Subdifferential representation formula and subdifferential criteria for the behavior of nonsmooth functions[☆]

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Abstract

Several kinds of behaviors of extended-real-valued lower semicontinuous functions are known to be equivalent to certain appropriate conditions in terms of the Clarke subdifferential. The paper provides a systematic study showing that any such condition with the Clarke subdifferential is valid if and only if it holds with any operator representing the Clarke subdifferential like in the subdifferential proximal formula.

Keywords: Subdifferential representation formula; Monotonicity; Cone decreasing property; Directionally Lipschitzian behavior; pln function; Convexly composite function

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1. Introduction

After the first finite dimensional proximal normal formula appeared in the paper [8] published by Clarke, the systematic study of this and the first proximal subdifferential formula for lower

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semicontinuous (lsc) functions began with the paper [37] of Rockafellar. For a lsc function $f : \mathbb{R}^p \to \mathbb{R} \cup \{+\infty\}$, Rockafellar introduced the concept of a proximal subgradient and defined the set $\partial_P f(x)$ of all proximal subgradients of f at x as the proximal subdifferential. He then investigated the limiting proximal subdifferential $\partial_P^L f(x)$ and the singular limiting proximal subdifferential $\partial_P^{L,\infty} f(x)$. A vector ζ is declared in [37] to be a limiting (resp. a singular limiting) proximal subgradient of f at x provided there exist $x_k \to x$ with $f(x_k) \to f(x)$ and $\zeta_k \in \partial_P f(x_k)$ such that $\zeta_k \to \zeta$ (resp. $\alpha_k \zeta_k \to \zeta$ for some $\alpha_k \to 0^+$). The proximal subdifferential formula in [37] states that the Clarke subdifferential $\partial_C f(x)$ (the lsc function f being finite at x) is related to the proximal subgradients by the equality

$$\partial_C f(x) = \overline{co}\,[\partial_P^L f(x) + \partial_P^{L,\infty} f(x)]. \tag{1.1}$$

Several applications of that formula for the study of the behavior of the Clarke subdifferential of optimal value functions $m(x) = \inf_{y \in \mathbb{R}^p} \varphi(x, y)$ illustrate in [38] and [39] the effectiveness of the formula. The general approach in [39] not only yields useful calculus rules for the Clarke subdifferential but it also implicitly clearly contains rich calculus rules for the limiting proximal subdifferential. As is known, the limiting (resp. singular limiting) proximal subdifferential of f coincides in $\mathbb{R}^p$ with the limiting (resp. singular limiting) Fréchet subdifferential of f (defined as above with the use of the Fréchet subdifferential $\partial_F f$ in place of the proximal one $\partial_P f$). Thus, the paper [39] contains other proofs of calculus rules established earlier by Mordukhovich [27] for the limiting Fréchet subdifferential (see also [21] and [28]). For other interesting proofs of calculus rules for the Mordukhovich subdifferential, we refer the reader to [18] where Ioffe integrates the use of Dini subgradients. Calculus rules with general abstract subdifferentials are contained in Jules [20] and references therein, and equivalences between some subdifferential calculus rules and some multidirectional mean value properties appeared in Lassonde [22] and references therein. Any formula of the type (1.1) is what we call in the present paper a *subdifferential representation formula*.

The extension of formula (1.1) to reflexive Banach spaces was achieved by Borwein and Strowjas in [7] and by Loewen in [25]. Borwein and Strojwas in their paper [7] carried out a thorough study of formula (1.1) for closed subsets of reflexive Banach spaces, that is, the case where the indicator function of the closed subset S is considered in place of f. Hence, the corresponding normal proximal formula takes the form

$$N_C(S; x) = \overline{co}\, N_P^L(S; x), \tag{1.2}$$

where $N_C(S; .)$ and $N_P^L(S; .)$ denote the Clarke normal cone and the limiting proximal normal cone, respectively. Borwein and Strojwas showed in [6] and [7] how many important properties of the geometry of closed sets in Banach spaces are strongly related to such normal representations. We can cite the following examples: the Bishop–Phelps property [4] concerning the density in the boundary of C of support points of closed convex sets C in Banach spaces, and the Lau theorem [24] relating to the existence of nearest points to nonconvex closed sets of reflexive Banach spaces. Formulae of types (1.1) and (1.2) with the Fréchet subdifferential in place of the proximal subdifferential have been established in Banach spaces admitting equivalent Fréchet differentiable (away from the origin) norms by Treiman in [50] where applications are also provided. As other important papers containing strong results on the subject in the infinite dimensional setting, we cite the papers [5] by Borwein and Giles, [19] by Ioffe, and the paper [29] of Mordukhovich and Shao, where related results in the Asplund space context can be found.

Note that [29] provides a detailed and complete analysis of several subdifferential properties of nonsmooth functions over Asplund spaces.

During the last decade, the study of several kinds of behaviors of the lsc function began, related to the Clarke subdifferential properties. The question concerns the identification of properties of the Clarke subdifferential that characterize the behavior of the function in which one is interested. The first significant result in this line was provided by Poliquin in [33] where he proved that a lsc function $f : \mathbb{R}^p \to \mathbb{R} \cup \{+\infty\}$ is convex if and only if its Clarke subdifferential (or proximal subdifferential) is a set-valued monotone operator. This result has been extended to Banach spaces for the Clarke subdifferential and some presubdifferentials by Correa et al. in [12], [13], and [14]. Since Poliquin's paper, interest in other kinds of behavior such as quasiconvexity, the Lipschitzian property, the Lipschitzian property up to a lsc convex function, the directionally Lipschitzian property, the decreasing property with respect to a convex cone, generalized convexity, approximate convexity, and bivariate behavior emerged in various papers such as [1,2,10,15–17,26,32,45,47–49], and some of their respective references. Another important behavior that garnered attention appeared in conjunction with the concept of primal lower nice (pln) functions. This class of functions appeared in the paper [34] by Poliquin. In [34], the author established the interest of this class of functions by providing several interesting applications. He then established the subdifferential characterization of such functions (defined on $\mathbb{R}^p$) with the Clarke and the proximal subdifferentials. This characterization has been shown to hold in any Hilbert space as proved by Levy et al. [23]. All the characterizations of the different aforementioned behaviors have also been studied by many other authors for several subdifferentials. All the subdifferentials involved have as a common point that the Zagrodny mean value theorem (see [51]) is valid for everyone (see [44]) in the appropriate space. The objective of the present paper is to show in a *direct way* that any such characterization with the Clarke subdifferential is valid if and only if this holds with any other operator δf for which the representation formula (1.1) is true with $\delta^L f$ and $\delta^L_\infty f$ in place of $\partial^L_P f$ and $\partial^{L,\infty}_P f$, respectively.

The paper is organized as follows. In Section 2, we consider the subdifferential representation formula (1.1) with an operator δf and we establish the equivalence of the monotonicity (resp. quasimonotonicity, hypomonotonicity and submonotonicity) of $\partial_C f$ with that of δf. A concept of asymptotic operator is associated in Section 3 with any set-valued operator between a Banach space and its topological dual space. This concept allows us to study, for $\partial_C f$ and δf, the equivalence of subdifferential criteria characterizing several properties of nonsmooth functions such as: the Lipschitzian property, the directionally Lipschitzian property, the decreasing property with respect to a convex cone and with respect to some function g that is either convex on a Banach space or pln on a Hilbert space. The case where the function g is convexly composite is studied in the last section.

2. Representation formula and various concepts of monotonicity

Let X be a real Banach space, X^* be its topological dual, and $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a lsc function with which we associate a set-valued operator $\delta f : X \rightrightarrows X^*$, which is empty valued at any point where f is not finite. We define the two set-valued operators $\delta^L f : X \rightrightarrows X^*$ and $\delta^L_\infty f : X \rightrightarrows X^*$ by

$$\delta^L f(x) := \{x^* \in X^* : \text{there exist } x_k \to_f x,\, x_k^* \in \delta f(x_k) \text{ such that } x_k^* \overset{*}{\rightharpoonup} x^*\}$$

$$\delta^L_\infty f(x) := \{x^* \in X^* : \text{there exist } \alpha_k \to 0^+,\, x_k \to_f x,\, x_k^* \in \delta f(x_k) \\ \text{such that } \alpha_k x_k^* \overset{*}{\rightharpoonup} x^*\},$$

where $x_k \to_f x$ means that the sequence x_k norm-converges to x together with $f(x_k) \to f(x)$ and "$\overset{*}{\rightharpoonup}$" denotes the w^* convergence in X^*. The operator $\delta^L f$ is the limiting operator associated with δf and $\delta^L_\infty f$ is the singular limiting operator. It is easily seen that $\delta f \subset \delta^L f$ and for any $x \in \operatorname{Dom} \delta^L_\infty := \{x \in X : \delta^L_\infty(x) \neq \emptyset\}$ the set $\delta^L_\infty(x)$ is a cone containing 0. Further, the inclusion $\operatorname{Dom} \delta^L \subset \operatorname{Dom} \delta^L_\infty$ also holds.

Definition 2.1. Given a lsc function $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$, we will say that the set-valued operator $\delta f : X \rightrightarrows X^*$ with $\delta f(\cdot) \subset \partial_C f(\cdot)$ satisfies the *representation formula* in an open subset U of X if there exist two operators $\delta^\Lambda f$, $\delta^\Lambda_\infty f$ with

$$\delta^\Lambda f(x) \subset \delta^L f(x)\ \forall x \in U \quad \text{and} \quad 0 \in \delta^\Lambda_\infty f(x) \subset \delta^L_\infty f(x)\ \forall x \in U \cap \operatorname{Dom} \delta^L_\infty f \tag{2.1}$$

such that

$$\overline{co}^{w^*}(\delta^\Lambda f(x) + \delta^\Lambda_\infty f(x)) = \partial_C f(x) \qquad \text{for all } x \in U, \tag{2.2}$$

where $\partial_C f(x)$ is the Clarke subdifferential of f at x, defined for $x \in \operatorname{dom} f := \{y \in X : f(y) < \infty\}$ by

$$\partial_C f(x) = \{x^* \in X^* : \langle x^*, h\rangle \leq f^\uparrow(x; h) \text{ for all } h \in X\}$$

and $\partial_C f(x) = \emptyset$ for $x \notin \operatorname{dom} f$. Here

$$f^\uparrow(x; h) = \sup_{\gamma>0} \inf_{\varepsilon>0} \sup_{\substack{x' \in B(x;\varepsilon) \\ |f(x)-f(x')|<\varepsilon \\ t \in]0,\varepsilon[}} \inf_{h' \in B(h;\gamma)} \frac{f(x'+th') - f(x')}{t},$$

$\overline{co}^{w^*}(S)$ denotes the w^*-closed convex hull of the set S in X^*, and $B(z; \varepsilon) = \{x \in X : \|x - z\| < \varepsilon\}$.

When the function f is locally Lipschitzian around x, $f^\uparrow(x; \cdot)$ reduces to

$$f^\uparrow(x; h) = \limsup_{\substack{x' \to x \\ t \to 0^+}} \frac{f(x'+th) - f(x')}{t} \tag{2.3}$$

and in such a case the function $f^\uparrow(\cdot; \cdot)$ is upper semicontinuous (see [9]).

Observe that (2.1) and (2.2) entail

$$U \cap \operatorname{Dom} \delta^\Lambda f = U \cap \operatorname{Dom} \partial_C f. \tag{2.4}$$

Example 2.1. Examples of functions f and operators δf for which (2.2) holds include:

- C^1 functions f on any Banach space, the usual derivative as operator δf, $\delta^\Lambda f = \{\nabla f\}$, and $\delta^\Lambda_\infty f = \{0\}$;
- lsc functions f on Asplund spaces, the Fréchet subdifferential $\partial_F f$ as operator δf, $\delta^\Lambda f = \partial^L_F f$ the limiting Fréchet subdifferential or the Murdukhovich subdifferential, and $\delta^\Lambda_\infty f = \partial^{L,\infty}_F f$ the singular limiting Fréchet subdifferential (see [29,50]);
- lsc functions f on reflexive Banach spaces, the proximal subdifferential $\partial_P f$ as operator δf, $\delta^\Lambda f = \partial^L_P f$ the limiting proximal subdifferential, and $\delta^\Lambda_\infty f = \partial^{L,\infty}_P f$ the singular limiting proximal subdifferential (see [7,25]);

- locally Lipschitzian functions on separable Banach spaces, the operator δf being given by $\delta f(x) = \{\nabla f(x)\}$ for any x at which f is Gâteaux differentiable and $\delta f(x) = \emptyset$ otherwise, $\delta^{\Lambda} f(x) = \{\lim \nabla f(x_k) : x_k \to x\}$ and $\delta^{\Lambda}_{\infty} f = \{0\}$ (see [43]);
- lsc functions f on any Banach space, the operator δf as the Clarke subdifferential $\partial_C f$, $\delta^{\Lambda} f = \partial_C f$, and $\delta^{\Lambda}_{\infty} f(x) = \{0\}$ for all $x \in \operatorname{Dom} \partial_C f$.

We will begin with the investigation of the monotonicity of $\partial_C f$ and δf under the subdifferential representation (2.2).

Definition 2.2. A set-valued operator $A : X \rightrightarrows X^*$ is called *monotone* over an open set U if for all $x, y \in U \cap \operatorname{Dom} A$, where $\operatorname{Dom} A = \{x \in X : A(x) \neq \emptyset\}$, one has

$$\langle x^* - y^*, x - y\rangle \geq 0 \qquad \text{for all } x^* \in A(x), y^* \in A(y).$$

The interest of the monotonicity of the Clarke subdifferential (also of several presubdifferentials; refer to [14]) lies in the fact that it characterizes the convexity of lsc functions over Banach spaces. The proposition below shows in particular that such a characterization holds for any operator for which the representation (2.2) is satisfied.

