Realization of a Choquet simplex as the set of invariant probability measures of a tiling system

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Abstract. In this paper we show that, for every Choquet simplex K and for every $d > 1$, there exists a $\mathbb{Z}^d$-Toeplitz system whose set of invariant probability measures is affine homeomorphic to K. Then, we conclude that K may be realized as the set of invariant probability measures of a tiling system $(\Omega_T, \mathbb{R}^d)$.

1. *Introduction*

The set of invariant probability measures of a dynamical system induced by a minimal continuous action of $\mathbb{Z}$ on the Cantor set may have the affine-topological structure of an arbitrary Choquet simplex. This was demonstrated by Downarowicz in [**4**] when he showed that every Choquet simplex may be realized as the set of invariant probability measures of a dyadic 0–1 Toeplitz flow. Later, Gjerde and Johansen in [**8**] provided an alternative proof of the realization of a Choquet simplex, using the theory of dimension groups and the characterization of the Bratteli–Vershik diagrams which corresponds to Toeplitz flows [**8**]. They stated that for any Choquet simplex K there exists a 0–1 Toeplitz flow with zero entropy and full rational spectrum such that its set of invariant probability measures is affine homeomorphic to K.

In this work we show that, for every Choquet simplex K and for every $d > 1$, there exists a dynamical system induced by a $\mathbb{Z}^d$-action on the Cantor set whose set of invariant probability measures is K. More precisely, we show that every Choquet simplex may be realized as the set of invariant probability measures of a 0–1 $\mathbb{Z}^d$-Toeplitz system. This $\mathbb{Z}^d$-Toeplitz system can be chosen to be an almost one-to-one extension of a product of d one-dimensional 2-odometers, or an almost one-to-one extension of a product of d one-dimensional universal odometers.

This paper is organized as follows. In §2, we give some basic definitions relevant for the study of general flows. In §3 we recall the characterization of one-dimensional Toeplitz flows given by Gjerde and Johansen in [**8**] in terms of Kakutani–Rohlin partitions. In §4 we show that for a certain class of one-dimensional Toeplitz flow, the set of invariant probability measures K can be expressed as an inverse limit using a special sequence of matrices that we use in §5 to construct a $\mathbb{Z}^d$-Toeplitz system ($d > 1$) whose set of invariant probability measures is K. We show that one can take the $\mathbb{Z}^d$-Toeplitz system to be an almost one-to-one extension of a d-dimensional 2-odometer or an almost one-to-one extension of a d-dimensional universal odometer. Finally, we conclude that every Choquet simplex can be seen as the set of invariant probability measures of a tiling system $(\Omega_T, \mathbb{R}^d)$.

Remark 1. At the time this paper was written, Downarowicz [**6**] announced that he had found an easy effective method to construct a $\mathbb{Z}^d$-Toeplitz system whose set of invariant probability measures was an arbitrary Choquet simplex. The method exposed here is different and independent of the Downarowicz method.

2. *Definition and background*

Let $d \geq 1$ and $\Gamma = \mathbb{Z}^d$ or $\mathbb{R}^d$. We denote by $\bar{v} = (v_1, \ldots, v_d)$ the elements in Γ (if $\Gamma = \mathbb{Z}$ we can use n instead of $\bar{n}$). By a *topological dynamical system* we mean a pair (X, Γ) such that X is a metric compact space and Γ acts continuously on X. Given $\bar{v} \in \Gamma$ and $x \in X$ we will identify $\bar{v}$ with the associated homeomorphism and we denote by $\bar{v} \cdot x$ the action of $\bar{v}$ on x (if $\Gamma = \mathbb{Z}$ we can use the usual notation $n \cdot x = T^n(x)$, where $T : X \to X$ is the function induced by the action of 1 on X). The *orbit* of $x \in X$ is the set $o(x) = \{\bar{v} \cdot x : \bar{v} \in \Gamma\}$, and the *set of return times* of x to a neighborhood V of x is $\mathcal{R}_V(x) = \{\bar{v} \in \Gamma : \bar{v} \cdot x \in V\}$. A topological dynamical system (X, Γ) is *minimal* if X is the orbit closure of each of its points, and it is *equicontinuous* if the collection of the maps defined by the group action is a uniformly equicontinuous family.

We say that (X, Γ) is an *extension* of (Y, Γ), or that (Y, Γ) is a factor of (X, Γ), if there exists a continuous surjection $\pi : X \to Y$ such that π preserves the action, that is, $\pi(\bar{v} \cdot x) = \bar{v} \cdot \pi(x)$ for every $x \in X$ and $\bar{v} \in \Gamma$. We call π a *factor map*. When the factor map is bijective, we say that (X, Γ) and (Y, Γ) are *conjugate*. The factor map π is an *almost one-to-one factor map* and (X, Γ) is an *almost one-to-one extension* of (Y, Γ) by π if the set of points having one pre-image is residual (contains a dense G_δ set) in Y. In the minimal case it is equivalent to the existence of a point with exactly one preimage.

An *invariant probability measure* of a topological dynamical system (X, Γ) is a regular probability measure μ defined on $\mathcal{B}(X)$, the Borel σ-algebra of X, such that $\mu(\bar{v} \cdot B) = \mu(B)$ for all $\bar{v} \in \Gamma$ and $B \in \mathcal{B}(X)$. We denote by $\mathcal{M}_\Gamma(X)$ the set of invariant probability measures of (X, Γ). It is well known that $\mathcal{M}_\Gamma(X)$ is a non-empty *Choquet simplex* [**9**], that is, a metrizable compact convex set K in a locally convex space such that each $p \in K$ is represented by a unique probability measure supported on the set of extreme points of K [**12**].

2.1. *Odometers.*

Let $d \geq 1$. A d-dimensional *odometer* (or a $\mathbb{Z}^d$-odometer) is a set G defined by

$$G = \left\{ (g_n)_{n \geq 0} \in \prod_{n \geq 0} \mathbb{Z}^d / Z_n : \pi_n(g_{n+1}) = g_n, \forall n \geq 0 \right\},$$

where $(Z_n)_{n\geq 0}$ is a decreasing sequence of subgroups isomorphic to $\mathbb{Z}^d$, and $\pi_n : \mathbb{Z}^d/Z_{n+1} \to \mathbb{Z}^d/Z_n$ is the projection. The set G is a subgroup of the product group $\prod_{n\geq 0} \mathbb{Z}^d/Z_n$. Moreover, it is a compact topological group if every $\mathbb{Z}^d/Z_n$ is endowed with the discrete topology and $\prod_{n\geq 0} \mathbb{Z}^d/Z_n$ with the product topology.

The projection $\tau : \mathbb{Z}^d \to \prod_{i\geq 0} \mathbb{Z}^d/Z_i$ given by

$$\tau(\bar{v}) = (\tau_n(\bar{v}))_{n\geq 0},$$

where $\tau_n : \mathbb{Z}^d \to \mathbb{Z}^d/Z_n$ is the canonical projection, defines a continuous action of $\mathbb{Z}^d$ on G by

$$\bar{v} \cdot (g_n)_{n\geq 0} = \tau(\bar{v}) + (g_n)_{n\geq 0} \quad \text{for every } \bar{v} \in \mathbb{Z}^d \text{ and } (g_n)_{n\geq 0} \in G.$$

The topological dynamical system $(G, \mathbb{Z}^d)$ is minimal and equicontinuous.

If $(G_1, \mathbb{Z}), \ldots, (G_d, \mathbb{Z})$ are d one-dimensional odometers, then the system $(G, \mathbb{Z}^d)$, where $G = G_1 \times \cdots \times G_d$ and the action of $\mathbb{Z}^d$ on G is given by

$$\bar{v} \cdot (g_1, \ldots, g_d) = (v_1 \cdot g_1, \ldots, v_d \cdot g_d),$$

is conjugate to a $\mathbb{Z}^d$-odometer.

When $d = 1$ and $Z_n = 2^n\mathbb{Z}$ for all $n \geq 0$, the odometer G is known as the *2-odometer* (or adding machine). If $Z_n = n!\mathbb{Z}$ for all $n \geq 0$, the odometer G is called the *universal odometer* and its set of continuous eigenvalues is $\mathbb{Q}$.

2.2. *Toeplitz arrays and Toeplitz systems.* Let Σ be a finite alphabet and let $d \geq 1$. An element $x = (x(\bar{v}))_{\bar{v}\in\mathbb{Z}^d} \in \Sigma^{\mathbb{Z}^d}$ is called a $\mathbb{Z}^d$-*Toeplitz array* if, for every $\bar{v} \in \mathbb{Z}^d$, there exists a subgroup $Z \subseteq \mathbb{Z}^d$ isomorphic to $\mathbb{Z}^d$ such that

$$x(\bar{v} + \bar{z}) = x(\bar{v}) \quad \text{for all } \bar{z} \in Z.$$

A subgroup $Z \subseteq \mathbb{Z}^d$ isomorphic to $\mathbb{Z}^d$ for which the set

$$\mathrm{Per}(x, Z) = \{\bar{v} \in \mathbb{Z}^d : x(\bar{v}) = x(\bar{v} + \bar{z}), \forall \bar{z} \in Z\}$$

is not empty is called a *group of periods of* x. If, in addition, the subgroup Z is such that $\mathrm{Per}(x, Z) \subseteq \mathrm{Per}(x, Z')$ implies $Z' \subseteq Z$, we say that Z is a *group generated by essential periods*† of x. For every group of periods Z there exists a group generated by essential periods Z' such that $\mathrm{Per}(x, Z) \subseteq \mathrm{Per}(x, Z')$ [**3**]. This ensures the existence of a *period structure* of x, that is, a sequence $(Z_n)_{n\geq 0}$ of groups generated by essential periods of x such that, for every $n \geq 0$, the group Z_{n+1} is contained in Z_n and such that $\mathbb{Z}^d = \bigcup_{n\geq 0} \mathrm{Per}(x, Z_n)$ (see [**3**] for more details). When $d = 1$, if $Z_n = p_n\mathbb{Z}$ then we write p_n instead of Z_n.

The *shift action* of $\mathbb{Z}^d$ on $\Sigma^{\mathbb{Z}^d}$ is defined by

$$\bar{w} \cdot (x(\bar{v}))_{\bar{v}\in\mathbb{Z}^d} = (x(\bar{v} + \bar{w}))_{\bar{v}\in\mathbb{Z}^d},$$

and it is continuous if we consider Σ endowed with the discrete topology and $\Sigma^{\mathbb{Z}^d}$ with the product topology. A subset $X \subseteq \Sigma^{\mathbb{Z}^d}$ is said to be *invariant* if for all $\bar{w} \in \mathbb{Z}^d$

† When $d = 1$, the generating element of a such group is called an essential period [**14**]. We use the term *group generated by essential periods* to be coherent with the case $d = 1$.

one has $\bar{w} \cdot X = X$. A *subshift* is the topological dynamical system induced by the restriction of the shift action on a closed invariant subset of $\Sigma^{\mathbb{Z}^d}$. A *Toeplitz system* is a subshift $(X, \mathbb{Z}^d)$ such that X is the orbit closure of a Toeplitz array. In general, we consider Toeplitz systems which are generated by non-periodic Toeplitz arrays.

For every $d \geq 1$, the family of $\mathbb{Z}^d$-Toeplitz systems coincides with the family of minimal subshifts which are almost one-to-one extensions of d-dimensional odometers [**3, 7**]. Moreover, if x is a Toeplitz array in a $\mathbb{Z}^d$-Toeplitz system $(X, \mathbb{Z}^d)$, then the maximal equicontinuous factor of $(X, \mathbb{Z}^d)$ is the odometer defined by any period structure of x. When $d = 1$, the Toeplitz arrays are called *Toeplitz sequences* and the Toeplitz systems are called *Toeplitz flows*. A Toeplitz flow which is an almost one-to-one extension of a 2-odometer is called a *dyadic Toeplitz flow*.

2.3. *Minimal Cantor systems and Kakutani–Rohlin partitions.* Let (X, T) be a minimal Cantor system. A clopen *Kakutani–Rohlin (CKR) partition* of (X, T) is a partition $\mathcal{P}$ of X of the following type:

$$\mathcal{P} = \{T^j C_k : 0 \leq j < h_k,\ 1 \leq k \leq l\},$$

where l is a positive integer, $C_1, \ldots, C_l$ are clopen subsets of X and $h_1, \ldots, h_k$ are positive integers. The set $\{T^j C_k : 0 \leq j < h_k\}$ is called the *kth tower* of $\mathcal{P}$, and the integer h_k is called the *height* of the kth tower.

In [**11**] and [**13**], the authors show that for every minimal Cantor system (X, T) there exists a *nested sequence of CKR partitions*, i.e. a sequence of CKR partitions

$$(\mathcal{P}_n = \{T^j C_{n,k} : 0 \leq j < h_{n,k},\ 1 \leq k \leq k_n\})_{n \geq 0}$$

verifying:

(KR1) $C_{n+1} \subseteq C_n$, where $C_n = \bigcup_{k=1}^{k_n} C_{n,k}$;
(KR2) $\mathcal{P}_{n+1}$ is finer than $\mathcal{P}_n$;
(KR3) $\bigcap_{n \geq 0} C_n$ contains a unique point;
(KR4) the sequence of partitions spans the topology of X;
(KR5) for all $1 \leq k \leq k_n$ and $1 \leq k' \leq k_{n+1}$ there exists $0 \leq j < h_{n+1,k'}$ such that $T^j C_{n+1,k'} \subseteq C_{n,k}$.

