High Frequency Solutions for the Singularly-Perturbed One-Dimensional Nonlinear Schrödinger Equation

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Abstract

This article is devoted to the nonlinear Schrödinger equation

$$\varepsilon^2 u'' - V(x)u + |u|^{p-1}u = 0,$$

when the parameter ε approaches zero. All possible asymptotic behaviors of bounded solutions can be described by means of envelopes, or alternatively by adiabatic profiles. We prove that for every envelope, there exists a family of solutions reaching that asymptotic behavior, in the case of bounded intervals. We use a combination of the Nehari finite dimensional reduction together with degree theory. Our main contribution is to compute the degree of each cluster, which is a key piece of information in order to glue them.

1. Introduction

In this paper we study the solutions of the one-dimensional nonlinear Schrödinger equation

$$\varepsilon^2 u'' - V(x)u + |u|^{p-1}u = 0, \tag{1.1}$$

where $p > 1$ and the potential V is positive and of class C^1. We treat the equation on a closed, bounded interval I with Neumann boundary conditions with no additional assumptions on the potential. We also discuss extensions to $\mathbb{R}$ assuming some extra hypotheses on the potential. The nonlinear Schrödinger equation serves as a model for various problems in physics, where usually the independent variable refers to the space variable. However, our results can also be interpreted in the context of dynamical systems, where the independent variable represents time.

As the parameter ε approaches zero, solutions to equation (1.1) become highly oscillatory and it is possible to describe their behavior by means of an envelope, which determines the asymptotic amplitude of the solutions. Alternatively such behavior can also be described by an adiabatic profile. Such phenomena has been discussed for the nonlinear Schrödinger equation in some special cases in [13]. In subsequent works more complete results have been obtained for various kinds of phase transition models in [14], [15] and [16].

On the other hand, having a prescribed envelope, a natural question is to ask if there are solutions of (1.1) exhibiting such asymptotic behavior. This question was studied in the case of the phase transition models in [15] and [16]. There, a Nehari method and a maximization scheme for the finite dimensional functional is developed to answer this question positively.

In this work we address the question for the nonlinear Schrödinger equation. A particularly interesting case is when the potential has several critical points. Near a positive local maximum of the potential we are able to construct positive (or negative) oscillatory solutions, called positive single clusters, and near a positive local minimum of the potential we can construct sign-changing single clusters. In order to glue these solutions, it seems hard to use a variational method as in [16], since these solutions correspond to saddle points of the functional, for which energy estimates are hard to obtain. Moreover, it certainly seems impracticable when we try to construct solutions in $\mathbb{R}$ having infinitely many clusters.

The main purpose of this article is to develop a degree theoretic approach to construct multiple clusters for the nonlinear Schrödinger equation. The key step in our approach is the construction of an appropriate potential, for which it is possible to estimate the degree of a cluster. Then, through homotopy, we obtain degree information for clusters of the original potential which allows us to glue them.

The degree theory approach for singularly-perturbed problems has been introduced by NAKASHIMA and TANAKA [23] for the Allen–Cahn equation and later used by DEL PINO, FELMER and TANAKA [12] for the nonlinear Schrödinger equation. In these papers, the construction of solutions with glued clusters, each having a number of oscillations independent of ε is addressed. The main step in [23] and [12] is the actual computation of the degree of a cluster, obtained via a homotopy with a fairly explicit function. In our paper the main difficulty is the computation of the degree, which is accomplished by constructing a suitable homotopy.

As mentioned above, to describe the asymptotic behavior of the solutions of (1.1) we may use the amplitude of the oscillations or the area enclosed by each loop in the phase space. These quantities give rise to the envelope and the adiabatic profile, respectively. The adiabatic profile is constant when the solution oscillates, that is, it corresponds to an adiabatic invariant of the system, that may jump when crossing the separatrix. In the description of our results, we may also use the energy function which describes the approximate energy of the oscillations.

We believe that our approach could be used for the description and the construction of solutions of Hamiltonian systems in $\mathbb{R}^2$ with a slowly-varying parameter in a very general setting. We emphasize that in our work, in the case of a bounded interval, we describe and construct solutions, including those crossing a separatrix. The phenomena of adiabatic jumps when solutions cross a separatrix has been studied by many authors, see for example the work by NEISHTADT [25, 26] and

BOURLAND & HABERMAN [9]. We think that our analysis could shed new light on this problem.

2. Statement of main results

Before stating our main results we introduce some necessary elements. Regarding $V \in I\!R_+$ and $y \in I\!R$ as parameters, we consider the equation

$$v''(s) - Vv(s) + v|v|^{p-1}(s) = 0, \quad s \in I\!R, \tag{2.1}$$

$$v(0) = y, \quad v'(0) = 0, \tag{2.2}$$

and denote its solution by $v = v(V, y; s)$. We let $y_0 = (\frac{p+1}{2}V)^{1/(p-1)}$ and $y_* = V^{1/(p-1)}$. If $y \in (0, \infty) \setminus \{y_0, y_*\}$, we denote by $T(V, y)$ one half of the period of v. In case $y = y_*$ we set $T(V, y) = 2\pi/\sqrt{pV}$ and we extend $T(V, y)$ as an even function, for negative values of y. It is convenient to define a frequency function

$$\omega(V, y) = \frac{1}{T(V, y)}.$$

We define $\omega(V, y_0) = \omega(V, -y_0) = \omega(V, 0) = 0$ and we observe that ω is continuous in all $I\!R_+ \times I\!R$, since $\lim_{|y| \to y_0} T(V, y) = \lim_{y \to 0} T(V, y) = \infty$. We finally see that $\lim_{|y| \to \infty} T(V, y) = 0$ and $\lim_{|y| \to \infty} \omega(V, y) = \infty$.

Next, we define the function

$$A(V, y) = \begin{cases} \frac{1}{2}\int_0^{T(V,y)} (v')^2 ds & \text{if } y > y_0, \\ \int_0^{T(V,y)} (v')^2 ds & \text{if } y_* \leqq y \leqq y_0, \end{cases} \tag{2.3}$$

which is a function of class C^1. We observe that when $y \in (y_*, y_0)$ then $A(V, y)$ represents the area enclosed by the corresponding orbit in the phase space and when $y \in [y_0, \infty)$ then $A(V, y)$ represents half of the area enclosed by the orbit.

Given our potential $V(x)$ on the bounded interval I, we let

$$e_0(x) = \left[\frac{(p+1)}{2}V(x)\right]^{\frac{1}{p-1}} \quad \text{and} \quad e_*(x) = [V(x)]^{\frac{1}{p-1}}. \tag{2.4}$$

We define the trivial action

$$\mathcal{A}_0(x) = A(V(x), e_0(x)), \tag{2.5}$$

which is a C^1 function on I.

Definition 2.1. We say that the function $\mathcal{A} : I \to (0, \infty)$ is an adiabatic profile (or action profile) if $\mathcal{A}$ is continuous and whenever $\mathcal{A}(x) \neq \mathcal{A}_0(x)$, we have $\dot{\mathcal{A}}(x) = 0$.

We define the support of an adiabatic profile as

$$\operatorname{supp}(\mathcal{A}) = \{x \in I \,/\, \mathcal{A}(x) \neq \mathcal{A}_0(x)\}.$$

In the study of the asymptotic behavior of systems with slowly-varying coefficients (or singularly-perturbed systems), one main feature is the oscillatory character of the solutions as ε approaches 0. This behavior is not arbitrary, as it has been shown in different situations by FELMER and TORRES [13], FELMER and MARTÍNEZ [14], and by FELMER, MARTÍNEZ and TANAKA [15, 16].

Given a family $\{u_\varepsilon\}$ of solutions to (1.1) we define an *approximate action* $\mathcal{A}_\varepsilon$. Consider $v = v_\varepsilon(x;\cdot)$, the solution to the initial-value problem

$$v''(s) - V(x)v(s) + |v|^{p-1}v(s) = 0, \tag{2.6}$$

$$v(0) = u_\varepsilon(x), \quad v'(0) = \varepsilon u'_\varepsilon(x). \tag{2.7}$$

We remark that this solution has constant energy equal to

$$E_\varepsilon(x) = \frac{\varepsilon^2}{2}|u'_\varepsilon(x)|^2 - \frac{1}{2}V(x)|u_\varepsilon(x)|^2 + \frac{1}{p+1}|u_\varepsilon(x)|^{p+1}, \tag{2.8}$$

and, depending on the values of this energy, it may be periodic (changing or fixed sign) or homoclinic. We let $T_\varepsilon(x)$ half of the period of v_ε and define the approximate action as

$$\mathcal{A}_\varepsilon(x) = \begin{cases} \frac{1}{2}\int_0^{T_\varepsilon(x)} (v'_\varepsilon)^2 ds & \text{if } E_\varepsilon(x) \geqq 0, \\ \int_0^{T_\varepsilon(x)} (v'_\varepsilon)^2 ds & \text{if } E_\varepsilon(x) < 0. \end{cases} \tag{2.9}$$

In Section 4 we prove the following theorem.

