

# Average adjacencies for tetrahedral skeleton-regular partitions

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## Abstract

For any conforming mesh, the application of a skeleton-regular partition over each element in the mesh, produces a conforming mesh such that all the topological elements of the same dimension are subdivided into the same number of child-elements. Every skeleton-regular partition has associated special constitutive (recurrence) equations. In this paper the average adjacencies associated with the skeleton-regular partitions in 3D are studied. In three-dimensions different values for the asymptotic number of average adjacencies are obtained depending on the considered partition, in contrast with the two-dimensional case [J. Comput. Appl. Math. 140 (2002) 673]. In addition, a priori formulae for the average asymptotic adjacency relations for any skeleton-regular partition in 3D are provided.

*Keywords:* Partitions; Adjacencies; Tetrahedral meshes

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## 1. Introduction and definitions

In the area of numerical methods a considerable effort has been made for designing and implementing methods able to construct meshes having a suitable distribution of points or elements over the problem domain. Very often we are interested in measuring the goodness of the partition (or the triangulation). In this sense several smoothing or improvement techniques have been developed (see, for example [8,13]).

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Some regularity measures for simplices (triangles in 2D, tetrahedra in 3D) and for the whole simplicial grid (*triangulation*) have been proposed in the literature [9,22]. Sometimes these regularity measures are not geometrical but topological. For example, Shimada [22] proposed a measure evaluating the absolute difference between the degree of each interior node of the mesh and the degree of the nodes of a regular lattice (6 in 2D, or 12 in 3D). Since the degree of a node  $N$  is the number of nodes connected to  $N$ , that measure is a topological measure, i.e. it considers adjacency relations, not geometrical quantities.

The adjacencies of triangular meshes based on a general class of simplex partitions, called skeleton-regular partitions, have been studied in [18]. In this paper we study the asymptotic behavior of different skeleton-regular simplex partitions in three-dimensions, when these partitions are repeatedly and globally used over any given mesh. We show that for each one of these partitions, the asymptotic average number of the adjacencies of the involved neighbor topological elements is constant. But although these average adjacencies are different from one tetrahedral partition to another, we show and prove some relations among the distinct adjacency relations that hold in all the 3D partitions. Some examples of equivalent and nonequivalent partitions are also given.

**Definition 1.1** (*conforming mesh*). Let  $\tau$  be any set of  $n$ -dimensional triangles ( $n = 2$  or  $3$ ) such that

- $\text{interior}(t) \neq \emptyset, \forall t \in \tau,$
- $\forall t_i, t_j \in \tau,$  with  $t_i \neq t_j,$  then  $t_i \cap t_j,$  if not empty, is an entire face, or a common edge, or a common vertex.

Then  $\tau$  is said to be a conforming triangulation.

Frequently, before obtaining a (conforming) triangulation we have an open (polygonal) domain  $\Omega \subset \mathbb{R}^n$  to be triangulated. The triangulation problem can be stated as follows: given an open domain  $\Omega \subset \mathbb{R}^n,$  and a fixed set of points in  $\Omega,$  find a suitable conforming triangulation  $\tau$  of  $\Omega,$  such that  $\bigcup_{t \in \tau} t = \Omega.$

Another related but essentially different problem from the classical triangulation problem is the triangulation refinement problem, which can be stated as follows: *given an acceptable triangulation of a polygonal region  $\Omega,$  a subregion  $R \in \Omega$  defining the refinement area, a condition over the diameter (longest-side) of the triangles, and a resolution parameter  $\varepsilon,$  find or construct a locally refined triangulation such that the diameter of the triangles that intersect the refinement area  $R$  is less than  $\varepsilon$  [20].*

It should be noted here that in the adaptive context the refinement area  $R$  is a changing set of triangles, so the mesh is dynamically constructed at each time when the front or singularity moves. However, the triangulation refinement problem for a given fixed refinement area  $R$  can be seen, in the context of nested meshes, comprised of two (separate) steps: uniform subdivision of all the triangles that intersect  $R$  until the resolution parameter  $\varepsilon$  is achieved, and extension of the refinement outside the refinement area  $R$  in order to assure the conformity of the new mesh. From this point of view, the study of geometrical and topological features of the partitions defining the triangular or tetrahedral refinement algorithms is of the most interest.

**Definition 1.2** (*skeleton*). Let  $\tau$  be any  $n$ -dimensional ( $n = 2$  or  $3$ ) conforming triangular mesh. The  $k$ -skeleton of  $\tau$  is the union of its  $k$ -dimensional faces. Furthermore, the  $(n - 1)$ -skeleton is called the skeleton [4,16].

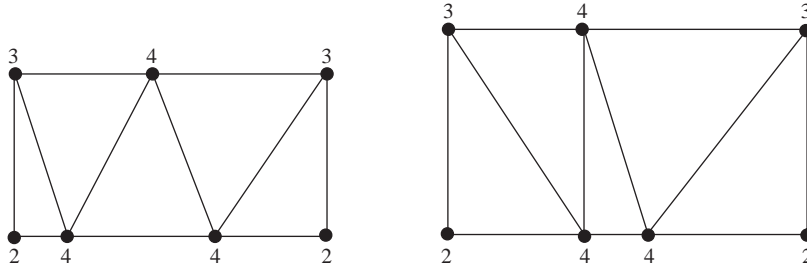


Fig. 1. Example of two topologically equivalent meshes.

The skeleton of a triangulation in three-dimensions is comprised of the faces of the tetrahedra, and in two-dimensions the skeleton is the set of the edges of the triangles.

**Definition 1.3** (*skeleton-regular partition*). For any triangle or tetrahedron  $t$ , a partition of  $t$  will be called *skeleton-regular* if the following properties hold:

1. All the topological elements of the same dimension are subdivided in the same number of child-elements.
2. The meshes obtained by application of the partition to any individual element in any conforming triangulation are conforming.

