Nonlinear problems with solutions exhibiting a free boundary on the boundary

Problèmes nonlinéaires avec frontière libre sur le bord du domaine

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Abstract

We prove existence of nonnegative solutions to $-\Delta u + u = 0$ on a smooth bounded domain Ω subject to the singular boundary derivative condition $\frac{\partial u}{\partial \nu} = -u^{-\beta} + \lambda f(x,u)$ on $\partial\Omega \cap \{u > 0\}$ with $0 < \beta < 1$. There is a constant λ^* such that for $0 < \lambda < \lambda^*$ every nonnegative solution vanishes on a subset of the boundary with positive surface measure. For $\lambda > \lambda^*$ we show the existence of a maximal positive solution. We analyze its linearized stability and its regularity. Minimizers of the energy functional related to the problem are shown to be regular and satisfy the equation together with the boundary condition.

Résumé

Nous démontrons l'existence de solutions $u \geqslant 0$ de l'équation $-\Delta u + u = 0$ sur un domaine borné régulier Ω avec la condition de Neumann singulière suivante : $\frac{\partial u}{\partial \nu} = -u^{-\beta} + \lambda f(x,u)$ sur $\partial\Omega \cap \{u > 0\}$ oú $0 < \beta < 1$. Il existe une constante λ^* telle que pour $0 < \lambda < \lambda^*$, toute solution $u \geqslant 0$ s'annule sur une partie du bord, de mesure (surfacique) strictement positive. Pour $\lambda > \lambda^*$, nous démontrons l'existence d'une solution maximale positive. Nous analysons ses propriétés de stabilité linéaire et de régularité. On démontre que les minimiseurs de la fonctionnelle d'énergie associée sont réguliers et vérifient l'équation ainsi que la condition de bord.

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1. Introduction

We study the existence and regularity of solutions of the following nonlinear boundary value problem

$$\begin{cases} -\Delta u + u = 0 & \text{in } \Omega, \\ u \geqslant 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = -u^{-\beta} + f(x,u) & \text{on } \partial\Omega \cap \{u > 0\}, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$, is a bounded domain with smooth boundary, $0 < \beta < 1$ and ν is the exterior unit normal vector to $\partial\Omega$. We assume that

$$f : \partial\Omega \times \mathbb{R} \to \mathbb{R} \quad \text{is } C^1 \text{ and } f \geqslant 0. \tag{2}$$

By a solution of (1) we mean a function $u \in H^1(\Omega) \cap C(\overline{\Omega})$ satisfying

$$\int_\Omega \nabla u \cdot \nabla \varphi + u\varphi = \int_{\partial\Omega \cap \{u>0\}} \big(-u^{-\beta} + f(x,u)\big)\varphi, \qquad \forall \varphi \in C_0^1\big(\Omega \cup \big(\partial\Omega \cap \{u > 0\}\big)\big). \tag{3}$$

An equivalent way to write problem (1) is

$$\begin{cases} -\Delta u + u = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = -u^{-\beta} + f(x,u) & \text{on } \Gamma_+(u), \\ u = 0 & \text{on } \Gamma_0(u), \end{cases} \tag{4}$$

where

$$\Gamma_+(u) = \big\{x \in \partial\Omega \colon u(x) > 0\big\}, \qquad \Gamma_0(u) = \big\{x \in \partial\Omega \colon u(x) = 0\big\}.$$

The last boundary condition in (4) is trivial by the definition of $\Gamma_0(u)$ itself. This notation emphasizes the fact that u satisfies a boundary condition of mixed type: a nonlinear Neumann condition on $\Gamma_+(u)$ and Dirichlet on $\Gamma_0(u)$. Observe that $\Gamma_+(u)$ and $\Gamma_0(u)$ form a partition of the boundary that depends on the solution u. In this sense u solves a free boundary problem on the boundary.

In principle one may try to find solutions of (1) which are positive on $\partial\Omega$, but it turns out that there are situations where no such a solution exists, and nonetheless there are nontrivial solutions of (1), see Theorem 1.8.

There are at least two approaches to tackle the question of existence of a solution: one is to work with a regularization of problem (1) and the second one is a variational formulation for (1), see (7) below.

In our first approach we consider the following regularization of (1)

$$\begin{cases} -\Delta u + u = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = -\frac{u}{(u+\varepsilon)^{1+\beta}} + f(x,u) & \text{on } \partial\Omega, \end{cases} \tag{5}$$

where $\varepsilon > 0$ is a parameter tending to zero.

The solutions of (5) have the following convergence property.

Theorem 1.1. *Suppose f satisfies* (2) *and*

$$\frac{f(x,u)}{u} \to 0 \quad \text{for } u \to \infty \quad \text{uniformly in } x. \tag{6}$$

Then Eq. (5) *possesses a maximal solution $\bar{u}^\varepsilon$ which is positive in $\overline{\Omega}$ and $u = \lim_{\varepsilon\to 0} \bar{u}^\varepsilon$ exists. The convergence is uniform in $\overline{\Omega}$ and u is a solution to the free boundary problem* (1).

Another approach to find solutions to (1) is to consider the functional $\phi : H^1(\Omega) \to \mathbb{R}$ given by

$$\phi(u) = \frac{1}{2}\int_{\Omega} \left(|\nabla u|^2 + u^2\right) + \int_{\partial\Omega} \frac{(u^+)^{1-\beta}}{1-\beta} - F(x, u^+), \tag{7}$$

where $F(x,u) = \int_0^u f(x,s)\,\mathrm{d}s$ and $u^+ = \max\{u, 0\}$.

Theorem 1.2. *Suppose that f satisfies* (2) *and* (6)*. Then ϕ attains its minimum in $H^1(\Omega)$ and any minimizer u of ϕ solves the free boundary problem* (1)*.*

Next we deal with the regularity of the maximal solution $\bar{u}^\varepsilon$ to (5) or a minimizer u of ϕ.

Theorem 1.3. *Suppose f satisfies* (2) *and* (6)*. Then there exists a constant C independent of ε such that the maximal solution $\bar{u}^\varepsilon$ to* (5) *satisfies*

$$|\nabla \bar{u}^\varepsilon| \leqslant C(\bar{u}^\varepsilon)^{-\beta} \quad \text{in } \Omega.$$

As a consequence we have

$$\|\bar{u}^\varepsilon\|_{C^{1/(1+\beta)}(\overline{\Omega})} \leqslant C.$$

Remark 1.4. A consequence of the previous theorem is that the convergence $\bar{u}^\varepsilon \to u$ of the maximal solution to (5) in Theorem 1.1 is in the norm of $C^\mu(\overline{\Omega})$ for all $0 < \mu < \frac{1}{1+\beta}$. And hence $u \in C^{\frac{1}{1+\beta}}(\overline{\Omega})$.

Theorem 1.5. *Suppose f satisfies* (2) *and* (6)*. Let u denote a minimizer of ϕ* (*cf.* (7))*. Then there exists a constant C such that*

$$|\nabla u| \leqslant C u^{-\beta} \quad \text{in } \Omega$$

and hence $u \in C^{1/(1+\beta)}(\overline{\Omega})$.

Remark 1.6. A prototype function describing the behavior of the solutions of (1) near a free boundary point is given by

$$u(r,\theta) = cr^\alpha \sin(\alpha\theta),$$

expressed in polar coordinates ($x_1 = r\cos\theta$, $x_2 = r\sin\theta$), where

$$\alpha = \frac{1}{1+\beta}.$$

The function u is harmonic in the upper half-plane $\mathbb{R}^2_+ = \{(x_1,x_2) \in \mathbb{R}^2 \mid x_2 > 0\}$ and satisfies $u = 0$ on $\{(x_1, 0)\colon x_1 \geqslant 0\}$ and

$$\frac{\partial u}{\partial \nu} = -u^{-\beta} \quad \text{on } \left\{(x_1,0)\colon x_1 < 0\right\},$$

for a suitable choice of the constant $c > 0$.

This example indicates that the regularity stated in Theorems 1.3 and 1.5 is optimal with respect to the Hölder exponent. A modification of this harmonic function u will be useful later in the proof of the regularity theorems.

For the proof of Theorem 1.5 we will use a Hardy type inequality.

Proposition 1.7. *Let $G \subset \mathbb{R}^n$ be a smooth domain (not necessarily bounded) and let $\Gamma \subset \partial G$ be a bounded, relatively open subset with smooth boundary $\partial\Gamma$ (the boundary is taken relative to ∂G).*

Let us define

$$d_{\Gamma^c}(x) = \operatorname{dist}(x, \Gamma^c), \quad \Gamma^c = \partial G \setminus \Gamma.$$

There exists a constant C_h such that

$$\int_\Gamma \frac{\psi^2}{d_{\Gamma^c}} \leqslant C_h \int_G |\nabla\psi|^2, \quad \forall\psi \in C_0^\infty(G \cup \Gamma), \tag{8}$$

where C_h depends on Γ and G.

Finally we address the question of whether there are in fact situations where the solutions that we construct in Theorems 1.1 and 1.2 are positive in $\overline{\Omega}$ or whether there are nontrivial solutions which are zero at some subset of $\partial\Omega$. For this purpose we consider (1) with f replaced by λf where $\lambda > 0$ is a parameter

$$\begin{cases} -\Delta u + u = 0 & \text{in } \Omega, \\ u \geqslant 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = -u^{-\beta} + \lambda f(x,u) & \text{on } \partial\Omega \cap \{u > 0\}. \end{cases} \tag{9}$$

Theorem 1.8. *Assume f satisfies* (2), (6) *and*

$$f(x,u) \text{ is increasing in } u \text{ and there is a } \xi > 0 \text{ such that } f(x,\xi) \not\equiv 0. \tag{10}$$

Then for any $\lambda > 0$ Eq. (9) *has a maximal solution $\bar{u}_\lambda$ and the map $\lambda \in (0,\infty) \mapsto \bar{u}_\lambda$ is nondecreasing. Moreover,*

(a) *There exists $\lambda^* > 0$ such that for $\lambda > \lambda^*$, $\bar{u}_\lambda > 0$ in $\overline{\Omega}$.*
(b) *For $0 < \lambda < \lambda^*$ all solutions must vanish in a nontrivial subset of $\partial\Omega$, that is, the surface measure of $\{x \in \partial\Omega\colon u(x) = 0\}$ is positive.*
(c) *The extremal solution $\bar{u}_{\lambda^*}$ is positive a.e. on $\partial\Omega$.*
(d) *For $\lambda > \lambda^*$, $\bar{u}_\lambda$ is stable in the sense that*

$$\Lambda(\bar{u}_\lambda) = \inf_{\varphi \in C^1(\overline{\Omega})} \frac{\int_\Omega |\nabla\varphi|^2 + \varphi^2 - \int_{\partial\Omega} (\beta \bar{u}_\lambda^{-1-\beta} + \lambda f_u(x, \bar{u}_\lambda))\varphi^2}{\int_{\partial\Omega} \varphi^2} > 0. \tag{11}$$

Remark 1.9. If in addition to (2) and (6) we assume that f is concave, then actually the stability condition (11) characterizes the maximal solution in a similar way as in [4].

Depending on the dimension, the maximal solution $\bar{u}_{\lambda^*}$ could be positive on $\partial\Omega$, not only a.e. The relation between β and n is $(3\beta + 1 + 2\sqrt{\beta^2 + \beta})/(\beta + 1) > n - 1$, which can only hold for some $0 < \beta < 1$ in dimensions $n = 2, 3, 4$. The proof of this assertion is related to the stability of $\bar{u}_\lambda$.

Proposition 1.10. *The extremal solution satisfies $\bar{u}_{\lambda^*} \geqslant c > 0$ on $\partial\Omega$ if $(3\beta + 1 + 2\sqrt{\beta^2 + \beta})/(\beta + 1) > n - 1$, where c is a constant.*

There are a few works dealing with a singular derivative boundary condition. For example in [5,6] the authors study an evolution equation in one space dimension with a Neumann condition involving the singular term $-u^{-\beta}$. In higher dimensions a similar evolution problem was addressed in [8] with a positive unbounded nonlinearity such as $1/(1-u)$ and with a time interval $[0,T)$ where $0 \leqslant u(t) < 1$. One of the main contributions of this paper is that we deal with the possibility that the solution of the stationary problem in dimension $n \geqslant 2$ vanishes on a large subset of $\partial\Omega$. In this situation the solution develops a free boundary, a phenomenon that can occur only when Ω is at least a two-dimensional domain.

Elliptic equations involving a nonlinear Neumann boundary condition have been studied elsewhere in the literature, see for instance [3,11,13] and [14].

The plan of the paper is the following. In Section 2 we present additional results and examples concerning the behavior of the maximal solution of (9) as λ varies. The proof of these assertions is postponed until Section 9. The Hardy-type inequality of Proposition 1.7 is proved in Section 3. Section 4 is devoted to introduce notations for later purposes. In Section 5 we construct a *local subsolution* [4,12] which is then used in Sections 6 and 7 for the proof of the regularity Theorems 1.3 and 1.5 respectively. In the proof of these theorems we employ frequently auxiliary results for linear equations that for convenience we have collected in the Appendix. Theorem 1.1 is proved in Section 6 and the proof of Theorem 1.2 is given in Section 7. Finally in Section 8 we prove Theorem 1.8 and Proposition 1.10.

2. Examples

In this section we give some examples illustrating the exact vanishing properties of the maximal solution of (9) when λ varies.