Theorem 2.1. *Let* $(u_\varepsilon)_{0<\varepsilon<\varepsilon_0}$ *be an* L^∞ *bounded family of solutions of* (1.1) *with a Neumann boundary condition on* ∂I. *Then after extracting a subsequence* $\varepsilon_n \to 0$, $\mathcal{A}_{\varepsilon_n}(x)$ *converges to an adiabatic profile* $\mathcal{A}$.

Remark 2.1. Let (a, b) be an isolated connected component of the support of $\mathcal{A}$ then,

(i) if $\mathcal{A}(x) < \mathcal{A}_0(x)$ in (a, b), $u_{\varepsilon_n}(x)$ is constant sign cluster in (a, b), and
(ii) if $\mathcal{A}(x) > \mathcal{A}_0(x)$ in (a, b), $u_{\varepsilon_n}(x)$ is sign-changing cluster in (a, b).

Constant sign clusters may be classified according to their sign (two cases), and sign-changing clusters may be classified according to the sign of the first and last critical point (four cases).

Remark 2.2. As we see in Section 4, the proof of this theorem is essentially given in [15, 16], where we studied a similar problem for the balanced and unbalanced Allen–Cahn equations. There we used the notion of envelopes instead of adiabatic profiles (see Section 3).

Conversely, given an adiabatic profile in I, with connected support we can use the techniques developed in [15] and [16] to construct clusters with an approximate action converging to the $\mathcal{A}$. Moreover, if the profile $\mathcal{A}$ possesses a support with several connected components, but satisfies $\mathcal{A}(x) \geqq \mathcal{A}_0(x)$ in I (or $\mathcal{A}(x) \leqq \mathcal{A}_0(x)$ in I), we can still apply the ideas of [15] and [16]. In fact, the corresponding finite-dimensional functional has to be minimized or maximized and the gluing process can be done without extra difficulty (see [16]).

However, when the adiabatic profile $\mathcal{A}$ has values above $\mathcal{A}_0$ and below $\mathcal{A}_0$ in I, then the corresponding finite-dimensional problem has a min–max structure. In principle, we could try to find a critical point by a min–max technique, but it would require very precise estimates, for a functional in a space of dimensions increasing as ε^{-1}. We do not know if this is possible. On the other hand, if we study the problem in $I\!R$ and try to construct solutions gluing an infinite number of clusters (for a periodic potential for example) the estimates would simply be impossible to obtain. Consequently a different approach is required.

It is the purpose of this article to develop a degree theoretic approach for the construction of solutions with multiple clusters. At the heart of our construction is the definition of a simple problem on which we can make degree computation. Then, through homotopy, we obtain information on the degree for our original problem. More precisely, we associate a given component of the support of the adiabatic profile a problem with a step potential, for which it is particularly easy to handle degree computations. Once we have estimated the degree for each component of the support of $\mathcal{A}$, the gluing process is fairly simple.

With this construction we can prove our main theorem

Theorem 2.2. *Assuming $p > 1$ and V is positive and of class C^1 in the interval I, then given any adiabatic profile $\mathcal{A}$ there exists a family u_ε of solutions to* (1.1) *with a Neumann boundary condition on ∂I, such that the approximate action $\mathcal{A}_\varepsilon$ associated to u_ε converges to $\mathcal{A}$.*

Remark 2.3. In order to keep the statement of Theorem 2.2 simple, we do not make precise considerations about the sign of the clusters that we construct. However, we can handle all possible situations as seen in Section 9. For the case of a bounded interval I, Theorem 2.2 and its extension in Section 9 gives account of all possible L^∞ bounded solutions.

Remark 2.4. In Section 9 we also consider the case of solutions of (1.1) in $I\!R$. Several situations are considered, but in general we cannot prove that every adiabatic profile is globally reached. Some uniform assumption on the potential, such as periodicity for example, is needed to give a global control. In this case various classes of chaotic solutions can be constructed.

Remark 2.5. In this article we only consider positive potentials. A potential with negative values can also be treated, but we did not just do it for simplicity in the description of the results.

Remark 2.6. Since our approach for the construction of solutions is based on a combination of the Nehari method with degree theory, we think it is possible to extend it to study some non-variational perturbations of (1.1). Specifically we think we can treat a problem like

$$\varepsilon u' = v + \varepsilon g_1(x, u, v) \tag{2.10}$$

$$\varepsilon v' = V(x)u - |u|^{p-1}u + \varepsilon g_2(x, u, v) \tag{2.11}$$

under appropriate hypotheses on the perturbation functions g_1 and g_2. However, we do not pursue this line of research.

Highly oscillatory solutions are very natural in the context of slowly-varying systems, however, as far as we know, not much is known in the literature about the rigorous construction of these solutions. In this direction we mention the earlier work by KURLAND [20] on the existence of highly-oscillatory solutions for unbalanced Allen–Cahn equation, and its connection with *adiabatic invariants*, see [7]. In [20], KURLAND constructs highly-oscillatory *local solutions* for which the oscillations stay away from homoclinic or heteroclinic orbits, in our terminology this is the case where supp $\mathcal{A} = I$. This allows a change of variables, transforming the system to action–angle variables. We also mention the work by AI in [1] and [2], where the author uses a shooting method to construct solutions for certain equations, having a number of oscillations of order ε^{-1}. Contributions are also given in [14, 15] and [16] for the Allen–Cahn equation and for the nonlinear Schrödinger equation in [13].

For solutions with prescribed Morse index, in the case of the balanced Allen–Cahn equation we have the work by NAKASHIMA [21, 22], and NAKASHIMA & TANAKA [23], and for unbalanced Allen–Cahn equation we have the work by ALIKAKOS, BATES and FUSCO [6], where some highly-oscillatory solutions with small amplitude are also constructed. We mention the work of HASTINGS and MC LEOD [18], where periodic solutions to a second-order equation with a slowly-varying force are found. This work motivated further research in systems with slowly-varying coefficients. In more recent work, AI and HASTINGS [4] and AI, CHEN and HASTINGS [3] construct solutions that combine peaks and transition layers. We also refer to the work of GEDEON, KOKUBU, MISCHAIKOW and OKA [17] where Conley index theory is used to construct solutions with oscillations prescribed in terms of symbolic sequences of integers, for a slowly-varying planar Hamiltonian system.

The problem of gluing concentrating solutions has received enormous attention during the last fifteen years. We mention the pioneering work of SÉRÉ [27] and COTI-ZELATI and RABINOWITZ [11], and subsequent papers of many others. Particularly interesting to our analysis is the work of ALESSIO and MONTECHIARI [5] and KANG & WEI [19]. In all these works a good understanding of the properties of the objects to be glued is needed, for instance uniqueness or non-degeneracy, which is expressed in analytical or topological terms. In the problem here such information seems more elusive, since the clusters we have in mind are solutions that do not survive in a reasonable manner in the limit procedure, as in the case of a single or multi-peak or transition layer.

3. Envelopes and adiabatic profiles

In earlier work [13, 16], we have defined the notion of an envelope, which is very useful in the analysis of our problem. The envelope represents the asymptotic amplitude of the solutions we are studying and it is closely related to the Nehari method, which consists of matching the amplitude of broken solutions.

On the other hand, in Section 2 we have introduced the adiabatic profiles in order to describe our results. These functions are easily defined and are very useful to describe our results, see [16].

In this section we make the connection between these two important concepts. Keeping the notation of Section 2, we define the auxiliary function

$$Q(V, y) = \frac{1}{T(V, y)} \int_0^{T(V,y)} v^2(V, y; s) ds,$$

when $T(V, y)$ is finite, and we set $Q(V, y) = 0$ if $y = -y_0, 0$ or y_0. Then we introduce the field

$$H(x, y) = \frac{V'(x)(y^2 - Q(V(x), y))}{2y(|y|^{p-1} - V(x))}.$$

Definition 3.1. We say that a continuous function $e : I \to I\!R$, such that $e(x) \geqq e_*(x)$ for all $x \in I$, is an envelope if it satisfies the ordinary differential equation

$$e' = H(x, e), \quad \text{for all } x \in I. \tag{3.1}$$

Among all envelopes, e_* and e_0 are distinguished and called trivial envelopes. We define the support of an envelope, the set

$$\text{supp}(e) = \{x \in I\!R / e(x) \neq e_0(x)\}.$$

Since the function A, defined in (2.3), is an increasing function for $y \geqq y_*$, the equation

$$A = A(V, y)$$

defines also a C^1 function $y = y(V, A)$. Then we have the key relation

Lemma 3.1. *If $\mathcal{A}$ and e are continuous functions and satisfy the equation*

$$\mathcal{A}(x) = A(V(x), e(x)), \quad x \in I, \tag{3.2}$$

then $\mathcal{A}$ is an adiabatic profile if and only if e is an envelope.