Proposition 2.1. *If $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ is a lsc function such that δf satisfies the representation formula (2.2) in U, then the Clarke subdifferential $\partial_C f$ is monotone in U if and only if δf is monotone in the same set.*

Proof. Suppose δf is monotone in U. Let $x, y \in U \cap \operatorname{Dom} \delta^L f$, take $x^* \in \delta^L f(x)$ and $y^* \in \delta^L f(y)$. Then there exist sequences $x_k \to_f x$, $y_k \to_f y$, $x_k^* \in \delta f(x_k)$ and $y_k^* \in \delta f(y_k)$ such that $x_k^* \overset{*}{\rightharpoonup} x^*$ and $y_k^* \overset{*}{\rightharpoonup} y^*$. For k sufficiently large, since δf is monotone in U, we can write

$$\langle x_k^* - y_k^*, x_k - y_k\rangle \geq 0,$$

and taking the limit over k we obtain the inequality that proves the monotonicity of $\delta^L f$ in U.

Now, we will show that for all $x, y \in \operatorname{Dom} \delta^L_{\infty} f \cap U$ and $x^* \in \delta^L_{\infty} f(x)$ one has

$$\langle x^*, x - y\rangle \geq 0,$$

and therefore we will also have $\langle -y^*, x-y\rangle \geq 0$ for all $x, y \in \operatorname{Dom} \delta^L_{\infty} f \cap U$, and $y^* \in \delta^L_{\infty} f(y)$. Hence for all $x, y \in \operatorname{Dom} \delta^L_{\infty} f \cap U$ we will obtain

$$\langle x^* - y^*, x - y\rangle \geq 0 \qquad \text{for all } x^* \in \delta^L_{\infty} f(x) \qquad \text{and} \qquad y^* \in \delta^L_{\infty} f(y),$$

that is, $\delta^L_{\infty} f$ is monotone in U.

Let $x, y \in \operatorname{Dom} \delta^L_{\infty} f \cap U$, $x^* \in \delta^L_{\infty} f(x)$ and $y^* \in \delta^L_{\infty} f(y)$. Then there exist $\alpha_k \to 0^+$, $x_k \to_f x$, $\beta_n \to 0^+$, $y_n \to_f y$, $x_k^* \in \delta f(x_k)$ and $y_n^* \in \delta f(y_n)$ such that $\alpha_k x_k^* \overset{*}{\rightharpoonup} x^*$ and $\beta_n y_n^* \overset{*}{\rightharpoonup} y^*$. According to the monotonicity of δf, we have for k and n sufficiently large

$$\langle \alpha_k x_k^* - \alpha_k y_n^*, x_k - y_n\rangle \geq 0,$$

and taking the limit over k and then over n we obtain $\langle x^*, x - y\rangle \geq 0$.

Therefore, the monotonicity of $\delta^L f$ and $\delta^L_{\infty} f$ implies that of $\delta^L f + \delta^L_{\infty} f$ and hence that of $\delta^{\Lambda} f + \delta^{\Lambda}_{\infty} f$; and we conclude recalling that, if a set-valued operator is monotone, then its pointwise w^*-closed convex hull is also monotone.

The opposite implication is trivial because of the inclusion $\delta f(x) \subset \partial_C f(x)$ for all $x \in U$. $\square$

Similar to the monotonicity, the quasimonotonicity of the Clarke subdifferential of lsc functions characterizes, on a Banach space, its quasiconvexity (see [1], [16] and [26]).

Let us recall the concept of quasimonotonicity.

Definition 2.3. A set-valued operator $A : X \rightrightarrows X^*$ is called *quasimonotone* over an open set U if, for all $x, y \in U \cap \operatorname{Dom} A$, $x^* \in A(x)$, $y^* \in A(y)$, one has

$$\langle x^*, y - x\rangle > 0 \Rightarrow \langle y^*, y - x\rangle \geq 0.$$

The following proposition establishes that the quasimonotonicity of $\partial_C f$ is equivalent to that of δf, provided (2.2) holds.

Proposition 2.2. *Let $f : X \longrightarrow \mathbb{R}\cup\{+\infty\}$ be a lsc function such that the representation formula (2.2) holds for δf. Then the Clarke subdifferential $\partial_C f$ is quasimonotone in U if and only if δf is quasimonotone in the same set.*

Proof. Suppose δf is quasimonotone in U. Let $x, y \in U \cap \operatorname{Dom} \delta^L f$, take $x^* \in \delta^L f(x)$ and $y^* \in \delta^L f(y)$. Then there exist $x_k \to_f x$, $y_k \to_f y$, $x_k^* \in \delta f(x_k)$ and $y_k^* \in \delta f(y_k)$ such that $x_k^* \xrightarrow{*} x^*$ and $y_k^* \xrightarrow{*} y^*$. For k sufficiently large, since δf is quasimonotone in U, we have

$$\langle x^*, y - x\rangle > 0 \Rightarrow \langle x_k^*, y_k - x_k\rangle > 0 \Rightarrow \langle y_k^*, y_k - x_k\rangle \geq 0 \Rightarrow \langle y^*, y - x\rangle \geq 0,$$

that is, $\delta^L f$ is quasimonotone in U.

We will now show that for $x, y \in U \cap \operatorname{Dom}(\delta^L f + \delta^L_\infty f)$ (note that, from the representation formula, we have the inclusion $U \cap \operatorname{Dom} \partial_C f \subset U \cap \operatorname{Dom}(\delta^L f + \delta^L_\infty f))$, $x_L^* \in \delta^L f(x)$, $x_\infty^* \in \delta^L_\infty f(x)$, $y_L^* \in \delta^L f(y)$ and $y_\infty^* \in \delta^L_\infty f(y)$, we have the implication

$$\langle x_L^*, y - x\rangle > 0 \quad \text{or} \quad \langle x_\infty^*, y - x\rangle > 0 \Rightarrow \langle y_L^*, y - x\rangle \geq 0 \quad \text{and} \quad \langle y_\infty^*, y - x\rangle \geq 0,$$

which will imply that $\delta^L f + \delta^L_\infty f$ is quasimonotone in U.

Suppose $\langle x_L^*, y - x\rangle > 0$. It is clear that $\langle y_L^*, y - x\rangle \geq 0$ because $\delta^L f$ is quasimonotone. Moreover, there exist $\beta_k \to 0^+$, $y_k \to_f y$, $y_k^* \in \delta f(y_k) \subset \delta^L f(y_k)$ such that $\beta_k y_k^* \xrightarrow{*} y_\infty^*$ and then for k sufficiently large, we have, because of the quasimonotonicity of $\delta^L f$ and of the inclusion $\delta f \subset \delta^L f$,

$$\begin{aligned}\langle x_L^*, y - x\rangle > 0 &\Rightarrow \langle x_L^*, y_k - x\rangle > 0 \Rightarrow \langle y_k^*, y_k - x\rangle \geq 0 \Rightarrow \langle \beta_k y_k^*, y_k - x\rangle \geq 0\\ &\Rightarrow \langle y_\infty^*, y - x\rangle \geq 0.\end{aligned}$$

Suppose now that $\langle x_\infty^*, y - x\rangle > 0$. Let $\alpha_k \to 0^+$, $x_k \to_f x$, $x_k^* \in \delta f(x_k) \subset \delta^L f(x_k)$ be such that $\alpha_k x_k^* \xrightarrow{*} x_\infty^*$. For k which is large enough, since $\delta^L f$ is quasimonotone in U, we obtain

$$\begin{aligned}\langle x_\infty^*, y - x\rangle > 0 &\Rightarrow \langle \alpha_k x_k^*, y - x_k\rangle > 0 \Rightarrow \langle x_k^*, y - x_k\rangle > 0 \Rightarrow \langle y_L^*, y - x_k\rangle \geq 0\\ &\Rightarrow \langle y_L^*, y - x\rangle \geq 0,\end{aligned}$$

and from the quasimonotonicity of δf in U we have

$$\begin{aligned}\langle x_\infty^*, y - x\rangle > 0 &\Rightarrow \langle \alpha_k x_k^*, y_k - x_k\rangle > 0 \Rightarrow \langle x_k^*, y_k - x_k\rangle > 0\\ &\Rightarrow \langle y_k^*, y_k - x_k\rangle \geq 0 \Rightarrow \langle \beta_k y_k^*, y_k - x_k\rangle \geq 0 \Rightarrow \langle y_\infty^*, y - x\rangle \geq 0.\end{aligned}$$

Finally, the fact that $\delta^L f + \delta^L_\infty f$ is quasimonotone in U implies the quasimonotonicity of $\delta^\Lambda f + \delta^\Lambda_\infty f$ and then of $x \mapsto \overline{co}^{w^*}(\delta^\Lambda f(x) + \delta^\Lambda_\infty f(x)) = \partial_C f(x)$.

The converse implication comes directly from the inclusion $\delta f(x) \subset \partial_C f(x)$ for all $x \in U$. $\square$

Several properties of functions which are convex up to a square over a Hilbert space have been studied by several authors. It can be seen (through Correa–Jofré–Thibault [13]) that such functions correspond to functions with hypomonotone Clarke subdifferentials. In Proposition 2.3 below, we will study the relationship between the hypomonotonicity of $\partial_C f$ and that of δf. Let us begin by clarifying the notion of hypomonotonicity (see [40]).

Definition 2.4. A set-valued operator $A : X \rightrightarrows X^*$ is called *hypomonotone* in an open set U if there exists $r \geq 0$ such that for all $x, y \in U \cap \operatorname{Dom} A$ one has

$$\langle x^* - y^*, x - y\rangle \geq -r\|x - y\|^2 \qquad \text{for all } x^* \in A(x), y^* \in A(y). \tag{2.5}$$

Proposition 2.3. *Let $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a lsc function such that δf verifies the representation formula (2.2). Then the Clarke subdifferential $\partial_C f$ is hypomonotone in U if and only if δf is hypomonotone in the same set.*

Proof. If we suppose δf is hypomonotone in U, then there exists $r \geq 0$ such that for all $x, y \in U \cap \operatorname{Dom} \delta f$ one has

$$\langle x^* - y^*, x - y\rangle \geq -r\|x - y\|^2 \qquad \text{for all } x^* \in \delta f(x), y^* \in \delta f(y).$$

Using an argument similar to that used in the proof of Proposition 2.1, we can prove that $\delta^L f$ is hypomonotone and that $\delta^L_\infty f$ is monotone in U. This will imply the hypomonotonicity of $\delta^L f + \delta^L_\infty f \supset \delta^\Lambda f + \delta^\Lambda_\infty f$ and then that of $\partial_C f$.

The converse implication is direct from the inclusion $\delta f(x) \subset \partial_C f(x)$ for all $x \in U$. $\square$

Remark 2.1. When X is a Hilbert space, a useful characterization of the hypomonotonicity in U of a set-valued operator A is the monotonicity of $A + rI$ in U, where I is the identity in X. This characterization cannot be generalized to a general Banach space by using the dual set-valued operator $I(x) = \{x^* \in X^* : \|x\|^2 = \|x^*\|^2_* = \langle x^*, x\rangle\} = \partial_C(1/2\|\cdot\|^2)(x)$. Nevertheless, if X admits an equivalent Gâteaux differentiable (away from zero) norm (for example, any reflexive space admits such a renormalization) that we use to define the dual operator I above, then we still obtain Proposition 2.3 provided we replace the definition of hypomonotonicity (2.5) by the monotonicity of $A + rI$. This is a direct consequence of the fact that in such a space, I is single valued and norm-weak* continuous.

We are going to consider now the concept of submonotonicity of a set-valued operator. This concept has been thoroughly studied in the papers [2], [15] and [42].

Definition 2.5. A set-valued operator $A : X \rightrightarrows X^*$ is called *submonotone* in an open set U if for each $x_0 \in U \cap \operatorname{Dom} A$ and $r > 0$, there exists $\varepsilon > 0$ with $B(x_0; \varepsilon) \subset U$ such that for all $x, y \in B(x_0; \varepsilon) \cap \operatorname{Dom} A$ one has

$$\langle x^* - y^*, x - y\rangle \geq -r\|x - y\| \qquad \text{for all } x^* \in A(x),\ y^* \in A(y).$$

Proposition 2.4. *If $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ is a lsc function such that δf verifies the representation formula (2.2), then the Clarke subdifferential $\partial_C f$ is submonotone in U if and only if δf is submonotone in the same set.*

Proof. Let $x_0 \in U$ and $r > 0$. From the submonotonicity of δf, there exists $\varepsilon > 0$ such that for all $x, y \in B(x_0; \varepsilon) \cap U \cap \operatorname{Dom} \delta f$ one has

$$\langle x^* - y^*, x - y\rangle \geq -r\|x - y\| \qquad \text{for all } x^* \in \delta f(x),\ y^* \in \delta f(y).$$

Using arguments similar to those used in the proof of Proposition 2.1, we can prove that $\delta^L f$ is submonotone and that $\delta^L_\infty f$ is monotone in U. This will imply the submonotonicity of $\delta^L f + \delta^L_\infty f$ and $\delta^\Lambda f + \delta^\Lambda_\infty f$, and later that of $\partial_C f$.

The converse implication is direct from the inclusion $\delta f(x) \subset \partial_C f(x)$ for all $x \in U$. $\square$

An important direct consequence of Propositions 2.1–2.4 concerns the convexity, the quasiconvexity, the convexity up to a square, and the approximate convexity behavior of nondifferentiable functions. We recall that a function f is quasiconvex in an open convex set U if for all $x, y \in U$ and $\lambda \in [0, 1]$ one has $f(\lambda x + (1-\lambda)y) \leq \max\{f(x), f(y)\}$. The function f is convex up to a square in U if there exists $r > 0$ such that $f + r\|\cdot\|^2$ is convex in U, and f is approximately convex in U if for all $x_0 \in U$ and $r > 0$, there exists $\varepsilon > 0$ such that for all $x, y \in B(x_0; \varepsilon) \cap U$ and $\lambda \in [0, 1]$ one has $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y) + r\lambda(1-\lambda)\|x - y\|$ (see [31]).

