) we consider a family of k functions $v_1, v_2, \dots, v_k \in H$ having compact supports, disjoint from each other. We define

$$A_k = \{v = \sum_{i=1}^k \alpha_i v_i / \|v\|_{p+1} = 1, \alpha_1, \dots, \alpha_k \in \mathbb{R}\},$$

and we see that there is a constant C_k so that

$$\int_0^\infty (|v'|^2 + V(0)v^2)r^{N-1}dr \leq C_k, \quad \text{for all } v \in A_k.$$

Next we consider the set $A_k^\varepsilon = \{v_\varepsilon / v_\varepsilon(x) = \varepsilon^{-N/(p+1)}v(x/\varepsilon), v \in A_k\}$, which belongs to A_k and whose elements $v_\varepsilon \in A_k$ satisfy

$$Q_\varepsilon(v_\varepsilon) = \frac{\varepsilon^{N(p-1)/(p+1)}}{2} \int_0^\infty (\varepsilon^2|v'|^2 + V(\varepsilon r)v^2)r^{N-1}dr \leq \varepsilon^{N(p-1)/(p+1)}C_k,$$

for small ε . From here 2) follows. \square

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