Before proving this lemma, we introduce the energy function, which provides yet another alternative for the description of our results. We define the energy function as

$$\mathcal{E}(x) = -V(x)\frac{e(x)^2}{2} + \frac{e(x)^{p+1}}{p+1}. \tag{3.3}$$

Direct differentiation with respect to x, using equation (3.1) for e, leads to

$$\dot{\mathcal{E}}(x) = -\frac{V'(x)}{2} R(V(x), \mathcal{E}(x)), \tag{3.4}$$

which is analogous to equation (3.1) for the envelope e. Here the function R is defined by

$$R(V, E) = Q(V, y),$$

where $E = -Vy^2/2 + y^{p+1}/(p+1)$. Thus $\mathcal{E}$ is a solution to (3.4) if and only if e is an envelope.

Proof of Lemma 3.1. By conservation of energy, for the solution $v = v(V(x), e(x); s)$ of (2.1)–(2.2), we have

$$\frac{(v'(s))^2}{2} = V(x)\frac{(v(s))^2}{2} - \frac{(v(s))^{p+1}}{p+1} + \mathcal{E}(x). \tag{3.5}$$

Then, differentiating (3.5) with respect to x, integrating from 0 to $T = T(V(x), e(x))$ and using the equation for v we get

$$2\int_0^T v'(v_x)' ds = V'(x)\int_0^T \frac{(v(s))^2}{2} ds + T\dot{\mathcal{E}}(x). \tag{3.6}$$

Thus, if e is an envelope, using the equation for $\mathcal{E}$ on $x \in \operatorname{supp}(e)$ we find that $\dot{\mathcal{A}}(x) = 0$. Conversely, if $\mathcal{A}$ is an adiabatic profile, the left-hand side of (3.6) is zero. Then we find

$$V'(x)Q(V(x), e(x)) - V'(x)\frac{e^2}{2} - (V(x)e - e^p)\dot{e} = 0,$$

from where the equation for e follows, proving e is an envelope. □

It follows from (3.6) that

$$\dot{\mathcal{A}}_0(x) = V'(x)\int_0^T \frac{v^2}{2} ds,$$

so that, in particular, the critical points of V and of $\mathcal{A}_0$ are the same. Thus, we can make a complete description of all possible adiabatic profiles.

In view of this observation and the relation between adiabatic profiles and envelopes given by Lemma 3.1, we obtain all possible solutions of the envelope equation in

$$\{(x, y) \,/\, x \in I,\, y > e_*(x)\}.$$

By proper reflection with respect to the x axis and also with respect to the graph of e_*, we can obtain the whole diagram of solutions of (3.1) in the plane.

Remark 3.1. From the connection between adiabatic profiles and envelopes we observe a bifurcation phenomenon occurring in the envelopes. Let $[a, b]$ be an interval such that $V'(x) > 0$ for all $x \in (a, b)$, then for every $\bar{x} \in (a, b)$ there is an envelope $e : I \to I\!R$ such that $(\bar{x}, e_0(\bar{x}))$ is a bifurcation point to the right, that is, in a neighborhood of $\bar{x}$, $e(x) = e_0(x)$ for $x < \bar{x}$ and $e(x) < e_0(x)$ for $x > \bar{x}$. This envelope is the only one bifurcating to the right.

On the other hand, there is an envelope e such that $(\bar{x}, e_0(\bar{x}))$ is a bifurcation point to the left, that is, in a neighborhood of $\bar{x}$, $e(x) = e_0(x)$ for $x > \bar{x}$ and $e(x) > e_0(x)$ for $x > \bar{x}$. This is envelope is the only one bifurcating to the left. Certainly, there is also an envelope e bifurcating to both sides, such that $e(\bar{x}) = e_0(\bar{x})$ and, in a neighborhood of $\bar{x}$, we have $e(\bar{x}) > e_0(\bar{x})$ if $x < \bar{x}$ and $e(\bar{x}) < e_0(\bar{x})$ if $x > \bar{x}$.

Analogous bifurcation phenomena hold true in the interval $[a, b]$ if we have $V'(x) < 0$ for all $x \in (a, b)$.

Remark 3.2. If $\bar{x}$ is an isolated local minimum of the potential, then $\bar{x}$ is a bifurcation point to the right and to the left, with bifurcating e staying locally below e_0. An analogous statement holds true at an isolated local maximum of the potential with bifurcation e staying locally above e_0.

4. Asymptotic behavior of solutions to (1.1)

This section reviews some basic facts about the asymptotic behavior of a family of solutions of (1.1), as the parameter ε approaches 0. The oscillatory character of the solutions and their asymptotic behavior, as described by an envelope, or an adiabatic profile, has been obtained in the case of the unbalanced Allen–Cahn equation in [16]. In this section, we provide proofs of these facts in the case of the nonlinear Schrödinger equation for completeness.

Assume we have functions $u_n : [a_n, b_n] \to I\!R$ satisfying

$$\varepsilon_n^2 u_n'' - V(x)u_n + |u_n|^{p-1}u_n = 0, \tag{4.1}$$

$$u_n'(a_n) = u_n'(b_n) = 0, \tag{4.2}$$

where $\varepsilon_n \to 0$ as $n \to \infty$. Assume further that $\lim_{n\to\infty} a_n = \bar{a}$, $\lim_{n\to\infty} b_n = \bar{b}$, and $\|u_n\|_{L^\infty(a_n,b_n)}$ is bounded.

Let $a_n < y_n^0 < y_n^1 < \cdots < y_n^{s_n-1} < y_n^{s_n} < b_n$ be the local maximum points of $|u_n|$ in $[a_n, b_n]$ and assume that $s_n \to \infty$ as $n \to \infty$. Considering a subsequence, if necessary, we define

$$\alpha = \lim_{n\to\infty} y_n^0 \quad \text{and } \beta = \lim_{n\to\infty} y_n^{s_n}.$$

In Proposition 5.2 from [13], it is proved that if V' does not vanish in $[\alpha, \beta]$ then in any given interval $[x_1, x_2] \subset (\alpha, \beta)$ there is n_0 so that for every $n \geqq n_0$ the solution u_n has at least one maximum point and one minimum point in $[x_1, x_2]$. The following result is crucial in our analysis.

Proposition 4.1.

1. *Assume V' is positive in $\left[\bar{a}, \bar{b}\right]$. Suppose that u_n is positive (or negative), and a_n and b_n are local minima (or local maxima) of u_n, then we have*
   (i) *if $\alpha > \bar{a}$ then $|u_n|(y_n^0) \to e_0(\alpha)$,*
   (ii) *if $y_n^{i_n} \to \bar{x} \in (\alpha, \bar{b}]$ then $\limsup_{n\to\infty} |u_n|(y_n^{i_n}) < e_0(\bar{x})$, and*
   (iii) $\bar{b} = \beta$.
2. *Assume V' is negative in $\left[\bar{a}, \bar{b}\right]$. Suppose that u_n changes sign and at a_n it satisfies $u_n(a_n) = 0$ and $u_n'(a_n) > 0$ (or $u_n'(a_n) < 0$) and at b_n we assume $u_n(b_n) = 0$ and $u_n'(b_n) < 0$ (or $u_n'(b_n) > 0$). Then*
   (i) *if $\alpha > \bar{a}$ then $|u_n|(y_n^0) \to e_0(\alpha)$,*
   (ii) *if $y_n^{i_n} \to \bar{x} \in (\alpha, \bar{b}]$ then $\limsup_{n\to\infty} |u_n|(y_n^{i_n}) > e_0(\bar{x})$, and*
   (iii) $\bar{b} = \beta$.

We can make analogous statements as in 1 if V' is negative, and as in 2 if V' is positive.

Proof.