The *incidence matrix* between $\mathcal{P}_n$ and $\mathcal{P}_{n+1}$ is the matrix $A_n \in \mathcal{M}_{k_n \times k_{n+1}}(\mathbb{N})$ defined by

$$A_n(i, j) = |\{0 \leq l < h_{n+1,j} : T^l(C_{n+1,j}) \subseteq C_{n,i}\}|,$$

for every $1 \leq i \leq k_n$, $1 \leq j \leq k_{n+1}$ and $n \geq 0$. We call $(A_n \in \mathcal{M}_{k_n \times k_{n+1}}(\mathbb{N}))_{n \geq 0}$ the *sequence of incidence matrices* associated to $(\mathcal{P}_n)_{n \geq 0}$.

3. *Kakutani–Rohlin partitions of Toeplitz flows*

Gjerde and Johansen in [**8**] show that the family of Toeplitz flows, in the one-dimensional case, coincides with the family of expansive minimal Cantor systems (X, T) which have a nested sequence of CKR partitions such that the heights of the towers belonging to a same level are equal. In order to show this result, given a Toeplitz sequence $x_0 \in \Sigma^{\mathbb{Z}}$ and its associated Toeplitz flow (X, T), they constructed a nested sequence of CKR partitions of X

$$(\mathcal{P}_n = \{T^j C_{n,k} : 0 \leq j < h_{n,k},\ 1 \leq k \leq k_n\})_{n \geq 0},$$

in the following way:

Let $(q_n)_{n\geq 1}$ be a period structure for x and let $p_{0,n}$ be an essential period of x such that $x[0, n] = x[kp_{0,n}, n + kp_{0,n}]$ for every $k \in \mathbb{Z}$ and $n \geq 0$. Let $(h_n)_{n\geq 1}$ be a new period structure of x defined as follows: h_1 is the least common multiple of $p_{0,0}$ and q_1. For $n > 0$, h_{n+1} is the least common multiple of p_{0,h_n-1} and q_{n+1}. Then the sequence of CKR partitions is given by $k_0 = 1$, $C_{0,1} = X$, $h_{0,1} = h_0 = 1$, and, for $n \geq 1$,

$$h_{n,k} = h_n \quad \text{and} \quad C_{n,k} = \{x \in C_n : x([0, h_n - 1]) = w_{n,k}\} \quad \text{for all } 1 \leq k \leq k_n,$$

where C_n is the closure of $\{T^{mh_n} x_0 : m \in \mathbb{Z}\}$ and $W_n = \{w_{n,1}, \ldots, w_{n,k_n}\}$ is the subset of words in x_0 of length h_n beginning with $x_0([0, h_{n-1} - 1]) = w_{n-1,1}$.

In this section we will focus our attention on the sequences of incidence matrices associated to this kind of nested sequences of CKR partitions. In the next lemma we will show that it is possible to make some little modifications to $(\mathcal{P}_n)_{n\geq 0}$ in order to obtain a new nested sequence of CKR partitions whose associated sequence of incidence matrices allows one to construct a $\mathbb{Z}^d$-Toeplitz system, with $d > 1$, whose set of invariant probability measures is affine homeomorphic to $\mathcal{M}_{\mathbb{Z}}(X)$. First, we introduce some definitions.

If $a_1, \ldots, a_n$ are n non-negative integers we set

$$\Lambda(a_1, \ldots, a_n) = \frac{(a_1 + \cdots + a_n)!}{a_1! \ldots a_n!}.$$

In other words, $\Lambda(a_1, \ldots, a_n)$ is the multinomial coefficient associated to $a_1, \ldots, a_n$ (it represents the number of different ways to choose $a_1, a_2, \ldots, a_n$ elements in a set with cardinal $\sum_{i=1}^n a_i$).

If A is a matrix in $\mathcal{M}_{s\times l}$, $N(A, k)$ denotes the number of different columns of A which are equal to its kth column, i.e.

$$N(A, k) = |\{1 \leq j \leq l : A(\cdot, j) = A(\cdot, k)\}|,$$

for all $1 \leq k \leq l$.

LEMMA 2. *Let (X, T) be a one-dimensional Toeplitz flow. There exists a nested sequence of nested CKR partitions of X*

$$(\mathcal{P}_n = \{T^j C_{n,k} : 0 \leq j < h_n,\ 1 \leq k \leq k_n\})_{n \geq 0},$$

such that $(h_n)_{n\geq 1}$ is a period structure of x and whose sequence of incidence matrices $(A_n \in \mathcal{M}_{k_n \times k_{n+1}}(\mathbb{Z}^+))_{n\geq 0}$ satisfies, for all $n \geq 0$,

(1) $k_{n+1} \geq 3$;

(2) *A_n is strictly positive;*

(3) $\sum_{i=1}^{k_n} A_n(i, k) = h_{n+1}/h_n > 1$, *for each* $1 \leq k \leq k_{n+1}$;

(4) *for every* $1 \leq k \leq k_{n+1}$,

$$N(A_n, k) \leq \Lambda(A_n(1, k) - 1, A_n(2, k), \ldots, A_n(k_n, k)).$$

Proof. Let $x \in X$ be a Toeplitz sequence and let

$$(\mathcal{P}_n = \{T^j C_{n,k} : 0 \le j < h_n,\ 1 \le k \le k_n\})_{n\ge 0}$$

be a nested sequence of CKR partitions of (X, T) constructed as in the beginning of this section. Let $(q_n)_{n\ge 1}$ and $(p_{0,n})_{n\ge 0}$ be the period structure and the sequence of essential periods used to define the sequence of CKR partitions.

Since x is not periodic, $k_n \ge 2$ for all $n > 0$. This implies that, for every $p \ge 0$, there exists $l_p > 0$ such that the number of words in x_0 of length l_p beginning with $x_0([0, p])$ is greater than or equal to three. In fact, let $m > 0$ be such that $h_{m-1} > p$. If we take $l_p = h_{m+1}$ then $w_{m+1,1}$ and $w_{m+1,2}$ belong to the set of words in x_0 of length l_p beginning with $x_0([0, h_{m-1} - 1]) = w_{m-1,1}$. Since $w_{m+1,1}$ and $w_{m+1,2}$ begin with $w_{m,1}$, and since there must be at least one word in such a set beginning with $w_{m,2}$, we deduce there are at least three words in x_0 of length l_p (or $l \ge l_p$) beginning with $w_{m-1,1}$ (and then with $x([0, p])$). Using this property, we choose a suitable subsequence $(q_{m_n})_{n\ge 1}$ to redefine the sequence $(\mathcal{P}_n)_{n\ge 0}$ in order that $k_n \ge 3$: we take $m_1 > 0$ such that the number of words in x_0 of length h'_1 beginning with $w_{0,1}$, where h'_1 is the least common multiple of $p_{0,0}$ and q_{m_1}, is greater than or equal to three. For $n \ge 1$, we take $m_{n+1} > m_n$ such that the number of words in x_0 of length h'_{n+1} beginning with $x([0, h'_n - 1])$, where h'_{n+1} is the least common multiple of p_{0,h'_n-1} and $q_{m_{n+1}}$, is greater than or equal to three. This choice ensures that the sequence $(\mathcal{P}_n)_{n\ge 0}$, defined with respect to $(q_{m_n})_{n\ge 0}$ as in the beginning of this section, satisfies $k_n \ge 3$ for every $n \ge 1$.

By minimality of (X, T), we can choose a subsequence of $(\mathcal{P}_n)_{n>0}$ such that its incidence matrices are strictly positive. Thus we get $k_{n+1} \ge 3$ and $A_n > 0$ for all $n \ge 0$.

Let A_n be the incidence matrix between $\mathcal{P}_n$ and $\mathcal{P}_{n+1}$. Since $h_{n,k} = h_n$ for every $1 \le k \le k_n$, the sum of all the entries of a column of A_n does not depend on the column. More precisely, for all $1 \le k \le k_{n+1}$ one has

$$\sum_{i=1}^{k_n} A_n(i, k) = \frac{h_{n+1}}{h_n} = d_n.$$

Since the incidence matrices are strictly positive and $k_n \ge 3$ for every $n > 0$, this implies that $d_n > 1$ for every $n \ge 0$.

Let $1 \le k \le k_{n+1}$. Every word in W_{n+1} is a concatenation of d_n words in W_n, where the first word in this concatenation is always $w_{n,1}$. Since $w_{n+1,j} \ne w_{n+1,k}$ for all $j \ne k$, it holds that $N(A_n, k)$ is smaller than or equal to the number of different possible $d_n - 1$ concatenations using exactly $A_n(i, k)$ copies of $w_{n,i}$, for each $1 \le i \le k_n$. This means that

$$\Lambda(A_n(1, k) - 1, A_n(2, k), \ldots, A_n(k_n, k)) \ge N(A_n, k). \qquad \square$$

From a nested sequence of CKR partitions of a Toeplitz flow one can determine its set of invariant probability measures and its maximal equicontinuous factor. Let (X, T) be a one-dimensional Toeplitz flow. If $(\mathcal{P}_n)_{n\ge 0}$ is a nested sequence of CKR partitions satisfying the conditions of Lemma 2, and h_n is the height of the towers in $\mathcal{P}_n$, then the maximal equicontinuous factor of (X, T) is the odometer

$$G = \left\{(g_n)_{n>0} \in \prod_{n>0} \mathbb{Z}/h_n\mathbb{Z} : \pi_n(g_{n+1}) = g_n,\ \forall n > 0\right\},$$

because $(h_n)_{n>0}$ is a period structure of x [**14**]. The set of invariant probability measures $\mathcal{M}_{\mathbb{Z}}(X)$ is affine-homeomorphic to

$$\left\{(\bar{v}_n)_{n\geq 0} \in \prod_{n\geq 0} \triangle_n : A_n \bar{v}_{n+1} = \bar{v}_n, \ \forall n \geq 0\right\},$$

where

$$\triangle_n = \left\{\bar{v} \in (\mathbb{R}^+)^{k_n} : \sum_{k=1}^{k_n} v_k = \frac{1}{h_n}\right\},$$

and $(A_n \in \mathcal{M}_{k_n \times k_{n+1}}(\mathbb{N}))_{n\geq 0}$ is the sequence of incidence matrices associated to $(\mathcal{P}_n)_{n\geq 0}$ [**10**].

Remark 3. This representation of $\mathcal{M}_{\mathbb{Z}}(X)$ can be deduced from Proposition 3.2 of [**10**], using the projective limit representation of (X, T) obtained from $(\mathcal{P}_n)_{n\geq 0}$.

Remark 4. In the rest of this paper we will use the sequence of nested CKR partitions $(\mathcal{P}_n)_{n\geq 1}$ instead of $(\mathcal{P}_n)_{n\geq 0}$, in order to skip the trivial partition $\mathcal{P}_0$. This is equivalent to taking $(\mathcal{P}')_{n\geq 0}$, where $\mathcal{P}'_n$ corresponds to the partition $\mathcal{P}_{n+1}$ in the original sequence.

4. *A Choquet simplex from a sequence of matrices*

In this section we give a special description of the set of invariant probability measures and the maximal equicontinuous factor of a one-dimensional Toeplitz flow (X, T). We express $\mathcal{M}_{\mathbb{Z}}(X)$ and the corresponding odometer, using a sequence $(d_n)_{n\geq 0}$ of positive integers. Furthermore, this sequence will be necessary to construct a $\mathbb{Z}^d$-Toeplitz system whose set of invariant probability measures is affine-homeomorphic to $\mathcal{M}_{\mathbb{Z}}(X)$ and such that its maximal equicontinuous factor is the $\mathbb{Z}^d$-odometer $G \times \cdots \times G$.

PROPOSITION 5. *Let $d > 1$. Let (X, T) be a one-dimensional Toeplitz flow and let G be the odometer which is its maximal equicontinuous factor. Let*

$$(\mathcal{P}_n = \{T^j C_{n,k} : 0 \leq j < h_n, \ 1 \leq k \leq k_n\})_{n\geq 0}$$

be a sequence of nested CKR partitions of X verifying conditions of Lemma 2. If, in addition, the sequence of partitions satisfies

for every $m, n \geq 0$ there exists $l \geq 0$ such that $h_n h_m$ divides h_l,

then there exist a sequence of positive integers $(d_n)_{n\geq 0}$ and a sequence of positive integer matrices $(A_n \in \mathcal{M}_{k_n \times k_{n+1}}(\mathbb{N}))_{n\geq 0}$ satisfying for all $n \geq 0$:

(C1) $k_n \geq 3$*;*

(C2) *for every* $1 \leq j \leq k_{n+1}$,

$$\sum_{i=1}^{k_n} A_n(i, j) = \left(\frac{d_{n+1}}{d_n}\right)^d \quad \textit{with } \frac{d_{n+1}}{d_n} \geq 6;$$

(C3) *for every* $1 \leq i \leq k_n$ *and* $1 \leq j \leq k_{n+1}$,

$$A_n(i, j) \geq \left(\frac{d_{n+1}}{d_n}\right)^d - \left(\frac{d_{n+1}}{d_n} - 2\right)^d;$$

(C4) *for every* $1 \leq k \leq k_{n+1}$,

$$N(A_n, k) \leq \Lambda(A_n(1, k) - 1, A_n(2, k), \ldots, A_n(k_n - 1, k)),$$

such that $\mathcal{M}_{\mathbb{Z}}(X)$ *is affine homeomorphic to*

$$\left\{(x_n)_{n\geq 0} \in \prod_{n\geq 0}\left\{\bar{v} \in (\mathbb{R}^+)^{k_n} : \sum_{i=1}^{k_n} v_i = \frac{1}{d_n^d}\right\} : A_n x_{n+1} = x_n, \forall n \geq 0\right\},$$

and G is conjugate to the odometer defined by

$$\left\{(g_n)_{n\geq 0} \in \prod_{n\geq 0} \mathbb{Z}/d_n\mathbb{Z} : \pi_n(g_{n+1}) = g_n,\ \forall n \geq 0\right\}.$$

To prove this proposition we need the following technical lemma.