**Definition 1.4** (*constitutive equations*). When a skeleton-regular partition is applied to some initial mesh, there exist recurrence relations between the number of topological elements in the refined mesh and the number of topological elements in the unrefined mesh. These recurrence equations will be called *constitutive equations* of the partition.

In the following we shall show that different partitions may have the same constitutive equations, and consequently they generate meshes having the same average numbers of topological adjacencies.

**Definition 1.5** (*topologically equivalent meshes*). Two meshes  $\tau$  and  $\tau^*$  are said to be topologically equivalent if there is a homeomorphism  $\phi$  such that  $\phi(\tau) = \tau^*$ .

Note that if two meshes are topologically equivalent, then the number of adjacencies of corresponding elements are the same. See Fig. 1 in which the degree of each node has been pointed out.

**Definition 1.6** (*topologically equivalent partitions and equivalent on average partitions*). Two partitions  $P_1$  and  $P_2$  of the same element  $t$  will be called *topologically equivalent* or simply *equivalent* if there is a homeomorphism  $\phi$  such that  $\phi(P_1(t)) = P_2(t)$ . Two partitions will be called *equivalent on average* or *topologically equivalent on average* if the meshes obtained by application of these partitions to any initial mesh show, on average, the same adjacency numbers.

For instance the 4T-LE partition and the 4T-SE partition introduced below are equivalent. However, the 4T-LE partition and the 4T similar partition are not equivalent but they are equivalent on average.

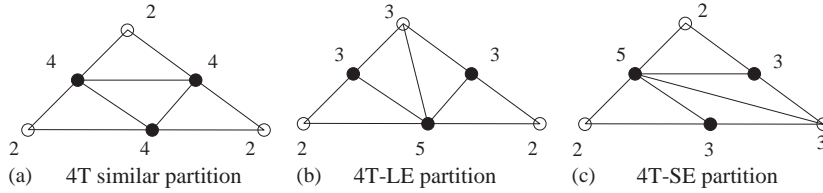


Fig. 2. Four triangle partitions in 2D: (a) 4T similar partition, (b) 4T-LE partition and (c) 4T-SE partition.

**Definition 1.7** (*4T similar partition*). The original triangle is divided into four subtriangles by connecting the midpoints of the father-triangle by straight line segments parallel to the sides.

Note that according to the definition, all the triangles are similar to the original one (see Fig. 2(a)). This is one of the simplest partitions of triangles considered in the literature (see for example [2]).

**Definition 1.8** (*4T-LE partition*) (Rivara [19]). The 4-Triangles Longest-Edge (4T-LE) partition bisects the triangle by its longest edge, and then the two resulting triangles are bisected by the midpoint of the common edge with the original triangle (see Fig. 2(b)).

**Definition 1.9** (*4T-SE partition*). The 4-Triangles Shortest-Edge (4T-SE) partition uses the shortest edge of the triangle to perform the first bisection, and then proceeds as in the 4T-LE partition (see Fig. 2(c)).

**Remark 1.10.** Different partitions can have the same recurrence-associated equations, because these equations depend only on the number of child-elements for each particular original element. For example, in 2D, the 4T-similar partition, the 4T-LE partition and the 4T-SE partition all have the following equations:

$$\begin{aligned}
 N_n &= N_{n-1} + E_{n-1}, \\
 E_n &= 2E_{n-1} + 3F_{n-1}, \\
 F_n &= 4F_{n-1},
 \end{aligned} \tag{1.1}$$

where  $N_n$ ,  $E_n$ , and  $F_n$  are, respectively, the number of nodes, edges, and triangles in the mesh  $\tau_n$ .

Note also that two partitions having the same recurrence-associated equations are equivalent on average or topologically equivalent on average since meshes obtained by application of these two partitions to the same initial mesh will have on average the same topological adjacency relations, since these average numbers depend only on the numbers of topological elements of the same dimension.

As a matter of example, see Fig. 3 in which three different barycentric partitions of a triangle are shown. Fig. 3(a) is for the so-called 0-barycentric partition, while Fig. 3(b) shows the 1-barycentric partition, and Fig. 3(c) is for the 2-barycentric partition. In general, in two-dimensions, the  $p$ -barycentric partition is defined as

**Definition 1.11** ( *$p$ -adic 2D Barycentric partition*). For any triangle  $t$  the  $p$ -adic barycentric partition of  $t$  is defined as follows:

1. Put a new node  $P$  in the barycenter of  $t$ , and put  $p$  nodes at equal distance on each of the edges of  $t$ .
2. Join the node  $P$  with the vertices of the edges, and with the nodes at the edges.

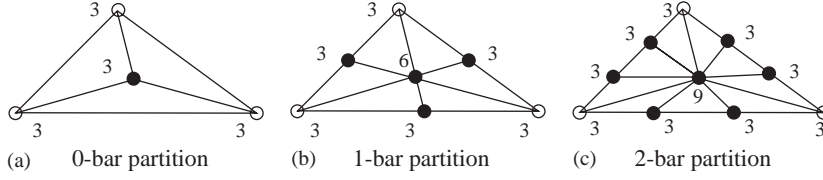


Fig. 3. Three  $p$ -barycentric partitions in 2D: (a) 0-bar partition, (b) 1-bar partition and (c) 2-bar partition.