(i) Since $\bar{a} < \alpha$, rescaling u_n around y_n^1 leads to a homoclinic orbit of the limiting equation, so that $u_n(y_n^1) \to e_0(\alpha)$. Moreover, if $\{y_n^{i_n}\}$ is a sequence of maximum points of u_n such that $y_n^{i_n} \to \alpha$ then $|u_n|(y_n^{i_n}) \to e_0(\alpha)$, as we see from

$$\left(-V(y)\frac{u_n^2(y)}{2} + \frac{u_n^{p+1}(y)}{p+1}\right)\Bigg|_{y_n^1}^{y_n^{i_n}} = -\int_{y_n^1}^{y_n^{i_n}} V'(x)\frac{u_n^2(x)}{2}dx. \tag{4.3}$$

(ii) If $\limsup_{n\to\infty} u_n(y_n^1) < e_0(\alpha)$, then the result follows from (4.3). We assume then, for contradiction, that for a subsequence $\lim_{n\to\infty} u_n(y_n^1) = e_0(\alpha)$ and $\lim_{n\to\infty} u_n(y_n^{i_n}) \to e_0(\bar{x})$. In view of (4.3), this implies that

$$\lim_{n\to\infty}\int_{y_n^1}^{y_n^{i_n}} V'(x)u_n^2(x)dx = 0 \tag{4.4}$$

and that for any sequence $\{l_n\}$, with $l_n \in \{1, 2, \ldots, i_n\} \equiv K_n$, such that $y_n^{l_n} \to \bar{y} \in [\alpha, \bar{x}]$, we have

$$\lim_{n\to\infty} u_n(y_n^{l_n}) = e_0(\bar{y}).$$

This last fact implies that, uniformly in the sequence $\{l_n\} \subset K_n$,

$$\lim_{n\to\infty}\frac{y_n^{l_n+1} - y_n^{l_n}}{\varepsilon_n} = \infty.$$

Next, let $r_0 > 0$ and v^* be the homoclinic solution of (2.1)–(2.2), with $V = V(x)$ and $y = e_0(x)$, for $x \in [\alpha, \bar{x}]$. Then there exists a positive constant A_1 such that

$$\int_{-r_0}^{r_0} (v^*)^2(s))ds \geqq A_1 > 0, \tag{4.5}$$

for all $x \in [\alpha, \bar{x}]$. Thus, letting $z_n^k < y_n^k < z_n^{k+1}$ be the minimum points enclosing y_n^k, we see that

$$\liminf_{n\to\infty} \frac{1}{\varepsilon_n} \int_{z_n^{l_n}-r_0\varepsilon_n}^{z_n^{l_n}+r_0\varepsilon_n} u_n^2(x)dx \geqq A_1, \tag{4.6}$$

uniformly in the sequence $\{l_n\} \subset K_n$. From here we find

$$\int_{y_n^1}^{y_n^{i_n}} V'(x)u_n^2(x))dx \geqq \varepsilon_n(i_n - 2)A_2, \tag{4.7}$$

for certain constant $A_2 > 0$. Thus, to get a contradiction between (4.7) and (4.4) we just need to prove that the sequence $\{\varepsilon_n i_n\}$ is bounded away from zero. From (4.3) and (4.7) we find a positive constant A_3 such that

$$-V(y_n^k)\frac{u_n^2(y_n^k)}{2} + \frac{u_n^{p+1}(y_n^k)}{p+1} \leqq -\varepsilon_n k A_3, \qquad \forall k \in K_n,$$

which implies that for any sequence $\{l_n\}$, with $l_n \in \{1, 2, \ldots, i_n\} \equiv K_n$, such that $y_n^{l_n} \to \bar{y} \in [\alpha, \bar{x}]$, we have

$$e_0(\bar{y}) - u_n(y_n^k) \geqq \varepsilon_n k A_4$$

for some A_4. From here we can get an upper bound for the period, that is, there is constant $\gamma_1 > 0$ such that for all $k \in K_n$

$$T(V(y_n^k), u_n(y_n^k)) \leqq -\gamma_1 \ln(\varepsilon_n k A_4) \tag{4.8}$$

(see Lemma 4.1 in [13] for a similar logarithmic estimate of the period function). Next we estimate $z_n^{k+1} - z_n^k$ in terms of the period. We let v_n be the solution of the equation

$$\varepsilon_n^2 v_n'' - V(y_n^k)v_n(x)) + v_n^p(x)) = 0,$$

with initial conditions $v_n'(y_n^k) = 0$ and $v_n(y_n^k) = u_n(y_n^k)$. By our hypothesis on V we have $V(x) \leqq V(y_n^k)$ for all $x \in \left[y_n^{k-1}, y_n^k\right]$. While u_n and v_n are decreasing, we define x_u and x_v as their inverses, respectively. Then we have

$$-\frac{\varepsilon^2}{2}\frac{d}{ds}\left(\frac{1}{(x_u')^2} - \frac{1}{(x_v')^2}\right) = (-V(x_u) + V(y_n^k))s,$$

and so $(x_v')^2 > (x_u')^2$. Let $\bar{x}_k \in \left[y_n^k - \varepsilon_n T(V(y_n^k), u_n(y_n^k)), y_n^k\right]$ so that $u_n(\bar{x}_k) = v_n(y_n^k - \varepsilon_n T(V(y_n^k), u_n(y_n^k)))$. We note that $(y_n^k - \bar{x}_k)/\varepsilon_n \leqq T(V(y_n^k), u_n(y_n^k))$ and, since $(\bar{x}_k - z_n^k)/\varepsilon_n$ is bounded, we find $(\bar{x}_k - z_n^k)/\varepsilon_n \leqq T(V(y_n^k), u_n(y_n^k))$. Thus

$$(y_n^k - z_n^k) \leqq 2\varepsilon_n T(V(y_n^k), u_n(y_n^k)), \qquad \forall k \in K_n.$$

Using similar arguments, we compare $u_n(x)$ with $u_n(2y_n^k - x)$ and find the same estimate for $z_n^{k+1} - y_n^k$, we conclude that

$$\frac{1}{2}(z_n^{k+1} - z_n^k) \leqq \varepsilon_n T(V(y_n^k), u_n(y_n^k))), \qquad \forall k \in K_n. \tag{4.9}$$

From here and (4.8), we obtain

$$z_n^{i_n} - z_n^1 \leqq 2\varepsilon_n \sum_{k=1}^{i_n} T(V(y_n^k), u_n(y_n^k)) \leqq -2\gamma_1 \varepsilon_n \sum_{k=1}^{i_n} \ln(\varepsilon_n k A_4).$$

Hence, using that $M! \geqq (rM)^M$ for a certain $r > 0$, we find

$$\frac{1}{2}(\bar{x} - \alpha) \leqq z_n^{i_n} - z_n^1 \leqq -2\gamma_1 \varepsilon_n i_n \ln(\varepsilon_n i_n r A_4),$$

from where we conclude that $\{\varepsilon_n i_n\}$ must be bounded away from zero, completing the proof of (ii). □

Remark 4.1. In Proposition 4.1 the way the oscillations take off from the trivial envelope are made precise. Moreover, Proposition 4.1 assures that the oscillations of positive solutions tend to concentrate where the potential has higher values and that the oscillations of sign-changing solutions concentrates at lower values of V.

We observe that it could happen that $\alpha = \beta$, in which case we have $\alpha = \beta = \bar{b}$.

With the information provided by Proposition 4.1 we can make precise statements about the asymptotic behavior of u_n on the interval $[\bar{a}, \bar{b}]$. We define the approximate upper envelope function $e_{\varepsilon_n} : [a_n, b_n] \to I\!R$ as follows: in the interval $[y_n^0, y_n^{s_n}]$ we consider

$$e_{\varepsilon_n}(x) = |u_n\left(y_n^k\right)| + \frac{|u_n\left(y_n^{k+1}\right)| - |u_n\left(y_n^k\right)|}{y_n^{k+1} - y_n^k}\left(y_n^{k+1} - x\right), x \in \left[y_n^k, y_n^{k+1}\right],$$

for $k = 1, \dots, s_n - 1$. If $\alpha > \bar{a}$ we extend e_{ε_n} as the trivial envelope e_0 to $[a_n, y_n^0 - \varepsilon_n]$, and in $(y_n^0 - \varepsilon_n, y_n)$ we interpolate linearly. In case $\alpha = \bar{a}$ we extend e_{ε_n} linearly to $[a_n, y_n^0]$. To the other extreme we extend e_{ε_n} linearly to $[y_n^{s_n}, b_n]$.

If we just assume $V'(\bar{a}) > 0$ and $V'(\bar{b}) < 0$, we can extend our analysis by working in subintervals where V' does not vanish and thus, we can thus construct an approximate envelope in $[a_n, b_n]$.

Proposition 4.2. *Under the conditions described above we have that the sequence* $\{e_{\varepsilon_n}\}$ *converges uniformly to a solution* e *of* (3.1) *in* $[\bar{a}, \bar{b}]$. *The envelope* e *is the trivial envelope* $e_0(x)$ *in the intervals* $[\bar{a}, \alpha)$ *and* $(\beta, \bar{b}]$.