Proposition 2.1. *Let Ω be a ball in $\mathbb{R}^n$ and assume that f satisfies* (2) *and* (6) (*the requirements of Theorem* 1.8) *and depends only on u. Then $\bar{u}_\lambda = 0$ for $0 < \lambda < \lambda^*$.*

Proposition 2.2. *For any smooth domain Ω and any function f satisfying the hypotheses of Theorem* 1.8 *we have that $\bar{u}_\lambda \equiv 0$ for λ sufficiently small.*

Example 2.3. Let Ω be a ball in $\mathbb{R}^n$. We construct a function $f = f(x)$ (depending only on x) such that $\bar{u}_\lambda \equiv 0$ for $0 < \lambda < \bar{\lambda}$ and $\bar{u}_\lambda \not\equiv 0$ for $\bar{\lambda} \leqslant \lambda \leqslant \lambda^*$ where $0 < \bar{\lambda} < \lambda^*$.

The construction of this example is presented in Section 9.

3. Hardy inequalities

In order to achieve our regularity results we need to establish the Hardy type inequality (8). For this aim we begin proving a Hardy inequality in a half space. Consider the upper half plane

$$\mathbb{R}^2_+ = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 > 0\}$$

and

$$\Gamma = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 < 0,\ x_2 = 0\}.$$

We use the standard notation r, θ for polar coordinates.

Proposition 3.1.

$$\inf_{\psi \in C_0^\infty(\mathbb{R}^2_+ \cup \Gamma)} \frac{\int_{\mathbb{R}^2_+} |\nabla\psi|^2}{\int_\Gamma \psi^2/r} = \frac{1}{\pi}. \tag{12}$$

Proof. Let $\psi \in C_0^\infty(\mathbb{R}^2_+)$. Then

$$\psi(r,\pi) = \int_0^\pi \frac{\partial \psi}{\partial \theta}(r,\theta)\,\mathrm{d}\theta \leqslant \pi^{1/2}\left(\int_0^\pi \left(\frac{\partial \psi}{\partial \theta}\right)^2 (r,\theta)\,\mathrm{d}\theta\right)^{1/2}.$$

Hence

$$\int_\Gamma \frac{\psi^2}{r} \leqslant \pi \int_0^\pi \int_0^\infty \frac{1}{r}\left(\frac{\partial \psi}{\partial \theta}\right)^2 \mathrm{d}r\,\mathrm{d}\theta \leqslant \pi \int_{\mathbb{R}^2_+} |\nabla \psi|^2.$$

For the opposite inequality we consider functions of the special form

$$\psi(r,\theta) = \varphi(r)\theta,$$

where $\varphi \in C_0^\infty(0,\infty)$. Observe that

$$\int_\Gamma \frac{\psi(r)^2}{r}\,\mathrm{d}r = \pi^2 \int_0^\infty \frac{\varphi(r)^2}{r}\,\mathrm{d}r$$

and

$$\int_{\mathbb{R}^2_+} |\nabla \psi|^2 = \frac{1}{3}\pi^3 \int_0^\infty \varphi'(r)^2 r\,\mathrm{d}r + \pi \int_0^\infty \frac{\varphi(r)^2}{r}\,\mathrm{d}r.$$

Hence

$$\frac{\int_{\mathbb{R}^2_+} |\nabla \psi|^2}{\int_\Gamma (\psi(r)^2)/r\,\mathrm{d}r} = \frac{\pi}{3}\frac{\int_0^\infty \varphi'(r)^2 r\,\mathrm{d}r}{\int_0^\infty ((\varphi(r)^2)/r)\,\mathrm{d}r} + \frac{1}{\pi}.$$

But

$$\inf_{\varphi \in C_0^\infty(0,\infty)} \frac{\int_0^\infty \varphi'(r)^2 r\,\mathrm{d}r}{\int_0^\infty (\varphi(r)^2/r)\,\mathrm{d}r} = 0,$$

which can be seen by taking

$$\varphi_\varepsilon(r) = \begin{cases} -\dfrac{\log\varepsilon}{\varepsilon} r & \text{if } 0 \leqslant r \leqslant \varepsilon, \\ -\log r & \text{if } \varepsilon \leqslant r \leqslant 1, \\ 0 & \text{if } r \geqslant 1. \end{cases} \tag{13}$$

In fact

$$\int_0^\infty \varphi_\varepsilon'(r)^2 r\,\mathrm{d}r = \frac{1}{2}\log^2\varepsilon - \log\varepsilon,$$

and

$$\int_0^\infty \frac{\varphi_\varepsilon(r)^2}{r}\,\mathrm{d}r = \frac{1}{2}\log^2\varepsilon - \frac{1}{3}\log^3\varepsilon.$$

Thus

$$\frac{\int_0^\infty \varphi_\varepsilon'(r)^2 r\,\mathrm{d}r}{\int_0^\infty (\varphi_\varepsilon(r)^2/r)\,\mathrm{d}r} \to 0 \quad \text{as } \varepsilon \to 0^+. \qquad \square$$

Remark 3.2. The fact that the functions ψ in (12) are required to vanish on a half of $\partial\mathbb{R}^2_+$ is important for the infimum to be positive, since

$$\inf_{\psi\in C_0^\infty(\bar{\mathbb{R}}^2_+\setminus\{0\})} \frac{\int_{\mathbb{R}^2_+}|\nabla\psi|^2}{\int_{\mathbb{R}}(\psi^2/r)}=0.$$

This can be seen by taking $\psi=\varphi_\varepsilon$ as defined in (13).

We prove now the Hardy inequality in a domain.

Proof of Proposition 1.7. First we claim that it suffices to prove (8) for functions in $C_0^\infty(G\cup\Gamma)$ which have support near $\partial\Gamma$. Indeed, let $\sigma>0$ and consider

$$(\partial\Gamma)^\sigma=\{x\in G:\ \mathrm{dist}(x,\partial\Gamma)<\sigma\}.$$

Let $\eta\in C^1(\overline{G})$ be such that $\eta\equiv 0$ in $G\setminus(\partial\Gamma)^\sigma$ and $\eta\equiv 1$ in $(\partial\Gamma)^{\sigma/2}$. Suppose that (8) is true for functions in $C_0^\infty(G\cup\Gamma)$ with support in $(\partial\Gamma)^\sigma$. Then for any $\psi\in C_0^\infty(G\cup\Gamma)$

$$\int_{(\partial\Gamma)^{\sigma/2}}\frac{\psi^2}{d_{\Gamma^c}}\leqslant\int_\Gamma\frac{(\eta\psi)^2}{d_{\Gamma^c}}\leqslant C\int_G\left|\nabla(\eta\psi)\right|^2$$

and therefore

$$\int_{(\partial\Gamma)^{\sigma/2}}\frac{\psi^2}{d_{\Gamma^c}}\leqslant C\int_{G\cap\mathrm{supp}(\eta)}|\nabla\psi|^2+\psi^2.$$

Since ψ vanishes on $\partial G\setminus\Gamma$, by Poincaré's inequality

$$\int_{G\cap\mathrm{supp}(\eta)}\psi^2\leqslant C\int_G|\nabla\psi|^2$$

and thus

$$\int_{(\partial\Gamma)^{\sigma/2}}\frac{\psi^2}{d_{\Gamma^c}}\leqslant C\int_G|\nabla\psi|^2. \tag{14}$$

On the other hand by the trace theorem we have

$$\int_\Gamma\psi^2\leqslant C\int_G|\nabla\psi|^2.$$

Since $1/d_{\Gamma^c}$ is bounded away from $\partial\Gamma$ and $\psi=0$ on $\partial G\setminus\Gamma$ we obtain

$$\int_{\partial G\setminus(\partial\Gamma)^{\sigma/2}}\frac{\psi^2}{d_{\Gamma^c}}\leqslant C\int_G|\nabla\psi|^2. \tag{15}$$

Combining (14) and (15) we see that (8) holds.

Using a partition of unity and the same argument as before we see that it is sufficient to consider the case of $\psi\in C_0^\infty(G\cup\Gamma)$ with support in a small ball $B_\sigma(x_0)$ centered at $x_0\in\partial\Gamma$. In this situation choose an open set $W\supset B_\sigma(x_0)$ and a change of variables $\varphi: W\to B_\sigma(0)$ which flattens the boundary of G, that is, $\varphi(W\cap G)=B_\sigma(0)\cap H$ where

$$H=\{(x',x_n):\ x'\in\mathbb{R}^{n-1},\ x_n>0\},$$

and $\varphi(W \cap \partial G) = B_\sigma(0) \cap \partial H$, $\varphi(W \setminus \overline{G}) = B_\sigma(0) \setminus \overline{H}$. We can also assume that $W \cap \Gamma$ is mapped into

$$\{(x_1, x_2, \ldots, x_n):\ x_1 > 0, x_n = 0\} \cap B_\sigma(0).$$

Then, applying Proposition 3.1 to $\psi \circ \varphi$ in the half plane $\{(x_1, x_2, \ldots, x_n):\ x_1 > 0, x_2 \in \mathbb{R}, x_3 = \cdots = x_n = 0\}$ and integrating in the variables $x_3, \ldots, x_n$ we see that (8) is valid. □

4. Notations

Let us choose and fix $\tau_0 > 0$ small enough so that for any $x_0 \in \partial\Omega$ there exists an open set W containing the ball $B_{\tau_0}(x_0)$ and a smooth diffeomorphism $\varphi:\ W \subset \mathbb{R}^n \to B_{\tau_0}(0)$ which flattens the boundary of Ω, that is

$$\varphi(W \cap \Omega) = B_{\tau_0}(0) \cap H,$$
$$\varphi(W \cap \partial\Omega) = B_{\tau_0}(0) \cap \partial H,$$
$$\varphi(W \setminus \overline{\Omega}) = B_{\tau_0}(0) \setminus \overline{H},$$

where

$$H = \{(x', x_n):\ x' \in \mathbb{R}^{n-1},\ x_n > 0\}.$$

We can also assume that $\varphi(x_0) = 0$, $\nabla\varphi(x_0) = I$ and φ preserves the normal direction on the surface $W \cap \partial\Omega$.

For $0 < \tau < \tau_0$ and $x_0 \in \partial\Omega$ let us adopt the notation

$$B_\tau^+ = B_\tau(x_0) \cap \Omega, \tag{16}$$

and let us decompose its boundary as $\partial B_\tau^+ = \Gamma^e \cup \Gamma^i$ (the external and internal boundaries of B_τ^+)

$$\Gamma^i = \partial B_\tau(x_0) \cap \Omega, \qquad \Gamma^e = B_\tau(x_0) \cap \partial\Omega. \tag{17}$$

We also decompose $\Gamma^e = \Gamma^1 \cup \Gamma^2$ with

$$\Gamma^1 = \varphi^{-1}\big(B_{\tau/2}(0)\big) \cap \partial\Omega, \qquad \Gamma^2 = \Gamma^e \setminus \Gamma^1. \tag{18}$$

In Fig. 1. we show the above defined sets. For convenience we have flattened Γ^1 and Γ^2, but in fact they are portions of the boundary $\partial\Omega$.

Let us introduce the rescaled variables $\tilde{x}$ and $\tilde{y}$ which allow us to work in the unit ball:

$$x = \tau\tilde{x} + x_0, \qquad \tilde{y} = \varphi_\tau(\tilde{x}) = \frac{1}{\tau}\varphi(\tau\tilde{x} + x_0) = \frac{1}{\tau}\varphi(x). \tag{19}$$

Henceforth we use the notation:

$$\widetilde{B}^+ = \frac{1}{\tau}(B_\tau^+ - x_0) = B_1(0) \cap \frac{1}{\tau}(\Omega - x_0), \qquad \widetilde{\Omega} = \frac{1}{\tau}(\Omega - x_0),$$
$$\widetilde{\Gamma}^i = \frac{1}{\tau}(\Gamma^i - x_0), \quad \widetilde{\Gamma}^e = \frac{1}{\tau}(\Gamma^e - x_0), \quad \widetilde{\Gamma}^k = \frac{1}{\tau}(\Gamma^k - x_0), \quad k = 1, 2. \tag{20}$$

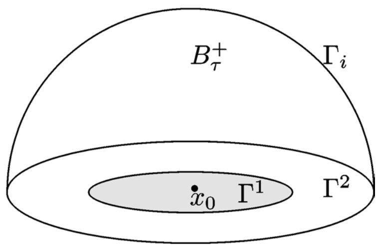

Fig. 1.

5. Construction of a local subsolution

Given $x_0 \in \partial\Omega$ and $0 < \tau < \tau_0$ we construct a special function $\tilde{v}$ in $\widetilde{B}^+$. This construction is inspired by the explicit solution of Remark 1.6.

Let $a \in L^\infty(\widetilde{B}^+)$, $a \geqslant 0$. For a parameter $s > 0$ consider the linear equation

$$\begin{cases} -\Delta\tilde{v} + a(\tilde{x})\tilde{v} = 0 & \text{in } \widetilde{B}^+, \\ \dfrac{\partial\tilde{v}}{\partial\nu}(\tilde{x}) = -\operatorname{dist}(\tilde{x}, \widetilde{\Gamma}^2)^{-\beta/(1+\beta)} & \tilde{x} \in \widetilde{\Gamma}^1, \\ \tilde{v}(\tilde{x}) = 0 & \tilde{x} \in \widetilde{\Gamma}^2, \\ \tilde{v}(\tilde{x}) = s\operatorname{dist}(\tilde{x}, \partial\widetilde{\Omega}) & \tilde{x} \in \widetilde{\Gamma}^i. \end{cases} \tag{21}$$

Remark 5.1. The above problem (21) has a solution. By Hardy's inequality of Lemma 1.7 the solution belongs to $H^1(\widetilde{B}^+)$ and by potential theory methods it can be shown that the solution is in $C^0(\overline{\widetilde{B}^+})$.