Table 1  
Adjacency relations in 2D

Constant relations	Non-constant
Vertices per edge = 2	Edges per vertex
Vertices per triangle = 3	Triangles per vertex
Edges per triangle = 3	
Triangles per edge = 2	

The constitutive equations for the  $p$ -barycentric partition in 2D are

$$\begin{aligned}
 N_n &= N_{n-1} + pE_{n-1} + F_{n-1}, \\
 E_n &= (p + 1)E_{n-1} + (3p + 3)F_{n-1}, \\
 F_n &= 3(p + 1)F_{n-1}.
 \end{aligned} \tag{1.2}$$

**Remark 1.12.** Since we are interested in the study of the adjacency relations, it can be noted first that some relations are held for every interior element of the mesh. For example, in 2D every internal edge is shared by two triangular faces (triangles), so the number of triangles per edge is always 2, no matter how the mesh is or what partition is applied to an initial mesh. These adjacency relations are called *constant*. Otherwise we say that the adjacency relation is *non-constant*.

In two dimensions all the adjacency relations are classified as given in Table 1.

**Theorem 1.13** (Plaza and Rivara [18]). *Let  $\tau$  be any (conforming) triangular mesh. For any skeleton-regular partition let  $N_n$ ,  $E_n$ ,  $T_n$  be, respectively, the total number of nodes, edges, and triangles after the  $n$ th partition application. Then the asymptotic average numbers of topological adjacencies are independent of the particular partition of each triangle and these numbers are*

$$\begin{aligned}
 \lim_{n \rightarrow \infty} Av\#(\text{triangles per node}) &= \lim_{n \rightarrow \infty} \frac{3 \times T_n}{N_n} = 6, \\
 \lim_{n \rightarrow \infty} Av\#(\text{edges per node}) &= \lim_{n \rightarrow \infty} \frac{2 \times E_n}{N_n} = 6.
 \end{aligned}$$

The previous result establishes that in 2D all the skeleton-regular partitions are asymptotically topologically equivalent on average; that is, the average adjacency numbers are the same for all the skeleton-regular

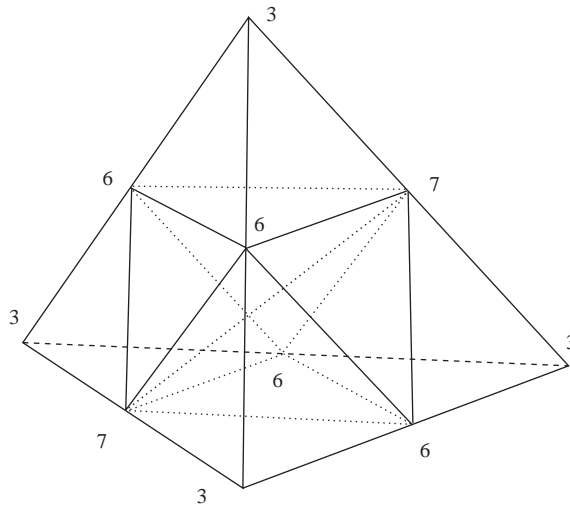


Fig. 4. Bey division: cutting off the corners and internal edge.

partitions considered, regardless of whether they have the same constitutive equations as those shown in Fig. 2, or not, as those in Fig. 3.

The focus of this paper is to investigate the situation of tetrahedral skeleton-regular partitions, and to prove some relations between the average number of topological adjacencies that always appear in these kinds of partitions in 3D.

## 2. 3D skeleton-regular partitions

In three-dimensions several techniques have been developed in recent years for refining (and coarsening) tetrahedral meshes by means of bisection. A general overview can be found in [7]. As in 2D, a refinement tetrahedral procedure consists of two main steps: firstly (uniform or global), partition of the elements presenting the highest error according to the error indicator and to the refinement strategy employed, and then (local or partial), refinement of the elements having hanging or non-conforming nodes.

Here, we resume briefly some of the main skeleton-regular partitions of tetrahedra used in the literature.

**Definition 2.1** (*3D Freudenthal–Bey partition*). The original tetrahedron is divided into eight sub-tetrahedra by cutting off the four corners by the midpoints of the edges (see Fig. 4), and the remaining octahedron is subdivided further into four tetrahedra by one of the three possible interior diagonals [5,6] (see Fig. 5).

Numbers in the figures indicate the degree of the nearest node.

**Definition 2.2** (*8T-LE partition*). The original tetrahedron is divided into eight sub-tetrahedra by performing the 4 T-LE partition of the faces, and then subdividing the interior of the tetrahedron in a manner consistent with the performed division in the 2-skeleton [16,17,21].

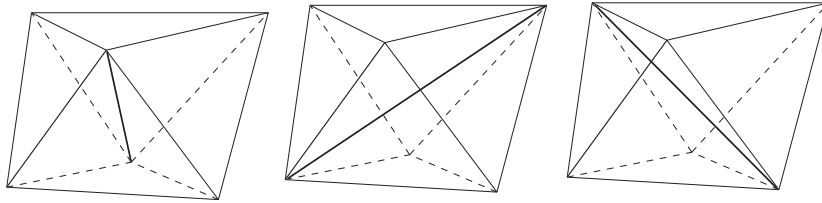


Fig. 5. The three possibilities for dividing the interior octahedron.