Proof. The proof of this proposition is similar to Theorem 6.1 in [13]. See also Proposition 3.4 in [16]. □

Remark 4.2. Let us denote by $N_n(x_0, x_1)$ the number of zeroes of u_n in $[x_0, x_1] \subset (\bar{a}, \bar{b})$ then, by a simple argument as in [14], we can prove that

$$\lim_{n\to\infty} N_n \varepsilon_n = \int_{x_0}^{x_1} \omega((V(x), e(x))dx, \tag{4.10}$$

where e is the envelope. The frequency function ω was defined in Section 2.

Remark 4.3. If u_n is positive and $u'(a_n) \neq 0$ and $u'(b_n) \neq 0$, but a_n and b_n are local minima and both $y_n^0 - a_n$ and $b_n - y_n^{s_n}$ stays away from zero, then Proposition 4.2 still holds true. A similar situation occurs if u_n is negative or sign changing.

To conclude this section we see how to obtain Theorem 2.1 from here. We already know from Proposition 4.2 that the sequence e_{ε_n} converges conclude up to a subsequence to an envelope e. Moreover, if $(a, b) \subset I$ is an isolated connected component of the support of e then

(i) If $e(x) < e_0(x)$ in (a, b), $u_n(x)$ is constant sign cluster in (a, b).
(ii) If $e(x) > e_0(x)$ in (a, b), $u_n(x)$ is sign-changing cluster in (a, b).

We can say more, the solution u_n oscillates near $x \in (a, b)$ with an amplitude approximately equal to $e(x)$ and with a frequency near $\omega(V(x), e(x))$.

In order to complete the proof of Theorem 2.1, we just need to make the connection between the approximate envelope e_{ε_n} with the approximate action as defined in (2.9). For this purpose we consider $x \in I$ and the function v_{ε_n} defined by (2.6)–(2.7). We define

$$\tilde{e}_{\varepsilon_n}(x) = \max_{s\in I\!R} |v_{\varepsilon_n}(x; s)|.$$

It is clear that the approximate action satisfies

$$\mathcal{A}_{\varepsilon_n}(x) = A(V(x), \tilde{e}_{\varepsilon_n}(x)),$$

so that we only need to prove that,

Lemma 4.1. *After extracting a subsequence,*

$$\lim_{n\to\infty} \tilde{e}_{\varepsilon_n}(x) = \lim_{n\to\infty} e_{\varepsilon_n}(x).$$

Proof. Consider $E_{\varepsilon_n}(x)$ as defined in (2.8). Then we have

$$\frac{d}{dx} E_{\varepsilon_n}(x) = -\frac{1}{2} V'(x)|u_\varepsilon(x)|^2,$$

so that $E_{\varepsilon_n}(x)$ is bounded in $W^{1,\infty}(I)$ as $n \to \infty$. In particular $E_{\varepsilon_n}(x)$ has a uniformly convergent subsequence. We also have that

$$E_{\varepsilon_n}(x) = -\frac{1}{2} V(x)\tilde{e}_{\varepsilon_n}(x)^2 + \frac{1}{p+1}\tilde{e}_{\varepsilon_n}(x)^{p+1}, \tag{4.11}$$

so that $\tilde{e}_{\varepsilon_n}(x)$ also has a uniformly convergent subsequence.

Let $x_0 \in int(I)$ and suppose that for $\delta > 0$, local maxima of $u_{\varepsilon_n}(x)$ are dense in $(x_0 - \delta, x_0 + \delta)$. Then we can easily see that $e_{\varepsilon_n}(x_0)$ and $\tilde{e}_{\varepsilon_n}(x_0)$ have a common limit. On the other hand, if for $\delta > 0$ local maxima of $u_\varepsilon(x)$ do not appear densely in $(x_0 - \delta, x_0 + \delta)$, by Proposition 4.2 we have $\lim_{n\to\infty} e_{\varepsilon_n}(x_0) = e_0(x_0)$. We also have $\lim_{n\to\infty} E_{\varepsilon_n}(x_0) = 0$ and thus $\tilde{e}_{\varepsilon_n}(x_0) \to e_0(x)$. □

5. Formulation. Nehari's method

In this section we formulate the finite-dimensional problem to which we reduce our existence result. This reduction is well known in various contexts, such as ordinary differential equations and geometry. In the case of nonlinear eigenvalue problems for differential equations this method was first used by NEHARI [24].

Consider an interval $[m, M]$ and an envelope $e : [m, M] \to I\!R$ such that $\mathrm{supp}(e) = (a, b)$ with $m < a < b < M$. Moreover, we assume that $V'(a) > 0$, $V'(b) < 0$ and $e(x) < e_0(x)$ in (a, b).

Associated e we have the number N_ε defined as

$$N_\varepsilon = \left\lfloor \frac{1}{\varepsilon} \int_m^M \omega(V(s), e(s))ds \right\rfloor, \tag{5.1}$$

with $\lfloor s \rfloor$ denoting the closest odd integer less than s; N_ε corresponds to the expected number of oscillations that the solution we are looking for has.

In order to formulate our finite-dimensional problem we need an auxiliary envelope $\tilde{e} : [m, M] \to I\!R_+$ such that $[a, b] \subset \mathrm{supp}(\tilde{e}) = (\tilde{a}, \tilde{b}) \subset \left[\tilde{a}, \tilde{b}\right] \subset (m, M)$. We define, for $x, y \in [m, M]$,

$$d(x, y) = \frac{1}{\varepsilon} \int_x^y \omega(V(s), \tilde{e}(s))ds \tag{5.2}$$

and we introduce the domain in $I\!R^{N_\varepsilon}$

$$\Delta_\varepsilon = \{(x_1, x_2, \dots, x_{N_\varepsilon}) \,/\, d(x_i, x_{i+1}) > 1,$$
$$i = 1, \dots, N_\varepsilon - 1,\ x_1 > \tilde{a} + \varepsilon t_0,\ x_{N_\varepsilon} < \tilde{b} - \varepsilon t_0\}, \tag{5.3}$$

with $t_0 > 0$. Also consider $x_0 = m, x_{N_\varepsilon+1} = M$. Since the period function is monotone increasing in e, for $e < e_0(x)$ as we see later in Proposition 5.1, we have that ω decreases in e and we obtain that

$$\int_m^M \omega(V(s), \tilde{e}(s))ds > \int_m^M \omega(V(s), e(s))ds.$$

As a consequence the set Δ_ε is not empty.

Under our conditions in V and e we are in a position to construct a positive cluster. Given $X = (x_1, x_2, \dots, x_{N_\varepsilon}) \in \Delta_\varepsilon$, using the fact that $d(x_i, x_{i+1}) \geqq 1$, we will prove later in Theorem 5.1 that for every $i = 0, \dots, N_\varepsilon$ the equation

$$\varepsilon^2 u_i'' - f(x, u_i) = 0, \qquad u_i'(x_i) = 0 = u_i'(x_{i+1}), \tag{5.4}$$

with the extra condition

$$(-1)^i u_i' > 0, u_i > 0 \quad \text{in } \left[x_i, x_{i+1}\right], \tag{5.5}$$

possesses a unique solution $u_i : \left[x_i, x_{i+1}\right] \to I\!R$. Here we have written $f(x, u) = V(x)u - |u|^{p-1}u$. Solutions to (5.4), for different i's are our building blocks and we call them *basic solutions*.

It will be notationally convenient to define an energy density as

$$\sigma_\varepsilon(x,u) = \varepsilon^2 \frac{u'^2(x)}{2} + F(x,u(x)),$$

where $F(x,u) = V(x)\frac{u^2}{2} - \frac{u^{p+1}}{p+1}$. We define our finite-dimensional functional $g_\varepsilon\colon \Delta_\varepsilon \to I\!R$ for $X \in \Delta_\varepsilon$ as:

$$g_\varepsilon(X) = \sum_{i=0}^{N_\varepsilon} \int_{x_i}^{x_{i+1}} \sigma_\varepsilon(x,u_i)dx. \tag{5.6}$$

We can easily check that

$$\frac{\partial g^\varepsilon}{\partial x_i}(X) = -F(x_i,u_i(x_i)) + F(x_i,u_{i-1}(x_i)), \quad 1 \leqq i \leqq N_\varepsilon.$$

Thus, if $\nabla g_\varepsilon(X) = 0$ then the function u_ε, defined as

$$u_\varepsilon(x) = u_i(x), \quad x \in \left[x_i, x_{i+1}\right], \quad i = 0,\ldots,N_\varepsilon, \tag{5.7}$$

is a solution of (1.1) in (m,M) with $u'_\varepsilon(m) = u'_\varepsilon(M) = 0$.

If we are looking for a negative cluster, we just need to change the signs in (5.5).