Corollary 2.1. *Let $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a lsc function with* $\operatorname{dom} f \neq \emptyset$ *and let δf satisfy the representation formula* (2.2) *in an open convex subset U of X. Then, the following characterizations hold:*

(a) *f is convex in U if and only if δf is monotone in U;*
(b) *f is quasiconvex in U if and only if δf is quasimonotone in U;*
(c) *when X is a Hilbert space, f is convex up to a square in U if and only if δf is hypomonotone in U;*
(d) *if f is locally Lipschitzian in U, then f is approximately convex in U if and only if δf is submonotone in the same set.*

Proof. (a) is a direct consequence of Proposition 2.1 and Theorem 2.4 in [14]. (b) is a direct consequence of Proposition 2.2 and Theorem 4.1 in [1]. From Remark 2.1, (c) is a direct consequence of Proposition 2.3 and Theorem 2.4 in [14] applied to the function $f + r\|\cdot\|^2$. (d) is a direct consequence of Proposition 2.4 and Theorem 2 in [15]. $\square$

3. Representation formula and local behavior of lsc functions

Several characterizations of the decreasing property with respect to a convex cone, Lipschitzian property, directional Lipschitzian behavior, pln property, etc. of lsc functions are provided in the literature with the Clarke subdifferential and some presubdifferentials. The objective of this section is to study similar characterizations with any operator δf for which (2.2) holds. In more general terms, we will examine the equivalence between the corresponding characterization with the Clarke subdifferential and the similar one with δf substituting for $\partial_C f$.

The next theorem makes use of the concepts of asymptotic operator and closedness of an operator at a point.

Definition 3.1. For a set-valued operator $\Gamma : X \rightrightarrows X^*$, we define *the asymptotic operator* $\Gamma_\infty : X \rightrightarrows X^*$ *of* Γ by

$$\Gamma_\infty(x) := \{x^* \in X^* : \text{there exist nets } \alpha_j \to 0^+, x_j \to x \text{ and } x_j^* \in \Gamma(x_j)$$
$$\text{such that } \{\alpha_j x_j^*\}_j \text{ is bounded and } \alpha_j x_j^* \xrightarrow{*} x^*\}.$$

Definition 3.2. We say that a set-valued operator $\Gamma : X \rightrightarrows X^*$ is $w_b^* - \|\cdot\|$ is closed at $x \in X$ if, for any net $x_j \to x$ and any bounded net $x_j^* \xrightarrow{*} x^*$ with $x_j^* \in \Gamma(x_j)$, one has $x^* \in \Gamma(x)$.

We will say that Γ is locally bounded at $x \in X$ if there exists $\varepsilon > 0$ such that the image of $B(x;\varepsilon)$ by Γ, given by $\bigcup_{x' \in B(x;\varepsilon)} \Gamma(x')$, is bounded.

We can now establish the following theorem concerning a general set-valued operator Γ. Several choices of this set-valued operator will allow us to derive various behaviors of functions.

Theorem 3.1. *Let $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a lsc function, D be a subset of* $\mathrm{Dom}\,(\delta^L)$ *and U be an open set in X. Let $\Gamma : X \rightrightarrows X^*$ be a $w_b^* - \|\cdot\|$ closed set-valued operator at any point of $U \cap D$ and K^0 be a w^*-closed convex cone in X^*. Then, the following assertions hold.*

(a) *If Γ is locally bounded at any point of $U \cap D$, then we have the equivalence*

$$\delta f(x) \subset K^0 + \Gamma(x) \quad \text{for all } x \in U \cap D \Leftrightarrow \delta^L f(x) + \delta^L_\infty f(x) \subset K^0 + \Gamma(x) \quad \text{for all } x \in U \cap D. \tag{3.1}$$

(b) *If K^0 and Γ satisfy*

- (i) *$\Gamma_\infty(x) \cap -K^0 = \{0\}$ for all $x \in U \cap D$;*
- (ii) *either $K^0 = \{0\}$ or there exists some w^*-compact subset S with $0 \notin S$, such that $K^0 = \mathbb{R}_+ S$;*
- (iii) *$\Gamma(x) + \Gamma_\infty(x) \subset \Gamma(x)$ for all $x \in U \cap D$,*

then we also have the equivalence (3.1).

(c) *If in* (a) *or* (b) *we assume that δf additionally satisfies* the representation formula (2.2) *in U, $D =$* $\mathrm{Dom}\,(\delta^\Lambda f)$*, and that Γ is convex valued, then we have the equivalence*

$$\delta f(x) \subset K^0 + \Gamma(x) \qquad \text{for all } x \in U \Leftrightarrow \partial_C f(x) \subset K^0 + \Gamma(x) \quad \text{for all } x \in U. \tag{3.2}$$

Proof. (a) The implication $\Leftarrow$ is evident because $\delta f(x) \subset \delta^L f(x) + \delta^L_\infty f(x)$ for all $x \in X$. In order to prove $\Rightarrow$ we first show that the left hand side of (3.1) implies that

$$\delta^L f(x) \subset K^0 + \Gamma(x) \qquad \text{for all } x \in U \cap D. \tag{3.3}$$

Let $x \in U \cap D$, $x^* \in \delta^L f(x)$, $x_k \to_f x$ and $x_k^* \in \delta f(x_k)$ be such that $x_k^* \overset{*}{\rightharpoonup} x^*$. Then, from the left hand side of (3.1), $x_k^* = a_k^* + b_k^*$ with $a_k^* \in K^0$ and $b_k^* \in \Gamma(x_k)$. Since b_k^* is bounded, it has a bounded subnet converging weakly-star to some b^*. Since Γ is $w_b^* - \|\cdot\|$ closed at x, we have that $b^* \in \Gamma(x)$ and $x^* - b^* \in K^0$. Therefore, we conclude that $x^* \in K^0 + \Gamma(x)$.

Let us now prove that

$$\delta^L_\infty f(x) \subset K^0 \qquad \text{for all } x \in U \cap D. \tag{3.4}$$

Let $x \in U \cap D$, $x^* \in \delta^L_\infty f(x)$, $x_k \to_f x$, $\alpha_k \to 0^+$ and $x_k^* \in \delta f(x_k)$ be such that $\alpha_k x_k^* \overset{*}{\rightharpoonup} x^*$. Then $x_k^* = a_k^* + b_k^*$ with $a_k^* \in K^0$ and $b_k^* \in \Gamma(x_k)$. Since b_k^* is bounded, we conclude that $\alpha_k a_k^* \overset{*}{\rightharpoonup} x^* \in K^0$.

Inclusions (3.3) and (3.4) and the fact that K^0 is a convex cone prove the result.

(b) As above, the implication $\Leftarrow$ is evident. In order to prove $\Rightarrow$ we first establish the inclusion (3.3). Let $x \in U \cap D$, $x^* \in \delta^L f(x)$, $x_k \to_f x$ and $x_k^* \in \delta f(x_k)$ be such that $x_k^* \overset{*}{\rightharpoonup} x^*$. Then $x_k^* = a_k^* + b_k^*$ with $a_k^* \in K^0$ and $b_k^* \in \Gamma(x_k)$. Let us show that the sequence a_k is bounded. Otherwise it has a subsequence with $\|a^*_{k(n)}\|_* \to +\infty$, and we define, for each n, some real number $\lambda_n > 0$ and $s_n^* \in S$ such that $a^*_{k(n)} = \lambda_n s_n^*$. As the set S is bounded, one has $\lambda_n \to +\infty$ and, according to the weak-star compactness of S, the bounded sequence s_n^* admits a bounded subnet converging weakly-star to some $s^* \in S$. Since $s_n^* \in K^0$, the w^*-closedness of K^0 entails $s^* \in K^0$. On the other hand, through the bounded subnet of s_n^* converging to s^* and the boundedness of $x_{k(n)}^*$, the equality $-s_n^* = \lambda_n^{-1} b_{k(n)}^* - \lambda_n^{-1} x_{k(n)}^*$ yields $-s^* \in \Gamma_\infty(x)$. Since $s^* \neq 0$, we get a contradiction with (i). Then, the bounded sequence a_k^* admits a bounded subnet converging weakly-star to some $a^* \in K^0$. In this way, the corresponding bounded subnet of $b_k^* = x_k^* - a_k^*$ will converge weakly-star to $x^* - a^*$ and, since Γ is $w_b^* - \|\cdot\|$ closed at x, we conclude that $x^* - a^* \in \Gamma(x)$, and then $x^* \in K^0 + \Gamma(x)$.

Let us prove now the inclusion

$$\delta_\infty^L f(x) \subset K^0 + \Gamma_\infty(x) \ \forall x \in U \cap D. \tag{3.5}$$

Let $x \in U \cap D$, $x^* \in \delta_\infty^L f(x)$, $x_k \to_f x$, $\alpha_k \to 0^+$ and $x_k^* \in \delta f(x_k)$ be such that $\alpha_k x_k^* \xrightharpoonup{*} x^*$. Then $x_k^* = a_k^* + b_k^*$ with $a_k^* \in K^0$ and $b_k^* \in \Gamma(x_k)$. As above, let us show that the sequence $\alpha_k a_k^*$ is bounded. Otherwise, it has a subsequence with $\|\alpha_{k(n)} a_{k(n)}^*\|_* \to +\infty$. Take for each n some $\lambda_n > 0$ and $s_n^* \in S$ such that $a_{k(n)}^* = \lambda_n s_{k(n)}^*$. As the set S is bounded one has $\alpha_{k(n)}\lambda_n \to +\infty$ and following the same arguments as were used in the proof of (3.3) above, we get a contradiction with (i). The bounded sequence $\alpha_k a_k^*$ then admits a bounded subnet converging weakly-star to some $a^* \in K^0$. Consequently, the corresponding bounded subnet of $\alpha_k b_k^* = \alpha_k x_k^* - \alpha_k a_k^*$ will converge weakly-star to $x^* - a^* \in \Gamma_\infty(x)$, that is $x^* \in K^0 + \Gamma_\infty(x)$.

Combining inclusions (3.3) and (3.5), hypothesis (iii), and the convexity of K^0 allows us to obtain the right hand side inclusion in (3.1) and to conclude (b).

(c) Observe first that, under the assumptions of (c) and for any $x \in U \cap D$, the second inclusion of (3.1) entails $\delta^\Lambda f(x) + \delta_\infty^\Lambda f(x) \subset K^0 + \Gamma(x)$. So for any $x \in U \cap D$ we obtain from (3.1) the implication

$$\delta f(x) \subset K^0 + \Gamma(x) \Rightarrow \delta^\Lambda f(x) + \delta_\infty^\Lambda f(x) \subset K^0 + \Gamma(x). \tag{3.6}$$

If the representation formula (2.2) holds in (a), from (3.6) the implication $\Rightarrow$ of (3.2) is evident when $x \in D$, because $K^0 + \Gamma(x)$ is the addition of a convex w^*-closed set and a convex w^*-compact set, which is always a convex w^*-closed set.

In (b), the convexity of $K^0 + \Gamma(x)$ is evident and the hypothesis (i) insures the w^*-closedness of this set for all $x \in D$. In fact, if $a_j^* + b_j^* \xrightharpoonup{*} x^*$ with $a_j^* \in K^0$ and $b_j^* \in \Gamma(x)$, from (i), with arguments similar to those used in this proof in the first part of (b), we can obtain the boundedness of these nets and we conclude that $x^* \in K^0 + \Gamma(x)$. From (3.6), the representation formula then allows us to obtain the implication $\Rightarrow$ of (3.2) for all $x \in U \cap D$.

The reverse implication $\Leftarrow$ of (3.2) being obvious, due to the inclusion $\delta f \subset \partial_C f$, we have proved the equivalence (3.2) for all $x \in U \cap D$ in either (a) or (b).

When $x \notin D$ the equivalence is trivial because $\delta f(x) = \partial_C f(x) = \emptyset$. □

Remark 3.1. If $\dim X < +\infty$, the hypothesis (ii) in part (b) of Theorem 3.1 always holds. Otherwise, as shown in the following lemma, a sufficient condition for this hypothesis is that $K^0 = \{x^* \in X^* : \langle x, x^*\rangle \le 0 \ \forall x \in K\}$ where $K \subset X$ is a convex cone with nonempty interior.

The following lemma is well known. We give a simple proof.

Lemma 3.1. *Let $K \subset X$ be a convex cone and*

$$K^0 = \{x^* \in X^* : \langle x^*, x\rangle \le 0 \ \forall x \in K\}.$$

If $K^0 \neq \{0\}$, a sufficient condition for the existence of a w^*-compact set S with $0 \notin S$ such that $K^0 = \mathbb{R}_+ S$ is that $\operatorname{int} K \neq \emptyset$.

Proof. For $p \in \operatorname{int} K$ and $r > 0$ such that $p + r\mathbb{B}_X \subset K$, where $\mathbb{B}_X = \{x \in X : \|x\| \leq 1\}$, we define the w^*-closed set

$$S = \{x^* \in K^0 : \langle x^*, p\rangle = -r\}.$$

Since $K^0 \neq \{0\}$, it is easy to see that $K^0 = \mathbb{R}_+ S$. To conclude, let us prove that S is bounded. For all $x^* \in S$ and $u \in \mathbb{B}_X$, we have $\langle x^*, p + ru\rangle \leq 0$, which implies that $\langle x^*, u\rangle \leq 1$ for all $u \in \mathbb{B}_X$ and $x^* \in S$, that is, $S \subset \mathbb{B}_{X^*} = \{x^* \in X^* : \|x^*\|_* \leq 1\}$. □

Observe that the condition of the lemma is easily seen to be also necessary whenever S is required to be in addition convex.