LEMMA 6. *Let* $a_1, \ldots, a_n$ *be n positive integers and let* $a = \sum_{i=1}^n a_i$. *If* $n \geq 3$ *and* $c > 2a$ *then*

$$\Lambda(ca_1, \ldots, ca_{n-1}) \geq \Lambda(a_1, \ldots, a_n).$$

Proof. It is well known that for all pairs of non-negative integers a and b it holds that

$$\Lambda(a', a - a')\Lambda(b', b - b') \leq \Lambda(a' + b', a + b - (a' + b')), \tag{4.1}$$

for every $0 \leq a' \leq a$ and $0 \leq b' \leq b$. Furthermore, if $a_1, \ldots, a_n$ are n non-negative integers such that $\sum_{i=1}^n a_i = a$, one has

$$\Lambda(a_1, \ldots, a_n) = \prod_{i=1}^{n} \Lambda\left(a_i, a - \sum_{j=1}^{i} a_j\right). \tag{4.2}$$

Suppose that $a_1, \ldots, a_n$ are $n \geq 3$ positive integers and $c \in \mathbb{N}$ is such that $c > 2a$, where $a = \sum_{i=1}^n a_i$.

Since $c > 2a \geq 2$, from (4.1) and (4.2) one gets

$$\Lambda(ca_1, \ldots, ca_{n-1}) \geq \Lambda(a_1, \ldots, a_n)\frac{\Lambda((c-1)a_1, \ldots, (c-1)a_{n-1})}{\Lambda(a_n, a - a_n)}. \tag{4.3}$$

Equation (4.2) implies

$$\begin{aligned}\Lambda((c-1)a_1, \ldots, (c-1)a_{n-1}) &= \prod_{i=1}^{n-1} \Lambda\left((c-1)a_i, (c-1)(a - a_n) - \sum_{j=1}^{i}(c-1)a_j\right)\\ &= \Lambda((c-1)a_1, (c-1)(a - a_n - a_1))\\ &\quad\times \prod_{i=2}^{n-1} \Lambda\left((c-1)a_i, (c-1)(a - a_n) - \sum_{j=1}^{i}(c-1)a_j\right).\end{aligned}$$

Because $\prod_{i=2}^{n-1} \Lambda((c-1)a_i, (c-1)(a - a_n) - \sum_{j=1}^{i}(c-1)a_j) \geq 1$, we get

$$\Lambda((c-1)a_1, \ldots, (c-1)a_{n-1}) \geq \Lambda((c-1)a_1, (c-1)(a - a_1 - a_n)). \tag{4.4}$$

By setting $a' = a_n$, $b' = (c-1)a_1 - a_n$ and $b = (c-1)(a - a_n) - a$ in Equation (4.1) (since $c > 2a$, we have $0 \leq b' \leq b$) one gets

$$\Lambda((c-1)a_1, (c-1)(a - a_1 - a_n)) \geq \Lambda(a_n, a - a_n). \tag{4.5}$$

Finally, by (4.3)–(4.5) we deduce the result. □

Proof of Proposition 5. Let $(B_n \in \mathcal{M}_{k_n \times k_{n+1}}(\mathbb{N}))_{n\geq 0}$ be the sequence of incidence matrices associated to $(\mathcal{P}_n)_{n\geq 0}$. Since this sequence satisfies conditions of Lemma 2, for every $n \geq 0$ one has:

(1) $k_n \geq 3$;

(2) B_n is strictly positive;

(3) $\sum_{i=1}^{k_n} B_n(i,k) = h_{n+1}/h_n > 1$, for each $1 \leq k \leq k_{n+1}$;

(4) for every $1 \leq k \leq k_{n+1}$,

$$N(B_n, k) \leq \Lambda(B_n(1,k) - 1, B_n(2,k), \ldots, B_n(k_n, k)).$$

We set $q_n = h_{n+1}/h_n$ for all $n \geq 0$. We choose the sequence $(d_n)_{n\geq 0}$ as a subsequence of $(h_n)_{n\geq 0}$ defined as follows.

For $n = 0$ we choose $n_0 \geq 0$ and we put $d_0 = h_{n_0}$. For $n > 0$, we set $d_n = h_{m_n}$, where h_{m_n} is such that $h_n d_{n-1}$ divides h_{m_n}, such that $h_{m_n}/d_{n-1} \geq 6$ and such that $l_{n-1} = (h_{n-1}/h_n)(h_{m_n}/d_{n-1})$ satisfies the following:

$$q_{n-1}^d - \left(q_{n-1} - \frac{2}{l_{n-1}}\right)^d \leq q_{n-1}^{d-1}.$$

Since $(d_n)_{n\geq 0}$ is a subsequence of $(h_n)_{n\geq 0}$, they determine the same odometer [**5**]. Thus, we get

$$G = \left\{(g_n)_{n\geq 0} \in \prod_{n\geq 0} \mathbb{Z}/d_n\mathbb{Z} : \pi_n(g_{n+1}) = g_n,\ \forall n \geq 0\right\}.$$

Let $n \geq 0$. Consider the diagonal matrix $M_n \in \mathcal{M}_{k_n \times k_n}(\mathbb{Q})$ given by

$$M_n(i,i) = m_n \quad \text{for all } 1 \leq i \leq k_n,$$

where

$$m_0 = \frac{h_0}{d_0^d} \quad \text{and} \quad m_{n+1} = \frac{m_n q_n}{(l_n q_n)^d}.$$

Let $A_n = M_n B_n M_{n+1}^{-1}$. We have

$$A_n(i,j) = \frac{m_n}{m_{n+1}} B_n(i,j) = l_n^d q_n^{d-1} B_n(i,j),$$

for every $1 \leq i \leq k_n$ and $1 \leq j \leq k_{n+1}$. We will show that A_n satisfies (C1), (C2), (C3) and (C4).

Since B_n is in $\mathcal{M}_{k_n \times k_{n+1}}(\mathbb{N})$, with $k_n \geq 3$, and $m_n/m_{n+1} = l_n^d q_n^{d-1}$ is a positive integer, the matrix A_n is in $\mathcal{M}_{k_n \times k_{n+1}}(\mathbb{N})$. This proves that A_n is a positive integer matrix that verifies (C1). Condition (C2) is also verified owing to

$$\sum_{i=1}^{k_n} A_n(i,k) = (l_n q_n)^d = \left(\frac{d_{n+1}}{d_n}\right)^d > 1,$$

for all $1 \leq k \leq k_{n+1}$.

Since the entries of B_n are greater than or equal to 1, for all $1 \leq i \leq k_n$ and $1 \leq j \leq k_{n+1}$,

$$A_n(i,k) = \frac{m_n}{m_{n+1}} B_n(i,j) \geq \frac{m_n}{m_{n+1}} = l_n^d q_n^{d-1}.$$

Thus,

$$\frac{A_n(i,k)}{(l_n q_n)^d - (l_n q_n - 2)^d} \geq \frac{(l_n)^d q_n^{d-1}}{(l_n q_n)^d - (l_n q_n - 2)^d} = \frac{q_n^{d-1}}{q_n^d - (q_n - (2/l_n))^d} \geq 1,$$

which shows that A_n satisfies (C3).

In order to show that A_n verifies (C4), observe that

$$A_n(\cdot, k) = A_n(\cdot, j) \quad \text{if and only if } B_n(\cdot, k) = B_n(\cdot, j),$$

i.e., the kth and jth columns of A_n coincide if and only if the kth and jth columns of B_n coincide. Hence, to get (C4) it is sufficient to prove that

$$\Lambda(A_n(1,k)-1, A_n(2,k), \ldots, A_n(k_n-1,k)) \geq \Lambda(B_n(1,k)-1, B_n(2,k), \ldots, B_n(k_n,k)).$$

This inequality can be expressed as

$$\Lambda(ca_1 - 1, ca_2, \ldots, ca_{k_n-1}) \geq \Lambda(a_1 - 1, a_2, \ldots, a_{k_n})$$

where $c = l_n^d q_n^{d-1}$, $a = q_n$ and $a_i = B_n(i,k)$ for all $1 \leq i \leq k_n$. Since $c > 2a$ and $k_n \geq 3$, by Lemma 6 we get $\Lambda(ca_1, ca_2, \ldots, ca_{k_n-1}) \geq \Lambda(a_1, a_2, \ldots, a_{k_n})$. On the other hand, we have

$$\Lambda(ca_1 - 1, ca_2, \ldots, ca_{k_n-1}) = \frac{a_1}{a - a_{k_n}} \Lambda(ca_1, ca_2, \ldots, ca_{k_n-1}),$$

and

$$\Lambda(a_1 - 1, a_2, \ldots, a_{k_n}) = \frac{a_1}{a} \Lambda(a_1, a_2, \ldots, a_{k_n}).$$

Because $a_1/(a - a_{k_n}) \geq a_1/a$, we obtain the desired. Therefore, (C4) is verified.

To conclude the proof of this proposition consider the set $\triangle_{n,i}$, $i = 1, 2$, consisting of the $\bar{x} \in (\mathbb{R}^+)^{k_n}$ such that

$$\sum_{k=1}^{k_n} x_k = \begin{cases} \dfrac{1}{h_n} & \text{if } i = 1, \\ \dfrac{1}{d_n^d} & \text{if } i = 2, \end{cases}$$

The map $M_n : \triangle_{n,1} \to \triangle_{n,2}$ is well defined. In fact, if $\bar{x} \in \triangle_{n,1}$ then the addition of the coordinates of $M_n \bar{x}$ is equal to m_n/h_n. In the case $n = 0$ we have $m_0/h_0 = 1/d_0^d$. So, $M_0 : \triangle_{0,1} \to \triangle_{0,2}$ is well defined. We suppose that $M_n : \triangle_{n,1} \to \triangle_{n,2}$ is well defined in order to prove that $M_{n+1} : \triangle_{n+1,1} \to \triangle_{n+1,2}$ is also well defined. We have

$$\frac{m_{n+1}}{h_{n+1}} = \frac{m_n q_n}{(l_n q_n)^d h_n q_n} = \frac{m_n}{h_n} \frac{1}{(l_n q_n)^d} = \frac{1}{d_n^d} \frac{1}{(d_{n+1}/d_n)^d} = \frac{1}{d_{n+1}^d}.$$

This shows that $M_n : \triangle_{n,1} \to \triangle_{n,2}$ is well defined for all $n \geq 0$. Moreover, the following diagram commutes:

$$\begin{array}{ccccccc}
\triangle_{0,1} & \xleftarrow{B_0} & \triangle_{1,1} & \xleftarrow{B_1} & \triangle_{2,1} & \xleftarrow{B_2} & \cdots \\
\downarrow{\scriptstyle M_0} & & \downarrow{\scriptstyle M_1} & & \downarrow{\scriptstyle M_2} & & \\
\triangle_{0,2} & \xleftarrow{A_0} & \triangle_{1,2} & \xleftarrow{A_1} & \triangle_{2,2} & \xleftarrow{A_2} & \cdots
\end{array}$$

which implies that

$$\lim_{\leftarrow n}(B_n, \triangle_{n,1}) = \left\{(x_n)_{n\geq 0} \in \prod_{n\geq 0} \triangle_{n,1} : B_n x_{n+1} = x_n,\ \forall n \geq 0\right\},$$

and

$$\lim_{\leftarrow n}(A_n, \triangle_{n,2}) = \left\{(x_n)_{n\geq 0} \in \prod_{n\geq 0} \triangle_{n,2} : A_n x_{n+1} = x_n, \forall n \geq 0\right\},$$

are affine homeomorphic. Since $\lim_{\leftarrow n}(B_n, \triangle_{n,1})$ is affine homeomorphic to $\mathcal{M}_{\mathbb{Z}}(X)$ we conclude the proof. □

5. *Construction of the $\mathbb{Z}^d$ Toeplitz system*

In this section we show that any Choquet simplex can be realized as the set of invariant probability measures of a $\mathbb{Z}^d$-Toeplitz system (Corollaries 9 and 10). We deduce this result using those given in [**4**] and [**8**], and using Theorem 7. In this theorem we prove that for every $d > 1$ there exists a $\mathbb{Z}^d$-Toeplitz system $(X, \mathbb{Z}^d)$ such that $\mathcal{M}_{\mathbb{Z}^d}(X)$ is affine homeomorphic to $\mathcal{M}_{\mathbb{Z}}(Y)$, where (Y, T) is a Toeplitz flow which admits a nested sequence of CKR partitions satisfying the conditions of Proposition 5.