The main goal of this section is to prove:

Lemma 5.2. *Let $a \in L^\infty(\widetilde{B}^+)$, $a \geqslant 0$. There exist $\tau_0 > 0$ (even smaller than that one in (16)) and $s_0 > 0$ such that if $0 < \tau < \tau_0$ and $s \geqslant s_0$ the solution of (21) is positive in $\widetilde{B}^+$ and satisfies*

$$\tilde{v}(\tilde{x}) \geqslant cs\operatorname{dist}(\tilde{x}, \widetilde{\Gamma}^2)^{1/(1+\beta)}, \quad \forall\tilde{x} \in \widetilde{\Gamma}^1, \tag{22}$$

where $c > 0$ is independent of x_0, τ and s (c depends only on Ω, n, β and $\|a\|_{L^\infty(\widetilde{B}^+)}$).

Remark 5.3. Note that in particular, for large s the solution $\tilde{v}$ to (21) satisfies

$$\frac{\partial\tilde{v}}{\partial\nu} \leqslant -\tilde{v}^{-\beta} \quad \text{on } \widetilde{\Gamma}^1. \tag{23}$$

For this reason we call $\tilde{v}$ a local subsolution. We will take s_0 larger if necessary so that (23) holds for $s \geqslant s_0$.

For the construction of the subsolution consider the function

$$\underline{u}(\tilde{y}) = \rho^\alpha\big(\sin(\gamma\theta) + b\theta^2\big), \quad \alpha = \frac{1}{1+\beta}, \tag{24}$$

where $\gamma, b > 0$ are constants to be chosen later and ρ, θ, ω are toroidal coordinates:

$$(\rho, \theta, \omega) \in (0, \infty) \times (0, 2\pi) \times S^{n-2} \mapsto \tilde{y} = \left(\frac{1}{2} + \rho\cos\theta\right)\omega + \rho\sin\theta e_n.$$

Here $w = (w_1, \dots, w_{n-1}, 0) \in S^{n-2}$ which is the unit sphere of $n-2$ dimensions, and $e_n = (0, \dots, 0, 1)$.

Abusing notation and using (19) we write

$$\underline{u}(\tilde{x}) = \underline{u}\big(\varphi_\tau(\tilde{x})\big). \tag{25}$$

Lemma 5.4. *Let $a \in L^\infty(\widetilde{B}^+)$. For $\sigma > 0$ define*

$$V_\sigma = \big\{\tilde{x} \in \widetilde{B}^+ \colon \operatorname{dist}(\tilde{x}, \partial\widetilde{\Gamma}^1) < \sigma\big\}.$$

There exist $\frac{1}{2} < \gamma < \alpha$ (recall that $\frac{1}{2} < \alpha < 1$) and $b > 0$, $\sigma > 0$, $\tau_0 > 0$ small so that for all $0 < \tau < \tau_0$ the function $\underline{u}(\tilde{x})$ defined by (24) and (25) satisfies

$$-\Delta_{\tilde{x}}\underline{u} + a(\tilde{x})\underline{u} \leqslant 0 \quad \text{in } V_\sigma.$$

Moreover

$$-C\,\mathrm{dist}(\tilde x,\widetilde\Gamma^2)^{-\beta/(1+\beta)} \leqslant \frac{\partial \underline{u}}{\partial \nu_{\tilde x}}(\tilde x) \leqslant -\frac{1}{C}\,\mathrm{dist}(\tilde x,\widetilde\Gamma^2)^{-\beta/(1+\beta)} \quad \forall \tilde x \in \widetilde\Gamma^1, \tag{26}$$

where $C > 0$. *The constants* $\gamma, b, \sigma, \tau_0, C$ *depend only on* Ω, n, β *and* $\|a\|_{L^\infty(\widetilde B^+)}$.

Proof. A calculation shows that the Laplacian of $\underline{u}$ with respect to $\tilde y$ is given by

$$\begin{aligned}\Delta_{\tilde y}\underline{u} &= \frac{\rho^{\alpha-2}}{1/2+\rho\cos(\theta)}\Big[\Big(\frac12+\rho\cos(\theta)\Big)\big((\alpha^2-\gamma^2)\sin(\gamma\theta)+b\theta^2+2b\big)\\ &\quad+\rho\big(\alpha\cos(\theta)\big(b\theta^2+\sin(\gamma\theta)\big)-\sin(\theta)\big(2b\theta+\gamma\cos(\gamma\theta)\big)\big)\Big]\\ &=\rho^{\alpha-2}\big((\alpha^2-\gamma^2)\sin(\gamma\theta)+b\theta^2+2b+O(\rho)\big),\end{aligned} \tag{27}$$

where $O(\rho)$ stands for a function bounded by a constant times ρ.

We fix γ such that $\frac12 < \gamma < \alpha$. Observe that

$$\frac{\partial \underline{u}}{\partial \nu_{\tilde y}} = \frac{1}{\rho}\frac{\partial \underline{u}}{\partial\theta}\Big|_{\theta=\pi} = \rho^{\alpha-1}\big(\gamma\cos(\gamma\pi)+2b\pi\big).$$

We choose now $b > 0$ small enough so that

$$\gamma\cos(\gamma\pi)+2b\pi < 0.$$

This ensures the validity of (26).

A computation shows that

$$\Delta_{\tilde x}\underline{u} = A_{ij}\partial_{\tilde y_i\tilde y_j}\underline{u} + B_i\partial_{\tilde y_i}\underline{u}, \tag{28}$$

where we have adopted the convention of summation over repeated indices. The functions A_{ij} and B_i are given by

$$A_{ij} = \frac{\partial \tilde y_i}{\partial \tilde x_k}\frac{\partial \tilde y_j}{\partial \tilde x_k}, \qquad B_i = \sum_k \frac{\partial^2 \tilde y_i}{\partial \tilde x_k^2}.$$

Since $\tilde y = \frac{1}{\tau}\varphi(\tau\tilde x + x_0)$, $\varphi(x_0) = 0$ and $\nabla\varphi(x_0) = I$ we have

$$\frac{1}{\tau}\varphi(\tau\tilde x + x_0) = \tilde x + \tau\frac12 D^2\varphi(0)\tilde x^2 + O(\tau^2).$$

Consequently,

$$A_{ij} = \delta_{ij} + O(\tau), \qquad B_i = O(\tau).$$

Thus from (28)

$$\Delta_{\tilde x}\underline{u} = \Delta_{\tilde y}\underline{u} + O(\tau)D^2_{\tilde y}\underline{u} + O(\tau)D_{\tilde y}\underline{u}.$$

On the other hand observe that

$$D^2_{\tilde y}\underline{u} = O(\rho^{\alpha-2}), \qquad D_{\tilde y}\underline{u} = O(\rho^{\alpha-1}).$$

Hence (27) implies

$$\Delta_{\tilde x}\underline{u} - a(\tilde x)\underline{u} \geqslant \rho^{\alpha-2}\big[2b + O(\rho) + O(\tau) + O(\tau\rho) - O(\rho^2)\big],$$

and thus, since $b > 0$ we can choose τ, σ small enough so that

$$\Delta_{\tilde x}\underline{u} - a(\tilde x)\underline{u} \geqslant 0 \quad \text{in } V_\sigma. \qquad \square$$

Proof of Lemma 5.2. Let us write $\tilde v = \tilde v_1 + \tilde v_2$ where $\tilde v_1$ is the solution of the following problem

$$\begin{cases} -\Delta \tilde v_1 + a(\tilde x)\tilde v_1 = 0 & \text{in } \widetilde B^+,\\ \dfrac{\partial \tilde v_1}{\partial \nu}(\tilde x) = -\operatorname{dist}(\tilde x, \widetilde\Gamma^2)^{-\beta/(1+\beta)} & \tilde x \in \widetilde\Gamma^1,\\ \tilde v_1(\tilde x) = 0 & \tilde x \in \widetilde\Gamma^2 \cup \widetilde\Gamma^i, \end{cases}$$

and $\tilde v_2$ satisfies

$$\begin{cases} -\Delta \tilde v_2 + a(\tilde x)\tilde v_2 = 0 & \text{in } \widetilde B^+,\\ \dfrac{\partial \tilde v_2}{\partial \nu}(\tilde x) = 0 & \tilde x \in \widetilde\Gamma^1,\\ \tilde v_2(\tilde x) = 0 & \tilde x \in \widetilde\Gamma^2,\\ \tilde v_2(\tilde x) = s\operatorname{dist}(\tilde x, \partial\widetilde\Omega) & \tilde x \in \widetilde\Gamma^i. \end{cases}$$

Observe that $\tilde v_1 \leqslant 0$ while $\tilde v_2 \geqslant 0$ in $\widetilde B^+$.

Let $\sigma > 0$ be small as in Lemma 5.4 and recall:

$$V_\sigma = \{\tilde x \in \widetilde B^+ \colon \operatorname{dist}(\tilde x, \partial\widetilde\Gamma^1) < \sigma\}.$$

For $\tilde x \in \widetilde B^+$ let $P(\tilde x) \in \partial\widetilde\Omega$ denote the closest point in $\partial\widetilde\Omega$ closest to $\tilde x$. If τ is small enough this projection is well defined and smooth on $\widetilde B^+$. Hence we fix τ_0 even smaller than that one in (16), so that this property holds for $0 < \tau < \tau_0$.

For $\tilde x \in \widetilde B^+$ define

$$g(\tilde x) = \begin{cases} \operatorname{dist}(\tilde x, \partial\widetilde\Omega) & \text{if } P(\tilde x) \in \widetilde\Gamma^2,\\ 1 & \text{if } P(\tilde x) \in \widetilde\Gamma^1. \end{cases} \tag{29}$$

By the strong maximum principle and Hopf's lemma applied to $\tilde v_2/s$ we have

$$\tilde v_2(\tilde x) \geqslant csg(\tilde x), \quad \forall \tilde x \in \widetilde B^+ \setminus V_\sigma,$$

where $c > 0$ depends only on Ω and σ. It follows that for s large

$$\tilde v(\tilde x) \geqslant csg(\tilde x), \quad \forall \tilde x \in \widetilde B^+ \setminus V_\sigma, \tag{30}$$

for a new constant $c > 0$ (here we use only the fact that $\tilde v_1$ is bounded from above in $\widetilde B^+ \setminus V_\sigma$).

Let $\underline{u}(\tilde x)$ be the function of Lemma 5.4. We are going to show that for large s there holds

$$s\underline{u} \leqslant C\tilde v \quad \text{in } V_\sigma,$$

where C is a constant. Indeed, first recall that $\Delta\underline{u} - a(\tilde x)\underline{u} \geqslant 0$ in V_σ. Also one has $\underline{u} \leqslant Cg$ on $\partial V_\sigma \cap \widetilde\Omega$ for some constant C. By (30) with s sufficiently large one obtains

$$s\underline{u} \leqslant C\tilde v \quad \text{on } \partial V_\sigma \cap \widetilde\Omega. \tag{31}$$

Estimate (26) implies

$$s\frac{\partial \underline{u}}{\partial \nu}(\tilde x) \leqslant -\frac{s}{C}\operatorname{dist}(\tilde x, \widetilde\Gamma^2)^{-\beta/(1+\beta)} \quad \forall \tilde x \in \partial V_\sigma \cap \widetilde\Gamma^1.$$

Hence choosing $s > 0$ large enough we have

$$s\frac{\partial \underline{u}}{\partial \nu} \leqslant C\frac{\partial \tilde v}{\partial \nu} \quad \text{on } \partial V_\sigma \cap \widetilde\Gamma^1. \tag{32}$$

On the other hand

$$\underline{u} = \tilde v = 0 \quad \text{on } \partial V_\sigma \cap \widetilde\Gamma^2. \tag{33}$$

Hence, by the maximum principle for s large

$$s\underline{u} \leqslant C\tilde{v} \quad \text{in } V_\sigma. \tag{34}$$

We fix s_0 sufficiently large such that (30)–(34) are valid for $s \geqslant s_0$. This shows that $\tilde{v}$ is positive in $\widetilde{B}^+$ and satisfies (22). □

6. Existence, regularity and convergence of $\bar{u}^\varepsilon$

In this section we prove Theorems 1.1 and 1.3. First we remark that there exists a maximal solution $\bar{u}^\varepsilon$ to (5) because 0 is a subsolution and a large constant is a supersolution by (10).

It is convenient to introduce some notation. Let $M > 0$ be such that

$$M \geqslant \sup_{x \in \partial\Omega} f\big(x, \bar{u}^\varepsilon(x)\big),$$

for all $0 < \varepsilon < 1$.

We need a Harnack inequality which, for completeness, we prove in the Appendix.