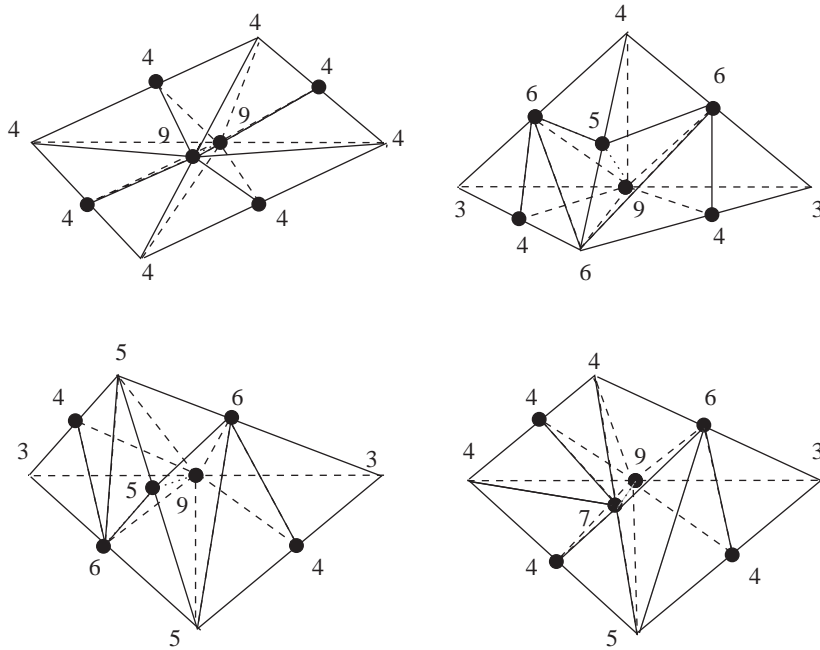


Fig. 6. Refinement patterns for the 8T-LE partition.

The 8T-LE partition and the associated local refinement algorithm [16] can also be explained by successive bisections by mid-point edges of the original tetrahedron, after classifying its edges based on their length (Fig. 6). In this sense this partition and its counterpart refinement algorithm generalize to three-dimensions, the 4T-LE partition and the respective local refinement. Other partitions in eight tetrahedra equivalent to the 8T-LE partition but not based on the length of the edges are those by Kossaczky [12], Maubach [14], Mukherjee [1,15], and Liu and Joe [13]. For a comparison of these partitions and the associated 3D local refinements see [16].

**Definition 2.3** (*3D Barycentric partition*). For any tetrahedron  $t$  the barycentric partition of  $t$  is defined as follows:

1. Put a new node  $P$  at the barycenter of  $t$ , put new nodes at the barycenters of the faces of  $t$ , and put new nodes at the midpoints of the edges of  $t$ .

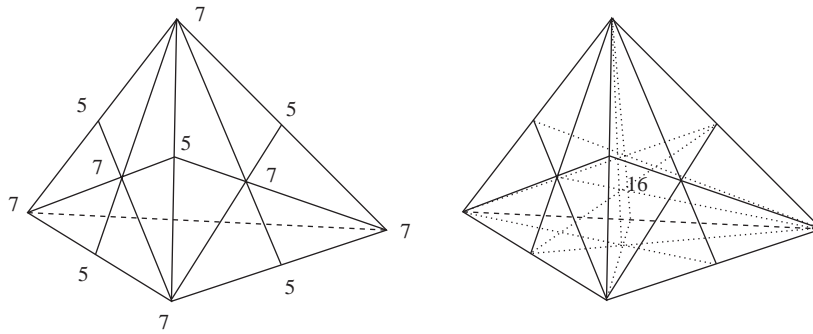


Fig. 7. 3D Barycentric partition.

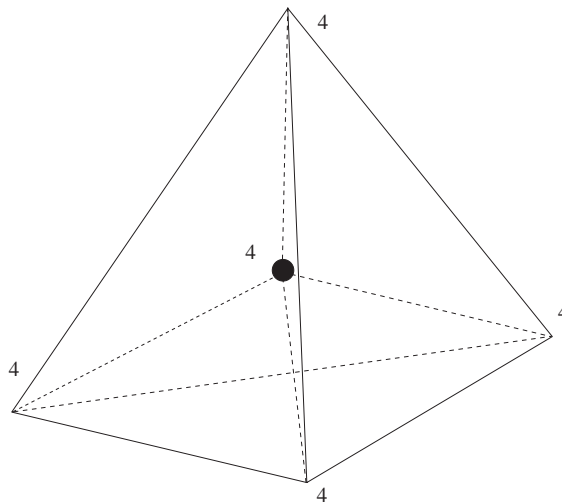


Fig. 8. 3D 4T Barycentric partition.

2. On each face of  $t$  perform the barycentric triangular partition (see Fig. 1(c) for the two-dimensional case).
3. Join the node  $P$  with all the vertices of  $t$ , and with all the new nodes just introduced (see Fig. 7).

Note that from a topological point of view, it is not relevant that the interior node  $P$  of initial tetrahedron  $t$  is located either at the barycenter or at any interior node of  $t$ . The same remark is valid for the barycenter of each face.

**Definition 2.4** (*4T partition*). For any tetrahedron  $t$  the 4T barycentric partition of  $t$  is defined by

1. Put a new node  $P$  at an interior point of  $t$  (for example at the barycenter of  $t$ ).
2. Join node  $P$  with all the vertices of  $t$  (see Fig. 8).



As in 2D, we can also introduce the *p*-adic barycentric partition in 3D as follows:

**Definition 2.5** (*p*-adic 3D Barycentric partition). For any tetrahedron  $t$  the *p*-adic barycentric partition of  $t$  is defined as follows:

1. Put a new node  $P$  at the barycenter of  $t$ , put new nodes at the barycenters of the faces of  $t$ , and put  $p$  new nodes at each one of the edges of  $t$ .
2. On each face of  $t$  perform the *p*-adic barycentric triangular partition (see Fig. 3(c) for the two-dimensional case).
3. Join node  $P$  with all the vertices of  $t$ , and with all the new nodes just introduced.

According to this definition, the 3D Barycentric partition is the 1-adic 3D Barycentric partition.

### 3. Asymptotic results of the adjacency relations in 3D

In three dimensions the adjacency relations are classified as given in Table 2.