THEOREM 7. *Let (Y, T) be a one-dimensional Toeplitz flow and let G be the odometer which is its maximal equicontinuous factor. Let*

$$(\mathcal{P}_n = \{T^j C_{n,k} : 0 \leq j < h_n,\ 1 \leq k \leq k_n\})_{n\geq 0}$$

be a sequence of nested CKR partitions of Y verifying conditions of Lemma 2. If, in addition, the sequence of partitions satisfies

for every $m, n \geq 0$ there exists $l \geq 0$ such that $h_n h_m$ divides h_l,

then for every $d > 1$ there exists a $\mathbb{Z}^d$-Toeplitz system $(X, \mathbb{Z}^d)$ such that $\mathcal{M}_{\mathbb{Z}^d}(X)$ is affine homeomorphic to $\mathcal{M}_{\mathbb{Z}}(Y)$ and such that the maximal equicontinuous factor of $(X, \mathbb{Z}^d)$ is the $\mathbb{Z}^d$-odometer $G \times \cdots \times G$.

The idea of the proof is as follows.

Let (Y, T) be a one-dimensional Toeplitz flow which satisfies the conditions of Proposition 7. From Proposition 5, there exists a sequence of positive integers $(d_n)_{n\geq 0}$ and a sequence of positive integer matrices $(A_n \in \mathcal{M}_{k_n \times k_{n+1}}(\mathbb{N}))_{n\geq 0}$ satisfying for all $n \geq 0$:

(C1) $k_n \geq 3$;

(C2) for every $1 \leq j \leq k_{n+1}$,

$$\sum_{i=1}^{k_n} A_n(i, j) = \left(\frac{d_{n+1}}{d_n}\right)^d \quad \text{with } \frac{d_{n+1}}{d_n} \geq 6;$$

(C3) for every $1 \leq i \leq k_n$ and $1 \leq j \leq k_{n+1}$,

$$A_n(i, j) \geq \left(\frac{d_{n+1}}{d_n}\right)^d - \left(\frac{d_{n+1}}{d_n} - 2\right)^d;$$

(C4) for every $1 \leq k \leq k_{n+1}$,

$$N(A_n, k) \leq \Lambda(A_n(1,k) - 1, A_n(2,k), \ldots, A_n(k_n - 1, k)),$$

such that $\mathcal{M}_{\mathbb{Z}}(Y)$ is affine homeomorphic to $\lim_{\leftarrow n}(A_n, \triangle_n)$, where

$$\triangle_n = \left\{ \bar{v} \in (\mathbb{R}^+)^{k_n} : \sum_{i=1}^{k_n} v_i = \frac{1}{d_n^d} \right\},$$

and the maximal equicontinuous factor G of (Y, T) is the odometer defined by

$$\left\{ (g_n)_{n\geq 0} \in \prod_{n\geq 0} \mathbb{Z}/d_n\mathbb{Z} : \pi_n(g_{n+1}) = g_n, \ \forall n \geq 0 \right\}.$$

We will use the sequences $(A_n)_{n\geq 0}$ and $(d_n)_{n\geq 0}$ to produce a $\mathbb{Z}^d$-Toeplitz system $(X, \mathbb{Z}^d)$ which admits a special nested sequence of clopen partitions $(\mathcal{F}_n)_{n\geq 0}$ of X. Every $\mathcal{F}_n$ will be a finite collection of clopen subsets of X

$$\mathcal{F}_n = \{\bar{v} \cdot F_{n,k} : \bar{v} \in D_n, \ 1 \leq k \leq k_n\},$$

where the set D_n will be a rectangle in $\mathbb{Z}^d$ centered in 0 with sides of length d_n, and the sets $F_{n,k}$ will be cylinders in X which fix the coordinates corresponding to D_n. That is,

$$F_{n,k} = \{x \in X : x(D_n) = B_{n,k}\},$$

for some $B_{n,k} \in \{0,1\}^{D_n}$, where $x(D_n)$ is the notation for $\{x(\bar{v}) : \bar{v} \in D_n\} \in \{0,1\}^{D_n}$. The choice of the blocks $B_{n,k}$ (where a block is an element in $\{0,1\}^D$, for any rectangle D in $\mathbb{Z}^d$) will ensure that $(\mathcal{F}_n)_{n\geq 0}$ is a nested sequence of clopen partitions of X spanning its topology such that the incidence matrix between $\mathcal{F}_n$ and $\mathcal{F}_{n+1}$ is A_n. Owing to these characteristics of $(\mathcal{P}_n)_{n\geq 0}$, we will be able to conclude that $\mathcal{M}_{\mathbb{Z}^d}(X)$ is affine homeomorphic to $\mathcal{M}_{\mathbb{Z}}(Y)$ and that the maximal equicontinuous factor of $(X, \mathbb{Z}^d)$ is $G \times \cdots \times G$ (Lemma 8).

Before giving the proof of Theorem 7, we will do the following: first, we will define the sequence $(D_n)_{n\geq 0}$ of rectangles of $\mathbb{Z}^d$. Next, we will give some sufficient conditions on the sequence of blocks $(B_{n,1}, \ldots, B_{n,k_n})_{n\geq 0}$ to construct a $\mathbb{Z}^d$-Toeplitz array whose associated $\mathbb{Z}^d$-Toeplitz system $(X, \mathbb{Z}^d)$ admits the sequence $(\mathcal{F}_n)_{n\geq 0}$ as a clopen covering. After that, in Lemma 8, we will impose an additional condition to $(B_{n,1}, \ldots, B_{n,k_n})_{n\geq 0}$ in order that the atoms of every $\mathcal{F}_n$ are disjoint. This will be sufficient to show that $(\mathcal{F}_n)_{n\geq 0}$ is a sequence of partitions of X spanning its topology, from which we will deduce that $\mathcal{M}_{\mathbb{Z}^d}(X)$ is affine-homeomorphic to $\mathcal{M}_{\mathbb{Z}}(Y)$ and that the maximal equicontinuous factor of $(X, \mathbb{Z}^d)$ is $G \times \cdots \times G$. In the proof of Theorem 7, we will construct an explicit sequence of blocks $(B_{n,1}, \ldots, B_{n,k_n})_{n\geq 0}$ satisfying all the necessary conditions that we will have introduced before.

We define $(D_n)_{n\geq 0}$ as an increasing sequence of rectangles in $\mathbb{Z}^d$ as follows. We set

$$D_0 = \{-a, \ldots, b\}^d,$$

where $a = d_0 - b - 1$ and

$$b = \begin{cases} \dfrac{d_0}{2} & \text{if } d_0 \text{ is even,} \\[2ex] \dfrac{d_0 - 1}{2} & \text{if } d_0 \text{ is odd.} \end{cases}$$

For $n \geq 0$ we put $q_n = d_{n+1}/d_n$, $l_n + r_n = q_n - 1$, where

$$r_n = \begin{cases} \dfrac{q_n}{2} & \text{if } q_n \text{ is even,} \\ \dfrac{q_n - 1}{2} & \text{if } q_n \text{ is odd.} \end{cases}$$

Since $q_n \geq 6$, we have $r_n, l_n \geq 2$.

In order to define the rectangle D_{n+1}, for $n \geq 0$, consider the sets

$$S_n = \{d_n \bar{v} : \bar{v} \in \mathbb{Z}^d \text{ such that } -l_n \leq v_i \leq r_n, \text{ for all } 1 \leq i \leq d\},$$

and

$$\partial S_n = \{d_n \bar{v} \in S_n : v_i \in \{r_n, -l_n\}, \text{ for some } 1 \leq i \leq d\}.$$

We define the set D_{n+1} as the disjoint union of the sets $D_{n,d_n\bar{v}}$, for $d_n \bar{v} \in S_n$, where $D_{n,\bar{z}}$ denotes the translated set $D_n + \bar{z}$ for all $\bar{z} \in \mathbb{Z}^d$.

Next, for all $n \geq 0$ we will choose the k_n different blocks $B_{n,1}, \ldots, B_{n,k_n}$ in $\{0,1\}^{D_n}$ such that they satisfy the following properties:

(i) $B_{n+1,k}(D_{n,\bar{v}}) \in \{B_{n,1}, \ldots, B_{n,k_n}\}$, for every $\bar{v} \in S_n$, for every $1 \leq k \leq k_{n+1}$;

(ii) $B_{n+1,k}(D_{n,0}) = B_{n,1}$, for every $1 \leq k \leq k_{n+1}$.

Once such a sequence of blocks is defined a $\mathbb{Z}^d$-Toeplitz array $x_0 \in \{0,1\}^{\mathbb{Z}^d}$ can be constructed directly such that $(\mathcal{F}_n)_{n\geq 0}$ is a covering of its orbit closure.

Consider, for all $n \geq 0$, the non-empty clopen set

$$I_n = \{x \in \{0,1\}^{\mathbb{Z}^d} : x(D_n) = B_{n,1}\}.$$

On the one hand, from (ii) we get $I_{n+1} \subseteq I_n$, which implies that $\bigcap_{n\geq 0} I_n \neq \emptyset$. On the other hand, the sequence $(D_n)_{n\geq 0}$ was chosen such that $\bigcup_{n\geq 0} D_n = \mathbb{Z}^d$, which allows one to conclude that there is at most one element x_0 in $\bigcap_{n\geq 0} I_n$. To check that x_0 is a Toeplitz array, notice that (i) implies that for every $\bar{v} \in d_n\mathbb{Z}^d$ there exists $1 \leq k \leq k_n$ such that $x_0(D_{n,\bar{v}}) = B_{n,k}$. Therefore, by (ii), $x_0(D_{n,\bar{v}}) = B_{n,1}$ for all $\bar{v} \in d_{n+1}\mathbb{Z}^d$. This means that $D_n \subseteq \mathrm{Per}(x_0, d_{n+1}\mathbb{Z}^d)$ for all $n \geq 0$, which proves that x_0 is a $\mathbb{Z}^d$-Toeplitz array. Recall that we have defined, for $n \geq 0$,

$$\mathcal{F}_n = \{\bar{v} \cdot F_{n,k} : \bar{v} \in D_n, \ 1 \leq k \leq k_n\},$$

where

$$F_{n,k} = \{x \in X : x(D_n) = B_{n,k}\},$$

for all $1 \leq k \leq k_n$. Since $x_0(D_n) = B_{n,1}$ for every $n \geq 0$, from condition (i) we get that the orbit of x_0 is included in $\bigcup_{k=1}^{k_n} \bigcup_{\bar{v}\in D_n} \bar{v} \cdot F_{n,k}$. Because this is a finite union of closed sets, it is closed. So, the orbit closure of x_0 is also included in $\bigcup_{k=1}^{k_n} \bigcup_{\bar{v}\in D_n} \bar{v} \cdot F_{n,k}$. This shows that $\mathcal{F}_n$ is a clopen covering of X, the orbit closure of x_0.

The following lemma sets down sufficient conditions on the sequence of blocks $(B_{n,1}, \ldots, B_{n,k_n})_{n\geq 0}$ in order that $(\mathcal{F}_n)_{n\geq 0}$ is a sequences of partitions spanning the topology of X. This will imply that $\mathcal{M}_{\mathbb{Z}^d}(X)$ is affine-homeomorphic to $\mathcal{M}_{\mathbb{Z}}(X)$ and that the maximal equicontinuous factor of $(X, \mathbb{Z}^d)$ is G^d.

LEMMA 8. *If, in addition to (i) and (ii), for every $n \geq 0$ the blocks $B_{n+1,1}, \ldots, B_{n+1,k_n}$ satisfy*

(iii) $B_{n+1,k}(D_{n,\bar{v}}) = B_{n+1,k'}(D_{n,\bar{v}})$ *for every* $\bar{v} \in \partial S_n$, *for every* $1 \leq k, k' \leq k_{n+1}$,

(iv) $|\{\bar{v} \in S_n : B_{n+1,k}(D_{n,\bar{v}}) = B_{n,i}\}| = A_n(i,k)$ *for all* $1 \leq i \leq k_n$,

(v) *If* $\bar{w} \in D_{n+1}$ *is such that for some* $1 \leq k, k' \leq k_{n+1}$, $B_{n+1,k}(\bar{v} + \bar{w}) = B_{n+1,k'}(\bar{v})$ *for all* $\bar{v} \in D_{n+1}$ *such that* $\bar{v} + \bar{w} \in D_{n+1}$, *then* $\bar{w} = 0$,

then the covering sets $\mathcal{F}_n$ *are partitions spanning the topology of* X. *This implies that* $\mathcal{M}_{\mathbb{Z}^d}(X)$ *is affine-homeomorphic to* $\mathcal{M}_{\mathbb{Z}}(Y)$ *and that the maximal equicontinuous factor of* $(X, \mathbb{Z}^d)$ *is the* d*-dimensional odometer* $G \times \cdots \times G$.