The choice of the Clarke subdifferential $\partial_C g(x)$, for some lsc function g in place of $\Gamma(x)$ requires the closedness of $\partial_C g$ at x. Such a property does not hold for any function g. The function g in [35] given by $g : \mathbb{R}^3 \longrightarrow \mathbb{R} \cup \{+\infty\}$ such that

$$g(x) = \psi(C; x) = \begin{cases} 0 & \text{if } x \in C \\ +\infty & \text{if } x \notin C, \end{cases} \tag{3.7}$$

where $C = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : |x_3| = x_1 x_2\}$ provides an example of a lsc function where $\partial_C g$ is not closed at $(0, 0, 0)$. In fact, it is easy to check that $\partial_C g(t, 0, 0) = \{0\} \times \mathbb{R} \times \mathbb{R}$ for all $t \neq 0$ and $\partial_C g(0, 0, 0) = \{0\} \times \{0\} \times \mathbb{R}$.

A primary general important class of not necessarily convex functions g, where $\partial_C g$ is $w^* - \|\cdot\|$ closed, is the class of directionally Lipschitzian functions.

Definition 3.3 (*[36]*). A lsc function $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ is *directionally Lipschitzian* at $x \in \operatorname{dom} f$ with respect to a vector $y \in X$ if

$$\limsup_{\substack{y' \to y \\ x' \to_f x \\ t \to 0^+}} \frac{f(x' + ty') - f(x')}{t} < +\infty.$$

We will say that f is directionally Lipschitzian at x if there is at least some vector y such that f is directionally Lipschitzian at x with respect to y. Note that f is locally Lipschitzian at x if and only if it is directionally Lipschitzian at x with respect to $y = 0$. Finally, we will say that f is directionally Lipschitzian if it is directionally Lipschitzian at each point of its effective domain $\operatorname{dom} f$.

It is well known that if f is a lsc function, directionally Lipschitzian at x, the Clarke subdifferential of f at x is characterized by (see [36])

$$\partial_C f(x) = \{x^* \in X^* : \langle x^*, h\rangle \leq f^0(x; h)\ \forall h \in X\} \tag{3.8}$$

where

$$f^0(x; h) = \limsup_{\substack{x' \to_f x \\ t \to 0^+}} \frac{f(x' + th) - f(x')}{t}. \tag{3.9}$$

The following lemma gives a useful semicontinuity property of the Clarke directional derivative $f^0(\cdot; h)$. It will allow us to easily derive in the proof of Theorem 3.2 the closedness of $\partial_C f$ at x when f is a directionally Lipschitzian function.

Lemma 3.2. *If $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ is a lsc function at x, then*

$$\limsup_{x' \to_f x} f^0(x'; h) \leq f^0(x; h) \ \forall h \in X. \tag{3.10}$$

Proof. Fix $h \in X$. Equality (3.9) can be written as

$$f^0(x'; h) = \inf_{\substack{\varepsilon>0 \\ \delta>0}} \sup_{\substack{z \in B_f(x';\varepsilon) \\ t \in]0,\delta[}} \frac{f(z+th) - f(z)}{t},$$

where $B_f(x'; \varepsilon) = \{z \in B(x'; \varepsilon) : |f(z) - f(x')| < \varepsilon\}$. Fix $\eta > 0$, $\varepsilon > 0$ and let $\gamma > 0$ be such that $B_f(x'; \varepsilon/2) \subset B_f(x; \varepsilon)$ for all $x' \in B_f(x; \gamma)$. Then

$$\begin{aligned} f^0(x'; h) &= \inf_{\substack{\nu>0 \\ \delta>0}} \sup_{\substack{z \in B_f(x';\nu) \\ t \in]0,\delta[}} \frac{f(z+th) - f(z)}{t} \leq \sup_{\substack{z \in B_f(x';\varepsilon/2) \\ t \in]0,\eta[}} \frac{f(z+th) - f(z)}{t} \\ &\leq \sup_{\substack{z \in B_f(x;\varepsilon) \\ t \in]0,\eta[}} \frac{f(z+th) - f(z)}{t} \end{aligned}$$

for all $x' \in B_f(x; \gamma)$. This implies

$$\limsup_{x' \to_f x} f^0(x'; h) \leq \sup_{\substack{z \in B_f(x;\varepsilon) \\ t \in]0,\eta[}} \frac{f(z+th) - f(z)}{t}$$

for all $\varepsilon > 0$ and $\eta > 0$. Taking the infimum over ε and η, we obtain (3.10). $\square$

When f is a lsc directionally Lipschitzian function in U, as a direct application of (3.8) and (3.10), we obtain the closedness of $\partial_C f$ and furthermore the equality

$$\partial_C f(x) = \partial_C^L f(x) + \partial_{C,\infty}^L f(x) \ \forall x \in U. \tag{3.11}$$

Although the inclusion of the first member of (3.11) in the second one always holds, the converse may fail. In fact, the function $f = g$, where g is given in (3.7) (see [35]) provides an example where (3.11) fails. In fact $(0, 1, 1) \in \partial_C^L f(0, 0, 0) + \partial_{C,\infty}^L f(0, 0, 0)$ but $(0, 1, 1) \notin \partial_C f(0, 0, 0)$. So, we cannot obtain for such a function f the representation formula (2.2) with $\delta^\Lambda f = \partial_C^\Lambda f$.

A second class of functions for which we can establish the closedness of the Clarke subdifferential is the class of primal lower nice functions. This class is also involved in Theorem 3.2. It has been introduced by Poliquin in [34] where he shows, in the finite dimensional setting, the integration property of such functions and studies their generalized second-order behavior.

Before stating the definition of primal lower nice functions, we need to recall the concept of proximal subgradient.

Definition 3.4. An element $x^* \in X^*$ is a *proximal subgradient* of a function f from X into $\mathbb{R} \cup \{+\infty\}$ at $x \in \operatorname{dom} f$ if, for some $t > 0$, the inequality

$$f(x') \geq f(x) + \langle x^*, x' - x \rangle - t\|x - x'\|^2$$

is valid for all x' in a neighborhood of x. We denote by $\partial_P f(x)$ the set of all proximal subgradients of f at x.

It is well known (and easy to see) that the inclusion $\partial_P f(x) \subset \partial_C f(x)$ always holds.

Definition 3.5 (*[34]*). We say that a lsc function $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ is *∂_P-primal-lower-nice* (∂_P-pln) at $\bar{x} \in \mathrm{cl}(\mathrm{dom}\, f)$ if the domain of $\partial_P f$ is a dense subset of some neighborhood of $\bar{x}$ intersected with the domain of f and if there exist positive scalars ε, c and τ such that if $t \geq \tau$, $\|x^*\|_* < ct$, $\|x - \bar{x}\| < \varepsilon$ and $x^* \in \partial_P f(x)$, then the inequality

$$f(x') \geq f(x) + \langle x^*, x' - x\rangle - \frac{t}{2}\|x' - x\|^2 \tag{3.12}$$

is valid for all x' with $\|x' - x\| < \varepsilon$. The function f is called ∂_P-primal-lower-nice if it is ∂_P-primal-lower-nice at all points in $\mathrm{cl}(\mathrm{dom}\, f)$.

We note that when X is a Hilbert space the proximal subdifferential is automatically nonempty on a dense subset of the domain of the lsc function f.

The above definition of the ∂_P-pln property can be obviously extended for any set-valued operator δf such that $\delta f(x) = \emptyset$ when $x \notin \mathrm{dom}\, f$.

Definition 3.6. We say that a lsc function $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ is *δ-primal-lower-nice* (δ-pln) at $\bar{x} \in \mathrm{cl}(\mathrm{dom}\, f)$ if the domain of δf is a dense subset of some neighborhood of $\bar{x}$ intersected with the domain of f and if there exist positive scalars ε, c and τ such that if $t \geq \tau$, $\|x^*\|_* < ct$, $\|x - \bar{x}\| < \varepsilon$ and $x^* \in \delta f(x)$, then the inequality (3.12) is valid for all x' with $\|x' - x\| < \varepsilon$. The function f is called δ-primal-lower-nice if it is δ-primal-lower-nice at all points in $\mathrm{cl}(\mathrm{dom}\, f)$.

We first establish a relation between $\partial_C f$ and $\partial_P f$ when δf satisfies (2.2).

Proposition 3.1. *Let $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a lsc function such that δf satisfies the representation formula (2.2) in U. If f is δ-pln, then we have the equality*

$$\partial_C f(x) = \mathrm{cl}_{w^*}(\partial_P f(x)) \qquad \textit{for all } x \in U.$$

Proof. For $x \in U \setminus \mathrm{Dom}\, \delta^{\Lambda} f = U \setminus \mathrm{Dom}\, \partial_C f$ (see (2.4)) the above equality is trivial. Let $x \in U \cap \mathrm{Dom}\, \delta^{\Lambda} f = U \cap \mathrm{Dom}\, (\delta^{\Lambda} f + \delta^{\Lambda}_{\infty} f)$, $x^*_{\Lambda} \in \delta^{\Lambda} f(x)$, $x^*_{\infty} \in \delta^{\Lambda}_{\infty} f(x)$, $x_k \to_f x$, $x^*_k \in \delta f(x_k)$, $\alpha_k \to 0^+$, $z_k \to_f x$, $z^*_k \in \delta f(z_k)$, be such that $x^*_k \overset{*}{\rightharpoonup} x^*_{\Lambda}$ and $\alpha_k z^*_k \overset{*}{\rightharpoonup} x^*_{\infty}$. On the other hand, from the representation formula we know that $x \in \mathrm{cl}(\mathrm{dom}\, f)$ and, thus, there exist positive constants τ, c and ε such that if $t \geq \tau$, $\|y^*\|_* < ct$, $\|x'' - x\| < \varepsilon$ and $y^* \in \delta f(x'')$, then

$$f(x') \geq f(x'') + \langle y^*, x' - x''\rangle - \frac{t}{2}\|x' - x''\|^2 \qquad \text{for all } x' \in B(x''; \varepsilon).$$

Since, for $t \geq \tau$ and k sufficiently large, we have $\|x^*_k\|_* < ct$, $\|x_k - x\| < \varepsilon/2$, $\|z^*_k\|_* < ct/\alpha_k$, $t/\alpha_k \geq \tau$ and $\|z_k - x\| < \varepsilon/2$, we can write

$$f(x') \geq f(x_k) + \langle x^*_k, x' - x_k\rangle - \frac{t}{2}\|x' - x_k\|^2 \qquad \text{for all } x' \in B(x; \varepsilon/2),$$

$$f(x') \geq f(z_k) + \langle z^*_k, x' - z_k\rangle - \frac{t}{2\alpha_k}\|x' - z_k\|^2 \qquad \text{for all } x' \in B(x; \varepsilon/2).$$

If $x' \in \mathrm{dom}\, f \cap B(x; \varepsilon/2)$ we multiply the second inequality by α_k, taking the lower limit over k in both inequalities and adding these two limits. This permits us to obtain the inequality

$$f(x') \geq f(x) + \langle x^*_{\Lambda} + x^*_{\infty}, x' - x\rangle - t\|x' - x\|^2.$$

The latter is evident when $x' \notin \mathrm{dom}\, f$. This yields $x^*_{\Lambda} + x^*_{\infty} \in \partial_P f(x)$ and therefore we have proved the inclusion

$$\delta^{\Lambda} f(x) + \delta^{\Lambda}_{\infty} f(x) \subset \partial_P f(x) \subset \partial_C f(x) \qquad \text{for all } x \in U.$$

Taking the w^*-closed convex hull in the above inclusion, we obtain the desired equality. $\square$

It is shown in [23] that when X is a Hilbert space and f is ∂_P-pln, then $\partial_P f = \partial_C f$. The following proposition, together with Proposition 3.1, will allow us to see that the equality still holds in the reflexive Banach context when f is ∂_P-pln.

For a subset $S \subset X^*$ (resp. $X \times X^*$) we will denote by $w_b^*\text{cl}\,(S)$ (resp. $\|\cdot\| - w_b^*\text{cl}\,(S)$) the set of all w^* limits (resp. $\|\cdot\| - w_b^*$ limits) of bounded nets of points in S.