Let  $\tau_0$  be an initial triangulation in three-dimensions, in which some skeleton-regular partition is recursively applied. If we denote by  $N_0$ ,  $E_0$ ,  $F_0$ , and  $T_0$ , respectively, the number of nodes, edges, faces (triangles) and tetrahedra in  $\tau_0$ , then the number of topological elements in the subsequent mesh levels  $\tau_n$  depend on the corresponding numbers of the previous mesh level  $\tau_{n-1}$ . In addition, the average of the adjacency relations depends on the number of topological elements in the mesh as the following lemma establishes:

**Lemma 3.1.** *Let  $\tau_n$  be some 3D triangulation with  $N_n$  nodes,  $E_n$  edges,  $F_n$  faces, and  $T_n$  tetrahedra. Then, the nonconstant adjacency relations averages are:*

$$\begin{aligned} \text{Av\#(tetrahedra per edge)} &= \frac{6T_n}{E_n}; & \text{Av\#(tetrahedra per node)} &= \frac{4T_n}{N_n}; \\ \text{Av\#(faces per edge)} &= \frac{3F_n}{E_n}; & \text{Av\#(faces per node)} &= \frac{3F_n}{N_n}; \\ \text{Av\#(edges per node)} &= \frac{2E_n}{N_n}. \end{aligned}$$

In order to calculate the asymptotic behavior of the average adjacencies of the topological elements of a particular skeleton-regular partition, we have to solve the constitutive equations associated to that partition. This can be done by means of generating functions [11]. The constitutive equations can also be solved easily by writing the equations in matrix form, if the associated matrix is diagonalizable [18], according to the following classic theorem [23]:

**Theorem 3.2.** *Let  $u_n = A^n u_0$  be a difference equation, in which matrix  $A$  is diagonalizable; i.e., there exists a non-singular matrix  $S$  such that  $A = SDS^{-1}$ , with  $D$  being a diagonal matrix. Then  $u_n = SD^n S^{-1} u_0$ .*

Then, equation  $u_n = SD^n S^{-1} u_0$  can be solved by using a symbolic calculus package like MAPLE or Mathematica [18].

Table 2  
Adjacency relations in 3D

Constant relations	Non-constant
Vertices per edge = 2	Edges per vertex
Vertices per triangle = 3	Triangles per vertex
Vertices per tetrahedron = 4	Tetrahedra per vertex
Edges per triangle = 3	Triangles per edge
Edges per tetrahedron = 6	Tetrahedra per edge
Triangles per tetrahedron = 4	
Tetrahedra per face = 2	

In 3D, the situation of the asymptotic behavior of the adjacency relations between the topological elements in the mesh is quite different from the situation in 2D. Now, different values for the average limit depending on the particular partition considered are obtained. Before reporting the different results for the average limits of adjacencies it should be noted that the nonconstant adjacency numbers are not independent as the following theorem establishes:

**Theorem 3.3.** *Let  $\tau_n$  be any 3D triangulation with  $N_n$  nodes,  $E_n$  edges,  $F_n$  triangular faces, and  $T_n$  tetrahedra. Then, the limits of the average of nonconstant adjacency relations verify*

$$\lim_{n \rightarrow \infty} Av\#(\text{tetrahedra per edge}) = \lim_{n \rightarrow \infty} Av\#(\text{faces per edge}), \quad (3.1)$$

$$\frac{3}{2} \lim_{n \rightarrow \infty} Av\#(\text{tetrahedra per node}) = \lim_{n \rightarrow \infty} Av\#(\text{faces per node}), \quad (3.2)$$

$$\frac{1}{2} \lim_{n \rightarrow \infty} Av\#(\text{tetrahedra per node}) + 2 = \lim_{n \rightarrow \infty} Av\#(\text{edges per node}). \quad (3.3)$$

**Proof.** Let us first consider Eq. (3.1). Since the number of internal edges is of order  $\mathcal{O}(N^2)$  while the number of external edges is of order  $\mathcal{O}(N)$ , the limits of the average numbers of tetrahedra per edge and faces per edge depend only on the limits involving the internal edges of the triangulations. Note that for internal edges the number of tetrahedra sharing each internal edge is equal to the number of faces sharing each internal edge (see Fig. 9). This proves (3.1).

For (3.2) consider that

$$\frac{4T_n}{N_n} = \frac{4 \cdot T_n \cdot F_n}{F_n \cdot N_n} = 2 \frac{F_n}{N_n},$$

hence, by Lemma 3.1

$$\begin{aligned} & \frac{3}{2} \times \lim_{n \rightarrow \infty} Av\#(\text{tetrahedra per node}) \\ &= \lim_{n \rightarrow \infty} \frac{3}{2} \cdot \frac{4 \cdot T_n}{N_n} = \lim_{n \rightarrow \infty} \frac{3}{2} \cdot \frac{2 \cdot F_n}{N_n} \\ &= \lim_{n \rightarrow \infty} \frac{3 \cdot F_n}{N_n} = \lim_{n \rightarrow \infty} Av\#(\text{faces per node}) \end{aligned}$$

so Eq. (3.2) is proved.

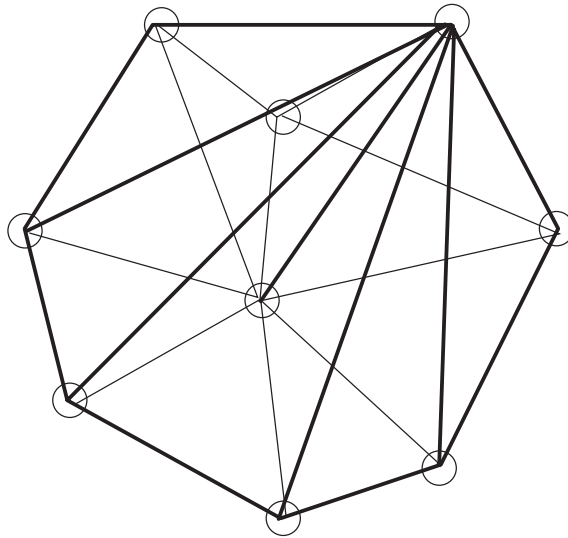


Fig. 9. Hull of tetrahedra sharing an internal edge.