Proof. Let $n > 0$. Since $\mathcal{F}_n$ is a covering of X, to conclude that it is a partition it remains to prove that its atoms are disjoint. A way to show this is by proving that the set of return times of x_0 to $F_n = \bigcup_{k=1}^{k_n} F_{n,k}$ is equal to $d_n\mathbb{Z}^d$. In fact, if $\mathcal{R}_{F_n}(x_0) = d_n\mathbb{Z}^d$ then $\mathcal{R}_{F_n}(x) = d_n\mathbb{Z}^d$ for all $x \in F_n$, because $(X, \mathbb{Z}^d)$ is minimal. Thus, if $\bar{v} \cdot F_{n,k} \cap \bar{w} \cdot F_{n,k'} \neq \emptyset$ for some $\bar{v}, \bar{w} \in D_n$ and $1 \leq k, k' \leq k_n$, then $\bar{v} - \bar{w} \in d_n\mathbb{Z}^d$, which implies that $\bar{w} - \bar{v} = 0$. This means that $F_{n,k} \cap F_{n,k'} \neq \emptyset$, but this is possible if and only if $B_{n,k} = B_{n,k'}$, i.e. when $F_{n,k} = F_{n,k'}$.

In order to show that $\mathcal{R}_{F_n}(x_0) = d_n\mathbb{Z}^d$, note that condition (i) ensures that $d_n\mathbb{Z}^d$ is included in $\mathcal{R}_{F_n}(x_0)$. Conversely, if $\bar{v}$ is a vector in $\mathcal{R}_{F_n}(x_0)$, then there exists $1 \leq k \leq k_n$ such that $\bar{v} \cdot x_0(D_n) = B_{n,k}$. Since any vector in $\mathbb{Z}^d$ can be written as a vector in $d_n\mathbb{Z}^d$ plus a vector in D_n, there exist $\bar{z} \in \mathbb{Z}^d$ and $\bar{w} \in D_n$ such that $\bar{v} = d_n\bar{z} + \bar{w}$. So, we have

$$(d_n\bar{z} + \bar{w}) \cdot x_0(D_n) = B_{n,k}. \tag{5.6}$$

Since $d_n\bar{z}$ is in $d_n\mathbb{Z}^d \subseteq \mathcal{R}_{F_n}(x_0)$, there exists $1 \leq k' \leq k_n$ such that

$$(d_n\bar{z} \cdot x_0)(D_n) = B_{n,k'}. \tag{5.7}$$

From (5.6) and (5.7) we get

$$B_{n,k'}(\bar{w} + \bar{v}) = B_{n,k}(\bar{v}) \quad \text{for all } \bar{v} \in D_n \text{ satisfying } \bar{v} + \bar{w} \in D_n. \tag{5.8}$$

From (5.8) and condition (v), we deduce that $\bar{w} = 0$. This implies that $\bar{v} \in d_n\mathbb{Z}^d$ and then that $\mathcal{R}_{F_n}(x_0) = d_n\mathbb{Z}^d$.

To show that $(\mathcal{F}_n)_{n\geq 0}$ spans the topology of X it is sufficient to prove that this sequence separates points. In other words, if x_1 and x_2 are two points in X which belong to the same atom of every $\mathcal{F}_n$, then $x_1 = x_2$. By using condition (iii), we will show that if x_1 and x_2 are in the same atom of $\mathcal{F}_n$, then $x_1(D_{n-1}) = x_2(D_{n-1})$. This determines that the sequence of partitions separates points because $(D_n)_{n\geq 0}$ is an increasing sequence of subsets converging to $\mathbb{Z}^d$.

Suppose that x_1 and x_2 are two points of X which belong to the same atom of $\mathcal{F}_n$. Namely, $x_1, x_2 \in \bar{v}_n \cdot F_{n,j_n}$ for some $\bar{v}_n \in D_n$ and $1 \leq j_n \leq k_n$. Let $y_1, y_2 \in F_{n,j_n}$ be such that $x_1 = \bar{v}_n \cdot y_1$ and $x_2 = \bar{v}_n \cdot y_2$.

Let $\bar{u} \in D_{n-1}$. We need to show that $x_1(\bar{u}) = x_2(\bar{u})$. In order to do that, we distinguish two cases: $\bar{v}_n + \bar{u} \in D_n$ and $\bar{v}_n + \bar{u} \notin D_n$.

The first case is direct. Since y_1 and y_2 are in F_{n,j_n}, they verify $y_1(\bar{v}) = y_2(\bar{v})$ for each $\bar{v} \in D_n$. In particular, if $\bar{v}_n + \bar{u} \in D_n$ then $y_1(\bar{v}_n + \bar{u}) = y_2(\bar{v}_n + \bar{u})$, which is equivalent to $x_1(\bar{u}) = x_2(\bar{u})$.

In the second case we need to use statement (iii). First, note that if $\bar{v}_n + \bar{u} \notin D_n$ then there exist $\bar{z} \in \mathbb{Z}^d \setminus \{0\}$ and $\bar{w} \in D_n$ such that $\bar{v}_n + \bar{u} = d_n\bar{z} + \bar{w}$. Since the set of return times to F_n for every point in F_n coincides with $d_n\mathbb{Z}^d$, we have that $d_n\bar{z} \cdot y_1$ and $d_n\bar{z} \cdot y_2$ are in F_n. This means that there exist $1 \leq l_1, l_2 \leq k_n$ such that $d_n\bar{z} \cdot y_1(D_n) = B_{n,l_1}$ and $d_n\bar{z} \cdot y_2(D_n) = B_{n,l_2}$, which implies

$$x_1(\bar{u}) = y_1(\bar{v}_n + \bar{u}) = y_1(d_n\bar{z} + \bar{w}) = d_n\bar{z} \cdot y_1(\bar{w}) = B_{n,l_1}(\bar{w}),$$
$$x_2(\bar{u}) = y_2(\bar{v}_n + \bar{u}) = y_2(d_n\bar{z} + \bar{w}) = d_n\bar{z} \cdot y_2(\bar{w}) = B_{n,l_2}(\bar{w}).$$

The relation $\bar{w} - \bar{u} = \bar{v}_n - d_n\bar{z}$ ensures that $\bar{w} - \bar{u} \notin D_n$ because $\bar{z} \neq 0$. Since $\bar{u} \in D_{n-1}$ and $\bar{w} \in D_n$, this is possible only if $\bar{w} \in D_{n-1,\bar{p}}$, for some $\bar{p} \in \partial S_{n-1}$. So, from condition (iii) we have $B_{n,l_1}(\bar{w}) = B_{n,l_2}(\bar{w})$ and then $x_1(\bar{u}) = x_2(\bar{u})$.

Now we will prove that $\mathcal{M}_{\mathbb{Z}^d}(X)$ is affine-homeomorphic to $\mathcal{M}_{\mathbb{Z}}(Y)$. First, we will show that any invariant measure in $\mathcal{M}_{\mathbb{Z}^d}(X)$ can be seen as an element in $\lim_{\leftarrow n}(A_n, \triangle_n)$, and later we will deduce the reciprocal affirmation.

As $(\mathcal{F}_n)_{n\geq 0}$ is a nested clopen partition spanning the topology of X, an invariant measure $\mu \in \mathcal{M}_{\mathbb{Z}^d}(X)$ is completely determined by the sequence $(\mu(F_{n,1}), \ldots, \mu(F_{n,k_n}))_{n\geq 0}$. Since $(\bar{v} \cdot F_n)_{n\geq 0}$ is a partition of X, we have

$$1 = \mu(X) = \sum_{\bar{v}\in D_n} \mu(\bar{v} \cdot F_n) = \sum_{\bar{v}\in D_n} \mu(\cdot F_n) = |D_n|\mu(F_n) = d_n^d \mu(F_n).$$

This relation implies that

$$\sum_{i=1}^{k_n} \mu(F_{n,k}) = \mu(F_n) = \frac{1}{d_n^d}, \tag{5.9}$$

which means that $(\mu(F_{n,1}), \ldots, \mu(F_{n,k_n}))$ is in $\triangle_n$. From condition (iv) we get that the number of $\bar{v} \in D_{n+1}$ satisfying $\bar{v} \cdot F_{n+1,k'} \subseteq F_{n,k}$ is $A_n(k, k')$, for every $1 \leq k \leq k_n$ and $1 \leq k' \leq k_{n+1}$. So, we obtain

$$\mu(F_{n,k}) = \sum_{k=1}^{k_{n+1}} A_n(k, k')\mu(F_{n+1,k'}). \tag{5.10}$$

From equations (5.9) and (5.10) we deduce that $(\mu(F_{n,1}), \ldots, \mu(F_{n,k_n}))_{n\geq 0}$ belongs to $\lim_{\leftarrow n}(A_n, \triangle_n)$.

Conversely, if $(x_{n,1}, \ldots, x_{n,k_n})_{n\geq 0}$ is an element in $\lim_{\leftarrow n}(A_n, \triangle_n)$, then $\mu(\bar{v} \cdot F_{n,k}) = x_{n,k}$ determines a probability measure on $(\mathcal{F}_n)_{n\geq 0}$, which extends to a unique probability measure on the Borel σ-algebra of X. It is not hard to see that this measure is also invariant. Thus, the map

$$\mu \to (\mu(F_{n,1}), \ldots, \mu(F_{n,k_n}))_{n\geq 0}$$

is an affine bijection from $\mathcal{M}_{\mathbb{Z}^d}(X)$ to $\lim_{\leftarrow n}(A_n, \triangle_n)$. Moreover, it is an homeomorphism if we consider $\mathcal{M}_{\mathbb{Z}^d}(X)$ equipped with the weak topology and $\lim_{\leftarrow n}(A_n, \triangle_n)$ with the product topology.

Finally, it remains to prove that the maximal equicontinuous factor of $(X, \mathbb{Z}^d)$ is the $\mathbb{Z}^d$-odometer G^d. We will show that there exists an almost one-to-one factor map from X to G^d. Since the odometers are minimal and equicontinuous, the existence of an almost one-to-one factor map from X to G^d implies that $(G^d, \mathbb{Z}^d)$ is the maximal equicontinuous factor of $(X, \mathbb{Z}^d)$ [**5**].

A way to express the odometer G^d is as follows,

$$G = \left\{ (g_n)_{n\geq 0} \in \prod_{n\geq 0} \mathbb{Z}^d/d_n\mathbb{Z}^d : \pi_n(g_{n+1}) = g_n, \forall n \geq 0 \right\},$$

where $\pi_n : \mathbb{Z}^d/d_{n+1}\mathbb{Z}^d \to \mathbb{Z}^d/d_n\mathbb{Z}^d$ is the projection, for all $n \geq 0$.

To every $x \in X$ we associate the sequence $(v_n(x))_{n\geq 0} \in \prod_{n\geq 0} \mathbb{Z}^d/d_n\mathbb{Z}^d$, where $v_n(x)$ is the class in $\mathbb{Z}^d/d_n\mathbb{Z}^d$ containing the vector $\bar{v}_n(x) \in D_n$ for which $x \in \bar{v}_n(x) \cdot F_n$. In fact, the sequence $(v_n(x))_{n\geq 0}$ is in G: since $\mathcal{F}_{n+1}$ is finer than $\mathcal{F}_n$ and since the intersection $\bar{v}_{n+1}(x) \cdot F_{n+1} \cap \bar{v}_n(x) \cdot F_n$ is not empty, it is necessary that F_{n+1} is included in $(-\bar{v}_{n+1}(x) + \bar{v}_n(x)) \cdot F_n$. From this inclusion and $F_{n+1} \subseteq F_n$, we obtain that the intersection $(-\bar{v}_{n+1}(x) + \bar{v}_n(x)) \cdot F_n \cap F_n$ is not empty. So, there exists a point in F_n whose set of return times to F_n contains $-\bar{v}_{n+1}(x) + \bar{v}_n(x)$. As this set of return times is always $d_n\mathbb{Z}^d$, the vectors $\bar{v}_{n+1}(x)$ and $\bar{v}_n(x)$ are two representing elements of the same class in $\mathbb{Z}^d/d_n\mathbb{Z}^d$. In other words, $\pi_n(v_{n+1}(x)) = v_n(x)$, which shows that $(v_n(x))_{n\geq 0}$ is in G.

Thus, the map $f : X \to G$ given by $f(x) = (v_n(x))_{n\geq 0}$ is well defined. This is continuous and commutes with the action, so it is a factor map. Moreover, since $\bigcap_{n\geq 0} F_n = \{x_0\}$, one has $f^{-1}\{0\} = x_0$. Thus, f is an almost one-to-one factor map. □

Proof of Theorem 7. We have seen that from a suitable sequence of blocks $(B_{n,1}, \ldots, B_{n,k_n})_{n\geq 0}$ it is always possible to construct a $\mathbb{Z}^d$-Toeplitz array $(X, \mathbb{Z}^d)$ whose maximal equicontinuous factor is the $\mathbb{Z}^d$-odometer G^d, and such that $\mathcal{M}_{\mathbb{Z}^d}(X)$ is affine-homeomorphic to $\mathcal{M}_{\mathbb{Z}}(Y)$. In the following, we focus on the existence of a sequence of blocks satisfying the conditions (i)–(v) introduced above. The proof is divided into two parts. In the first, we define an explicit sequence of blocks $(B_{n,1}, \ldots, B_{n,k_n})_{n\geq 0}$. From the definition, it will be direct to check that the sequence satisfies the first four conditions. The second part is devoted to proving that the sequence verifies the statement (v).