Lemma 6.1. *Suppose that $u \in H^1(B_3 \cap \widetilde{\Omega})$, $u \geqslant 0$ satisfies*

$$\begin{cases} -\Delta u + a(\tilde{x})u = 0 & \text{in } B_3 \cap \widetilde{\Omega}, \\ \dfrac{\partial u}{\partial \nu} \leqslant N & \text{on } \widetilde{\Gamma}^e, \end{cases}$$

where N is a constant. Then there is a constant $c_k > 0$ such that

$$u(\tilde{x}) \geqslant c_k \operatorname{dist}(\tilde{x}, \widetilde{\Gamma}^e)\big(c_k u(\tilde{x}_1) - N\big), \quad \forall \tilde{x} \in \widetilde{B}^+ \quad \text{and } \forall \tilde{x}_1 \in B_{1/2} \cap \widetilde{B}^+.$$

The constant c_k can be chosen independent of $x_0 \in \partial\Omega$ and of $0 < \tau < \tau_0$ ($\tau_0 > 0$ was introduced in Section 4).

Let τ_0 and s_0 be the constants in the statement of Lemma 5.2 and Remark 5.3. Let us fix a large constant $\widetilde{C} > 0$, independent of $0 < \varepsilon < 1$, such that

$$s_0 < \frac{1}{2} c_k^2 \widetilde{C}, \tag{35}$$

$$\|\bar{u}^\varepsilon\|_{L^\infty(\Omega)}^{1+\beta} < \tau_0 \widetilde{C}^{1+\beta}, \tag{36}$$

$$\|\bar{u}^\varepsilon\|_{L^\infty(\Omega)}^{\beta} < \frac{c_k \widetilde{C}^{1+\beta}}{2M}. \tag{37}$$

Next we fix C_0 large enough such that

$$\left(\frac{C_0}{\widetilde{C}}\right)^{1+\beta} \geqslant 6. \tag{38}$$

For the sake of notation, from this point on we write $u = \bar{u}^\varepsilon$.
Given a point $x_0 \in \partial\Omega$ and $0 < \tau < \tau_0$ we define

$$\tilde{u}(\tilde{x}) = \tau^{-1/(1+\beta)} u(\tau\tilde{x} + x_0), \quad \tilde{x} \in \widetilde{\Omega} = \frac{1}{\tau}(\Omega - x_0), \tag{39}$$

which satisfies

$$\begin{cases} -\Delta\tilde{u} + \tau^2\tilde{u} = 0 & \text{in } \widetilde{\Omega}, \\ \dfrac{\partial \tilde{u}}{\partial \tilde{\nu}} = g_\tau^\varepsilon(\tilde{x}, \tilde{u}) & \text{on } \partial\widetilde{\Omega}, \end{cases} \tag{40}$$

where $\tilde{\nu}$ is the exterior unit normal vector to $\partial\widetilde{\Omega}$ and g_τ^ε is given by

$$g_\tau^\varepsilon(\tilde{x},\tilde{u}) = \tau^{\beta/(1+\beta)} g^\varepsilon(\tau\tilde{x}+x_0, \tau^{1/(1+\beta)}\tilde{u}),$$

and

$$g^\varepsilon(x,u) = -\frac{u}{(u+\varepsilon)^{1+\beta}} + f(x,u).$$

Lemma 6.2. *Let $x_1 \in \Omega$ and assume that*

$$u(x_1) \geqslant C_0\delta(x_1)^{1/(1+\beta)}. \tag{41}$$

Fix

$$\tau = \left(\frac{u(x_1)}{\widetilde{C}}\right)^{1+\beta}, \tag{42}$$

and let $x_0 \in \partial\Omega$ be such that

$$\operatorname{dist}(x_1, \partial\Omega) = |x_0 - x_1|. \tag{43}$$

Then $\tau < \tau_0$ and $\tilde{u}$ (defined in (39)) verifies

$$\tilde{u}(\tilde{x}) \geqslant s_0 \operatorname{dist}(\tilde{x}, \partial\widetilde{\Omega}), \quad \forall \tilde{x} \in \widetilde{\Gamma}^i.$$

Proof. By (36) we have $\tau < \tau_0$. Let $\tilde{x}_1$ denote the point $\frac{1}{\tau}(x_1 - x_0)$ which satisfies

$$|\tilde{x}_1| \leqslant \frac{1}{6} \tag{44}$$

by (38), (41)–(43). Observe that by the choice of τ we have

$$\tilde{u}(\tilde{x}_1) = \widetilde{C}. \tag{45}$$

Using Harnack's Lemma 6.1 and (45) we obtain

$$\tilde{u}(\tilde{x}) \geqslant c_k \operatorname{dist}(\tilde{x}, \partial\widetilde{\Omega})\left(c_k\widetilde{C} - \sup_{\widetilde{\Gamma}^e} \frac{\partial\tilde{u}}{\partial\nu}\right), \quad \forall \tilde{x} \in \widetilde{B}^+. \tag{46}$$

From the boundary condition in (40) and the definition of M

$$\sup_{\widetilde{\Gamma}^e} \frac{\partial\tilde{u}}{\partial\nu} \leqslant \tau^{\beta/(1+\beta)} M.$$

Notice that from (37) it follows that

$$u(x_1)^\beta \leqslant \frac{c_k\widetilde{C}^{1+\beta}}{2M}$$

and therefore

$$\tau^{\beta/(1+\beta)} M = M\left(\frac{u(x_1)}{\widetilde{C}}\right)^\beta \leqslant \frac{1}{2}c_k\widetilde{C}.$$

Inserting this in (46) and recalling (35) we find

$$\tilde{u}(\tilde{x}) \geqslant \frac{1}{2}c_k^2\widetilde{C} \operatorname{dist}(\tilde{x}, \partial\widetilde{\Omega}) \geqslant s_0 \operatorname{dist}(\tilde{x}, \partial\widetilde{\Omega}), \quad \forall \tilde{x} \in \widetilde{\Gamma}^i. \qquad \square$$

Proof of Theorem 1.3. Let x_1 be a point in Ω. We distinguish two cases.

Case 1. Assume

$$u(x_1) \geqslant C_0 \delta(x_1)^{1/(1+\beta)}. \tag{47}$$

Let τ be given by (42) and $x_0 \in \partial\Omega$ be such that $\operatorname{dist}(x_1, \partial\Omega) = |x_0 - x_1|$. Let $\tilde{v}$ be the solution of problem (21) with $s = s_0$ and $a(\tilde{x}) = \tau^2$. We know by Lemma 5.2 that $\tilde{v}$ satisfies

$$\tilde{v}(\tilde{x}) \geqslant c s_0 \operatorname{dist}(\tilde{x}, \widetilde{\Gamma}^2)^{1/(1+\beta)} \quad \forall \tilde{x} \in \widetilde{\Gamma}^1. \tag{48}$$

By Lemma 6.2

$$\tilde{u} \geqslant \tilde{v} \quad \text{on } \widetilde{\Gamma}^i. \tag{49}$$

We claim that $\tilde{u} \geqslant \tilde{v}$ on $\widetilde{B}^+$. To see this, define

$$\widetilde{U}(\tilde{x}) = \begin{cases} \tilde{u}(x) & \text{for } \tilde{x} \in \widetilde{\Omega} \setminus B_1(0), \\ \max(\tilde{u}(x), \tilde{v}(x)) & \text{for } \tilde{x} \in \widetilde{\Omega} \cap B_1(0) = \widetilde{B}^+. \end{cases}$$

Then the function

$$U(x) = \tau^{1/(1+\beta)} \widetilde{U}\left(\frac{1}{\tau}(x - x_0)\right)$$

is a subsolution of Eq. (5) (cf. (23)) and since u is the maximal solution we have $U \leqslant u$ in Ω and as consequence

$$\tilde{v} \leqslant \tilde{u} \quad \text{in } \widetilde{B}^+.$$

This fact in combination with (48) yields the estimate

$$\tilde{u}(\tilde{x}) \geqslant c s_0 \operatorname{dist}(\tilde{x}, \widetilde{\Gamma}^2)^{1/(1+\beta)}, \quad \forall \tilde{x} \in \widetilde{\Gamma}^1. \tag{50}$$

From the boundary condition in (40) we deduce that

$$\left|\frac{\partial \tilde{u}}{\partial \tilde{\nu}}\right| \leqslant C \operatorname{dist}(\tilde{x}, \widetilde{\Gamma}^2)^{-\beta/(1+\beta)} + \tau^{\beta/(1+\beta)} M \quad \text{on } \widetilde{\Gamma}^1, \tag{51}$$

and therefore, on a smaller set we obtain an estimate

$$\left|\frac{\partial \tilde{u}}{\partial \tilde{\nu}}\right| \leqslant C \quad \text{on } B_{1/3} \cap \partial\widetilde{\Omega}, \tag{52}$$

with a constant C independent of ε. We will deduce from this an estimate of the form

$$\left|\nabla \tilde{u}(\tilde{x}_1)\right| \leqslant C, \tag{53}$$

with C independent of ε. From the definition of $\tilde{u}$ it will follow that

$$\left|\nabla u(x_1)\right| \leqslant C u(x_1)^{-\beta}.$$

Let us prove (53). For this purpose choose $p > n$ and take $n < r < \frac{np}{n-1}$. By Lemma 9.3

$$\|\tilde{u}\|_{W^{1,r}(B_{1/4} \cap \widetilde{\Omega})} \leqslant C\left(\left\|\frac{\partial \tilde{u}}{\partial \tilde{\nu}}\right\|_{L^p(B_{1/3} \cap \partial\widetilde{\Omega})} + \|\tilde{u}\|_{L^1(B_{1/3} \cap \widetilde{\Omega})}\right),$$

and by the embedding $W^{1,r} \subset C^\mu$ we have for some $0 < \mu < 1$

$$\|\tilde{u}\|_{C^\mu(B_{1/4} \cap \widetilde{\Omega})} \leqslant C\left(\left\|\frac{\partial \tilde{u}}{\partial \tilde{\nu}}\right\|_{L^p(B_{1/3} \cap \partial\widetilde{\Omega})} + \|\tilde{u}\|_{L^1(B_{1/3} \cap \widetilde{\Omega})}\right).$$

By the assumption (2) and the lower bound (50) we see that the right-hand side of the boundary condition in (40) satisfies

$$\left\|g_\tau^\varepsilon(\tilde{x}, \tilde{u})\right\|_{C^\mu(B_{1/4} \cap \partial\widetilde{\Omega})} \leqslant C\left(\left\|\frac{\partial \tilde{u}}{\partial \tilde{\nu}}\right\|_{L^p(B_{1/3} \cap \partial\widetilde{\Omega})} + \|\tilde{u}\|_{L^1(B_{1/3} \cap \widetilde{\Omega})}\right).$$

Using Schauder estimates (see e.g. [9]) we deduce

$$\|\tilde u\|_{C^{1,\mu}(B_{1/5}\cap\widetilde\Omega)}\leqslant C\left(\left\|\frac{\partial\tilde u}{\partial\tilde\nu}\right\|_{L^p(B_{1/3}\cap\partial\widetilde\Omega)}+\|\tilde u\|_{L^1(B_{1/3}\cap\widetilde\Omega)}\right).$$

Recalling that $|\tilde x_1|\leqslant\frac16$ by (44) we obtain

$$\left|\nabla\tilde u(\tilde x_1)\right|\leqslant C\left(\left\|\frac{\partial\tilde u}{\partial\tilde\nu}\right\|_{L^p(B_{1/3}\cap\partial\widetilde\Omega)}+\|\tilde u\|_{L^1(B_{1/3}\cap\widetilde\Omega)}\right).$$

By (52) we can assert that

$$\left\|\frac{\partial\tilde u}{\partial\tilde\nu}\right\|_{L^p(B_{1/3}\cap\partial\widetilde\Omega)}\leqslant C$$

with C independent of ε. It suffices then to find an estimate for $\|\tilde u\|_{L^1(B_{1/3}\cap\widetilde\Omega)}$. Using (51) we see that

$$\left|\frac{\partial\tilde u}{\partial\tilde\nu}\right|\leqslant C\quad\text{on } B_{5/12}\cap\partial\widetilde\Omega$$

and therefore, using Lemma 9.5 we find

$$\int\limits_{B_{1/3}\cap\widetilde\Omega}\tilde u\leqslant C\big(\tilde u(\tilde x)+1\big),\quad\forall\tilde x\in B_{1/2}\cap\widetilde\Omega.$$

Putting $\tilde x=\tilde x_1$ in the latter estimate and recalling (44) and (45) we obtain the desired conclusion.

Case 2. Assume $u(x_1)\leqslant C_0\delta(x_1)^{1/(1+\beta)}$.