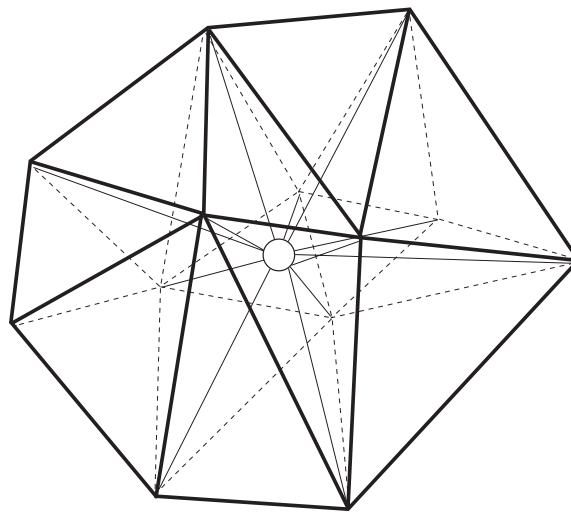


Fig. 10. Hull of tetrahedra sharing an internal node.

For Eq. (3.3) consider the Euler–Poincaré formula applied to the surface triangulation of the hull of each interior node. Fig. 10 shows the set of tetrahedra sharing an interior node. If the number of nodes of such a node-hull is noted by  $n$ , the number of edges by  $e$  and the number of triangular faces by  $f$ , then the Euler–Poincaré formula says that  $n - e + f = 2$ .

On the other hand, the number of edges incident with the interior node  $P$  is equal to the number of nodes of the node-hull of  $P$ ,  $n$ ; the number of faces incident with  $P$  is equal to the number of edges of the hull  $e$ , and the number of tetrahedra having node  $P$  as a vertex is the number of triangular faces of its

Table 3  
Asymptotic average of adjacency relations in 3D

Partition	$\lim_{n \rightarrow \infty} Av\#(\text{tet per edge})$	$\lim_{n \rightarrow \infty} Av\#(\text{tet per node})$
8T-LE	$\frac{36}{7}$	24
3D-Bey	$\frac{36}{7}$	24
Barycentric	$\frac{66}{13}$	22
4T	$\frac{9}{2}$	12
$p$ -adic Barycentric	$33(p+1)/(6p+7)$	$44(p+1)/(p+3)$
$\lim_{p \rightarrow \infty} p$ -adic Bar	$\frac{33}{6} = \frac{11}{2}$	44

node-hull. This reasoning leads us to another relation between the adjacency numbers

$$Av\#(\text{edges/node}) - Av\#(\text{faces/node}) + Av\#(\text{tetrahedra/node}) = 2.$$

Taking into account Eq. (3.2) we have

$$Av\#(\text{edges/node}) - \frac{3}{2}Av\#(\text{tets/node}) + Av\#(\text{tets/node}) = 2$$

which is equivalent to Eq. (3.3).  $\square$

In view of Eq. (3.2), Eq. (3.3) is equivalent to

$$\frac{1}{3} \lim_{n \rightarrow \infty} Av\#(\text{faces per node}) + 2 = \lim_{n \rightarrow \infty} Av\#(\text{edges per node}).$$

This is because Eq. (3.2) is equivalent to

$$\frac{1}{2} \lim_{n \rightarrow \infty} Av\#(\text{tetrahedra per node}) = \frac{1}{3} \lim_{n \rightarrow \infty} Av\#(\text{faces per node})$$

and replacing the first term of Eq. (3.3) by  $\frac{1}{3} \lim_{n \rightarrow \infty} Av\#(\text{faces per node})$  gives the result.

**Remark 3.4.** Since the 3D Freudenthal–Bey partition is equivalent on average to the 8T-LE partition, then both have the same asymptotic adjacencies. Note that since on average each vertex is shared by 24 tetrahedra, it can be said that these partitions lead to *almost regular triangulations* (see [10]) when they are applied successively over any initial triangulation.

Table 3 shows the results for the partitions presented in Section 2.

We shall prove the results for the 3D 4T partition following [11] by basic maneuvers with the generating functions associated with the corresponding constitutive equations.

**Theorem 3.5.** *Let  $\tau$  be a (conforming) initial tetrahedral mesh in which the 3D 4T partition is recursively applied. Then the asymptotic average adjacencies are the following:*

$$\begin{aligned}\lim_{n \rightarrow \infty} Av\#(\text{tetrahedra per edge}) &= \frac{9}{2} = \lim_{n \rightarrow \infty} Av\#(\text{faces per edge}), \\ \lim_{n \rightarrow \infty} Av\#(\text{tetrahedra per node}) &= 12, \\ \lim_{n \rightarrow \infty} Av\#(\text{faces per node}) &= 18, \\ \lim_{n \rightarrow \infty} Av\#(\text{edges per node}) &= 8.\end{aligned}$$

**Proof.** From Definition 2.4 and Fig. 8, it is easy to get the recurrence equations associated with this partition

$$\begin{aligned}T_n &= 4 \cdot T_{n-1} + T_0 \cdot 1_0(n), \\ F_n &= F_{n-1} + 6 \cdot T_{n-1} + F_0 \cdot 1_0(n), \\ E_n &= E_{n-1} + 4 \cdot T_{n-1} + E_0 \cdot 1_0(n), \\ N_n &= N_{n-1} + T_{n-1} + N_0 \cdot 1_0(n)\end{aligned}$$

for  $n \geq 0$ , where  $1_0(n)$  is equal to zero if  $n \neq 0$  and equal to 1 if  $n = 0$ ; so in this form, initial conditions are included in the equations.