First part: definition of $(B_{n,1}, \ldots, B_{n,k_n})_{n\geq 0}$. We start with the definition of the blocks $B_{0,1}, \ldots, B_{0,k_0}$. In the proof of Proposition 5, the first term of the sequence $(d_n)_{n\geq 0}$ was chosen as $d_0 = h_{n_0}$, for an arbitrary $n_0 \geq 0$. Here we take $n_0 \geq 0$ such that $d_0/2 - 1 \geq k_0$.

Now, for $1 \leq k \leq k_0$, we define $B_{0,k} \in \{0, 1\}^{D_0}$ by

$$B_{0,k}(\bar{v}) = \begin{cases} 1 & \text{if there is } 1 \leq i \leq d \text{ such that } k_0 - (k-1) \leq |v_i|, \\ 0 & \text{otherwise,} \end{cases} \tag{5.11}$$

where $\bar{v} \in D_0$. Figure 1 shows an example.

Let $n \geq 0$. In order to define the blocks $B_{n+1,1}, \ldots, B_{n+1,k_{n+1}}$, for every $1 \leq k \leq k_{n+1}$ we shall introduce a subset $W_{n+1,k}$ of the blocks in $\{0, 1\}^{D_{n+1}}$ which satisfy the conditions (i)–(iv). We will verify that it is possible to choose a block $B_{n+1,k}$ in $W_{n+1,k}$ such that $B_{n+1,k} \neq B_{n+1,k'}$ if $k \neq k'$. After that, we will have defined k_{n+1} different blocks $B_{n+1,1}, \ldots, B_{n+1,k_{n+1}}$ satisfying the first four required conditions. With the objective to introduce the sets $W_{n+1,1}, \ldots, W_{n+1,k_{n+1}}$, for every $1 \leq k \leq k_{n+1}$ we construct two disjoint subsets $S_{n,k,1}$ and $S_{n,k,2}$ of S_n as follows.

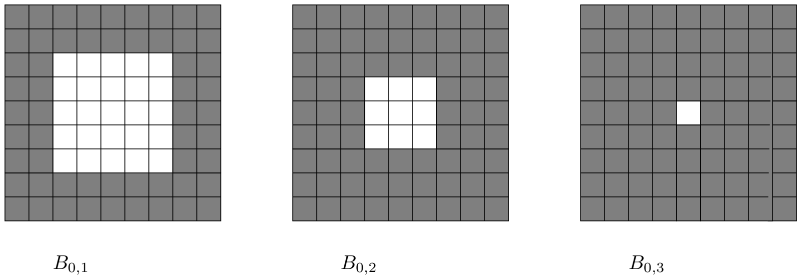

FIGURE 1. The blocks $B_{0,1}$, $B_{0,2}$ and $B_{0,3}$ for the case $d = 2$, $q_0 = 9$ and $k_0 = 3$.

Using (C1)–(C3) of Proposition 5, we get $A_n(k_n, k) \leq (q_n - 2)^d$. From this relation and $q_n^d - (q_n - 2)^d \geq 2(q_n - 2)^{d-1}$, it follows that

$$A_n(k_n, k) - (q_n^d - (q_n - 2)^d) \leq (q_n - 4)(q_n - 2)^{d-1},$$

which implies that there exist $0 \leq m_{n,k_n,k} \leq q_n - 4$ and $0 \leq h_{n,k_n,k} < (q_n - 2)^{d-1}$ such that

$$A_n(k_n, k) - (q_n^d - (q_n - 2)^d) = m_{n,k_n,k}(q_n - 2)^{d-1} + h_{n,k_n,k}.$$

Now, we define

$$S_{n,k,1} = \{d_n \bar{v} \in S_n \backslash (\partial S_n \cup \{0\}) : r_n - m_{n,k_n,k} \leq v_d \leq r_n - 1\},$$

and $S_{n,k,2}$ as any subset of $\{d_n \bar{v} \in S_n \backslash (\partial S_n \cup \{0\}) : v_d = r_n - m_{n,k_n,k} - 1\}$ verifying

$$|S_{n,k,2}| = \begin{cases} h_{n,k_n,k} & \text{if } r_n - m_{n,k_n,k} > 0, \\ h_{n,k_n,k} + 1 & \text{otherwise.} \end{cases}$$

Because $m_{n,k_n,k} \leq q_n - 4$, the points $d_n \bar{v}$ in S_n satisfying $v_d = -d_n(l_n - 1)$ do not belong to $S_{n,k,1} \cup S_{n,k,2}$.

Let $1 \leq k \leq k_{n+1}$. We call $W_{n+1,k}$ the set consisting of all the blocks $B \in \{0, 1\}^{D_{n+1}}$ which satisfy:

(i) $B(D_{n,\bar{v}}) \in \{B_{n,1}, \ldots, B_{n,k_n}\}$ for all $\bar{v} \in S_n$;
(ii) $B(D_{n,0}) = B_{n,1}$;
(iii) $B(D_{n,\bar{v}}) = B_{n,k_n}$ for all $\bar{v} \in \partial S_n \cup S_{n,k,1} \cup S_{n,k,2}$;
(iv) $|\{\bar{v} \in S_n : B(D_{n,\bar{v}}) = B_{n,i}\}| = A_n(i, k)$ for all $1 \leq i \leq k_n$.

Figure 2 gives an idea about the blocks in $W_{n+1,k}$. From (iii) and (iv) one gets that any block B in $W_{n+1,k}$ verifies

$$B(D_{n,\bar{v}}) = B_{n,k_n} \quad \text{if and only if } \bar{v} \in \partial S_n \cup S_{n,k,1} \cup S_{n,k,2}. \tag{5.12}$$

Using this equation and (ii), a simple computation yields that $W_{n+1,k}$ contains exactly $\Lambda(A_n(1, k) - 1, A_n(2, k), \ldots, A_n(k_n - 1, k))$ elements. Since A_n satisfies (C4) of Proposition 5, there are at least as many elements in $W_{n,k}$ as in $N(A_n, k)$ (the number of columns of A which are equal to its kth column). Because $W_{n+1,k} = W_{n+1,k'}$ if and only if $A_n(\cdot, k) = A_n(\cdot, k')$ (if $W_{n+1,k} \neq W_{n+1,k'}$ then they are disjoint), this ensures it is possible to choose $B_{n+1,k}$ in $W_{n+1,k}$ such that $B_{n+1,k} \neq B_{n+1,k'}$ if $k \neq k'$.

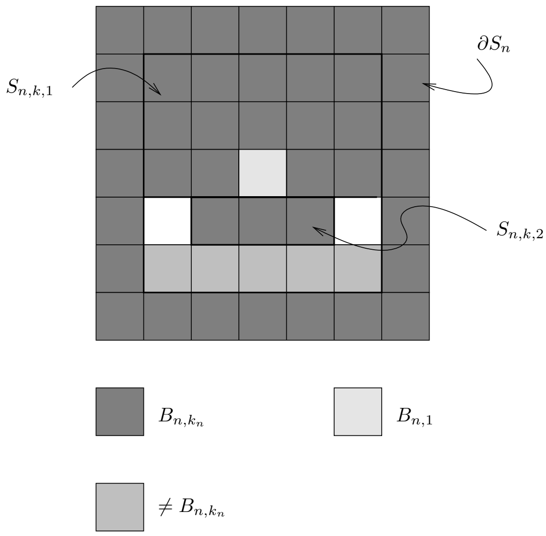

FIGURE 2. The blocks in $W_{n+1,k}$.

Second part: to verify that $(B_{n,1},\ldots,B_{n,k_n})_{n\geq 0}$ *satisfies condition (v).* We have to prove that for every $n \geq 0$, if there exists $\bar{w} \in D_n$ such that for some $1 \leq k,k' \leq k_n$, $B_{n,k}(\bar{v}+\bar{w}) = B_{n,k'}(\bar{v})$ for all $\bar{v} \in D_n$ such that $\bar{v}+\bar{w} \in D_n$, then $\bar{w} = 0$. We will show this by induction on n.

Let $n = 0$. Suppose that $\bar{w} \in D_0$ satisfies

$$B_{0,k}(\bar{w}+\bar{v}) = B_{0,k'}(\bar{v}) \quad \text{for all } \bar{v} \in D_0 \text{ such that } \bar{w}+\bar{v} \in D_0. \tag{5.13}$$

In particular, this is valid for $\bar{v} = 0$. Then, from (5.11) and (5.13) we get $B_{0,k}(\bar{w}) = B_{0,k'}(0) = 0$. From (5.11), this implies

$$|w_i| < k_0 - (k-1) \quad \text{for all } 1 \leq i \leq d. \tag{5.14}$$

Let $1 \leq i \leq d$. We call j the integer such that

$$w_i + j = \begin{cases} k_0 - (k-1) & \text{if } w_i > 0, \\ -(k_0 - (k-1)) & \text{if } w_i \leq 0. \end{cases}$$

Since $0 < |j| \leq k_0 - (k-1)$, the vectors je_i and $-je_i$, where e_i is the unitary vector in the i-coordinate, are in D_0. Furthermore, we have $\bar{w}+je_i \in D_0$ and $\bar{w}-je_i \in D_0$. Thus, from (5.13) we get

$$B_{0,k}(\bar{w}+je_i) = B_{0,k'}(je_i), \tag{5.15}$$

$$B_{0,k}(\bar{w}-je_i) = B_{0,k'}(-je_i). \tag{5.16}$$

As $|w_i + j| = k_0 - (k-1)$, from (5.11) we obtain $B_{0,k}(\bar{w}+je_i) = 1$. By equation (5.15), this means that $B_{0,k'}(je_i) = 1$. From (5.11), it follows that $k_0 - (k'-1) \leq |j|$ and then $B_{0,k'}(-je_i) = 1$, which implies, by (5.16), that $B_{0,k}(\bar{w}-je_i) = 1$. Again from (5.11), this implies there exists a coordinate of $\bar{w} - je_i$ whose absolute value is greater than or equal to $k_0 - (k-1)$. From (5.14) we deduce that

$$k_0 - (k-1) \leq |w_i - j|. \tag{5.17}$$

On the other hand, we have

$$|w_i - j| \leq \max\{|w_i|, |j|\} \leq k_0 - (k-1). \tag{5.18}$$

From (5.17), (5.18) and (5.14) we conclude that $|j| = k_0 - (k-1)$, which implies, by definition of j, that $w_i = 0$.

Let $n > 0$. We suppose that if $\bar{w} \in D_{n-1}$ is such that, for some $1 \leq k, k' \leq k_{n-1}$, $B_{n-1,k}(\bar{v} + \bar{w}) = B_{n-1,k'}(\bar{v})$ for all $\bar{v} \in D_{n-1}$ with $\bar{v} + \bar{w} \in D_{n-1}$, then $\bar{w} = 0$. We use this as the induction hypothesis.

Let $\bar{w} \in D_n$ be such that $B_{n,k}(\bar{v} + \bar{w}) = B_{n,k'}(\bar{v})$ for all $\bar{v} \in D_n$ with $\bar{v} + \bar{w} \in D_n$. In order to show that $\bar{w} = 0$, first we prove that $\bar{w} \in S_{n-1} \setminus \partial S_{n-1}$. Next, we use this result to study the case $\bar{w} = ke_d$, where e_d is the unitary vector in the d-coordinate and $k \in \mathbb{Z}$. Finally, we deduce that $\bar{w} = 0$ in the general case.

Let $\bar{u} \in D_{n-1}$ and $\bar{r} \in \mathbb{Z}^d$ be such that $\bar{w} = d_{n-1}\bar{r} + \bar{u}$. We have

$$B_{n,k}(\bar{v} + d_{n-1}\bar{r} + \bar{u}) = B_{n,k'}(\bar{v})$$

for all $\bar{v} \in D_n$ such that $\bar{v} + d_{n-1}\bar{r} + \bar{u}$ is in D_n. Since $\bar{w} \in D_n$, it is necessary that $d_{n-1}\bar{r} \in S_{n-1}$. This implies that $d_{n-1}\bar{r} + \bar{u} + \bar{v} \in D_n$ for every $\bar{v}$ in D_{n-1} such that $\bar{u} + \bar{v} \in D_{n-1}$. Thus, we deduce that

$$B_{n,k}(\bar{v} + d_{n-1}\bar{r} + \bar{u}) = B_{n,k'}(\bar{v}),$$

for every $\bar{v}$ in D_{n-1} such that $\bar{v} + \bar{u}$ is in D_{n-1}. In other words,

$$B_{n-1,l}(\bar{v} + \bar{u}) = B_{n-1,1}(\bar{v}),$$

for every $\bar{v} \in D_{n-1}$ such that $\bar{v} + \bar{u} \in D_{n-1}$, where $B_{n-1,l}$ is the block such that $B_{n,k}(D_{n-1,d_{n-1}\bar{r}}) = B_{n-1,l}$ (condition (i) ensures the existence of this block) and $B_{n-1,1}$ corresponds to $B_{n,k'}(D_{n-1,0})$ (this is true by condition (ii)). So, by the induction hypothesis we have $\bar{u} = 0$ and then $\bar{w} \in d_{n-1}\mathbb{Z}^d \cap D_n = S_{n-1}$.