Define $\tilde u(\tilde x)=\tau^{-1/(1+\beta)}u(\tau\tilde x+x_1)$, where $\tau=\frac12\delta(x_1)$. Then $-\Delta\tilde u+\tau^2\tilde u=0$ in $B_1(0)$, $\tilde u\geqslant0$ in $B_1(0)$ and $\tilde u(0)\leqslant 2^{1/(1+\beta)}C_0$. Since $\tilde u\geqslant0$, by elliptic estimates we have $|\nabla\tilde u(0)|\leqslant C$, where C depends only on n, β, C_0, τ_0. This implies $|\nabla u(x_1)|\leqslant C\tau^{-\beta/(1+\beta)}\leqslant Cu(x_1)^{-\beta}$. □

Proof of Theorem 1.1. Note that $\bar u^\varepsilon$ decreases as ε decreases to 0 and by the uniform estimate of Theorem 1.3 there exists $u\in C^{1/(1+\beta)}(\overline\Omega)$ such that $\bar u^\varepsilon\to u$ in $C^\mu(\overline\Omega)$ for $0<\mu<\frac{1}{1+\beta}$. Let $\varphi\in C_0^1(\Omega\cup(\partial\Omega\cap\{u>0\}))$. It is easy to pass to the limit in

$$\int\limits_\Omega\nabla\bar u^\varepsilon\cdot\nabla\varphi+\bar u^\varepsilon\varphi=\int\limits_{\partial\Omega}\left(-\frac{\bar u^\varepsilon}{(\bar u^\varepsilon+\varepsilon)^{1+\beta}}+f(x,\bar u^\varepsilon)\right)\varphi\tag{54}$$

for all terms, except possibly for $\int_\Omega\nabla\bar u^\varepsilon\cdot\nabla\varphi$. But $\bar u^\varepsilon$ is bounded in $H^1(\Omega)$ since, by taking $\varphi=\bar u^\varepsilon$ in (54) we get

$$\int\limits_\Omega|\nabla\bar u^\varepsilon|^2+(\bar u^\varepsilon)^2\leqslant\int\limits_{\partial\Omega}f(x,\bar u^\varepsilon)\bar u^\varepsilon\leqslant C.$$

Thus for a sequence $\varepsilon_k\to0$, we conclude that $\bar u^{\varepsilon_k}$ converges weakly in $H^1(\Omega)$ to a function which necessarily coincides with u. Then by weak convergence $\int_\Omega\nabla\bar u^\varepsilon\cdot\nabla\varphi\to\int_\Omega\nabla u\cdot\nabla\varphi$. This shows that u is a solution of (1). □

7. Regularity for minimizers of ϕ

Let us recall the functional corresponding to problem (1):

$$\phi(u)=\frac12\int\limits_\Omega\big(|\nabla u|^2+u^2\big)+\int\limits_{\partial\Omega}\frac{(u^+)^{1-\beta}}{1-\beta}-F(x,u^+).$$

Proof of Theorem 1.5. Suppose that $u \in H^1(\Omega)$ is a minimizer of ϕ. First observe that $u \geqslant 0$. Indeed u^+ is also a minimizer and $\phi(u^+) \leqslant \phi(u)$. But if $u^+ \not\equiv u$, then $\phi(u^+) < \phi(u)$, which is an absurd. Next remark that u satisfies $-\Delta u + u = 0$ in Ω. This follows by observing that for any $\varphi \in C_0^\infty(\Omega)$ the function $s \mapsto \phi(u + s\varphi)$ is differentiable and attains its minimum value at $s = 0$. Thus u is smooth in Ω and the objective now is to show that

$$\left|\nabla u(x_1)\right| \leqslant C u(x_1)^{-\beta}, \quad \forall x_1 \in \Omega, \tag{55}$$

for some constant C.

The argument follows the same scheme as in the proof of Theorem 1.3: given $x_1 \in \Omega$ we distinguish two cases:

Case 1. $u(x_1) \geqslant C_0 \delta(x_1)^{1/(1+\beta)}$, and

Case 2. $u(x_1) \leqslant C_0 \delta(x_1)^{1/(1+\beta)}$.

In Case 2, the argument is exactly the same as in the proof of Theorem 1.3.

Suppose that Case 1 occurs. Let τ be given by (42) and $x_0 \in \partial\Omega$ be such that $\operatorname{dist}(x_1, \partial\Omega) = |x_0 - x_1|$. Let $\tilde{v}$ be the solution of problem (21) with $s = s_0$. By Lemma 6.2 we deduce (49), that is

$$\tilde{u} \geqslant \tilde{v} \quad \text{on } \widetilde{\Gamma}^i.$$

We claim that

$$\tilde{u} \geqslant \tilde{v} \quad \text{in } \widetilde{B}^+. \tag{56}$$

To prove this let $v(x) = \tau^{1/(1+\beta)} \tilde{v}(\frac{1}{\tau}(x - x_0))$ and define

$$U(x) = \begin{cases} u(x) & \text{for } x \in \Omega \setminus B_\tau(x_0), \\ \max\bigl(u(x), v(x)\bigr) & \text{for } x \in B_\tau(x_0) \cap \Omega. \end{cases}$$

Then U satisfies

$$\begin{cases} -\Delta U + U \leqslant 0 & \text{in } \Omega, \\ \dfrac{\partial U}{\partial \nu} \leqslant -U^{-\beta} + f(x, U) & \text{on } \partial\Omega \cap \{U > 0\}. \end{cases} \tag{57}$$

We shall prove that if τ is small enough then $\phi(U) < \phi(u)$ unless $u \equiv U$. First note that

$$\phi(U) - \phi(u) = -\frac{1}{2} \int_\Omega \bigl(\left|\nabla(U - u)\right|^2 + (U - u)^2\bigr) + \int_\Omega \nabla U \cdot \nabla(U - u) + U(U - u) + \int_{\partial\Omega} \frac{1}{1 - \beta}(U^{1-\beta} - u^{1-\beta}) - F(x, U) + F(x, u). \tag{58}$$

Next we multiply (57) by $U - u \geqslant 0$ and integrate by parts to obtain

$$\int_\Omega \nabla U \cdot \nabla(U - u) + U(U - u) \leqslant \int_{\partial\Omega} \frac{\partial U}{\partial \nu}(U - u) \leqslant \int_{\partial\Omega \cap \{U > 0\}} \bigl(-U^{-\beta} + f(x, U)\bigr)(U - u). \tag{59}$$

Combining (58) and (59)

$$\phi(U) - \phi(u) \leqslant -\frac{1}{2} \int_\Omega \left|\nabla(U - u)\right|^2 + \int_{\partial\Omega \cap \{U > 0\}} \frac{U^{1-\beta}}{1 - \beta} - \frac{u^{1-\beta}}{1 - \beta} - U^{-\beta}(U - u) - \int_{\partial\Omega \cap \{U > 0\}} F(x, U) - F(x, u) - f(x, U)(U - u).$$

We claim that

$$\frac{U^{1-\beta}}{1-\beta} - \frac{u^{1-\beta}}{1-\beta} - U^{-\beta}(U-u) \leqslant CU^{-1-\beta}(U-u)^2 \tag{60}$$

and

$$\left|F(x,U) - F(x,u) - f(x,U)(U-u)\right| \leqslant CU^{-1-\beta}(U-u)^2. \tag{61}$$

To verify (60) we consider first the case $U \leqslant 2u$. By the mean value theorem there is a $\xi \geqslant 0$ such that $u \leqslant \xi \leqslant U$ and

$$\frac{U^{1-\beta}}{1-\beta} - \frac{u^{1-\beta}}{1-\beta} - U^{-\beta}(U-u) = \frac{1}{2}\beta\xi^{-1-\beta}(U-u)^2 \leqslant \frac{\beta}{2}u^{-1-\beta}(U-u)^2 \leqslant CU^{-1-\beta}(U-u)^2.$$

In the second case, $U \geqslant 2u$, we have $U \leqslant 2(U-u)$ and therefore

$$\frac{U^{1-\beta}}{1-\beta} - \frac{u^{1-\beta}}{1-\beta} - U^{-\beta}(U-u) \leqslant \frac{1}{1-\beta}U^{1-\beta} = \frac{1}{1-\beta}U^{-1-\beta}U^2 \leqslant \frac{4}{1-\beta}U^{-1-\beta}(U-u)^2.$$

To prove (61) observe that

$$F(x,U) - F(x,u) - f(x,U)(U-u) = \frac{1}{2}f_u(x,\xi)(U-u)^2,$$

for some $u \leqslant \xi \leqslant U$ and (61) follows because $f_u(x,\xi)$ is bounded on $\partial\Omega \times [0, \max_{\partial\Omega} U]$.

Hence

$$\phi(U) - \phi(u) \leqslant -\frac{1}{2}\int_{\Omega} \left|\nabla(U-u)\right|^2 + C \int_{\partial\Omega\cap\{U>0\}} U^{-1-\beta}(U-u)^2. \tag{62}$$

Define $\widetilde{U}(\tilde{x}) = \tau^{-1/(1+\beta)}U(\tau\tilde{x} + x_0)$. Using that $\widetilde{U} \equiv \tilde{u}$ in $\widetilde{\Omega} \setminus \widetilde{B}^+$ we can rewrite (62) as

$$\phi(U) - \phi(u) \leqslant \tau^{n-2\beta/(1+\beta)}\left(-\frac{1}{2}\int_{\widetilde{B}^+} \left|\nabla(\widetilde{U}-\tilde{u})\right|^2 + C\int_{\widetilde{\Gamma}^e\cap\{\widetilde{U}>0\}} \widetilde{U}^{-1-\beta}(\widetilde{U}-\tilde{u})^2\right).$$

Using the explicit lower bound (22) we obtain

$$\phi(U) - \phi(u) \leqslant \tau^{n-2\beta/(1+\beta)}\left(-\frac{1}{2}\int_{\widetilde{B}^+} \left|\nabla(\widetilde{U}-\tilde{u})\right|^2 + \frac{C}{s_0^{1+\beta}}\int_{\widetilde{\Gamma}^1} \operatorname{dist}(\tilde{x},\widetilde{\Gamma}^2)^{-1}(\widetilde{U}-\tilde{u})^2\right),$$

where we have also used the fact that if $\tilde{v}(\tilde{x}) = 0$ then $\widetilde{U}(\tilde{x}) - \tilde{u}(\tilde{x}) = 0$, which allows us to restrict the integral to $\widetilde{\Gamma}^1$. By Hardy's inequality (cf. (8))

$$\phi(U) - \phi(u) \leqslant \tau^{n-2\beta/(1+\beta)}\left(-\frac{1}{2} + \frac{CC_h}{s_0^{1+\beta}}\right)\int_{\widetilde{B}^+} \left|\nabla(\widetilde{U}-\tilde{u})\right|^2.$$

We can choose s_0 larger if necessary in order to make $(-1/2 + CC_h/s_0^{(1+\beta)}) < 0$. Thus, we see that $\phi(U) < \phi(u)$ unless $\widetilde{U} \equiv \tilde{u}$, which implies our claim (56).

The rest of the argument continues in exactly the same manner as in the proof of Theorem 1.3. □

Proof of Theorem 1.2. Since f is sublinear and $0 < \beta < 1$ the functional ϕ attains its minimum in $H^1(\Omega)$. Let u be a minimizer of ϕ. We have shown at the beginning of the proof of Theorem 1.5 that u is smooth in Ω and solves

$$-\Delta u + u = 0 \quad \text{in } \Omega.$$

By Theorem 1.5 we know also that $u \in C^{1/(1+\beta)}(\overline{\Omega})$ and hence $\partial\Omega \cap \{u > 0\}$ is an open subset of $\partial\Omega$. To verify the boundary condition in (1) observe that

$$\frac{\mathrm{d}}{\mathrm{d}s}\phi(u + s\varphi)|_{s=0} = 0, \quad \forall \varphi \in C_0^1\big(\Omega \cup \big(\partial\Omega \cap \{u > 0\}\big)\big)$$

which is equivalent to (3). □

8. Proof of Theorem 1.8 and Proposition 1.10

Proof of Theorem 1.8. By Theorem 1.1 for every $\lambda > 0$ there is a solution of (1). The solution $\bar{u}_\lambda^\varepsilon$ of (21) is unique, since f is sublinear. Furthermore $\bar{u}_\lambda = \lim_{\varepsilon\to 0} \bar{u}_\lambda^\varepsilon$ is the maximal solution of (1). Indeed, if v is a solution of (1), then it is a subsolution to (21), so $v \leqslant \bar{u}_\lambda^\varepsilon$. As $\varepsilon \to 0$ one obtains $v \leqslant \bar{u}_\lambda$. This solution $\bar{u}_\lambda$ is also nondecreasing with respect to λ because the same is true for (21), that is, $\lambda_1 \leqslant \lambda_2$ implies that $\bar{u}_{\lambda_1}^\varepsilon \leqslant \bar{u}_{\lambda_2}^\varepsilon$, since $f(x,u)$ is increasing in u.

For $\lambda > 0$ small enough there is no nontrivial solution, see the proof of Proposition 2.2 in Section 9.

For $\lambda > 0$ large enough we will see that there exists a positive solution. To prove this we will follow the method of sub-super solutions. By a subsolution of (9) we mean a function $\underline{u} \in H^1(\Omega) \cap C(\overline{\Omega})$ such that the surface measure of $\{x \in \partial\Omega\colon u(x) = 0\}$ is zero and that verifies

$$\begin{cases} -\Delta\underline{u} + \underline{u} \leqslant 0 & \text{in } \Omega, \\ \dfrac{\partial\underline{u}}{\partial\nu} \leqslant -\underline{u}^{-\beta} + \lambda f(x,\underline{u}) & \text{on } \partial\Omega, \end{cases} \tag{63}$$

A supersolution is a function $\bar{u} \in H^1(\Omega) \cap C(\overline{\Omega})$ the surface measure of $\{x \in \partial\Omega\colon u(x) = 0\}$ is zero satisfying the above (63) with the inequality signs reversed. Our aim is to find a subsolution $\underline{u} \geqslant c > 0$ for some constant c and a supersolution $\bar{u}$ such that $\underline{u} \leqslant \bar{u}$, implying the existence of a solution u such that $\underline{u} \leqslant u \leqslant \bar{u}$, see [1].