To solve the system of equations, we begin by noting that the generating function associated with the first equation is

$$\begin{aligned}T(z) &= \sum_{n=0}^{\infty} T_n z^n = \sum_{n=1}^{\infty} 4T_{n-1} z^n + \sum_{n=0}^{\infty} T_0 1_0(n) z^n \\ &= 4 \sum_{n=0}^{\infty} T_n z^{n+1} + T_0 = 4zT(z) + T_0\end{aligned}$$

from which we can conclude  $T(z) = (T_0/1 - 4z)$ .

In an analogous way for solving the equation for the faces we observe the following:

$$\begin{aligned}F(z) &= \sum_{n=0}^{\infty} F_n z^n = \sum_{n=1}^{\infty} F_{n-1} z^n + \sum_{n=1}^{\infty} 6T_{n-1} z^n + \sum_{n=0}^{\infty} F_0 1_0(n) z^n, \\ (1 - z)F(z) &= \frac{6T_0 z}{1 - 4z} + F_0, \\ F(z) &= \frac{6T_0 z}{(1 - 4z)(1 - z)} + \frac{F_0}{1 - z}.\end{aligned}$$

For the edges we obtain

$$E(z) = \sum_{n=0}^{\infty} E_n z^n = \sum_{n=1}^{\infty} E_{n-1} z^n + \sum_{n=1}^{\infty} 4T_{n-1} z^n + \sum_{n=0}^{\infty} E_0 1_0(n) z^n,$$

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$$(1 - z)E(z) = \frac{4T_0z}{1 - 4z} + E_0,$$

$$E(z) = \frac{4T_0z}{(1 - 4z)(1 - z)} + \frac{E_0}{1 - z}.$$

Finally, for the nodes we get

$$N(z) = \sum_{n=0}^{\infty} N_n z^n = \sum_{n=1}^{\infty} N_{n-1} z^n + \sum_{n=1}^{\infty} T_{n-1} z^n + \sum_{n=0}^{\infty} N_0 1_0(n) z^n,$$

$$(1 - z)N(z) = \frac{T_0z}{1 - 4z} + N_0,$$

$$N(z) = \frac{T_0z}{(1 - z)(1 - 4z)} + \frac{N_0}{1 - z}.$$

Once the generating functions have been obtained from their power series expansions, we obtain the values for the unknowns  $T_n$ ,  $F_n$ ,  $E_n$  and  $N_n$

$$T_n = 4^n T_0,$$

$$F_n = F_0 + 2(4^n - 1)T_0,$$

$$E_n = E_0 + \frac{4(4^n - 1)}{3}T_0,$$

$$N_n = N_0 + \frac{4^n - 1}{3}T_0.$$

Now taking limits in the corresponding quotients given in Lemma 3.1 the asymptotic adjacency relations are obtained.  $\square$

The next theorem establishes explicit formulae for the nonconstant asymptotic adjacencies for any skeleton-regular tetrahedral partition in 3D. First note that the recurrence equations of any skeleton-regular tetrahedral partition in 3D are given by an upper triangular matrix  $A \in \mathbb{R}^{4 \times 4}$ , where the entries of the matrix  $a_{ij}$  for  $0 \leq i \leq j \leq 4$  are the numbers of  $i$ -dimensional topological elements arising in each previous  $j$ -dimensional element. As an example, the recurrence equations of the 3D 4T partition studied before can be written in matrix form as

$$u_n = \begin{pmatrix} N_n \\ E_n \\ F_n \\ T_n \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 4 \end{pmatrix} \cdot \begin{pmatrix} N_{n-1} \\ E_{n-1} \\ F_{n-1} \\ T_{n-1} \end{pmatrix} = A \cdot u_{n-1}.$$

**Theorem 3.6.** *Let  $\tau_0$  be a (conforming) tetrahedral mesh. For any skeleton-regular partition let  $N_n$ ,  $E_n$ ,  $F_n$ , and  $T_n$  be the total number of nodes, edges, triangular faces, and tetrahedra, respectively, after the  $n$ th partition application to  $\tau_0$  of the considered partition. Let  $A \in \mathbb{R}^{4 \times 4}$  be the upper triangular matrix,*

whose entries are nonnegative integers, defining the recurrence equations of the partition considered,  $u_n = Au_{n-1}$

$$\begin{pmatrix} N_n \\ E_n \\ F_n \\ T_n \end{pmatrix} = \begin{pmatrix} 1 & a & b & c \\ 0 & d & e & f \\ 0 & 0 & g & h \\ 0 & 0 & 0 & i \end{pmatrix} \cdot \begin{pmatrix} N_{n-1} \\ E_{n-1} \\ F_{n-1} \\ T_{n-1} \end{pmatrix}. \quad (3.4)$$

If the partition is not trivial, which means that the coefficients  $c, f, h$  are positive integers and  $i > 1$ ; then, the asymptotic average of nonconstant adjacency numbers are

$$\lim_{n \rightarrow \infty} Av\#(\text{tetrahedra per edge}) = \frac{3h + 4e}{2e + f},$$

$$\lim_{n \rightarrow \infty} Av\#(\text{tetrahedra per node}) = \frac{4(i - d)(i - 1)}{a(2e + f) + (2b + c)(i - d)}.$$

**Proof.** From the constitutive equations in matrix form (3.4) we obtain

$$u_n = A^2 \cdot u_{n-2} = \dots = A^n \cdot u_0 = A^n \cdot (N_0, E_0, F_0, T_0)^T, \quad (3.5)$$

where  $N_0, E_0, F_0$ , and  $T_0$  are the initial values for the number of nodes, edges, faces, and tetrahedra, respectively. The entries of matrix  $A$  verify

- $d = \#(\text{edges per edge}) = a + 1 \geq 1$ , since  $a \geq 0$ ,
- $g = \#(\text{triangles per triangle}) = a + 1 + \frac{2e}{3} \geq d$ , since  $3g = 3(a + 1) + 2e$ ,
- $i = \#(\text{tetrahedra per tetrahedron}) = g + \frac{h}{2} = a + 1 + \frac{2e}{3} + \frac{h}{2} > g$ ,
- $g = 3a + 2b + 1$ ,
- $e = 3a + 3b$ .