Suppose that $\bar{v} \in S_{n-1}$ is such that $\bar{v} + \bar{w} \in S_{n-1}$. Since S_{n-1} is a subset of D_n, we have $B_{n,k}(\bar{v} + \bar{w}) = B_{n,k'}(\bar{v})$. Because for every $\bar{u} \in D_{n-1}$ it holds that $\bar{v} + \bar{u} + \bar{w}$ and $\bar{v} + \bar{u}$ are in D_n, we conclude that

$$B_{n,k}(D_{n-1,\bar{w}+\bar{v}}) = B_{n,k'}(D_{n-1,\bar{v}}), \tag{5.19}$$

for all $\bar{v} \in S_{n-1}$ such that $\bar{v} + \bar{w} \in S_{n-1}$. In particular, this is valid for $\bar{v} = 0$. Using condition (ii) we get

$$B_{n,k}(D_{n-1,\bar{w}}) = B_{n,k'}(D_{n-1,0}) = B_{n-1,1},$$

which implies by (iii) that $\bar{w} \in S_{n-1} \setminus \partial S_{n-1}$. This means that w_i satisfies $-d_{n-1}l_{n-1} < w_i < d_{n-1}r_{n-1}$ for every $1 \leq i \leq d$.

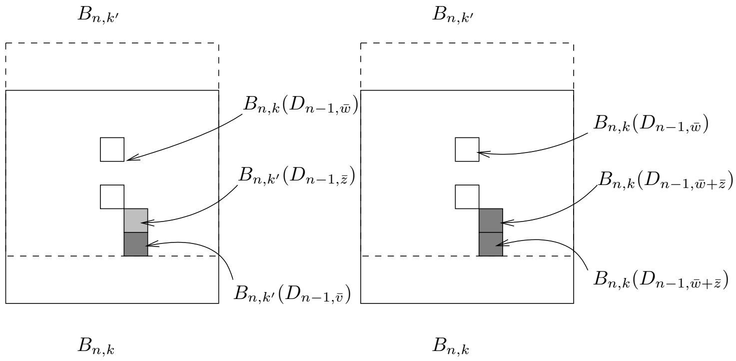

FIGURE 3. Contradiction in the case $\bar{w} = ke_d$, with $k > 0$.

Now we consider the case $\bar{w} = ke_d$. We suppose that $k \neq 0$ to obtain a contradiction. Let $1 \leq i < d$. We set

$$\bar{v} = \begin{cases} d_{n-1}e_i - d_{n-1}l_{n-1}e_d & \text{if } k > 0, \\ d_{n-1}e_i - d_{n-1}l_{n-1}e_d - ke_d & \text{if } k < 0, \end{cases}$$

and $\bar{z} = \bar{v} + d_{n-1}e_d$.

When $k > 0$ we have $\bar{v} \in \partial S_{n-1}$ and $\bar{z} \in S_{n-1} \setminus (\partial S_{n-1} \cup S_{n-1,k',1} \cup S_{n-1,k',2})$. By (5.12), this implies

$$B_{n,k'}(D_{n-1,\bar{v}}) = B_{n-1,k_{n-1}} \quad \text{and} \quad B_{n,k'}(D_{n-1,\bar{z}}) \neq B_{n-1,k_{n-1}}. \tag{5.20}$$

Note that $k > 0$ implies $-d_{n-1}l_{n-1} < w_d + v_d$. Because $r_{n-1} - l_{n-1} \in \{0, 1\}$ and $w_d < d_{n-1}r_{n-1}$, we get

$$\bar{w} + \bar{v} \in S_{n-1} \setminus \partial S_{n-1}. \tag{5.21}$$

Since $\bar{w} + \bar{v}$ is in S_{n-1}, from (5.19) and (5.20) we obtain

$$B_{n,k}(D_{n-1,\bar{v}+\bar{w}}) = B_{n,k'}(D_{n-1,\bar{v}}) = B_{n-1,k_{n-1}}.$$

On the one hand, the previous relation, (5.12) and (5.21) ensure that

$$\bar{v} + \bar{w} \in S_{n-1,k,1} \cup S_{n-1,k,2}, \tag{5.22}$$

which implies that

$$\bar{w} + \bar{z} \in S_{n-1,k,1} \cup \partial S_{n-1}, \tag{5.23}$$

because $w_d + z_d = w_d + v_d + d_{n-1}$ and $\bar{w} + \bar{z} \neq 0$ (the i-coordinate is equal to d_{n-1}). On the other hand, (5.12) and (5.23) imply that

$$B_{n,k}(D_{n-1,\bar{w}+\bar{z}}) = B_{n-1,k_{n-1}}.$$

Nevertheless, from this relation and (5.19) we have

$$B_{n,k}(D_{n-1,\bar{w}+\bar{z}}) = B_{n,k'}(D_{n-1,\bar{z}}) = B_{n-1,k_{n-1}},$$

which contradicts (5.20) (see Figure 3).

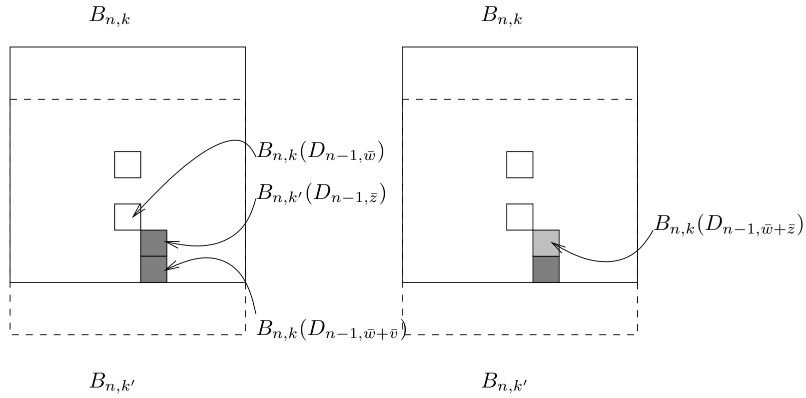

FIGURE 4. Contradiction in the case $\bar{w} = ke_d$, with $k < 0$.

If $k < 0$ then $\bar{v} = d_{n-1}e_i - d_{n-1}l_{n-1}e_d - ke_d$ and $\bar{z} = d_{n-1}e_i - d_{n-1}(l_{n-1} - 1)e_d - ke_d$. Since $-l_{n-1}d_{n-1} < k < 0$, we have $-l_{n-1}d_{n-1} < v_d < 0$, which implies that $\bar{v}$ is in $S_{n-1} \setminus \partial S_{n-1}$. As $\bar{w} + \bar{v} \in \partial S_{n-1}$, from (5.12) and (5.19) we get

$$B_{n,k}(D_{n-1,\bar{v}+\bar{w}}) = B_{n,k'}(D_{n-1,\bar{v}}) = B_{n-1,k_{n-1}}.$$

By (5.12), this ensures that $\bar{v}$ is in $\partial S_{n-1} \cup S_{n-1,k',1} \cup S_{n-1,k',2}$, which implies that $\bar{v}$ is in $\bar{v} \in S_{n-1,k',1} \cup S_{n-1,k',2}$ because $\bar{v}$ is not in ∂S_{n-1}. From this inclusion it follows that $\bar{z}$ is in $S_{n-1,k',1} \cup \partial S_{n-1}$, because $\bar{z} \neq 0$ and $z_d = v_d + d_{n-1}$. On the one hand, from (5.12) we obtain

$$B_{n,k'}(D_{n-1,\bar{z}}) = B_{n-1,k_{n-1}}. \tag{5.24}$$

On the other hand, as $w_d + z_d = -d_{n-1}(l_{n-1} - 1)$, we have that $\bar{w} + \bar{z}$ is in $S_{n-1} \setminus (\partial S_{n-1} \cup S_{n-1,k,1} \cup S_{n-1,k,2})$. Thus, from (5.12) we get

$$B_{n,k}(D_{n-1,\bar{w}+\bar{z}}) \neq B_{n-1,k_{n-1}},$$

and from (5.19)

$$B_{n,k}(D_{n-1,\bar{w}+\bar{z}}) = B_{n,k'}(D_{n-1,\bar{z}}) \neq B_{n-1,k_{n-1}},$$

which contradicts (5.24) (see Figure 4).

Thus, we have shown that

$$\text{if } \bar{w} = ke_d \quad \text{then } k = 0. \tag{5.25}$$

Now, we consider the general case. In order to show that $\bar{w} = 0$, first we suppose that $w_d > 0$ to obtain a contradiction. Next, using the fact that $w_d \leq 0$, we suppose that among the first $d - 1$ coordinates of $\bar{w}$ there is one which is not zero. We deduce that this is not possible and from (5.25) we obtain the final result.

Suppose that $w_d > 0$. We set $\bar{v} = -d_{n-1}(l_{n-1} - 1)e_d$ and $\bar{z} = -d_{n-1}l_{n-1}e_d$. Both vectors are in S_{n-1}. In particular,

$$\bar{v} \in S_{n-1} \setminus (\partial S_{n-1} \cup S_{n-1,k',1} \cup S_{n-1,k',2}) \quad \text{and} \quad \bar{z} \in \partial S_{n-1}.$$

From (5.12) we get

$$B_{n,k'}(D_{n-1,\bar{v}}) \neq B_{n-1,k_{n-1}}, \tag{5.26}$$

$$B_{n,k'}(D_{n-1,\bar{z}}) = B_{n-1,k_{n-1}}. \tag{5.27}$$

Note that $\bar{w}+\bar{v}$ and $\bar{w}+\bar{v}$ are in S_{n-1}. In fact, since $r_{n-1}-l_{n-1} \in \{0,1\}$ and w_d satisfies $0 < w_d < d_{n-1}r_{n-1}$, we have

$$-d_{n-1}l_{n-1} < w_d + z_d < w_d + v_d \leq d_{n-1} < d_{n-1}r_{n-1},$$

which implies that $\bar{w}+\bar{v}$ and $\bar{w}+\bar{z}$ are in $S_{n-1} \setminus \partial S_{n-1}$. Thus, using (5.19), (5.26) and (5.27), we get

$$B_{n,k}(D_{n-1,\bar{w}+\bar{v}}) = B_{n,k'}(D_{n-1,\bar{v}}) \neq B_{n-1,k_{n-1}}, \tag{5.28}$$

$$B_{n,k}(D_{n-1,\bar{w}+\bar{z}}) = B_{n,k'}(D_{n-1,\bar{z}}) = B_{n-1,k_{n-1}}. \tag{5.29}$$

From (5.12), the equation (5.29) ensures that $\bar{w}+\bar{z}$ is in $\partial S_n \cup S_{n-1,k,1} \cup S_{n-1,k,2}$, which implies that $\bar{w}+\bar{z}$ is in $S_{n-1,k,1} \cup S_{n-1,k,2}$, because $\bar{w}+\bar{z}$ is not in ∂S_{n-1}. Since $w_d + v_d = w_d + z_d + d_{n-1}$ and $\bar{w}+\bar{v} \notin \partial S_{n-1}$, the previous inclusion implies that $\bar{w}+\bar{v}$ is in $S_{n,k,1} \cup \{0\}$. If $\bar{w}+\bar{v}$ is in $S_{n,k,1}$ then from (5.12) we get $B_{n,k}(D_{n-1,\bar{w}+\bar{v}}) = B_{n-1,k_{n-1}}$, which contradicts (5.28). So, the only possibility is $\bar{w}+\bar{v} = 0$. Nevertheless, this means that $\bar{w} = ke_d$ for some $k \in \mathbb{Z}$, which contradicts (5.25). Thus we conclude that $w_d \leq 0$.

Finally, using the fact that $w_d \leq 0$, suppose there exists $1 \leq i < d$ such that $w_i \neq 0$. We set

$$\bar{v} = \begin{cases} r_{n-1}d_{n-1}e_i - d_{n-1}(l_{n-1}-1)e_d - w_d e_d & \text{if } w_i < 0, \\ -l_{n-1}d_{n-1}e_i - d_{n-1}(l_{n-1}-1)e_d - w_d e_d & \text{if } w_i > 0. \end{cases}$$

In both cases the vector $\bar{v}$ is in ∂S_{n-1}. So, from (5.12) we get

$$B_{n,k'}(D_{n-1,\bar{v}}) = B_{n-1,k_{n-1}}. \tag{5.30}$$

Since $-d_{n-1}l_{n-1} < w_i < d_{n-1}r_{n-1}$, we deduce that $\bar{w}+\bar{v}$ is in S_n, which implies, by (5.19) and (5.30),

$$B_{n,k}(D_{n-1,\bar{v}+\bar{w}}) = B_{n,k'}(D_{n-1,\bar{v}}) = B_{n-1,k_{n-1}}. \tag{5.31}$$

As $w_d + v_d = -d_{n-1}(l_{n-1}-1)$, we have that $\bar{w}+\bar{v}$ is not in $S_{n-1,k,1} \cup S_{n,k,2}$. Furthermore, because $w_i \neq 0$ we get that $\bar{w}+\bar{v}$ is not in ∂S_{n-1}. From (5.12) it follows that

$$B_{n,k}(D_{n-1,\bar{v}+\bar{w}}) \neq B_{n-1,k_{n-1}},$$

which contradicts (5.31) (see Figure 5). So, it is necessary that $w_i = 0$ for every $1 \leq i < d$, which means that $\bar{w} = ke_d$ for some $k \in \mathbb{Z}$. Thus from (5.25) we get $\bar{w} = 0$. □

In the proof of the two following corollaries, we use the fact that there exists a factor map $\pi : G_1 \to G_2$ between the $\mathbb{Z}^d$-odometers G_1 and G_2, defined by the sequences of groups $(Z_n)_{n\geq 0}$ and $(Y_n)_{n\geq 0}$, respectively, if and only if for every $n \geq 0$ there exists $k \geq 0$ such that $Z_k \subseteq Y_n$ **[3, 5]**.