Construction of the subsolution for sufficiently large λ: Let Y be the solution of

$$\begin{cases} -\Delta Y + Y = 0 & \text{in } \Omega, \\ \dfrac{\partial Y}{\partial\nu} = 1 & \text{on } \partial\Omega. \end{cases} \tag{64}$$

Let ξ_0 be as in the assumption (10), that is, such that $f(x,\xi_0) \not\equiv 0$ and solve

$$\begin{cases} -\Delta v + v = 0 & \text{in } \Omega, \\ \dfrac{\partial v}{\partial\nu} = f(x,\xi_0) & \text{on } \partial\Omega. \end{cases} \tag{65}$$

By the maximum principle and Hopf's lemma v is positive in $\overline{\Omega}$. Let us fix $\varepsilon > 0$ such that

$$b := \inf_{\overline{\Omega}}(v - \varepsilon Y) > 0.$$

Define

$$\underline{u} = k(v - \varepsilon Y),$$

where we choose k large enough in such a way that

$$kb \geqslant \xi_0 \quad \text{and} \quad b^{-\beta} \leqslant \varepsilon k^{1+\beta}.$$

Subsequently we choose $\lambda \geqslant k$. This results in

$$kf(x,\xi_0) \leqslant \lambda f(x,kb) \quad \text{and} \quad k^{-\beta}(v - \varepsilon Y)^{-\beta} \leqslant k\varepsilon,$$

implying the normal derivative inequality in (63).

Construction of the ordered supersolution: Let Y be the solution to (64) and define $\bar{u} = AY$, where $A > 0$ is a large constant such that $A \geqslant \lambda f(x, A)$ for every $x \in \partial\Omega$. This makes $\bar{u}$ a supersolution of (9). We may take A even larger in order to have $\underline{u} \leqslant \bar{u}$. □

We now proceed with the proof of the remaining items of Theorem 1.8.

Proof of Theorem 1.8(a) and (b). Define

$$\lambda^* = \inf\{\lambda > 0\colon\ \bar{u}_\lambda > 0 \text{ in } \overline{\Omega}\}.$$

Observe that $0 < \lambda^* < \infty$. Let $0 < \lambda' < \lambda^*$ and suppose that $\bar{u}_{\lambda'} > 0$ a.e. on $\partial\Omega$. Fix λ such that $\lambda' < \lambda < \lambda^*$. Let ζ be the solution of

$$\begin{cases} -\Delta\zeta + \zeta = 0 & \text{in } \Omega, \\ \dfrac{\partial\zeta}{\partial\nu} = f(x, \bar{u}_{\lambda'}) & \text{on } \partial\Omega. \end{cases} \tag{66}$$

Clearly $\zeta > 0$ in Ω since $f(x, \bar{u}_{\lambda'}) \geqslant 0$ and $f(x, \bar{u}_{\lambda'}) \not\equiv 0$ on $\partial\Omega$. Let $0 < \varepsilon \leqslant \lambda - \lambda'$. Then the function $w = \bar{u}_{\lambda'} + \varepsilon\zeta$ is positive in $\overline{\Omega}$ and satisfies

$$\frac{\partial w}{\partial\nu} = -\bar{u}_{\lambda'}^{-\beta} + \lambda' f(x, \bar{u}_{\lambda'}) + \varepsilon f(x, \bar{u}_{\lambda'}) \leqslant -w^{-\beta} + \lambda f(x, w).$$

Hence w is a subsolution of (9) corresponding to λ, thus $\bar{u}_\lambda \geqslant w$ in Ω. Since $\lambda < \lambda^*$ this contradicts the definition of λ^*. □

Proof of Theorem 1.8(c). We have $\int_{\partial\Omega} \bar{u}_\lambda^{-\beta} \leqslant C$ as λ decreases to λ^*. In fact, integrating Eq. (9) in Ω we find

$$\int\limits_{\partial\Omega} \bar{u}_\lambda^{-\beta} \leqslant \lambda \int\limits_{\partial\Omega} f(x, \bar{u}_\lambda).$$

But $\bar{u}_\lambda$ decreases to $\bar{u}_{\lambda^*}$ and hence $\bar{u}_\lambda^{-\beta}$ increases to $\bar{u}_{\lambda^*}^{-\beta}$ as λ decreases to λ^*, and by monotone convergence we deduce that $\int_{\partial\Omega} \bar{u}_{\lambda^*}^{-\beta} \leqslant C$. In particular $\bar{u}_{\lambda^*} > 0$ a.e. on $\partial\Omega$. □

Proof of Theorem 1.8(d). Fix $\lambda > \lambda^*$. From now on we drop the dependence on λ and write $u = \bar{u}_\lambda$, we also write $u_\varepsilon = \bar{u}_\lambda^\varepsilon$. Our aim is to prove that $\Lambda(u) > 0$, see the definition in (11). First we prove that for $\lambda > \lambda^*$ the maximal solution u is weakly stable [2,4] in the sense that

$$\Lambda(u) \geqslant 0. \tag{67}$$

In fact, we know that the maximal solution u_ε of (5) is weakly stable, that is

$$\int\limits_{\partial\Omega} \left(\frac{\beta u_\varepsilon - \varepsilon}{(u_\varepsilon + \varepsilon)^{2+\beta}} + \lambda f_u(x, u_\varepsilon)\right)\varphi^2 \leqslant \int\limits_{\Omega} |\nabla\varphi|^2 + \varphi^2, \quad \forall\varphi \in C^1(\overline{\Omega}). \tag{68}$$

Since $u_\varepsilon \geqslant c$ for some $c > 0$ independent of ε one can let $\varepsilon \to 0$ in (68) and obtain (67).

Let us show that $\Lambda(u) > 0$. We introduce a new parameter $\theta \leqslant 0$ and consider the family of problems

$$\begin{cases} -\Delta u + u = 0 & \text{in } \Omega, \\ u \geqslant 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial\nu} = -u^{-\beta} + \lambda f(x, u) + \theta & \text{on } \partial\Omega \cap \{u > 0\}. \end{cases} \tag{69}$$

The main observations to conclude are:

there exists $\theta_0 < 0$ such that for $\theta > \theta_0$ (69) has a positive maximal solution $\bar{u}_\theta$, and (70)

$$\text{for } \theta > \theta_0 \text{ we have } \Lambda(\bar{u}_\theta) \geqslant 0 \text{ and } \Lambda(\bar{u}_\theta) \text{ is strictly increasing with } \theta. \tag{71}$$

Indeed, assuming (70) and (71) we have a maximal solution $\bar{u}_\theta$ to (69) for some $\theta < 0$. But then

$$0 \leqslant \Lambda(\bar{u}_\theta) < \Lambda(\bar{u}_0)$$

and finally note that $\bar{u}_0$ is just the maximal solution of (9). □

Proof of (70). Fix λ' such that $\lambda^* < \lambda' < \lambda$ and let $0 < \varepsilon \leqslant \lambda - \lambda'$. Let ζ be the solution of (66) and Y be the solution of (64). Take $\theta_0 < 0$ with $|\theta_0|$ small enough so that

$$|\theta_0| Y \leqslant \frac{1}{2}\varepsilon\zeta.$$

For $\theta > \theta_0$ set

$$w = \bar{u}_{\lambda'} + \varepsilon\zeta + \theta Y,$$

and observe that $w > \bar{u}_{\lambda'}$. We compute

$$\frac{\partial w}{\partial \nu} = -\bar{u}_{\lambda'}^{-\beta} + \lambda' f(x, \bar{u}_{\lambda'}) + \varepsilon f(x, \bar{u}_{\lambda'}) + \theta \leqslant -w^{-\beta} + \lambda f(x, w) + \theta.$$

Thus w is a positive subsolution, and by the method of sub and supersolutions we can find a maximal solution $\bar{u}_\theta$. □

Proof of (71). Let $\theta_0 < \theta_1 < \theta_2$ and let $\bar{u}_{\theta_1}$, $\bar{u}_{\theta_2}$ denote the maximal solution of (69) with parameters θ_1 and θ_2. Note that

$$\bar{u}_{\theta_1} < \bar{u}_{\theta_2}. \tag{72}$$

Let ψ_1 and ψ_2 denote the first eigenfunctions, i.e.,

$$\begin{cases} -\Delta\psi_i + \psi_i = 0 & \text{in } \Omega, \\ \dfrac{\partial \psi_i}{\partial \nu} = \beta u^{-\beta-1}\psi_i + \lambda f_u(x, u)\psi_i + \Lambda(\bar{u}_{\theta_i})\psi_i & \text{on } \partial\Omega, \end{cases} \tag{73}$$

$i = 1, 2$, normalized so that $\|\psi_i\|_{L^2(\partial\Omega)} = 1$. Then

$$\begin{aligned} \Lambda(\bar{u}_{\theta_1}) &= \int_\Omega |\nabla\psi_1|^2 + \psi_1^2 - \int_{\partial\Omega} \big(\beta(\bar{u}_{\theta_1})^{-\beta-1} + \lambda f_u(x, \bar{u}_{\theta_1})\big)\psi_1^2 \\ &\leqslant \int_\Omega |\nabla\psi_2|^2 + \psi_2^2 - \int_{\partial\Omega} \big(\beta(\bar{u}_{\theta_1})^{-\beta-1} + \lambda f_u(x, \bar{u}_{\theta_1})\big)\psi_2^2 \\ &< \int_\Omega |\nabla\psi_2|^2 + \psi_2^2 - \int_{\partial\Omega} \big(\beta(\bar{u}_{\theta_2})^{-\beta-1} + \lambda f_u(x, \bar{u}_{\theta_2})\big)\psi_2^2 \\ &= \Lambda(\bar{u}_{\theta_2}), \end{aligned}$$

where the last inequality is strict because $\psi_2 > 0$ and (72). □

Proof of Proposition 1.10. We shall prove that there exists a constant $c > 0$ independent of λ such that

$$\bar{u}_\lambda \geqslant c \quad \text{on } \partial\Omega, \ \forall \lambda > \lambda^*. \tag{74}$$

The conclusion follows by letting $\lambda \searrow \lambda^*$.

Multiply the equation by $\varphi \in C^1(\overline{\Omega})$ and integrate in Ω:

$$-\int_{\Omega} \frac{\partial \bar{u}_\lambda}{\partial \nu}\varphi + \int_{\Omega} \nabla \bar{u}_\lambda \cdot \nabla\varphi + \int_{\Omega} \bar{u}_\lambda \varphi = 0. \tag{75}$$

Take $\varphi = \bar{u}_\lambda^{-\gamma}$ where $\gamma > 1$ will be specified later. Notice that φ is well defined because $\bar{u}_\lambda > 0$ in $\overline{\Omega}$ for $\lambda > \lambda^*$. We have $\nabla\varphi = -\gamma \bar{u}_\lambda^{-\gamma-1}\nabla u$ and using (75)

$$\int_{\partial\Omega} \bar{u}_\lambda^{-\beta-\gamma} - \lambda \int_{\partial\Omega} f(x, \bar{u}_\lambda)\bar{u}_\lambda^{-\gamma} - \gamma \int_{\Omega} \bar{u}_\lambda^{-\gamma-1}|\nabla \bar{u}_\lambda|^2 + \int_{\Omega} \bar{u}_\lambda^{1-\gamma} = 0.$$

Using the hypothesis $f(x,u) \geqslant 0$ we find

$$\frac{4\gamma}{(1-\gamma)^2}\int_{\Omega} |\nabla \bar{u}_\lambda^{(1-\gamma)/2}|^2 \leqslant \int_{\partial\Omega} \bar{u}_\lambda^{-\beta-\gamma} + \int_{\Omega} \bar{u}_\lambda^{1-\gamma}. \tag{76}$$

We use the stability condition (67) with $\varphi = \bar{u}_\lambda^{(1-\gamma)/2}$ and the fact that f_u is bounded to obtain

$$\beta \int_{\partial\Omega} \bar{u}_\lambda^{-\beta-\gamma} \leqslant \int_{\Omega} |\nabla \bar{u}_\lambda^{(1-\gamma)/2}|^2 + \int_{\Omega} \bar{u}_\lambda^{1-\gamma} + C\int_{\partial\Omega} \bar{u}_\lambda^{1-\gamma}. \tag{77}$$

Let $\varepsilon > 0$ be a small constant to be fixed later. Then from (76) and (77) we obtain

$$\frac{4\gamma(1-\varepsilon)}{(1-\gamma)^2}\left(\beta\int_{\partial\Omega} \bar{u}_\lambda^{-\beta-\gamma} - \int_{\Omega} \bar{u}_\lambda^{1-\gamma} - C\int_{\partial\Omega} \bar{u}_\lambda^{1-\gamma}\right) + \frac{4\gamma\varepsilon}{(1-\gamma)^2}\int_{\Omega} |\nabla \bar{u}_\lambda^{(1-\gamma)/2}|^2 \leqslant \int_{\partial\Omega} \bar{u}_\lambda^{-\beta-\gamma} + \int_{\Omega} \bar{u}_\lambda^{1-\gamma}$$

which is equivalent to

$$\left(\frac{4\gamma\beta(1-\varepsilon)}{(1-\gamma)^2} - 1\right)\int_{\partial\Omega} \bar{u}_\lambda^{-\beta-\gamma} + \frac{4\gamma\varepsilon}{(1-\gamma)^2}\int_{\Omega} |\nabla \bar{u}_\lambda^{(1-\gamma)/2}|^2 \leqslant C\int_{\Omega} \bar{u}_\lambda^{1-\gamma} + C\int_{\partial\Omega} \bar{u}_\lambda^{1-\gamma}. \tag{78}$$

We need the following versions of Sobolev's inequality and trace inequality.