Proposition 3.2. *Let $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a lsc function. If f is δ-pln, then*

$$\|\cdot\| - w_b^*\text{cl}\,(\text{gph}\,\delta f) \subset \text{gph}\,\partial_P f,$$

where gph A *is the graph of the set-valued operator $A : X \rightrightarrows X^*$ defined by*

$$\text{gph}\,A = \{(x, x^*) \in X \times X^* : x^* \in A(x)\}.$$

Proof. Let $(x, x^*) \in \|\cdot\| - w_b^*\text{cl}\,(\text{gph}\,\delta f)$. Thus, there exist a net $x_j \to x$ and a bounded net $x_j^* \overset{*}{\rightharpoonup} x^*$ such that $x_j^* \in \delta f(x_j)$. Since $x \in \text{cl}\,(\text{Dom}\,\delta f) \subset \text{cl}\,(\text{dom}\,f)$, f is δ-pln at x. We know that there exist positive scalars ε, c, τ such that if $t \geq \tau$, $\|y^*\|_* < ct$, $\|x'' - x\| < \varepsilon$ and $y^* \in \delta f(x'')$, then

$$f(x') \geq f(x'') + \langle y^*, x' - x''\rangle - \frac{t}{2}\|x' - x''\|^2 \qquad \text{for all } x' \in B(x''; \varepsilon).$$

Then, for t sufficiently large we have $\|x_j^*\|_* < ct$ for all j and there exists j_0 such that $\|x_j - x\| < \varepsilon/2$ for all $j \geq j_0$. In this way, we can write for all $j \geq j_0$

$$f(x') \geq f(x_j) + \langle x_j^*, x' - x_j\rangle - \frac{t}{2}\|x' - x_j\|^2 \text{ for all } x' \in B(x; \varepsilon/2) \subset B(x_j; \varepsilon) \tag{3.13}$$

and taking the lower limit over j we obtain

$$f(x') \geq f(x) + \langle x^*, x' - x\rangle - \frac{t}{2}\|x' - x\|^2 \qquad \text{for all } x' \in B(x; \varepsilon/2),$$

which implies that $x^* \in \partial_P f(x)$. $\square$

Corollary 3.1. *Assume that the space X is reflexive and f is ∂_P-pln at $\bar{x}$. Then for all x near $\bar{x}$, one has the equality*

$$\partial_C f(x) = \partial_P f(x). \tag{3.14}$$

Proof. The space X being reflexive, we know that the representation formula (2.2) holds for $\delta f = \partial_P f$. Then for all x near $\bar{x}$ we have by Proposition 3.1 and the reflexivity of X

$$\partial_C f(x) = \text{cl}_{w^*}(\partial_P f(x)) = \text{cl}_{\|\cdot\|}(\partial_P f(x)).$$

Applying Proposition 3.2 with $\delta f = \partial_P f$ we obtain $\partial_C f(x) \subset \partial_P f(x)$ and hence $\partial_C f(x) = \partial_P f(x)$ for x near $\bar{x}$. $\square$

The following theorem examines some special examples of the set-valued mapping Γ involved in Theorem 3.1. These examples, which are of great interest in the rest of the paper, correspond to the choice $\Gamma = \partial_C g + \gamma \mathbb{B}_{X^*}$ with the classes of functions g introduced in Definitions 3.3 and 3.5. The well known classes of convex and locally Lipschitzian functions are also considered.

Theorem 3.2. *Let $f : X \longrightarrow \mathbb{R}\cup\{+\infty\}$ be a lsc function such that δf verifies the representation formula (2.2) in U and let $K^0 \subset X^*$ be a w^*-closed convex cone. The following assertions hold.*

(a) *If $g : X \longrightarrow \mathbb{R}$ is a locally Lipschitzian function, then*

$$\delta f(x) \subset K^0 + \partial_C g(x) + \gamma\mathbb{B}_{X^*}\ \forall x \in U \Leftrightarrow \partial_C f(x) \subset K^0 + \partial_C g(x) + \gamma\mathbb{B}_{X^*}\quad \forall x \in U, \tag{3.15}$$

where $\gamma \geq 0$.

(b) *Assume that $g : X \longrightarrow \mathbb{R}\cup\{+\infty\}$ is a function that satisfies one of the following properties:*
 (1) *g is a lsc convex function,*
 (2) *g is a lsc directionally Lipschitzian function that is continuous relative to* Dom $\partial_C g$,
 (3) *X is a reflexive space and g is ∂_P-pln,*
 and assume also that
 (i) *$-K^0 \cap (\partial_C g)_\infty(x) = \{0\}$ for all $x \in U$, and*
 (ii) *either $K^0 = \{0\}$ or there exists a w^*-compact set S with $0 \notin S$ such that $K^0 = \mathbb{R}_+ S$.*
 Then, the equivalence (3.15) still holds.

Proof. (a) This is a consequence of Theorem 3.1 part (c) with the hypothesis of part (a). Indeed, since g is locally Lipschitzian, the set-valued operator $\partial_C g$ is $w_b^* - \|\cdot\|$ closed and locally bounded in X, and hence it is easily seen that the same holds for the set-valued operator Γ with $\Gamma(x) = \partial_C g(x) + \gamma\mathbb{B}_{X^*}$.

(b) We proceed to prove that this is a consequence of Theorem 3.1 part (c) with the hypothesis of part (b). In fact, we will show under conditions (1), (2) or (3), that $\partial_C g$ is a $w_b^* - \|\cdot\|$ closed operator in $D \cap U$, where $D :=$ Dom $(\delta^\Lambda f + \delta^\Lambda_\infty f) =$ Dom $\delta^\Lambda f$, and hence $\Gamma(x) := \partial_C g(x) + \gamma\mathbb{B}_{X^*}$ will be a $w_b^* - \|\cdot\|$ closed operator in $D \cap U$, and that

$$\partial_C g(x) + (\partial_C g)_\infty(x) \subset \partial_C g(x) \qquad \text{for all } x \in X.$$

The above inclusion and the fact that $\Gamma_\infty(x) = (\partial_C g)_\infty(x)$ will give us hypothesis (iii) in Theorem 3.1 and will allow us to conclude.

(1) The lower semicontinuity of the convex function g and the classical characterization of the Clarke subdifferential for convex functions

$$\partial_C g(x) = \{x^* \in X^* : g(x) + \langle x^*, y - x\rangle \leq g(y) \text{ for all } y \in X\}, \tag{3.16}$$

imply that $\Gamma(x) := \partial_C g(x) + \gamma\mathbb{B}_{X^*}$ is a $w_b^* - \|\cdot\|$ closed set-valued operator in X. To conclude, let us now prove the inclusion

$$\partial_C g(x) + (\partial_C g)_\infty(x) \subset \partial_C g(x) \qquad \text{for all } x \in X.$$

Let $x^* \in \partial_C g(x)$, $x^*_\infty \in (\partial_C g)_\infty(x)$, $\alpha_j \to 0^+$, $x_j \to x$ and $x_j^* \in \partial_C g(x_j)$ be such that $\alpha_j x_j^*$ is bounded and $\alpha_j x_j^* \overset{*}{\rightharpoonup} x^*_\infty$. Then

$$g(y) \geq g(x) + \langle x^*, y - x\rangle \qquad \text{for all } y \in X \tag{3.17}$$

$$g(y) \geq g(x_j) + \langle x_j^*, y - x_j\rangle \qquad \text{for all } y \in X. \tag{3.18}$$

If we multiply (3.18) by α_j and take the lower limit over j we obtain, using the positivity of α_j and the lower semicontinuity of g, the inequality

$$\langle x^*_\infty, y - x\rangle \leq 0 \qquad \text{for all } y \in \operatorname{dom} g;$$

and if we add (3.17) to this, we obtain

$$g(y) \geq g(x) + \langle x^* + x^*_\infty, y - x\rangle \qquad \text{for all } y \in X,$$

that is, $x^* + x^*_\infty \in \partial_C g(x)$.

(2) Assume now that g is a lsc directionally Lipschitzian function, that is continuous relative to $\operatorname{Dom} \partial_C g$. First, (3.10) allows us to write

$$\limsup_{x' \to_g x} g^0(x'; h) \leq g^0(x; h) \qquad \text{for all } h \in X.$$

Then, the characterization (3.8) of the Clarke subdifferential of g, the inequality (3.10), and the continuity of g relative to $\operatorname{Dom} \partial_C g$ imply the $w^*_b - \|\cdot\|$ closedness of $\partial_C g$ at all points of $\operatorname{Dom} \partial_C g$ and in particular in $D = \operatorname{Dom}(\delta^\Lambda f + \delta^\Lambda_\infty f)$.

Now, let $x \in \operatorname{Dom} \partial_C g \subset \operatorname{dom} g$, $x^* \in \partial_C g(x)$, $x^*_\infty \in (\partial_C g)_\infty(x)$, $\alpha_j \to 0^+$, $x_j \to x$ and $x^*_j \in \partial_C g(x_j)$ be such that $\alpha_j x^*_j$ is bounded and $\alpha_j x^*_j \overset{*}{\rightharpoonup} x^*_\infty$. From (3.8) we have

$$\langle x^*, h\rangle \leq g^0(x; h) \qquad \text{for all } h \in X$$
$$\langle x^*_j, h\rangle \leq g^0(x_j; h) \qquad \text{for all } h \in X.$$

If $g^0(x; h)$ is finite, we multiply the second inequality by α_j and take the upper limit over j taking (3.10) into account. Then, we add it to the first one and we obtain the inequality

$$\langle x^* + x^*_\infty, h\rangle \leq g^0(x; h).$$

Otherwise, the above inequality is evident and hence we have

$$\langle x^* + x^*_\infty, h\rangle \leq g^0(x; h) \qquad \text{for all } h \in X,$$

that is, $x^* + x^*_\infty \in \partial_C g(x)$. The desired inclusion

$$\partial_C g(x) + (\partial_C g)_\infty(x) \subset \partial_C g(x) \qquad \text{for all } x \in X$$

is then valid.

(3) We now suppose that g is ∂_P-pln. Using Proposition 3.2, it is not difficult to see that the convex set $\partial_P g(x)$ is w^*-closed. By Proposition 3.1, the reflexivity of X, and (3.14) we have the equality $\partial_P g(x) = \partial_C g(x)$ for all $x \in U$. Proposition 3.2 also yields that the operator $\partial_C g$ is $\|\cdot\| - w^*_b$ closed.

To finish, we will prove now the inclusion

$$\partial_C g(x) + (\partial_C g)_\infty(x) \subset \partial_C g(x) \qquad \text{for all } x \in X. \tag{3.19}$$

Let $x \in \operatorname{Dom} \partial_P g$, $x^* \in \partial_P g(x)$, $x^*_\infty \in (\partial_P g)_\infty(x)$, $\alpha_j \to 0^+$, $x_j \to x$ and $x^*_j \in \partial_P g(x_j)$ be such that $\alpha_j x^*_j$ is bounded and $\alpha_j x^*_j \overset{*}{\rightharpoonup} x^*_\infty$. As above, we know that there exists j_0 such that for all $t \geq T$ sufficiently large and $j \geq j_0$ we have $\|x^*_j\|_* < ct/\alpha_j$, $t/\alpha_j \geq T$, $\|x_j - x\| < \lambda/2$, and then

$$g(x') \geq g(x_j) + \langle x^*_j, x' - x_j\rangle - \frac{t}{2\alpha_j}\|x' - x_j\|^2 \qquad \text{for all } x' \in B(x; \lambda/2) \subset B(x_j; \lambda).$$

If we multiply this inequality by α_j and we take the lower limit over j, we obtain

$$0 \geq \langle x^*_\infty, x' - x\rangle - \frac{t}{2}\|x' - x\|^2 \qquad \text{for all } x' \in \operatorname{dom} g \cap B(x; \lambda/2).$$

On the other hand, we know that there exist $r > 0$ and $0 < \varepsilon \leq \lambda/2$ such that

$$g(x') \geq g(x) + \langle x^*, x' - x\rangle - r\|x' - x\|^2 \qquad \text{for all } x' \in B(x; \varepsilon).$$

The addition of the two last inequalities allows us to write

$$g(x') \geq g(x) + \langle x^* + x^*_\infty, x' - x\rangle - \left(\frac{t}{2} + r\right)\|x' - x\|^2 \qquad \text{for all } x' \in B(x; \varepsilon)$$

which proves the inclusion (3.19). □

A first important consequence of Theorem 3.2 concerns the locally Lipschitzian behavior of functions.

Corollary 3.2. *Let $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a lsc function. Assume that δf satisfies the representation formula in an open convex set U with $U \cap \operatorname{dom} f \neq \emptyset$. Then f is Lipschitzian in U with modulus $\gamma \geq 0$ if and only if*

$$\delta f(x) \subset \gamma \mathbb{B}_{X^*} \qquad \textit{for all } x \in U.$$

Proof. According to (a) in Theorem 3.2, one has

$$\delta f(x) \subset \gamma \mathbb{B}_{X^*} \qquad \text{for all } x \in U \Leftrightarrow \partial_C f(x) \subset \gamma \mathbb{B}_{X^*} \qquad \text{for all } x \in U.$$

So, the equivalence of the Corollary follows from the characterization of Lipschitzian functions in terms of Clarke subdifferentials (see for example [49]). □

More generally, we proceed with the study of the γ-nondecreasing property of a function with respect to a convex cone $K \subset X$. The exact ($\gamma = 0$) nondecreasing property in terms of subdifferentials has been investigated by Clarke et al. in [10] with the proximal subdifferential. Here, we examine the general case for any γ in relation to any operator δf satisfying the representation formula. In the next two propositions the set $K^0 \subset X^*$ will be the negative polar cone of some cone K, that is,

$$K^0 = \{x^* \in X^* : \langle x^*, x\rangle \leq 0 \text{ for all } x \in K\}.$$

We state now a version of the Zagrodny approximate mean value theorem [51] in the form that we will use it in the next two propositions.

Theorem 3.3. *Let a, b two points in an open convex subset U of a Banach space X (with $a \neq b$) and let $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be lsc on U and finite at a. Then for each real number $\rho < f(b)$, there exist a point $x_0 \in [a, b[:= \{tb + (1-t)a : t \in [0, 1[\}$, a sequence x_k converging to x_0, and points $x^*_k \in \partial_C f(x_k)$ such that*

(a) $\liminf_{k\to+\infty}\langle x^*_k, b - x_k\rangle \geq \frac{(\rho - f(a))}{\|b-a\|}\|b - x_0\|$ *and*
(b) $\liminf_{k\to+\infty}\langle x^*_k, b - a\rangle \geq \rho - f(a)$.