If we are looking for sign-changing clusters, we assume that the envelope e: $[m,M] \to I\!R$ is such that $\operatorname{supp}(e) = (a,b)$ and $e(x) > e_0(x)$ in (a,b). Moreover, we assume that $V'(a) < 0$, $V'(b) < 0$.

Depending on the type of sign-changing cluster (among the four possible ones) that we are willing to construct, we need to redefine N_ε and change condition (5.5) accordingly. If for example, we want a sign-changing cluster, starting with a maximum and ending with a minimum, then N_ε needs to be chosen even and condition (5.5) should be replaced by

$$(-1)^i u'_i > 0, \text{ in } \left[x_i, x_{i+1}\right],\ u_0 > 0,\ u_{N_\varepsilon} < 0, \tag{5.8}$$

and u_i changing sign in $\left[x_i, x_{i+1}\right]$ if $1 \leqq i \leqq N_\varepsilon - 1$. Moreover, in this case the auxiliary envelope $\tilde{e} : [m,M] \to I\!R_+$ is such that $[a,b] \subset \operatorname{supp}(\tilde{e}) = (\tilde{a},\tilde{b}) \subset \left[\tilde{a},\tilde{b}\right] \subset (m,M)$ and $e(x) > e_0(x)$ in (a,b).

We now discuss the existence and uniqueness of basic solutions. For this purpose we need monotonicity properties of the period function $T(\cdot,\cdot)$ defined in Section 2. The next proposition provides the properties we need about the period function $T(y) = T(V,y)$.

Proposition 5.1. *For the period function T defined in $[y_*, y_0) \cup (y_0,\infty)$ we have*

$$\frac{dT}{dy}(y) > 0 \quad y \in (y_*, y_0) \quad \text{and} \quad \frac{dT}{dy}(y) < 0 \quad y \in (y_0,\infty).$$

Proof. If $y \in (y_0,\infty)$ then the result follows from the uniqueness theorem for the Dirichlet problem as given in Theorem 7 and Lemma 8 of [8]. When $y \in [y_*, y_0)$ the result is a consequence of Proposition 3.1 of [10]. □

Based on the monotonicity of the period we obtain a non-degeneracy property of the linearized equation associated to (2.1) with Neumann boundary conditions. We observe that the solution $v = v(V(x), y; s)$ of (2.1)–(2.2) satisfies the Neumann boundary condition

$$v'(0) = v'(T(V(x), y)) = 0,$$

and $v'(s) < 0$ for $s \in (0, T(V(x), y))$. When the initial value is $e_0(x)$, then the period function is infinity and $v_0 = v(V(x), y; s)$ is a homoclinic orbit satisfying the boundary condition $v_0'(0) = 0$, $\lim_{s\to\infty} v_0(s) = 0$, with $v_0'(s) < 0$ for all $s > 0$. We have

Lemma 5.1. *The equations*

$$h'' - V(x)h + p|v|^{p-1}h = 0,\ h'(0) = h'(T(V(x), y)) = 0,$$

and

$$h'' - V(x)h + p|v_0|^{p-1}h = 0,\ \ h'(0) = \lim_{s\to\infty} h'(s) = 0,$$

have only the trivial solution.

From these non-degeneracy properties of the linearized frozen equations, we can prove the existence and uniqueness theorem for basic solutions of our equation. We remark that $\lim_{y\to\infty} T(V(x), y) = 0$ and $l_0(x) := \lim_{y\to y_*} T(V(x), y) = 2\pi/\sqrt{pV(x)}$. We have

Theorem 5.1. *Given $\delta > 0$, there exists ε_0 such that, for every $\varepsilon \in (0, \varepsilon_0)$, $x_0 \in [m, M]$, $x_0 + \varepsilon l \leqq M$ and letting $l_0(x_0) + \delta \leqq l$, the equation*

$$u'' - V(x_0 + \varepsilon s)u(s) + |u(s)|^{p-1}u(s) = 0, \tag{5.9}$$

$$u'(0) = u'(l) = 0,\ \ u' > 0,\ \ u > 0 \tag{5.10}$$

has a unique solution which is differentiable in x_0 and l.

Sign-changing solutions can also be found: replacing $l_0(x_0) + \delta$ by δ and (5.10) *by*

$$u'(0) = u'(l) = 0,\ u' > 0,\ u(0)u(l) < 0. \tag{5.11}$$

Naturally, in this theorem, the other combinations of positive/negative, increasing/decreasing can also be considered.

The proof is completely analogous to Theorem 4.1 in [16] so we omit it. There is one special case where the arguments in [16] may not be applicable directly, which is the case of a sign-changing solution which is getting close to homoclinic (from outside). In this case we can use Lemma 2.1 proved in [12].

6. Existence of basic solutions for a step potential

This section is devoted to the study of existence and uniqueness of basic solutions for a step potential. We discuss in detail the case of a positive cluster and at the end we mention the changes necessary to include the other cases.

We consider a given positive number $V_0 > 0$. We consider equation (2.1)–(2.2) with $V = V_0$ and we denote by $v(y; \cdot)$ its solution. According to Proposition 5.1, the period function of positive periodic orbits of this equation is monotone increasing with a minimum value that we denote by l_0. For $T > l_0$ let $y_T \in (y_*, y_0)$ such that $v'(y_T; -T) = 0$ and $v'(y_T; s) > 0$ for $s \in (-T, 0)$. We write $\hat{y}_T = v(y_T; -T)$ and define the curve

$$\gamma_T(y) = (v(y; T), v'(y; T))$$

for $y \in (0, \hat{y}_T]$. Next we consider a value $V_1 > V_0$ and define $y_*^1 = (V_1)^{1/(p-1)}$, $y_0^1 = (\frac{p+1}{2} V_1)^{1/(p-1)}$ and $v^1(y; \cdot)$ as the solution to the equation (2.1)–(2.2) with $V = V_1$. Associated with V_1 there is a minimum period $l_0(V_1) < l_0$. Assuming that $t > l_0(V_1)$ we define $y_t^1 \in (y_*^1, y_0^1)$ such that $(v^1)'(y_t^1; -t) = 0$ and

$$\gamma_t^1(y) = (v^1(y; -t), (v^1)'(y; -t)), \qquad y \in (y_t^1, y_0^1).$$

Our goal is to prove existence and uniqueness of solutions for the equation

$$w'' - \mathcal{V}(x)w + |w|^{p-1}w = 0, \quad w' > 0, \quad w(0) < y_0^1, \tag{6.1}$$

$$w'(0) = w'(-T - t) = 0, \tag{6.2}$$

with the step potential $\mathcal{V}$ defined as

$$\mathcal{V}(x) = \begin{cases} V_0 & \text{if } x \in [-T - t, -t) \\ V_1 & \text{if } x \in [-t, 0]. \end{cases} \tag{6.3}$$

We have

Lemma 6.1. *There exists t_0 such that for all $t \geqq t_0$, for all $T > l_0$ and for all $V_1 \in [V_0, V_0 + 1]$, the equation* (6.1)–(6.2) *has exactly one solution.*

Proof. If t is large enough then the curves γ_t^1 and γ_T intersect, that is, there are $\underline{y} \in (y_t^1, y_0^1)$ and $\bar{y} \in (0, \hat{y}_T)$ such that

$$\gamma_t^1(\underline{y}) = \gamma_T(\bar{y}). \tag{6.4}$$

Defining

$$w(x) = \begin{cases} v^1(\underline{y}; x) & \text{if } x \in [-t, 0] \\ v(\bar{y}; x + T + t) & \text{if } x \in [-T - t, -t] \end{cases} \tag{6.5}$$

we obtain a C^1 solution to equation (6.1)–(6.2).