Lemma 8.1. *For any $\mu > 0$ and $d_1 > 0$, $d_2 > 0$ there exists $C = C(\Omega, \mu, d_1, d_2)$ such that*

$$\int_{\partial\Omega} \varphi^2 \leqslant \mu\int_{\Omega} |\nabla\varphi|^2 + C\left(\int_{\partial\Omega} |\varphi|^{d_1}\right)^{2/d_1}, \quad \forall\varphi \in C^1(\overline{\Omega}), \quad \textit{and} \tag{79}$$

$$\int_{\Omega} \varphi^2 \leqslant \mu\int_{\Omega} |\nabla\varphi|^2 + C\left(\int_{\Omega} |\varphi|^{d_2}\right)^{2/d_2}, \quad \forall\varphi \in C^1(\overline{\Omega}). \tag{80}$$

Proof of Proposition 1.10 continued. We use (79) and (80) with $\varphi = \bar{u}_\lambda^{(1-\gamma)/2}$ and $d_1 > 0$, $d_2 > 0$ to be fixed shortly, and combine with (78)

$$\left(\frac{4\gamma\beta(1-\varepsilon)}{(1-\gamma)^2} - 1\right)\int_{\partial\Omega} \bar{u}_\lambda^{-\beta-\gamma} + \frac{2\gamma\varepsilon}{\mu(1-\gamma)^2}\left(\int_{\Omega} \bar{u}_\lambda^{1-\gamma} + \int_{\partial\Omega} \bar{u}_\lambda^{1-\gamma}\right)$$
$$\leqslant C\int_{\Omega} \bar{u}_\lambda^{1-\gamma} + C\int_{\partial\Omega} \bar{u}_\lambda^{1-\gamma} + \frac{2\gamma\varepsilon C}{\mu(1-\gamma)^2}\left(\int_{\partial\Omega} \bar{u}_\lambda^{d_1(1-\gamma)/2} + \int_{\Omega} \bar{u}_\lambda^{d_2(1-\gamma)/2}\right).$$

We fix μ small enough such that $2\gamma\varepsilon/(\mu(1-\gamma)^2)=C$ and deduce that

$$\left(\frac{4\gamma\beta(1-\varepsilon)}{(1-\gamma)^2}-1\right)\int\limits_{\partial\Omega}\bar{u}_\lambda^{-\beta-\gamma}\leqslant C\int\limits_{\partial\Omega}\bar{u}_\lambda^{d_1(1-\gamma)/2}+C\int\limits_{\Omega}\bar{u}_\lambda^{d_2(1-\gamma)/2}.$$

Observe that by taking $\varphi\equiv 1$ in (11) we deduce

$$\int\limits_{\partial\Omega}\bar{u}_\lambda^{-\beta-1}\leqslant C$$

with a constant C independent of λ, for λ in a bounded interval, say, $\lambda^*<\lambda\leqslant\lambda_0$. For this reason we take $d_1>0$ so that $d_1\frac{1-\gamma}{2}=-\beta-1$ and conclude

$$\left(\frac{4\gamma\beta(1-\varepsilon)}{(1-\gamma)^2}-1\right)\int\limits_{\partial\Omega}\bar{u}_\lambda^{-\beta-\gamma}\leqslant C+C\int\limits_{\Omega}\bar{u}_\lambda^{d_2(1-\gamma)/2}, \tag{81}$$

where C independent of λ for $\lambda^*<\lambda\leqslant\lambda_0$.

To proceed further we have to bound the integral $\int_\Omega\bar{u}_\lambda^{d_2(1-\gamma)/2}$ by a constant independent of λ, $\lambda^*<\lambda\leqslant\lambda_0$. By Lemma 9.1,

$$\bar{u}_\lambda(x)\geqslant c\,\mathrm{dist}(x,\partial\Omega)\int\limits_{\partial\Omega}\bar{u}_\lambda,$$

where c depends only on Ω. Since $\bar{u}_\lambda\geqslant\bar{u}_{\lambda^*}>0$ a.e. we see that there is a constant C independent of $\lambda>\lambda^*$ such that

$$\int\limits_{\Omega}\bar{u}_\lambda^{d_2(1-\gamma)/2}\leqslant C\int\limits_{\Omega}\mathrm{dist}(x,\partial\Omega)^{d_2(1-\gamma)/2}\,\mathrm{d}x<\infty, \tag{82}$$

if we fix $d_2>0$ small so that $d_2\frac{1-\gamma}{2}>-1$.

Finally, from (81) and (82) we deduce

$$\left(\frac{4\gamma\beta(1-\varepsilon)}{(1-\gamma)^2}-1\right)\int\limits_{\partial\Omega}\bar{u}_\lambda^{-\beta-\gamma}\leqslant C.$$

The latter estimate is useful if $4\gamma\beta(1-\varepsilon)/(1-\gamma)^2-1>0$ for some $\varepsilon>0$, that is, if $4\gamma\beta/(1-\gamma)^2>1$. This is the case for $1<\gamma<1+2\beta+2\sqrt{\beta+\beta^2}$.

In summary, if $0<p<1+3\beta+2\sqrt{\beta+\beta^2}$ then there is a constant C independent of $\lambda^*<\lambda\leqslant\lambda_0$ such that

$$\int\limits_{\partial\Omega}\bar{u}_\lambda^{-p}\leqslant C.$$

Let $v_\lambda=1/\bar{u}_\lambda$. Then v_λ satisfies

$$\begin{cases}-\Delta v_\lambda+v_\lambda\leqslant 0 & \text{in } \Omega,\\ \dfrac{\partial v_\lambda}{\partial\nu}\leqslant v_\lambda^{2+\beta} & \text{on } \partial\Omega.\end{cases}$$

The proof of (74) will be completed with the aid of the following lemma.

Lemma 8.2. *Suppose $v\in C^2(\overline{\Omega})$ satisfies*

$$\begin{cases}-\Delta v+v\leqslant 0 & \textit{in } \Omega,\\ \dfrac{\partial v}{\partial\nu}\leqslant v^{q} & \textit{on } \partial\Omega,\end{cases}$$

where $q > 1$. If

$$\int_{\partial\Omega} v^p < K$$

for some $p > (n-1)(q-1)$ then

$$\|v\|_{L^\infty(\partial\Omega)} \leqslant C,$$

where C depends only on Ω, p, q and K.

Proof of Lemma 8.1. We only prove (79) the other inequality being analogous. By Hölder's inequality we can assume that $d < \frac{2(n-1)}{n-2}$, the Sobolev exponent for the trace inequality. For the sake of contradiction suppose that there exists a sequence $\varphi_n \in C^1(\overline{\Omega})$ such that

$$1 = \int_{\partial\Omega} \varphi_n^2 > \mu \int_\Omega |\nabla\varphi_n|^2 + n\left(\int_{\partial\Omega} |\varphi_n|^d\right)^{2/d}.$$

Then φ_n is bounded in $H^1(\Omega)$ and up to a subsequence $\varphi_n \rightharpoonup \varphi$ weakly in $H^1(\Omega)$. By the compact embedding $H^1(\Omega) \subset L^2(\partial\Omega)$ we have $\int_{\partial\Omega} \varphi^2 = 1$. But the embedding $H^1(\Omega) \subset L^d(\partial\Omega)$ is also compact and therefore we also have $\int_{\partial\Omega} |\varphi|^d = 0$, a contradiction. □

Proof of Lemma 8.2. We have $v^q \in L^{p/q}(\partial\Omega)$. By L^p theory $v \in W^{1,r}(\Omega)$ for all $1 \leqslant r < pn/(q(n-1))$. By the trace inequality $v \in W^{1-1/r,r}(\partial\Omega)$ and by Sobolev's embedding $v \in L^t(\partial\Omega)$ for $\frac{1}{t} = \frac{1}{r}\frac{n}{n-1} - \frac{1}{n-1}$. It follows that $v \in L^t(\partial\Omega)$ for $t < t^*$ with $\frac{1}{t^*} = \frac{q}{p} - \frac{1}{n-1}$. But $t^* > p$. Repeating this process a finite number of times (bootstrap) we obtain the conclusion. □

9. Proofs for Section 2

Proof of Proposition 2.1. If the domain is a ball and $f = f(u)$, the maximal solutions $\bar{u}^\varepsilon$ to (5) are radial and hence $\bar{u}_\lambda$ is radial. This means that $\bar{u}_\lambda$ is a constant on $\partial\Omega$ and this constant is either positive or zero. □

Proof of Proposition 2.2. Assume $\bar{u}_\lambda \not\equiv 0$ for a sufficiently small $\lambda > 0$. Since $\Delta\bar{u}_\lambda = \bar{u}_\lambda \geqslant 0$, then $\bar{u}_\lambda$ attains it maximum at a point $y \in \partial\Omega$. Thus $\frac{\partial\bar{u}_\lambda}{\partial\nu}(y) \geqslant 0$, implies $\frac{1}{\lambda} \leqslant \bar{u}_\lambda(y)^\beta f(y, \bar{u}_\lambda(y))$. But $\bar{u}_\lambda(y)^\beta f(y, \bar{u}_\lambda(y)) \leqslant K$ where $K > 0$ is a constant independent of λ for λ small, because the map $\lambda \mapsto \bar{u}_\lambda$ is nondecreasing. This is an absurd. □

Construction of Example 2.3. Let $\Omega = B_R(0)$ and pick a point $x_0 \in \partial B_R$. We take f smooth such that $0 \leqslant f \leqslant 1$, $f \equiv 1$ in $B_{r_0}(x_0)$ and $f \equiv 0$ in $\mathbb{R}^n \setminus B_{2r_0}(x_0)$ for a fixed $r_0 > 0$ and consider the problem

$$\begin{cases} -\Delta u + u = 0 & \text{in } B_R, \\ \dfrac{\partial u}{\partial\nu} = -u^{-\beta} + \lambda f(x) & \text{on } \partial B_R \cap \{u > 0\}. \end{cases} \tag{83}$$

We will proceed in two steps. First we will prove that for large R and λ the maximal solution of (83) is nontrivial. Then we fix such a large $\lambda = \bar{\lambda}$ and take R even larger in order to prove that the maximal solution vanishes on a subset of $\partial B_R(0)$ with positive surface measure.

Claim 1. *If λ is large enough then for any R large the maximal solution of* (83) *is nontrivial.*

In fact, we will construct a nontrivial subsolution of (83) for λ large. Let $D = B_{3r_0}(x_0) \cap B_R(0)$ and write its boundary as $\partial D = \Gamma \cup \Upsilon$ where $\Gamma = B_{r_0}(x_0) \cap \partial B_R(0)$ and $\Upsilon = \partial D \setminus \Gamma$.

Let $w = w_1 + w_2$ where w_1 solves

$$\begin{cases} -\Delta w_1 + w_1 = 0 & \text{in } D, \\ \dfrac{\partial w_1}{\partial \nu}(x) = -\operatorname{dist}(x,\Upsilon)^{-\beta/(1+\beta)} & \text{on } \Gamma, \\ w_1 = 0 & \text{on } \Upsilon, \end{cases}$$

and w_2 satisfies

$$\begin{cases} -\Delta w_2 + w_2 = 0 & \text{in } D, \\ \dfrac{\partial w_2}{\partial \nu}(x) = \lambda & \text{on } \Gamma, \\ w_2 = 0 & \text{on } \Upsilon. \end{cases}$$

For $\sigma > 0$ define $W_\sigma = \{x \in D\colon \operatorname{dist}(x, \partial\Gamma) < \sigma\}$. A similar calculation as in Lemma 5.4 implies that for R large enough there is a function $\underline{u}$ defined in W_σ that satisfies

$$\begin{cases} -\Delta \underline{u} + \underline{u} \leqslant 0 & \text{in } W_\sigma, \\ \dfrac{\partial \underline{u}}{\partial \nu}(x) \leqslant -\frac{1}{C}\operatorname{dist}(x,\Upsilon)^{-\beta/(1+\beta)} & \text{on } \Gamma \cap \partial W_\sigma, \\ \underline{u} = 0 & \text{on } \Upsilon \cap \partial W_\sigma. \end{cases}$$

(Taking R large here corresponds to work with small τ in Lemma 5.4.) Moreover, as in (22)

$$\underline{u} \geqslant c \operatorname{dist}(x,\Upsilon)^{1/(1+\beta)} \quad \text{on } \Gamma \cap \partial W_\sigma,$$

where $c > 0$. Here C and c are positive constants that are independent of R and λ.

Following the argument of Lemma 5.4 the following assertions hold:

(1) $w \geqslant c\lambda \underline{u}$ in $\partial W_\sigma \cap D$ for λ large enough,
(2) $w = c\lambda\underline{u} = 0$ on $\partial W_\sigma \cap \Upsilon$,
(3) $c\lambda \frac{\partial \underline{u}}{\partial \nu}(x) \leqslant -\lambda \operatorname{dist}(x,\Upsilon)^{-\beta/(1+\beta)} \leqslant \frac{\partial w}{\partial \nu}(x)$ for every $x \in \partial W_\sigma \cap \Gamma$ and λ large, see (26).

By the maximum principle

$$c\lambda\underline{u} \leqslant w \quad \text{in } W_\sigma.$$

Therefore

$$w(x) \geqslant c\lambda \operatorname{dist}(x,\Upsilon)^{1/(1+\beta)} \quad \text{on } \Gamma.$$

Hence on Γ

$$\frac{\partial w}{\partial \nu}(x) = -\operatorname{dist}(x,\Upsilon)^{-\beta/(1+\beta)} + \lambda \leqslant -c^\beta \lambda^\beta w^{-\beta} + \lambda.$$

Thus, if λ is sufficiently large, w is a subsolution in D.