Proposition 3.3. *Let $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a lsc function. If K^0 is the negative polar of a convex cone $K \subset X$ containing the origin and if δf satisfies the representation formula in an open convex set U of X, then*

$$\delta f(x) \subset K^0 + \gamma \mathbb{B}_{X^*} \qquad \textit{for all } x \in U$$

if and only if f is γ-nondecreasing over U with respect to the cone K, that is,

$$f(y) \leq f(x) + \gamma\|x - y\| \qquad \textit{for all } x, y \in U, \textit{with } y - x \in K.$$

Proof. If we define in part (a) of Theorem 3.2, $g(x) = 0$ for all $x \in X$, we obtain the equivalence

$$\delta f(x) \subset K^0 + \gamma \mathbb{B}_{X^*}\ \forall x \in U \Leftrightarrow \partial_C f(x) \subset K^0 + \gamma \mathbb{B}_{X^*}\ \forall x \in U.$$

Then, it suffices to prove that

$$\begin{cases} \partial_C f(x) \subset K^0 + \gamma \mathbb{B}_{X^*} \\ \text{for all } x \in U \end{cases} \Leftrightarrow \begin{cases} f(y) \leq f(x) + \gamma \|x - y\| \\ \text{for all } x, y \in U, \\ \text{such that } y - x \in K \end{cases} \tag{3.20}$$

($\Rightarrow$) Let $x, y \in U$ be such that $y - x \in K$. If $x \notin \operatorname{dom} f$ the result is evident. Otherwise, by the Zagrodny mean value theorem, for any real number $\rho < f(y)$, there exist $x_0 \in [x, y[$, $x_k \to_f x_0$, $x_k^* \in \partial_C f(x_k)$ such that

$$\rho - f(x) \leq \liminf_{k \to +\infty} \langle x_k^*, y - x \rangle.$$

From the left hand side of (3.20) and the convexity of U, we can write, for k sufficiently large, $x_k^* = q_k^* + \gamma b_k^*$ with $q_k^* \in K^0$ and $b_k^* \in \mathbb{B}_{X^*}$. Hence

$$\rho - f(x) \leq \liminf_{k \to +\infty} \langle q_k^* + b_k^*, y - x \rangle \leq \liminf_{k \to +\infty} \langle b_k^*, y - x \rangle \leq \gamma \|x - y\|,$$

which proves the implication.

($\Leftarrow$) We will first prove that the right hand side of (3.20) implies that

$$f^{\uparrow}(x; h) \leq \psi(K, h) + \gamma \|h\| \qquad \text{for all } x \in U \text{ and } h \in X, \tag{3.21}$$

where $\psi(K; \cdot)$ is the indicator function of K defined by

$$\psi(K, h) = \begin{cases} 0 & \text{if } h \in K \\ +\infty & \text{if } h \notin K. \end{cases}.$$

Let $h \in K$, $x \in U$ and $\lambda > 0$ be such that $x + th \in U$ for all $t \in]0, \lambda[$. Then we have

$$\frac{f(x' + th) - f(x')}{t} \leq \gamma \|h\| \qquad \text{for all } x' \text{ near } x \text{ and } t \in]0, \lambda[,$$

and hence

$$\begin{aligned} f^{\uparrow}(x; h) &= \sup_{\eta > 0} \inf_{\varepsilon > 0} \sup_{\substack{x' \in B(x;\varepsilon) \\ |f(x) - f(x')| < \varepsilon \\ t \in]0, \varepsilon[}} \inf_{h' \in B(h; \eta)} \frac{f(x' + th') - f(x')}{t}, \\ &\leq \inf_{\varepsilon > 0} \sup_{\substack{x' \in B(x;\varepsilon) \\ |f(x) - f(x')| < \varepsilon \\ t \in]0, \varepsilon[}} \frac{f(x' + th) - f(x')}{t} \leq \gamma \|h\|, \end{aligned}$$

that is, (3.21) holds. Moreover, for $x \in U$ and for all $x^* \in \partial_C f(x)$, (3.21) implies that

$$\langle x^*, y - x \rangle \leq \psi(K, y - x) + \gamma \|y - x\| \qquad \text{for all } y \in X.$$

Hence $x^* \in \partial_C(\psi(K, \cdot - x) + \gamma \| \cdot - x \|)(x) = K^0 + \gamma \mathbb{B}_{X^*}$. This equality is direct from the characterization (3.16) of the Clarke subdifferential for lsc convex functions and the equality

$$\partial_C(\psi(K, \cdot - x) + \gamma \| \cdot - x \|)(y) = \partial_C(\psi(K, \cdot - x))(y) + \partial_C(\gamma \| \cdot - x \|)(y)$$
$$\text{for all } y \in X$$

which is a consequence of the continuity of the norm (according to the Moreau–Rockafellar Theorem [30] concerning the subdifferential of the sum of two convex functions). $\square$

Remark 3.2. As a direct generalization of implication $\Rightarrow$ in the last Proposition, we can prove that

$$\begin{cases}\delta f(x) \subset K^0 + \gamma(x)\mathbb{B}_{X^*} \\ \text{for all } x \in U\end{cases} \Rightarrow \begin{cases}\text{for all } x, y \in U, y \in \operatorname{dom} f \text{ with } y - x \in K \\ \text{there exists } c \in [x, y[\text{ such that} \\ f(y) \leq f(x) + \gamma(c)\|x - y\|\end{cases}$$

where γ is a given real-valued continuous function in U.

A property along the lines of the γ-nondecreasing property but involving in addition a convex function g can also be analyzed. Before stating the property, recall that a locally Lipschitz function $g : X \longrightarrow \mathbb{R}$ is Clarke regular at a point x provided the directional derivative $g'(x; \cdot)$ exists and coincides with $g^{\uparrow}(x; \cdot)$. The directional derivative of any function $\varphi : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ at $x \in \operatorname{dom}\varphi$ is given by $\varphi'(x; h) = \lim_{t \to 0^+} \frac{\varphi(x+th)-\varphi(x)}{t}$ when it exists.

We must mention that the proof of the convex part of the theorem makes use of many ideas of Thibault and Zagrodny [45] and [46].

Theorem 3.4. *Let K^0 be the negative polar of some cone $K \subset X$ and $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a lsc function such that δf satisfies the representation formula in an open convex set U of X with $U \cap \operatorname{dom} f \neq \emptyset$. Then, the following assertions hold.*

(a) *If $g : X \longrightarrow \mathbb{R}$ is a Clarke directionally regular locally Lipschitzian function in U, then*

$$\begin{cases}\delta f(x) \subset K^0 + \partial_C g(x) + \gamma\mathbb{B}_{X^*} \\ \text{for all } x \in U\end{cases} \Rightarrow \begin{cases}g(x) + f(y) \leq f(x) + g(y) + \gamma\|x - y\| \\ \text{for all } x, y \in U \text{ such that} \\ y - x \in K\end{cases} \tag{3.22}$$

(b) *Assume that the cone K is open and $g : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ is a lsc convex function such that $-K^0 \cap (\partial_C g)_\infty(x) = \{0\}$ for all $x \in U$. Then we also have implication (3.22).*

Proof. Observe that under condition (b), if $K \neq X$ then by Lemma 3.1 there exists some w^*-compact set S with $0 \notin S$ such that $K^0 = \mathbb{R}_+ S$. So condition (ii) in (b) of Theorem 3.2 holds. So, from Theorem 3.2, we see that the inclusion $\delta f(x) \subset K^0 + \partial_C g(x) + \gamma\mathbb{B}_{X^*}$ implies, under (2.2) and the assumptions above, the inclusion

$$\partial_C f(x) \subset K^0 + \partial_C g(x) + \gamma\mathbb{B}_{X^*} \qquad \text{for all } x \in U. \tag{3.23}$$

Therefore, we need to obtain the conclusion of (3.22) under (3.23) and under the assumption that g is either a Clarke directionally regular locally Lipschitzian function or a lsc convex function. Observe that the nonemptiness of $U \cap \operatorname{dom} f$ entails that $U \cap \operatorname{Dom} \partial_C f$ and $U \cap \operatorname{dom} g$ are nonempty. Fix any $x \in U \cap \operatorname{dom} f$ and $y \in U$ with $y - x \in K$, $x \neq y$, and take any $\varepsilon > 0$. Fix also any $u \in [x, y[\cap \operatorname{dom} f$.

(I) We proceed first to show that there exists some $r \in]0, 1[$ such that

$$f(u + r(y - u)) - f(u) \leq g(u + r(y - u)) - g(u) + (\gamma + \varepsilon)\|r(y - u)\|. \tag{3.24}$$

Case (a). g is locally Lipschitzian and Clarke directionally regular.

By the definition of directional derivative, we know that there exists $t_0 \in]0, 1[$ such that

$$g'(u; y - u) \leq \frac{g(u + t(y - u)) - g(u)}{t} + \frac{\varepsilon}{2}\|y - u\| \qquad \text{for all } t \in]0, t_0[$$

and if we put $w_t := u + t(y - u)$ the above inequality can be written in the form

$$g'(u; w_t - u) \leq g(w_t) - g(u) + \frac{\varepsilon}{2}\|w_t - u\|. \tag{3.25}$$

On the other hand, the upper semicontinuity of $g'(\cdot; y-u) = g^0(\cdot; y-u)$ implies that there exists $r \in \,]0, t_0]$ such that

$$\frac{g'(w_s; y-u)}{\|y-u\|} \le \frac{g'(u; y-u)}{\|y-u\|} + \frac{\varepsilon}{2} \qquad \text{for all } s \in [0, r]. \tag{3.26}$$

From (b) in the Zagrodny mean value theorem, for any real number $\rho < f(w_r)$ there exist $x_0 \in [u, w_r[$, $x_k \to x_0$, and $x_k^* \in \partial_C f(x_k)$ such that $\rho - f(u) \le \liminf_{k\to+\infty}\langle x_k^*, w_r - u\rangle$. From (3.23), we can write $x_k^* = q_k^* + z_k^* + \gamma b_k^*$ with $q_k^* \in K^0$, $z_k^* \in \partial_C g(x_k)$, and $b_k^* \in \mathbb{B}_{X^*}$, and since $w_r - u = r(y-u) \in K$ (according to $y - x \in K$) we obtain using the upper semicontinuity of $g'(\cdot; w_r - u)$

$$\begin{aligned}
\rho - f(u) &\le \liminf_{k\to+\infty}\langle z_k^*, w_r - u\rangle + \gamma\|w_r - u\| \\
&\le \liminf_{k\to+\infty} g'(x_k; w_r - u) + \gamma\|w_r - u\| \\
&\le g'(x_0; w_r - u) + \gamma\|w_r - u\| \\
&\le \|w_r - u\|\left[\frac{g'(u; y-u)}{\|y-u\|} + \gamma + \frac{\varepsilon}{2}\right],
\end{aligned}$$

where the last inequality follows from (3.26), given that $x_0 = w_s$ for some $s \in [0, r[$. In this way, we can write, using (3.25),

$$\begin{aligned}
\rho - f(u) &\le g'(u; w_r - u) + \left(\gamma + \frac{\varepsilon}{2}\right)\|w_r - u\| \\
&\le g(w_r) - g(u) + (\gamma + \varepsilon)\|w_r - u\|.
\end{aligned}$$

The parameter $\rho < f(w_r)$ being arbitrary, we obtain the desired inequality (3.24).

Case (b). g is a lsc and convex.

Here in a first step, we suppose, in addition to $y - x \in K$ with $x \neq y$, that $x \in U \cap \operatorname{Dom}\partial_C f$ and $y \in \operatorname{dom} g$; hence $x \in U \cap \operatorname{Dom}\partial g$. Below, we will follow several arguments from [45].

Fix some $a^* \in \partial g(x)$ and consider the convex function $\varphi : \mathbb{R} \longrightarrow \mathbb{R} \cup \{+\infty\}$ with

$$\varphi(s) = \begin{cases} g(x) + s\langle a^*, y - x\rangle & \text{if } s \in \,]-\infty, 0[\\ g(x + s(y-x)) & \text{if } s \in [0, 1] \\ +\infty & \text{if } s \in \,]1, +\infty[. \end{cases}$$

Since $g(x)$ and $g(y)$ are finite, this convex function φ is finite on $]-\infty, 1[$ and thus it is locally Lipschitzian on $]-\infty, 1[$. Therefore the restriction to $[0, 1[$ of the function $s \mapsto g'(x + s(y - x); y - x)$ is finite and upper semicontinuous.

Put now $w_t := u + t(y - u)$, where $u \in [x, y[\cap \operatorname{dom} f$ as above. The convexity of g entails for all $t > 0$

$$g'(u; y-u) \le \frac{g(u + t(y-u)) - g(u)}{t}$$

and hence

$$g'(u; w_t - u) \le g(w_t) - g(u) \qquad \text{for all } t \ge 0. \tag{3.27}$$

As the restriction to $[0, 1[$ of the function $s \mapsto g'(w_s; y - x)$ is, according to the foregoing, finite and upper semicontinuous, we know that there exists $r \in \,]0, 1[$ such that

$$\frac{g'(w_r; y-u)}{\|y-u\|} \le \frac{g'(u; y-u)}{\|y-u\|} + \varepsilon. \tag{3.28}$$

From (a) in the Zagrodny mean value theorem, for any real number $\rho < f(w_r)$ there exist $x_0 \in [u, w_r[$, $x_k \to x_0$, and $x_k^* \in \partial_C f(x_k)$ such that

$$\frac{\|w_r - x_0\|}{\|w_r - u\|}(\rho - f(u)) \leq \liminf_{k\to+\infty}\langle x_k^*, w_r - x_k\rangle.$$

By (3.23), we may write $x_k^* = q_k^* + z_k^* + \gamma b_k^*$ with $q_k^* \in K^0$, $z_k^* \in \partial g(x_k)$, and $b_k^* \in \mathbb{B}_{X^*}$, and since $w_r - u = r(y-u) \in K$ we obtain $w_r - x_k \in K$ (because K is open) and then

$$\frac{\|w_r - x_0\|}{\|w_r - u\|}(\rho - f(x)) \leq \liminf_{k\to+\infty}\langle z_k^*, w_r - x_k\rangle + \gamma\|w_r - x_0\|. \tag{3.29}$$

On the other hand, using (3.28) and (3.27), we obtain

$$\begin{aligned}
\liminf_{k\to+\infty}\langle z_k^*, w_r - x_k\rangle &\leq \liminf_{k\to+\infty}(g(w_r) - g(x_k)) \leq g(w_r) - g(x_0) \leq g'(w_r; w_r - x_0)\\
&= \|w_r - x_0\| g'\left(w_r; \frac{y-u}{\|y-u\|}\right) \leq \|w_r - x_0\|\\
&\quad \times \left[g'\left(u; \frac{y-u}{\|y-u\|}\right) + \varepsilon\right]\\
&= \frac{\|w_r - x_0\|}{\|w_r - u\|}[g'(u; w_r - u) + \varepsilon\|w_r - u\|]\\
&\leq \frac{\|w_r - x_0\|}{\|w_r - u\|}[g(w_r) - g(u) + \varepsilon\|w_r - u\|].
\end{aligned}$$

Then, from (3.29) we can write the inequality

$$\rho - f(u) \leq g(w_r) - g(u) + (\gamma + \varepsilon)\|w_r - u\|$$

which implies (3.24) when $x \in U \cap \operatorname{Dom} \partial_C f$.