Assuming that there is a sequence $\{t_n\}$ diverging to infinity so that (6.1)–(6.2) possesses more than one solution, we can construct a non-trivial solution to equation

$$h'' - V_1 h + p|v_*|^{p-1}h = 0, \;\; h'(0) = \lim_{s \to \infty} h'(s) = 0,$$

where v_* is the positive homoclinic orbit of (2.1) with $V = V_1$, for some $V_1 \in [V_0, V_0 + 1]$, contradicting Lemma 5.1. □

Keeping T fixed, we may consider $\underline{y}$ obtained in (6.4) as function of t, for $t \geqq t_0$. We have

Lemma 6.2. *Enlarging t_0 if necessary we have that the function $\underline{y} = \underline{y}(t)$ is strictly increasing, actually*

$$\frac{d\underline{y}}{dt} > 0, \quad \textit{for } t \geqq t_0.$$

Proof. As a first step we find the asymptotic behavior of $\gamma_T(y)$ as $y \to 0$. From conservation of energy for system (2.1) with $V = V_0$ we have

$$T = \int_1^{v(y,T)/y} \frac{dt}{\sqrt{V_0(t^2-1) - 2y^{p-1}(t^{p+1}-1)/(p+1)}},$$

from where it follows that

$$\lim_{y \to 0} \frac{v(y,T)}{y} = \bar{\alpha},$$

where $\bar{\alpha}$ is given by

$$T = \int_1^{\bar{\alpha}} \frac{dt}{\sqrt{V_0(t^2-1)}}.$$

Next we consider $y \in (y_*^1, y_0^1)$ and denote by $T(V_1, y)$ half of the period of the solution of system (2.1)–(2.2) with $V = V_1$, and we recall that $T(V_1, y)$ is monotone in y and it approaches infinity as $y \to y_0^1$. Given $\underline{y} \in (y_*^1, y_0^1)$ we solve the equation $\gamma^1_{t(\underline{y})}(\underline{y}) = \gamma_T(\bar{y})$ to get a solution $w(x)$ as in (6.5), with

$$t(\underline{y}) = T(V_1, \underline{y}) - t_1(\underline{y}), \tag{6.6}$$

where $t_1(\underline{y})$ is the time it takes system (2.1) with $V = V_1$ to go from $\gamma_T(\bar{y})$ to $(\bar{y}, 0)$. However, we see that

$$t_1(\underline{y}) = \int_1^{v(\bar{y},T)/\bar{y}} \frac{dt}{\sqrt{V_1(t^2-1) - 2\bar{y}^{p-1}(t^{p+1}-1)/(p+1)}}$$

and then

$$\lim_{\underline{y} \to y_0^1} t_1(\underline{y}) = \int_1^{\bar{\alpha}} \frac{dt}{\sqrt{V_1(t^2-1)}}.$$

Now the result follows from (6.6), since t approaches infinity if and only if $\underline{y}$ approaches y_0^1. $\square$

Next we construct solutions with a variable step potential, as will appear in the next section in the homotopy argument. For this purpose we first state a nondegeneracy property of the linearization of (6.1)–(6.2).

Lemma 6.3. *For $t \geqq t_0$ and $T \geqq l_0$, with $\mathcal{V}$ as in* (6.3) *and w as in* (6.5)*, the equation*

$$h'' - \mathcal{V}(x)h + p|w|^{p-1}h = 0, \tag{6.7}$$

$$h'(0) = h'(-T-t) = 0, \tag{6.8}$$

has only the trivial solution.

Here we may consider also the case when t or T are infinity, making the obvious modifications.

Proof. The function w constructed as in (6.5) can be differentiated with respect to the initial value. We obtain in such a way, a solution h_1 to (6.7), which satisfies the initial value $h_1(0) = 1$ and $h_1'(0) = 0$. Moreover, it follows from Lemma 6.2 that $h_1'(-T-t) \neq 0$.

On the other hand, we construct a second solution as $h_2(x) = w'(x)$, for $x \in (-t, 0]$ extended as the solution to

$$h'' - V_0 h + p|w|^{p-1}h = 0, \quad h(-t) = w'(-t), h'(-t) = w''(-t),$$

for $x \in [-T-t, -t]$. This function h_2 is a C^1 solution of (6.7), with initial values $h_2(0) = 0$ and $h_2'(0) \neq 0$.

If h is a C^1 solution of (6.7)–(6.8) then h is a linear combination of h_1 and h_2. Then the boundary conditions imply that $h = 0$. □

Remark 6.1. If in the construction of w we consider, instead of V_1, any value $V_\lambda \in [V_0, V_1]$, then the conclusion of the lemma still holds.

In Section 8 we consider a potential depending on a parameter $V : (-\infty, r] \times [0, 1] \to I\!R^+$, $r > 0$, with the following properties:

(i) $V(x, \lambda) = V_0$ for all $x < 0$.

(ii) $V(0, \lambda) = V_\lambda \in [V_0, V_1]$ for all $\lambda \in [0, 1]$.

(iii) V is C^1 in $[0, r)$.

Under these general conditions, using the non-degeneracy properties of the solutions of the step potential just proved in Lemma 6.3, we can follow the arguments given in Theorem 4.1 in [16] to prove the following theorem.

Theorem 6.1. *Given $R > 0$, there exists $\varepsilon_0 > 0$ such that, for every $\varepsilon \in (0, \varepsilon_0)$, $t > \varepsilon l_0$, $t < r$, $\lambda \in [0, 1]$ the equation*

$$\varepsilon^2 u'' - V(s, \lambda)u(s) + |u(s)|^{p-1}u(s) = 0, \tag{6.9}$$

$$u'(-R) = u'(\varepsilon t) = 0, \ \ u'(s) > 0, \ \ u(s) > 0 \tag{6.10}$$

has a unique solution which is differentiable in t.

Remark 6.2. Up to this point we considered a potential with a step up and we constructed positive increasing solutions, with $u(0) < y_0^1$. In this situation a negative decreasing solutions is similarly obtained. By reflection, the same construction allows us to consider a step down and a positive decreasing solution, and the corresponding negative increasing one. These solutions will allow us to construct positive (and negative) clusters.

In the construction of sign-changing clusters we need to obtain positive increasing solutions for a step down, satisfying $u(0) > y_0^1$. This construction can be done with only minor modifications to the arguments just given, in particular in the proof of Lemma 6.2. The negative decreasing solution is readily obtained by the same argument, and the other two solutions are obtained by reflection.

7. Computing the degree of a cluster for a step potential

In this section we define a special step potential in an interval, in such a way that equation (1.1) possesses a unique positive (or sign-changing) cluster. When this solution is considered as a critical point of (5.6) we prove it has index 1 or -1. We will consider in detail the case of a positive cluster.

Given $m < M \in I\!R$ and $V_0 < V_1 \leqq V_0 + 1$ fixed we define the step potential V as

$$\bar{V}(x) = \begin{cases} V_0 & x \in I\!R \setminus [m - \ell + 1, M + \ell - 1] \\ V_1 & x \in [m - \ell + 1, M + \ell - 1], \end{cases} \tag{7.1}$$

where ℓ is a positive constant.

Lemma 7.1. *If ℓ is large enough then there exists a unique positive cluster for the equation*

$$\varepsilon^2 u'' - \bar{V}(x)u + |u|^{p-1}u = 0 \quad \text{in } (m - \ell, M + \ell), \tag{7.2}$$

$$u'(m - \ell) = u'(M + \ell) = 0, \tag{7.3}$$

having N_ε oscillations in the interval $(m - \ell, M + \ell)$.

Proof. We start considering $T = \varepsilon^{-1}$ and $t \geqq t_0$, where t_0 is given in Lemmas 6.1 and 6.2. Under this assumptions equation (6.1)–(6.2) has a unique solution w_ε and we write $\underline{y}^\varepsilon(t) = w_\varepsilon(0)$. Consider v_ε, the unique solution of (2.1) with potential $V = V_1$ and $v_\varepsilon(0) = \underline{y}^\varepsilon(t)$, $v_\varepsilon{}'(0) = 0$.

Then, provided ε, t, ℓ satisfy

$$T(V_1, \underline{y}^\varepsilon(t)) = \frac{2\ell + M - m - 2 - 2\varepsilon t}{\varepsilon(N_\varepsilon - 1)}, \tag{7.4}$$

the function u_ε defined as

$$u_\varepsilon(x) = \begin{cases} w_\varepsilon((x - m_\ell - \varepsilon t)/\varepsilon) & x \in [m_\ell - 1, m_\ell + \varepsilon t] \\ v_\varepsilon((x - m_\ell - \varepsilon t)/\varepsilon) & x \in [m_\ell + \varepsilon t, M_\ell - \varepsilon t] \\ w_\varepsilon((-x - \varepsilon t + M_\ell)/\varepsilon) & x \in [M_\ell - \varepsilon t, M_\ell + 1], \end{cases}$$

for $m_\ell = m - \ell + 1$ and $M_\ell = M + \ell - 1$, is the unique solution of (6.1)–(6.2).

To prove that for every ε small and ℓ large there is exactly one $t \geqq t_0$ so that (7.4) is actually satisfied, we assume that $\ell > 0$ is such that

$$\ell > \lim_{\varepsilon \to 0} \frac{\varepsilon N_\varepsilon}{2} T(V_1, \underline{y}^\varepsilon(t_0)) + 1 - \frac{(M-m)}{2}. \tag{7.5}$$

Therefore, the lemma follows from Lemma 6.2 and the monotonicity of $T(V_1, \cdot)$. □

Associated with equation (7.2)–(7.3) we have a functional g_ε as in (5.6) defined in an appropriate set $\Delta_\varepsilon \subset I\!R^{N_\varepsilon}$. In what follows we would like to show that $deg(g_\varepsilon, \Delta_\varepsilon, 0) = -1$.