Next, we extend w by zero to $B_R(0) \setminus D$ and this is a nontrivial subsolution of (83).

Claim 2. *We fix a sufficiently large $\lambda = \bar{\lambda}$ and prove that for R large enough the maximal solution vanishes on a subset of ∂B_R with positive surface measure.*

Define $v(y) = u(Ry)$ and $y = \frac{x}{R}$ for $y \in B_1$ then v satisfies

$$\begin{cases} -R^{-2}\Delta v + v = 0 & \text{in } B_1, \\ \dfrac{\partial v}{\partial \nu} = -v^{-\beta}R + \lambda R f(Ry) & \text{on } \partial B_1 \cap \{v > 0\}. \end{cases}$$

Integration over B_1 gives

$$R^{-2}\int_{\partial B_1}\frac{\partial v}{\partial n}=\int_{B_1}v$$

and using the boundary condition

$$\int_{B_1}v=R^{-1}\biggl(\int_{\partial B_1\cap\{v>0\}}\bigl(-v^{-\beta}+\lambda f(Ry)\bigr)+\int_{\partial B_1\cap\{v=0\}}\frac{\partial v}{\partial n}\biggr).$$

But $\frac{\partial v}{\partial n}\leqslant 0$ in the set $\partial B_1\cap\{v=0\}$ and therefore

$$\int_{B_1}v+R^{-1}\int_{\partial B_1\cap\{v>0\}}v^{-\beta}\leqslant\lambda R^{-1}\int_{\partial B_1}f(Ry)\leqslant C\lambda R^{-n}. \tag{84}$$

To proceed further we need to estimate v on ∂B_1. Define $Y(r)=\frac{1}{2R}r^2+\frac{n}{R}$, which satisfies

$$\begin{cases}-\Delta Y+Y\geqslant 0 & \text{in } B_R,\\ Y'(R)=\dfrac{\partial Y}{\partial\nu}=1 & \text{on } \partial B_R.\end{cases}$$

The function $U=MY$ is a supersolution of (83) if one takes M large enough (independently of λ). Therefore $u\leqslant CY$ on ∂B_R. Notice that $Y(R)=\frac{R}{2}+\frac{n}{R}$ and hence $v\leqslant CR$ on ∂B_1. Using this estimate in (84) we deduce

$$CR^{-\beta}\bigl|\partial B_1\cap\{v>0\}\bigr|\leqslant\int_{\partial B_1\cap\{v>0\}}v^{-\beta}\leqslant\lambda CR^{-n+1}.$$

We conclude that for R large enough

$$\bigl|\partial B_1\cap\{v>0\}\bigr|\leqslant\lambda CR^{-n+1+\beta}<|\partial B_1|.$$

This implies that u vanishes on subset of ∂B_R of positive surface measure. □

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Appendix. Auxiliary results

In this section we collect a number of auxiliary results for linear equations that we use in the paper.

Lemma 9.1. *Let $D\subset\mathbb{R}^n$ be a bounded, smooth domain, $a\in L^\infty(D)$ and $a\geqslant 0$. Suppose $u\in H^1(D)$, $u\geqslant 0$ satisfies*

$$-\Delta u+a(x)u\geqslant 0\quad\text{in } D.$$

Then there is a constant $C>0$ independent of u such that

$$u(x)\geqslant C\biggl(\int_{\partial D}u\biggr)\operatorname{dist}(x,\partial D),\quad\forall x\in D. \tag{85}$$

Proof. Let $D' \subset\subset D$ be a nonempty open set, with smooth boundary. By a weak version of Harnack's inequality (see [7, Theorem 8.18])

$$u(x) \geqslant C \int_{D'} u \quad \forall x \in D',$$

where $C > 0$. We proceed to verify that

$$\int_{\partial D} u \leqslant C \int_{D'} u.$$

Let w be the solution of

$$\begin{cases} -\Delta w + a(x)w = \chi_{D'} & \text{in } D, \\ w = 0 & \text{on } \partial D. \end{cases}$$

Then $w \geqslant 0$ and

$$\begin{aligned} \int_{D'} u &= \int_D (-\Delta w + a(x)w)u = -\int_D \frac{\partial w}{\partial \nu} u + \int_D \big(\nabla w \cdot \nabla u + a(x)wu\big) \\ &= -\int_D \frac{\partial w}{\partial \nu} u + \int_D w \frac{\partial u}{\partial \nu} + \int_D (-\Delta u + a(x)u)w \geqslant -\int_{\partial D} \frac{\partial w}{\partial \nu} u. \end{aligned}$$

But $\frac{\partial w}{\partial \nu} \leqslant -C$, then $\int_{D'} u \geqslant C \int_{\partial D} u$.

From here we deduce that (85) holds for $x \in D'$. If $x \in D \setminus D'$ we argue as follows. Let z solve

$$\begin{cases} -\Delta z + a(x)z = 0 & \text{in } D \setminus \overline{D'}, \\ z = 0 & \text{on } \partial D, \\ z = 1 & \text{on } \partial D'. \end{cases}$$

By Hopf's lemma $z(x) \geqslant c \operatorname{dist}(x, \partial D)$ for all $x \in D \setminus D'$, where $c > 0$. Then $v(x) = u(x)/(C \int_{\partial D} u)$ satisfies $v \geqslant z$ on $\partial(D \setminus D')$. By the maximum principle $v(x) \geqslant z(x)$ $\forall x \in D \setminus D'$. So

$$u(x) \geqslant C\left(\int_{\partial D} u\right) z(x) \geqslant C\left(\int_{\partial D} u\right) \operatorname{dist}(x, \partial D) \quad \forall x \in D \setminus D'. \qquad \square$$

The next estimate follows from [10].

Lemma 9.2. *Let $D \subset \mathbb{R}^n$ be a bounded, smooth domain, $a \in L^\infty(D)$ and $a \geqslant 0$. Suppose $u \in H^1(D)$ satisfies*

$$\begin{cases} -\Delta u + a(x)u = 0 & \text{in } D, \\ u = g & \text{on } \partial D, \end{cases}$$

where $g \in L^p(\partial\Omega)$ and $p \geqslant 1$. Let $1 \leqslant r < \frac{np}{n-1}$ Then there exists C independent of g and u such that

$$\|u\|_{L^r(D)} \leqslant C\|g\|_{L^p(\partial D)}.$$

We state the next results in this section in a form suitable for the proof of the regularity results. Therefore we recall the notation introduced in Section 4:

$$\begin{aligned} &\widetilde{\Omega} = \frac{1}{\tau}(\Omega - x_0) \quad \text{where } x_0 \in \partial\Omega,\ 0 < \tau < \tau_0, \\ &\widetilde{B}^+ = \widetilde{\Omega} \cap B_1(0), \quad \widetilde{\Gamma}^e = \partial\widetilde{\Omega} \cap B_1(0). \end{aligned} \tag{86}$$

The constants that appear in the next lemmas can be chosen independently of $x_0 \in \partial\Omega$ and $0 < \tau < \tau_0$.

The following estimate follows from [10].

Lemma 9.3. *Let $a \in L^\infty(\widetilde{B}^+)$. Suppose $u \in H^1(\widetilde{B}^+)$ satisfies*

$$\begin{cases} -\Delta u + a(x)u = 0 & \text{in } \widetilde{B}^+, \\ \dfrac{\partial u}{\partial \nu} = g & \text{on } \widetilde{\Gamma}^e, \end{cases}$$

where $g \in L^p(\partial\Omega)$ and $p \geqslant 1$. Let $1 \leqslant r < \frac{np}{n-1}$. Then there exists C independent of g and u such that

$$\|u\|_{W^{1,r}(B_{3/4}\cap\widetilde{\Omega})} \leqslant C\big(\|g\|_{L^p(\widetilde{\Gamma}^e)} + \|u\|_{L^1(\widetilde{B}^+)}\big).$$

Lemma 9.4. *Let $a \in L^\infty(\widetilde{B}^+)$ and suppose that $u \in H^1(\widetilde{B}^+)$ satisfies*

$$\begin{cases} -\Delta u + a(x)u \leqslant 0 & \text{in } \widetilde{B}^+, \\ \dfrac{\partial u}{\partial \nu} \leqslant N & \text{on } \widetilde{\Gamma}^e, \end{cases}$$

where N is a constant. If $p > 1$ then there is a constant $C > 0$ independent of u, N such that

$$u(x) \leqslant C\big(\|u^+\|_{L^p(B_{3/4}\cap\widetilde{B}^+)} + N\big), \quad \forall x \in B_{1/2} \cap \widetilde{B}^+.$$

Proof. Use Moser's iteration scheme, see e.g. [7, p. 195]. □

Proof of Lemma 6.1. By (86) we may assume that there is a smooth domain $D \subset \widetilde{B}^+$ such that $(B_2 \cap \widetilde{\Omega}) \subset D \subset (B_3 \cap \widetilde{\Omega})$. Denote $x = \tilde{x}$ and $x_1 = \tilde{x}_1$. By Lemma 9.4

$$u(x_1) \leqslant C\|u\|_{L^r(B_{3/4}\cap\widetilde{B}^+)} + C\,N, \tag{87}$$

where we fix $1 < r < \frac{n}{n-1}$. On the other hand, by Lemma 9.1

$$u(x) \geqslant c\bigg(\int_{\partial D} u\bigg)\operatorname{dist}(x, \partial D) \quad \forall x \in D. \tag{88}$$

By L^p estimates (Lemma 9.2)

$$\|u\|_{L^r(B_{3/4}\cap\widetilde{B}^+)} \leqslant C\int_{\partial D} u,$$

and by (88) and the fact that $\operatorname{dist}(x, \partial D) = \operatorname{dist}(x, \widetilde{\Gamma}^e)$ $\forall x \in \widetilde{B}^+$, we find

$$u(x) \geqslant c\|u\|_{L^r(B_{3/4}\cap\widetilde{B}^+)}\operatorname{dist}(x, \widetilde{\Gamma}^e) \quad \forall x \in \widetilde{B}^+.$$

Using (87) we obtain

$$u(x) \geqslant c\operatorname{dist}(x, \widetilde{\Gamma}^e)\big(u(x_1) - C\,N\big). \qquad \square$$

Lemma 9.5. *Let $a \in L^\infty(\widetilde{B}^+)$ and suppose that $u \in H^1(\widetilde{B}^+)$, $u \geqslant 0$ satisfies*

$$\begin{cases} -\Delta u + a(x)u \geqslant 0 & \text{in } \widetilde{B}^+, \\ \dfrac{\partial u}{\partial \nu} \geqslant -N & \text{on } \widetilde{\Gamma}^e, \end{cases}$$

where N is a constant. Then there is a constant $C > 0$ independent of u, N such that

$$\int_{B_{3/4}\cap\widetilde{B}^+} u \leqslant C\big(u(x) + N\big) \quad \forall x \in B_{1/2} \cap \widetilde{B}^+.$$

Proof. Use Moser's iterations, see e.g. [7, p. 195]. □

References

[1] H. Amann, Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces, SIAM Rev. 18 (1976) 620–709.
[2] H. Brezis, J.L. Vazquez, Blow-up solutions of some nonlinear elliptic problems, Rev. Mat. Univ. Complut. Madrid 10 (1997) 443–469.
[3] K. Bryan, M. Vogelius, Singular solutions to a nonlinear elliptic boundary value problem originating from corrosion modeling, Quart. Appl. Math. 60 (2002) 675–694.
[4] J. Dávila, M. Montenegro, Positive versus free boundary solutions to a singular equation, J. Anal. Math. 90 (2003) 303–335.
[5] M. Fila, J.S. Guo, Complete blow-up and incomplete quenching for the heat equation with a nonlinear boundary condition, Nonlinear Anal. 48 (2002) 995–1002.
[6] M. Fila, H.A. Levine, Quenching on the boundary, Nonlinear Anal. 21 (1993) 795–802.
[7] D. Gilbarg, N. Trudinger, Elliptic Partial Differential Equations of Second Order, second ed., Grundlehren Math. Wiss., vol. 224, Springer-Verlag, Berlin, 1983.
[8] H.A. Levine, G.M. Lieberman, Quenching of solutions of parabolic equations with nonlinear boundary conditions in several dimensions, J. Reine Angew. Math. 345 (1983) 23–38.
[9] G.M. Lieberman, Boundary regularity for solutions of degenerate elliptic equations, Nonlinear Anal. 12 (1988) 1203–1219.
[10] J.L. Lions, E. Magenes, Problemi ai limiti non omogenei. V, Ann. Scuola Norm Sup. Pisa 16 (1962) 1–44.
[11] K. Medville, M. Vogelius, Blow up behaviour of planar harmonic functions satisfying a certain exponential Neumann boundary condition, preprint.
[12] D. Phillips, A minimization problem and the regularity of solutions in the presence of a free boundary, Indiana Univ. Math. J. 32 (1983) 1–17.
[13] F. Santosa, M. Vogelius, J.-M. Xu, An effective nonlinear boundary condition for a corroding surface. Identification of the damage based on steady state electric data, Z. Angew. Math. Phys. 49 (1998) 656–679.
[14] M. Vogelius, J.-M. Xu, A nonlinear elliptic boundary value problem related to corrosion modeling, Quart. Appl. Math. 56 (1998) 479–505.