The first equation shows the relation between the number of edges per edge ( $d$ ) and the number of nodes per edge ( $a$ ). The second one comes from the counting of the edges in a triangular face once a triangular face has been divided. If the number of triangles per triangle is “ $g$ ”, there will be “ $3g$ ” edges. The same number is obtained by counting the edges arising by dividing the 3 edges of the original triangle “ $3d$ ”, and adding up the edges arising inside the triangle which have to be counted twice, “ $2e$ ”. So,  $3g = 3d + 2e$ .

The third relation comes from the counting of the faces arising by dividing a single tetrahedron. The number of faces will be “ $4i$ ”. The same number is obtained by summing up the number of faces arising by the partition of the 4 faces of the initial tetrahedron, “ $4g$ ”; and the number of internal faces counted twice, “ $2h$ ”. So,  $4i = 4g + 2h$ .

The last two relations are a consequence of the following property [3, Theorem 9.1]: “Let  $P$  be a set of  $n$  points in a triangle, not all collinear, and let  $k$  denote the number of points in  $P$  on the edges of the triangle. Then, any triangulation of  $P$  has  $2n - 2 - k$  triangles and  $3n - 3 - k$  edges”. In our case  $n = 3 + 3a + b$  and  $k = 3 + 3a$ . Note that relations of the second, fourth and fifth equations are dependent.

We will have in mind these relations in the calculus below.

Matrix  $A$  is diagonalizable, and

$$A = \begin{pmatrix} 1 & a & b & c \\ 0 & a+1 & e & f \\ 0 & 0 & a+1+\frac{2e}{3} & h \\ 0 & 0 & 0 & a+1+\frac{2e}{3}+\frac{h}{2} \end{pmatrix} = S \cdot D \cdot S^{-1},$$

$D$  being the diagonal matrix with the diagonal of matrix  $A$  and  $S$  a nonsingular matrix. Therefore,  $A^n = SD^nS^{-1}$ . Then, by using MAPLE or Mathematica, we get the following value for  $A^n$ , where, as usual,  $\mathcal{O}(g^n)$  designates a quantity whose limit as  $n \rightarrow \infty$  divided by  $g^n$  is a constant

$$A^n = \begin{pmatrix} 1 & d^n - 1 & \mathcal{O}(g^n) & 6\frac{12ae+6af+8be+6bh+4ce+3ch}{24ae+18ah+16e^2+24eh+9h^2}i^n + \mathcal{O}(g^n) \\ 0 & d^n & \frac{3}{2}(g^n - d^n) & 6\frac{2e+f}{4e+3h}i^n + \mathcal{O}(g^n) \\ 0 & 0 & g^n & 2(i^n - g^n) \\ 0 & 0 & 0 & i^n \end{pmatrix}. \quad (3.6)$$

Now, from (3.5) and (3.6) the asymptotic average adjacency relations are

$$\begin{aligned} \lim_{n \rightarrow \infty} Av\#(\text{tetrahedra per edge}) &= \lim_{n \rightarrow \infty} \frac{6T_n}{E_n} \\ &= \lim_{n \rightarrow \infty} \frac{6 \cdot i^n \cdot T_0}{6\frac{2e+f}{3h+4e} \cdot i^n \cdot T_0} = \frac{3h+4e}{2e+f} = \frac{6(i-d)}{2e+f}, \end{aligned}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} Av\#(\text{tetrahedra per node}) &= \lim_{n \rightarrow \infty} \frac{4T_n}{N_n} \\ &= \lim_{n \rightarrow \infty} \frac{4 \cdot i^n \cdot T_0}{6\frac{12ae+6af+8be+6bh+4ce+3ch}{24ae+18ah+16e^2+24eh+9h^2} \cdot i^n \cdot T_0} \\ &= \frac{4[6a \cdot 6(i-d) + (6(i-d))^2]}{6[6a(2e+f) + 6(i-d)(2b+c)]} \\ &= \frac{4(i-d)(i-1)}{a(6a+6b+f) + (2b+c)(i-d)} = \frac{4(i-d)(i-1)}{a(2e+f) + (2b+c)(i-d)}. \end{aligned}$$

Note that in the last equation, relations among the entries of matrix  $A$  have been applied.  $\square$

This theorem establishes the asymptotic behavior of any skeleton-regular tetrahedral partition. This result is a generalization to 3D of the behavior of the skeleton-regular partitions in 2D (Theorem 1.13).

#### 4. Concluding remarks

In this paper we have studied the average adjacencies for tetrahedral skeleton-regular partitions. Although in 2D all skeleton-regular triangular partitions verify the same average asymptotic relations, in 3D



different values are obtained depending on the considered partition. However, some independent relations between these numbers have been proved here.

The study of the asymptotic behavior of the partitions based on their recurrence equation system could be a clue in the proof of the non-degeneracy or stability properties of some local refinement algorithms in 3D and higher dimensions based on these partitions. The study presented here can be applied to other polyhedral or polygonal partitions of the space, not only simplicial partitions, and can also be generalized to higher dimensions.

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