(II) Put now

$$\begin{aligned}
\sigma := \sup\{t \in]0,1] : f(x + t(y-x)) - f(x) \leq g(x + t(y-x))\\
- g(x) + (\gamma + \varepsilon)\|t(y-x)\|\}
\end{aligned}$$

and observe that the latter set is nonempty according to (3.24). Using the lsc property of f and the continuity of the restriction of g on the segment $[x, y]$, it is easily seen that the supremum above is attained. We claim that $\sigma = 1$. Otherwise, for $v := x + \sigma(y - x)$, we have $v \in [x, y[$ and $v \in \operatorname{dom} f$ according to the definition of σ and the finiteness of $g(v)$. Applying (3.24) with v in place of u, we obtain some $r \in]0, 1[$ such that

$$f(v + r(y-v)) - f(v) \leq g(v + r(y-v)) - g(v) + (\gamma + \varepsilon)\|y - v\|. \tag{3.30}$$

Further, we also have, according to the definition of σ which is attained,

$$f(v) - f(x) \leq g(v) - g(x) + (\gamma + \varepsilon)\|v - x\|$$

and adding this inequality and (3.30) we arrive at the inequality

$$f(v + r(y-x)) - f(x) \leq g(v + r(y-x)) - g(x) + (\gamma + \varepsilon)\|y - x\|,$$

which is easily seen to be in contradiction with the definition of σ. So $\sigma = 1$ and $\varepsilon > 0$ being arbitrary, we get the inequality

$$f(y) + g(x) \leq f(x) + g(y) + \gamma\|y - x\| \tag{3.31}$$

in its full generality in case (a) and for $x \in \operatorname{Dom} \partial_C f$ in case (b), since the inequality is obvious for $x = y$.

(III) Now, we complete the proof for the convex case. Suppose that $x \in U \cap \operatorname{dom} f$. We know from the graphical density of $\operatorname{Dom} \partial_C f$ in $\operatorname{dom} f$ that there exists a sequence $x_k \to_f x$ with $x_k \in \operatorname{Dom} \partial_C f$. As $y - x_k \in K$, for k large enough (because K is open), we have from (3.31)

$$f(y) + g(x_k) \le f(x_k) + g(y) + \gamma \|y - x_k\|.$$

Using the lsc property of g and passing to the limit, we obtain

$$f(y) + g(x) \le f(x) + g(y) + \gamma \|y - x\|.$$

The latter inequality still obviously holds for either $y \notin \operatorname{dom} g$ or $x \notin \operatorname{dom} f$. The proof is then complete. □

A similar result also holds for ∂_P-pln functions in Hilbert spaces. Before stating the result, we recall that for the Fréchet subdifferential $\partial_F f(x)$ a vector $x^* \in \partial_F f$ provided

$$\liminf_{x' \to x} \frac{1}{\|x' - x\|}[f(x') - f(x) - \langle x^*, x' - x\rangle] \ge 0.$$

Proposition 3.4. *Assume that X is a Hilbert space and let $f, g : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be lsc functions such that g is ∂_P-pln at $\bar{x} \in \operatorname{dom} f$ and δf satisfies the representation formula near $\bar{x}$. Assume also that K^0 is the negative polar of some convex cone $K \subset X$ such that*

(i) $-K^0 \cap (\partial_C g)_\infty(x) = \{0\}$ *for all x near to $\bar{x}$;*
(ii) *either $K = X$ or there exists S, w^*-compact with $0 \notin S$, such that $K^0 = \mathbb{R}_+ S$;*
(iii) *there exists $\gamma > 0$ such that*

$$\delta f(x) \subset K^0 + \partial_C g(x) + \gamma \mathbb{B}_{X^*}$$

for all x in a neighborhood of $\bar{x}$.

Then for any $\gamma' > \gamma$ there exists some neighborhood $U_{\gamma'}$ of $\bar{x}$ such that for all $x, y \in U_{\gamma'}$ with $y - x \in K$ one has

$$g(x) + f(y) \le f(x) + g(y) + \gamma' \|x - y\|.$$

Proof. From Theorem 3.2 (part (b)(3) and Eq. (3.15) and (iii) we have

$$\partial_C f(x) \subset K^0 + \partial_C g(x) + \gamma \mathbb{B}_{X^*}$$

for all x in a neighborhood of $\bar{x}$. We introduce $\partial f(x) := \partial_P^L f(x) = \partial_F^L f(x)$. The second equality is due to the fact that X is a Hilbert space (see for example [19]). Recall that since g is ∂_P-pln one has $\partial_P g = \partial_C g = \partial g$. So, there exists $\alpha > 0$ such that

$$\partial f(x) \subset K^0 + \partial g(x) + \gamma \mathbb{B}_{X^*} \ \forall x \in B(\bar{x}; \alpha). \tag{3.32}$$

Following the proof of Theorem 3.7 in [3] we may suppose that $\bar{x} = 0$, $f(0) = 0$, and $g(0) = 0$. Take $\varepsilon > 0$ and $c > 0$ corresponding to the ∂_P-pln property of g and satisfying conditions (14a) and (14b) in [3] with in addition $\varepsilon < \alpha$. Take $\gamma' > \gamma$ and $0 < \varepsilon' < \min\{\varepsilon, c, c(\gamma' - \gamma)/\gamma'\}$. As in the same proof, we obtain for all $\lambda > 0$ small enough and all $u \in B(0; \varepsilon/4)$

$$P_\lambda \bar{f}(u) \subset (I + \lambda T^f_{t_\lambda})^{-1}(u) \tag{3.33}$$

$$\partial_F e_\lambda \bar{f}(u) \subset \lambda^{-1}[I - (I + \lambda T^f_{t_\lambda})^{-1}](u), \tag{3.34}$$

where, for $B[0;\varepsilon]$ denoting the closed ball with radius ε centered at the origin, $\bar f := f + \psi(B[0;\varepsilon],\cdot)$, $t_\lambda := \varepsilon/\lambda$,

$$e_\lambda \bar f(u) := \inf_{x\in X}\left\{\bar f(x) + \frac{1}{2\lambda}\|u-x\|^2\right\} \qquad \text{(Moreau's envelope of } \bar f\text{)},$$

$$P_\lambda \bar f(u) := \operatorname*{Argmin}_{x\in X}\left\{\bar f(x) + \frac{1}{2\lambda}\|u-x\|^2\right\},$$

and $T^f_{t_\lambda}$ is the set-valued operator whose graph is

$$\operatorname{gph} T^f_{t_\lambda} := \{(x,x^*)\in \operatorname{gph}\partial f : \|x-\bar x\| < \varepsilon \text{ and } \|x^*\|_* \le t_\lambda\}.$$

We will prove in the lemma below that there is some $\eta > 0$ for which for every $r \in]0,1]$ there exists $\rho > 0$ such that

$$\partial f(x)\cap r\mathbb{B}_{X^*} \subset \partial_C g(x)\cap(r+\gamma+\rho)\mathbb{B}_{X^*} + K^0 + \gamma\mathbb{B}_{X^*}\ \forall x\in B(\bar x;\eta). \tag{3.35}$$

Hence, by (3.32) and (3.35) we have the existence of some $\eta\in]0,\varepsilon/8[$ such that for any λ small enough we can find $\rho_\lambda > 0$ for which $T^f_{t_\lambda} \subset T^g_{t_\lambda+\gamma+\rho_\lambda} + K^0 + \gamma\mathbb{B}_{X^*}$ on $B(0;\eta)$, and then we follow the proof of Theorem 3.7 [3] in order to obtain

$$\partial_F(e_\lambda \bar f - e_\lambda \bar g) \subset K^0 + \gamma'\mathbb{B}_{X^*} \qquad \text{on } B(0;\eta)$$

where $\bar g := g + \psi(B[0;\varepsilon],\cdot)$. Observe that $e_\lambda \bar f$ and $e_\lambda \bar g$ are finite everywhere. Therefore, from Proposition 3.3 we have for all $x, y\in B(0,\eta)$ with $y-x\in K$

$$e_\lambda \bar f(x) - e_\lambda \bar g(x) \le e_\lambda \bar f(y) - e_\lambda \bar g(y) + \gamma'\|x-y\|.$$

As $e_\lambda \bar f$ and $e_\lambda \bar g$ converge pointwise to $\bar f$ and $\bar g$ respectively as $\lambda\to 0^+$ (see, e.g., [3]) one has the desired inequality

$$g(x) + f(y) \le f(x) + g(y) + \gamma'\|x-y\|.$$

We only have to prove the lemma that shows (3.35).

Lemma 3.3. *Let suppose the same hypothesis as for Proposition 3.4. Then there exists some $\eta > 0$ such that for every $r > 0$ there exists $\rho > 0$ for which*

$$\partial f(x)\cap r\mathbb{B}_{X^*} \subset \partial_C g(x)\cap(r+\gamma+\rho)\mathbb{B}_{X^*} + K^0 + \gamma\mathbb{B}_{X^*}\ \forall x\in B(\bar x;\eta).$$

Proof. The lemma is obvious with the assumption $K = X$ in (ii). So we assume in (ii) that K has the compact base S as stated in (ii). Suppose in this case that the inclusion of the lemma does not hold. Fix a sequence $\eta_k \downarrow 0$ with $\eta_k < \alpha$. Then for each $k\in\mathbb{N}$, there exist $r_k\in]0,1]$, $x_k\in B(\bar x;1/k)$, and $x^*_k \in \partial f(x_k)\cap r_k\mathbb{B}_{X^*}$ such that $x^*_k \notin \partial g(x_k)\cap(r_k+\gamma+k)\mathbb{B}_{X^*} + K^0 + \gamma\mathbb{B}_{X^*}$. By (3.32) one has that

$$x^*_k = z^*_k + p^*_k + q^*_k \tag{3.36}$$

where $z^*_k\in\partial g(x_k)$, $p^*_k\in K^0$, and $q^*_k\in\gamma\mathbb{B}_{X^*}$. So p^*_k is not bounded due to the boundedness of x^*_k and q^*_k and the fact that $z^*_k\notin(r_k+\gamma+k)\mathbb{B}_{X^*}$. By (ii), taking a subsequence if necessary, we can write $p^*_k = \beta_k s^*_k$ with $\beta_k\to\infty$ and $s^*_k\in S$. Then by (3.36) there exist subsequences such that $s^*_{k_n} \xrightarrow{*} s^*\in S$ and $z^*_{k_n}/\beta_{k_n} \xrightarrow{*} z^*$. Since $x_k\to\bar x$, we have $(\partial g)_\infty(\bar x)\ni z^* = -s^*\in -K^0\setminus\{0\}$ which is a contradiction with (i). $\square$

Then we have finished the proof of the proposition. $\square$

Characterizations of directionally Lipschitzian property of lsc functions in terms of some subdifferentials have been established by Treiman in [49] (with the Clarke subdifferential) and Thibault and Zlateva in [47] (with several other subdifferentials). We show in Proposition 3.6 that such characterizations generally hold with any operator satisfying the representation formula (2.2). First, let us recall the result with the Clarke subdifferential.

Proposition 3.5 ([49]). *A lsc function $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ is directionally Lipschitzian at $\bar{x} \in \operatorname{dom} f$ with respect to a vector h if and only if there exist $\varepsilon > 0, r \in \mathbb{R}$, and $\gamma > 0$ such that*

$$(P_{\partial_C f}) \begin{cases} \langle x^*, h\rangle + \gamma \|x^*\|_* \leq r \\ \textit{for all } x \in B(\bar{x}; \varepsilon), x^* \in \partial_C f(x) \end{cases} \tag{3.37}$$

is satisfied.

Proposition 3.6. *Let $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a lsc function such that δf satisfies the representation formula in U. Then, f is directionally Lipschitzian at $\bar{x} \in U \cap \operatorname{dom} f$ with respect to a vector h if and only if there exist $\varepsilon > 0$, $r \in \mathbb{R}$ and $\gamma > 0$ such that*

$$(P_{\delta f}) \begin{cases} \langle x^*, h\rangle + \gamma \|x^*\|_* \leq r \\ \textit{for all } x \in B(\bar{x}; \varepsilon), x^* \in \delta f(x) \end{cases} \tag{3.38}$$

is satisfied.

Proof. From Proposition 3.5, it suffices to show that $(P_{\delta f})$ in (3.38) is equivalent to $(P_{\partial_C f})$ in (3.37).