COROLLARY 9. *Let K be a Choquet simplex and let $d > 1$. There exists a $\mathbb{Z}^d$-Toeplitz system $(X, \mathbb{Z}^d)$ such that $\mathcal{M}_{\mathbb{Z}^d}(X)$ is affine-homeomorphic to K, and such that its maximal equicontinuous factor is a product of d one-dimensional 2-odometers.*

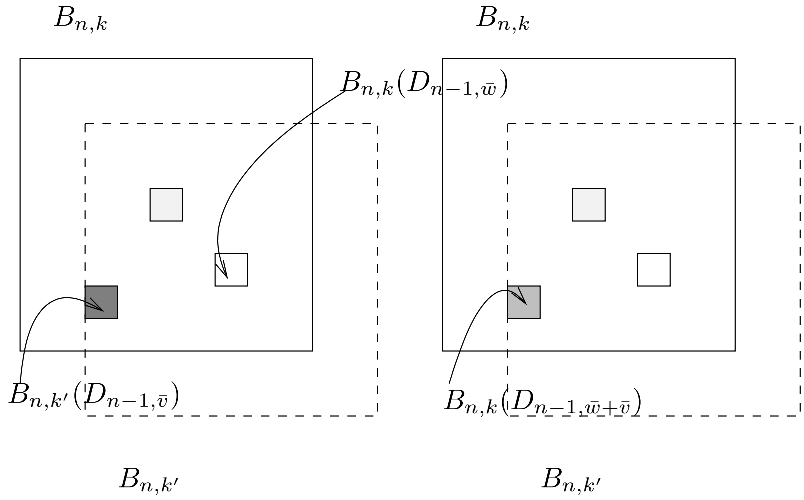

FIGURE 5. Contradiction in the case $w_d \leq 0$.

Proof. Let K be a Choquet simplex. From [**4**], there exists a dyadic Toeplitz flow (Y, T) such that $\mathcal{M}_{\mathbb{Z}}(Y)$ is affine-homeomorphic to K. Let

$$(\mathcal{P}_n = \{T^j C_{n,k} : 0 \leq j \leq h_n,\ 1 \leq k \leq k_n\})_{n\geq 0}$$

be a nested sequence of CKR partitions of Y which satisfies the conditions of Lemma 2. Since $\{(g_n)_{n\geq 0} \in \prod_{n\geq 0} \mathbb{Z}_{h_n} : \pi_n(g_{n+1}) = g_n\}$ is conjugate to the odometer which is the maximal equicontinuous factor of (Y, T), one has that for every $n \geq 0$ there exist $m_1, m_2 \geq 0$ and $k_1, k_2 \in \mathbb{Z}$ such that $h_n = k_1 2^{m_1}$ and $2^n = k_2 h_{m_2}$. This implies that for every $n \geq 0$ there exists $m_n \geq 0$ such that $h_n = 2^{m_n}$. Because $(h_n)_{n\geq 0}$ is an increasing sequence, we get that for every $n, m \geq 0$ there exists $l \geq 0$ such that $h_n h_m$ divides h_l. Thus, from Proposition 7, we conclude there exists a $\mathbb{Z}^d$-Toeplitz system $(X, \mathbb{Z}^d)$ such that $\mathcal{M}_{\mathbb{Z}^d}(X)$ is affine-homeomorphic to K and such that its maximal equicontinuous factor is a product of d one-dimensional 2-odometers. □

COROLLARY 10. *Let K be a Choquet simplex and let $d > 1$. There exists a $\mathbb{Z}^d$-Toeplitz system $(X, \mathbb{Z}^d)$ such that $\mathcal{M}_{\mathbb{Z}^d}(X)$ is affine-homeomorphic to K and such that its maximal equicontinuous factor is a product of d one-dimensional universal odometers. Hence, the set of continuous eigenvalues of $(X, \mathbb{Z}^d)$ is $\mathbb{Q}^d$.*

Proof. Let K be a Choquet simplex. From [**8**], there exists a Toeplitz flow (Y, T) which is an almost one-to-one extension of the universal odometer G, such that $\mathcal{M}_{\mathbb{Z}}(Y)$ is affine-homeomorphic to K. Let

$$(\mathcal{P}_n = \{T^j C_{n,k} : 0 \leq j \leq h_n,\ 1 \leq k \leq k_n\})_{n\geq 0}$$

be a nested sequence of CKR partitions of Y which satisfies conditions of Lemma 2. Since $\{(g_n)_{n\geq 0} \in \prod_{n\geq 0} \mathbb{Z}_{h_n} : \pi_n(g_{n+1}) = g_n\}$ is conjugate to G, one has that for every $n \geq 0$ there exist $m \geq 0$ and $k \in \mathbb{Z}$ such that $h_n = km!$. Because $(h_n)_{n\geq 0}$ is an increasing sequence, we get that for every $n, m \geq 0$ there exists $l \geq 0$ such that $h_n h_m$ divides h_l. Thus, from Proposition 7, we conclude there exists a $\mathbb{Z}^d$-Toeplitz system $(X, \mathbb{Z}^d)$ such that $\mathcal{M}_{\mathbb{Z}^d}(X)$ is affine-homeomorphic to K and such that its maximal equicontinuous factor is the d-dimensional odometer $G \times \cdots \times G$. Since the set of continuous eigenvalues of the Toeplitz system coincides with the set of continuous eigenvalues of its maximal equicontinuous factor, we conclude that this set is equal to $\mathbb{Q}^d$. □

6. *A Choquet simplex as the set of invariant probability measures of a tiling system*

A *tiling* of $\mathbb{R}^d$ is a countable collection $T = (t_n)_{n \geq 0}$ of closed subsets of $\mathbb{R}^d$ (which are known as *tiles*) whose union is the whole space and their interiors are pairwise disjoint. It is often assumed that the tiles are homeomorphic to closed balls and that they belong, up to translation, to a finite collection of closed subsets of $\mathbb{R}^d$ whose elements are called *prototiles*. Sometimes it is useful to consider every prototile as a closed set endowed with a label.

The translation of the tiling T by a vector $\bar{v} \in \mathbb{R}^d$ is the tiling $T + \bar{v}$ that results after translating every tile of T by $\bar{v}$. The tiling T is said to be *aperiodic* (or *non-periodic*) if $T + \bar{v} = T$ implies $\bar{v} = 0$. A *patch* of T is a finite sub-collection of tiles of T. The tiling T satisfies the *finite pattern condition* (FPC) if, for any $r > 0$, there are up to translation only finitely many patches with diameter smaller than r. This condition is automatically satisfied in the case of a tiling whose tiles are polyhedra that meet face-to-face. A tiling is *repetitive* if for any patch in T there exists $r > 0$ such that, for every $x \in \mathbb{R}^d$, there exists a translate of this patch which is in T and in the ball $B_r(x)$. The non-periodic repetitive tilings that satisfy the FPC are called *perfect* tilings. An easy way to produce perfect tilings is from non-periodic uniformly recurrent arrays in $\Sigma^{\mathbb{Z}^d}$: to any array $x \in \Sigma^{\mathbb{Z}^d}$ we can associate a tiling $T_x = (t_{\bar{v}})_{\bar{v} \in \mathbb{Z}^d}$, where the support of $t_{\bar{v}}$ is the translation by $\bar{v}$ of $[0, 1]^d$ and its label is $x(\bar{v})$, for all $\bar{v} \in \mathbb{Z}^d$. The tiling T_x always satisfies the FPC condition. It is repetitive and non-periodic if and only if x is uniformly recurrent and non-periodic, respectively.

Given a finite collection of prototiles M, we denote by $T(M)$ (*full tiling space*) the space of all the tilings of $\mathbb{R}^d$ whose tiles are equivalent to some element in M. When $T(M) \neq \emptyset$, the group $\mathbb{R}^d$ acts on $T(M)$ by translations:

$$(\bar{v}, T) \to T + \bar{v} \quad \text{for } \bar{v} \in \mathbb{R}^d \text{ and } T \in T(M).$$

Furthermore, this action is continuous with the topology induced by the following distance. Let A be the set of $\varepsilon \in (0, 1)$ such that there exist v and v' in $B_\varepsilon(0)$ such that $(T + v) \cap B_{1/\varepsilon}(0) = (T' + v') \cap B_{1/\varepsilon}(0)$. Then

$$d(T, T') = \begin{cases} \inf A & \text{if } A \neq \emptyset, \\ 1 & \text{if } A = \emptyset. \end{cases}$$

The orbit closure of a tiling T in $T(M)$ is called the *continuous Hull* of T and it is denoted by Ω_T. When T satisfies the FPC, its continuous Hull is compact. If, in addition, T is repetitive, $(\Omega_T, \mathbb{R}^d)$ is a minimal topological dynamical system. Thus, if T is a perfect tiling then $(\Omega_T, \mathbb{R}^d)$ is a free minimal topological dynamical system, and it is called a *tiling system* (see [**1**] and [**2**] for more details on tiling systems).

The examples of Toeplitz arrays constructed in the proof of Theorem 7 are non-periodic and uniformly recurrent, so the associated tilings are perfect. Furthermore, if x is a Toeplitz array and $(X, \mathbb{Z}^d)$ its associated Toeplitz system, the tiling system $(\Omega_{T_x}, \mathbb{R}^d)$ is the suspension of $(X, \mathbb{Z}^d)$. So, there is an affine-homeomorphism between the spaces $\mathcal{M}_{\mathbb{Z}^d}(X)$ and $\mathcal{M}_{\mathbb{R}^d}(\Omega_{T_x})$. Thus, we conclude the following.

COROLLARY 11. *Every Choquet simplex can be realized as the set of invariant probability measures of a tiling system.*

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REFERENCES

[1] J. Bellissard, R. Benedetti and J.-M. Gambaudo. Spaces of tilings, finite telescopic, approximations and gap-labelling. *Commun. Math. Phys.* **261** (2006), 1–41.

[2] R. Benedetti and J.-M. Gambaudo. On the dynamics of $\mathbb{G}$-solenoids. Applications to Delone sets. *Ergod. Th. & Dynam. Sys.* **23**(3) (2003), 673–691.

[3] M. I. Cortez. $\mathbb{Z}^d$-Toeplitz arrays. *Discrete Contin. Dynam. Sys. A* **15**(3) (2006), 859–881.

[4] T. Downarowicz. The Choquet simplex of invariant measures for minimal flows. *Israel J. Math.* **74** (1991), 241–256.

[5] T. Downarowicz. Survey of odometers and Toeplitz flows. *Algebraic and Topological Dynamics (May 1–July 31, 2004, Max-Planck-Institut Fur Mathematik, Bonn, Germany)* (*Contemporary Mathematics, 385*). Eds. S. Kolyada, Y. Manin and T. Ward. American Mathematical Society, Providence, RI, 2005, pp. 7–28.

[6] T. Downarowicz. Private communication.

[7] T. Downarowicz and Y. Lacroix. Almost 1–1 extensions of Furstenberg-Weiss type and applications to Toeplitz flows. *Studia Math.* **130**(2) (1998), 149–170.

[8] R. Gjerde and O. Johansen. Bratteli-Vershik models for Cantor minimal systems: applications to Toeplitz flows. *Ergod. Th. & Dynam. Sys.* **20**(6) (2000), 1687–1710.

[9] E. Glasner. *Ergodic Theory via Joinings (Mathematical Surveys and Monographs, 101)*. American Mathematical Society, Providence, RI, 2003.

[10] J.-M. Gambaudo and M. Martens. Algebraic topology for minimal Cantor sets. *Annales de l'I.H.P. Physique Théorique* **7**(3) (2006), 423–446.

[11] R. Herman, I. Putnam and C. Skau. Ordered Bratteli diagrams, dimension groups and topological dynamics. *Int. J. Math.* **3**(6) (1992), 827–864.

[12] R. Phelps. *Lectures on Choquet's Theorems*, 2 edn (*Lectures Notes in Mathematics, 1757*). Springer, Berlin, 2001.

[13] I. F. Putnam. The C^*-algebras associated with minimal homeomorphisms of the Cantor set. *Pacific J. Math.* **136**(2) (1989), 329–353.

[14] S. Williams. Toeplitz minimal flows which are not uniquely ergodic. *Z. Wahrs. Verw. Gebiete* **67**(1) (1984), 95–107.