Suppose $(P_\delta f)$. Let $0 < \varepsilon' \leq \varepsilon$ be such that $B(\bar{x}; \varepsilon') \subset U$. Fix any $x \in B(\bar{x}; \varepsilon')$ and $x^*_\Lambda \in \delta^\Lambda f(x)$. There then exist $x_k \to_f x$, $x^*_k \in \delta f(x_k)$ such that $x^*_k \overset{*}{\rightharpoonup} x^*_\Lambda$. By (3.38) and the lower semicontinuity of $\|\cdot\|_*$ with respect to the w^*-topology, we can write

$$\langle x^*_k, h\rangle + \gamma \|x^*_k\|_* \leq r \Rightarrow \langle x^*_\Lambda, h\rangle + \gamma \|x^*_\Lambda\|_* \leq r,$$

and hence the property

$$(P_{\delta^\Lambda f}) \begin{cases} \langle x^*_\Lambda, h\rangle + \gamma \|x^*_\Lambda\|_* \leq r \\ \text{for all } x \in B(\bar{x}; \varepsilon'), x^*_\Lambda \in \delta^\Lambda f(x) \end{cases} \tag{3.39}$$

is valid.

Now, consider any $x^*_\infty \in \delta^\Lambda_\infty f(x)$. There then exist $\alpha_k \to 0^+$, $x_k \to_f x$, $x^*_k \in \delta f(x_k)$ such that $\alpha_k x^*_k \overset{*}{\rightharpoonup} x^*_\infty$. Using (3.38) we conclude that

$$\langle x^*_\infty, h\rangle + \gamma \|x^*_\infty\|_* \leq 0.$$

Furthermore, from (3.39) we obtain that

$$\langle x^*_\Lambda + x^*_\infty, h\rangle + \gamma (\|x^*_\infty\|_* + \|x^*_\Lambda\|_*) \leq r$$

is satisfied for all $x \in B(\bar{x}; \varepsilon')$, $x^*_\Lambda \in \delta^\Lambda f(x)$ and $x^*_\infty \in \delta^\Lambda_\infty f(x)$ which entails in particular the inequality $\langle x^*_\Lambda + x^*_\infty, h\rangle + \gamma \|x^*_\infty + x^*_\Lambda\|_* \leq r$. That implies (passing to the w^*-closed convex hull of $\delta^\Lambda f(x) + \delta^\Lambda_\infty f(x)$) that $(P_{\partial_C f})$ is valid and hence f is directionally Lipschitzian at $\bar{x}$ with respect to h.

The opposite implication is direct because of the inclusion $\delta f(x) \subset \partial_C f(x)$ for all $x \in U$. $\square$

4. The convexly composite case

In this section, we analyze the behavior of another significant class of functions, the composition of a lower semicontinuous convex function h, defined on a real Banach space Y with values in $\mathbb{R} \cup \{+\infty\}$, and a continuously differentiable operator $F : X \longrightarrow Y$. The function $g = h \circ F$ is called *convexly composite*. Convexly composite functions are omnipresent in optimization theory and nonsmooth analysis. Many problems commonly encountered in optimization can be reformulated in terms of these functions. In [34] Poliquin showed, in the finite dimensional case, that when F is twice continuously differentiable and a natural qualification condition is satisfied, then the function g is ∂_P-pln. In general Banach spaces, Thibault and Zagrodny [45] gave a partial extension of this result. In order to extend the set of functions for which the result of Theorem 3.2 holds, we will first prove the $w_b^* - \|\cdot\|$ closedness of $\partial_C g$.

We will say that the convexly composite function $g = h \circ F$ satisfies the Robinson constraint qualification at $x \in X$ if

$$\mathbb{R}_+[\operatorname{dom} h - F(x)] - \nabla F(x)(X) = Y. \tag{R}$$

The following two lemmas will be useful in the reformulation of (R) and in the proof of Proposition 4.1 which is the key for the proof of the closedness of $\partial_C g$.

Lemma 4.1. *Let E be a real linear space, A and B two convex sets in E such that $0 \in A \cap B$. Then, $\mathbb{R}_+(A + B) = \mathbb{R}_+ A + \mathbb{R}_+ B$.*

Proof. Let αa and βb be such that $\alpha, \beta \in \mathbb{R}_+$, $a \in A$ and $b \in B$. Without loss of generality, we suppose that $\beta \geq \alpha$. If $\beta = 0$ it is clear that $\alpha a + \beta b \in \mathbb{R}_+(A + B)$. Otherwise, since A is convex and $0 \in A$, we have $(\alpha/\beta)a \in A$, and hence $\alpha a + \beta b = \beta((\alpha/\beta)a + b) \in \mathbb{R}_+(A + B)$. The opposite inclusion is evident. $\square$

The next lemma recalls a classical result (see for example Lemma 12.1 in [41]).

Lemma 4.2 (*[41]*). *Let E be a Banach space, $C \subset E$ a closed convex set. Then, $\mathbb{R}_+ C = E$ if and only if C is a neighborhood of $0 \in E$.*

If $F(x) \in \operatorname{dom} h$, from Lemma 4.1, we note that condition (R) can be written as

$$\mathbb{R}_+([\operatorname{dom} h - F(x)] \cap \mathbb{B}_Y - \nabla F(x)(\mathbb{B}_X)) = Y,$$

and then, from Lemma 4.2, the Robinson constraint qualification at $x \in X$ is equivalent to saying that $[\operatorname{dom} h - F(x)] \cap \mathbb{B}_Y - \nabla F(x)(\mathbb{B}_X)$ is a neighborhood of $0 \in Y$, that is,

$$s\mathbb{B}_Y \subset [\operatorname{dom} h - F(x)] \cap \mathbb{B}_Y - \nabla F(x)(\mathbb{B}_X) \qquad \text{for some } s > 0. \tag{R$'$}$$

We denote by $\nabla F(x)^* y^*$ the adjoint of $\nabla F(x)$ evaluated in $y^* \in Y^*$.

Proposition 4.1. *Assume that the convexly composite function $g = h \circ F$ satisfies the condition (R) at x and $F(x) \in \operatorname{dom} h$. If $x_j \to x$, $y_j^* \in \partial h(F(x_j))$ and $\alpha_j > 0$ are three nets such that $\alpha_j \nabla F(x_j)^* y_j^*$ and α_j are eventually bounded, then there exist $K > 0$ and j_0 such that $\|\alpha_j y_j^*\|_{Y^*} \leq K$ for all $j \geq j_0$.*

Proof. First, let us prove that

$$\mathbb{R}_+(\operatorname{dom} h - F(x)) = \mathbb{R}_+(\operatorname{dom} h \cap L_{1+h(F(x))} - F(x)),$$

where $L_{1+h(F(x))} = \{v \in Y : h(v) \leq 1 + h(F(x))\}$.

Let $y \in \mathbb{R}_+(\operatorname{dom} h - F(x))$. Then $y = \alpha(z - F(x))$ with $\alpha > 0$ and $z \in \operatorname{dom} h$. Assume that $1 + h(F(x)) < h(z)$; otherwise, it is clear that $y \in \mathbb{R}_+(\operatorname{dom} h \cap L_{1+h(F(x))} - F(x))$. Without loss of generality, we may suppose $h(F(x)) \geq 0$. Putting $\beta = \alpha h(z)$ and $\bar{y} = (1 - 1/h(z))F(x) + (1/h(z))z$, we have $y = \beta(\bar{y} - F(x))$ and furthermore, according to the convexity of h,

$$h(\bar{y}) \leq (1 - 1/h(z))h(F(x)) + 1 \leq h(F(x)) + 1,$$

that is, $y \in \mathbb{R}_+(\operatorname{dom} h \cap L_{1+h(F(x))} - F(x))$. The opposite inclusion is evident. Then, with the same arguments as were used to show that (R) is equivalent to (R′), we can see that (R) is equivalent to

$$s\mathbb{B}_Y \subset [\operatorname{dom} h \cap L_{1+h(F(x))} - F(x)] \cap \mathbb{B}_Y - \nabla F(x)(\mathbb{B}_X) \qquad \text{for some } s > 0. \tag{4.1}$$

Let us now prove that $\alpha_j y_j^*$ is eventually bounded. Choose some j_0 and some $\gamma \geq 0$ such that $\alpha_j + \|\alpha_j \nabla F(x_j)^* y_j^*\|_* \leq \gamma$ for all $j \geq j_0$ and

$$\|\nabla F(x_j) - \nabla F(x)\| \leq s/2 \qquad \text{for all } j \geq j_0.$$

Take any $b \in \mathbb{B}_Y$. By (4.1) we may write $sb = v_b - F(x) - \nabla F(x)(u_b)$ with $h(v_b) \leq 1 + h(F(x))$ and $u_b \in \mathbb{B}_X$. Then the inclusion $y_j^* \in \partial h(F(x_j))$ gives

$$\begin{aligned} s\langle y_j^*, b\rangle &= \langle y_j^*, v_b - F(x) - \nabla F(x)(u_b)\rangle \\ &= \langle y_j^*, v_b - F(x)\rangle - \langle y_j^*, \nabla F(x_j)(u_b)\rangle + \langle y_j^*, [\nabla F(x_j) - \nabla F(x)](u_b)\rangle \\ &\leq h(v_b) - h(F(x)) + \|\nabla F(x_j)^* y_j^*\|_* + \|y_j^*\|_{Y^*}\|\nabla F(x_j) - \nabla F(x)\|. \end{aligned}$$

Then the inequality $h(v_b) \leq 1 + h(F(x))$ and the choice of γ and j_0 entail for all $j \geq j_0$

$$s\langle \alpha_j y_j^*, b\rangle \leq \gamma + (s/2)\|\alpha_j y_j^*\|_{Y^*}$$

and hence $s\|\alpha_j y_j^*\|_{Y^*} \leq \gamma + (s/2)\|\alpha_j y_j^*\|_{Y^*}$. In conclusion, we obtain $\|\alpha_j y_j^*\|_{Y^*} \leq 2\gamma/s$ for all $j \geq j_0$. □

Recall that (see [11]) $\partial_C g = \nabla F(\cdot)^* \partial h(F(\cdot))$ whenever the convexly composite function $g = h \circ F$ satisfies the condition (R).

Corollary 4.1. *If the convexly composite function $g = h \circ F : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ satisfies the Robinson constraint qualification (R) at $x \in \operatorname{dom} g$, then the Clarke's subdifferential of g, $\partial_C g$, is $w_b^* - \|\cdot\|$ closed at x.*

Proof. Let x_j be a net converging to x and x_j^* a bounded net converging to x^* with $x_j^* \in \partial_C g(x_j)$. Then, there exists $y_j^* \in \partial h(F(x_j))$ such that $x_j^* = \nabla F(x_j)^* y_j^*$. From the above proposition, putting $\alpha_j = 1$, we obtain that for some j_0, the set $\{y_j^* : j \geq j_0\}$ is bounded, implying the existence of a bounded subnet w^*-converging to some $y^* \in Y^*$. Since ∂h is $w_b^* - \|\cdot\|$ closed at $F(x)$, according to the convexity of the lsc function h, we have $y^* \in \partial h(F(x))$ and then $x^* = \nabla F(x)^* y^* \in \partial_C g(x)$, which proves the result. □

We can now prove our theorem concerning convexly composite functions.

Theorem 4.1. *Let $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a lsc function such that δf verifies the representation formula (2.2) in U and let $K^0 \subset X^*$ be a w^*-closed convex cone. Assume that the convexly composite function $g = h \circ F$ satisfies the Robinson constraint qualification (R) at all points of U and*

(i) $-K^0 \cap (\partial_C g)_\infty(x) = \{0\}$ *for all* $x \in U$, *and*
(ii) *either* $K^0 = \{0\}$ *or there exists a* w^**-compact set* S *with* $0 \notin S$, *such that* $K^0 = \mathbb{R}_+ S$.

Then one has

$$\delta f(x) \subset K^0 + \partial_C g(x) + \gamma \mathbb{B}_{X^*}\ \forall x \in U \Leftrightarrow \partial_C f(x) \subset K^0 + \partial_C g(x) + \gamma \mathbb{B}_{X^*}\ \forall x \in U.$$

Proof. In order to use Theorem 3.1, we will first prove the inclusion

$$\partial_C g(x) + (\partial_C g)_\infty(x) \subset \partial_C g(x) \qquad \text{for all } x \in X. \tag{4.2}$$

Using Proposition 4.1, we directly check that $(\partial_C g)_\infty(x) = \nabla F(x)^*(\partial h)_\infty(F(x))$. Indeed, the second member being obviously included in the first one, fix any $x^* \in (\partial_C g)_\infty(x)$. Let $\alpha_j \to 0^+$, $x_j \to x$, and $x_j^* \in \partial_C g(x_j)$ be such that the net $\alpha_j x_j^*$ is bounded and w^*-convergent to x^*. Choose some $y_j^* \in \partial h(F(x_j))$ such that $x_j^* = \nabla F(x_j)^* y_j^*$. From Proposition 4.1 we know that the set $\{\alpha_j y_j^* : j \geq j_0\}$ is bounded for some j_0. Then we conclude that $\alpha_j y_j^* \xrightarrow{*} y^* \in (\partial h)_\infty(F(x))$ and $x^* = \nabla F(x)^* y^* \in \nabla F(x)^*(\partial h)_\infty(F(x))$.

Finally, we obtain the desired inclusion (4.2) using the relation

$$\partial h(y) + (\partial h)_\infty(y) \subset \partial h(y) \qquad \text{for all } y \in Y,$$

proved in part (b) (1) of Theorem 3.2. Further, by Corollary 4.1, $\partial_C g$ is $w_b^* - \|\cdot\|$ closed and hence the proof is complete. □

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