We set $x_0 = m - \ell$ and $x_{N_\varepsilon+1} = M + \ell$ and define the domain in $I\!R^{N_\varepsilon}$

$$\begin{aligned}\Delta_\varepsilon = \{(x_1, x_2, \ldots, x_{N_\varepsilon}) \,/\, x_{i+1} - x_i > \tilde{d}\varepsilon,\ i = 1, \ldots, N_\varepsilon - 1 \\ x_1 > m - \ell + 1 + t_0\varepsilon,\ x_{N_\varepsilon} < M + \ell - 1 - t_0\varepsilon, \},\end{aligned} \tag{7.6}$$

where

$$\ell_0 < \tilde{d} < \lim_{\varepsilon \to 0} \frac{M - m + 2\ell - 2}{\varepsilon N_\varepsilon}.$$

We set the functional $g_\varepsilon \colon \Delta_\varepsilon \to I\!R$ for $X = (x_1, x_2, \ldots, x_{N_\varepsilon}) \in \Delta_\varepsilon$ as in (5.6) replacing $V(x)$ by $\bar{V}(x)$. As before if $\nabla g_\varepsilon(X) = 0$ then the function u, defined as

$$u(x) = u_i(x), \quad x \in \left[x_i, x_{i+1}\right], \quad i = 0, \ldots, N_\varepsilon, \tag{7.7}$$

is a solution of (7.2) with $u'(x_0) = u'(x_{N_\varepsilon+1}) = 0$.

We observe that from our discussion above and by the choice of ℓ in (7.4), the function u_ε is corresponding to the unique critical point of g_ε in Δ_ε. Moreover, the functional g_ε does not have any critical point in $\partial\Delta_\varepsilon$.

Proposition 7.1. *Under the conditions discussed in this section, there exist* $\bar{\varepsilon} > 0$ *such that if* $0 < \varepsilon < \bar{\varepsilon}$, *then* u_ε *has index* -1 *and*

$$deg(\nabla g_\varepsilon, \Delta_\varepsilon, 0) = -1.$$

Proof. We compute $D^2 g_\varepsilon(X)$ at a point $X = (x_1, \ldots, x_{N_\varepsilon})$ in Δ_ε, a critical point of g_ε. We let $e^+ = u_\varepsilon(x_i)$ if i is odd and $e^- = u_\varepsilon(x_i)$ if i is even, and note that $F(e^\pm)$ does not depend explicitly on x_i. For every odd i, except for $i = 1$ or $i = N_\varepsilon$, we have

$$\frac{\partial^2 g_\varepsilon}{\partial x_i^2}(X) = 2\frac{f(e^+)}{T'(e^+)}$$

and

$$\frac{\partial^2 g_\varepsilon}{\partial x_i \partial x_{i-1}}(X) = \frac{\partial^2 g_\varepsilon}{\partial x_i \partial x_{i+1}}(X) = -\frac{f(e^+)}{T'(e^+)},$$

where $T'(e^+)$ is the derivative of the period function defined for (2.1), with $V = V_1$, which is positive as proved in Proposition 5.1.

For an even i, we repeat the computation, noticing that $F(e^-) = F(e^+)$, and obtain the same result. For $i = 1$ we find

$$\frac{\partial^2 g_\varepsilon}{\partial x_1^2}(X) = \frac{f(e^+)}{T'(e^+)} + \frac{f(e^+)}{T_1'} \quad \text{and} \quad \frac{\partial^2 g_\varepsilon}{\partial x_1 \partial x_2}(X) = -\frac{f(e^+)}{T'(e^+)}$$

where $T_1' = 1/\frac{dy}{dt}$, as given in Lemma 6.2. For $i = N_\varepsilon$ we obtain a similar result, with the obvious changes. Thus,

$$D^2 g_\varepsilon(X) = \left[\frac{f(e^+)}{T'(e^+)}\right]^{N_\varepsilon} A,$$

where A is a tridiagonal matrix with -1 in the upper and lower diagonal, and with 2 in the diagonal, except at the top and lower corner where we have the value

$$\alpha = 1 + \frac{T'(e^+)}{T_1'}.$$

It is important to note that $\alpha > 1$ thanks to Lemma 6.2. Through a recursive formula it is not hard to prove that

$$\det(A) = (N_\varepsilon - 1)\left((\alpha - 1)^2 + \frac{2\alpha - 2}{N_\varepsilon - 1}\right),$$

so that $\det(A) > 0$. Since $f(e^+) < 0$ and $T'(e^+) > 0$, then $\det(D^2 g_\varepsilon(X)) < 0$, because N_ε is odd. We can repeat this argument for every principal matrix, just changing α by 2. We conclude that $D^2 g_\varepsilon(X)$ is negative definite. □

Remark 7.1. In our analysis we have considered positive solutions. The case of negative solutions which give rise to negative clusters can be treated in a completely analogous way.

The case of sign-changing clusters requires some slight changes in the arguments, but we do not provide the details. We should emphasize that in this case sign-changing solutions correspond to minima of the functional g_ε in Δ_ε.

8. Computing the degree of a cluster

In this section we will compute the degree of the functional g_ε in the set $\Delta_\varepsilon \subset I\!R^{N_\varepsilon}$, with N_ε, g_ε and Δ_ε as defined in Section 5 and $\varepsilon > 0$ small. We recall that we are given a prescribed envelope $e : [m, M] \to I\!R$ such that $\text{supp}(e) = (a, b)$, with $m < a < b < M$, and an auxiliary envelope $\tilde{e}$ with

$$[a, b] \subset \text{supp}(\tilde{e}) = (\tilde{a}, \tilde{b}) \subset \left[\tilde{a}, \tilde{b}\right] \subset (m, M).$$

Our goal in this section is to prove the following

Proposition 8.1. *Assume that $V'(a) > 0$ and $V'(b) < 0$ and that $\tilde{a}$ and $\tilde{b}$ are close enough to a and b, respectively. Then there is $\varepsilon_0 > 0$ such that*

$$deg(\nabla g_\varepsilon, \Delta_\varepsilon, 0) = -1, \quad 0 < \varepsilon \leqq \varepsilon_0.$$

This proposition is the crucial ingredient to glue clusters in the next section. However, for the equation

$$\varepsilon^2 u'' - V(x)u + |u|^{p-1}u = 0, \quad \text{in } [m, M], \tag{8.1}$$

$$u'(m) = u'(M) = 0, \tag{8.2}$$

we can already obtain an existence theorem for a single cluster, that we state at the end of this section.

We will prove Proposition 8.1 by means of a series of lemmas. The idea is that through various homotopies, we obtain the required degree information starting from the step potential.

Our first step is to compute the degree of the gradient of an auxiliary functional g_ε^1 considered in a set Δ_ε^1, defined through an extension of the potential V to the interval $[m-\ell, M+\ell]$, with $\ell > 0$ large.

Set $0 < V_0 < \min_{x\in[m,M]} V(x)$ fixed and choose V_1 with

$$\min_{x\in[m,M]} V(x) > V_1 > V_0, \; V_1 \leqq V_0 + 1,$$

as in Lemma 6.1 and choose ℓ satisfying (7.5), so that Proposition 7.1 holds. We start by extending V as a C^1 function in $[m-1, M+1]$, with $V(m-1) = V(M+1) > V_1$, $V'(m-1) = -V'(M+1) > 0$. Then we extend V to the interval $[m-\ell, M+\ell]$ as a positive C^1 function satisfying: $V(m-\ell+2) = V_1$, $V(x) = V_0$ in $[m-\ell, m-\ell+1]$; $V'(x) > 0$ in $[m-\ell+1, m-1]$; $V'(x) = C/\ell$ in $[m-\ell+2, m-2]$; and $V(M+x) = V(m-x)$ if $x \in [1, \ell]$. The shape of V in $[m-\ell, m-\ell+2] \cup [M+\ell-2, M+\ell]$ is independent of ℓ.

Next we consider the homotopy $V_\lambda = \lambda V + (1-\lambda)\bar{V}$, where $\bar{V}$ was defined in (7.1). It can be easily checked that if ℓ is large enough, then there exists a solution $\hat{e}_\lambda$ of the equation (3.1) considered with the potential $V = V_\lambda$, with

$$\hat{e}_\lambda(m-\ell+2) < \left(\frac{p+1}{2}V_1\right)^{\frac{1}{p-1}}$$

and satisfying

$$\int_{m-\ell+1}^{M+\ell-1} \omega(V_\lambda(s), \hat{e}_\lambda(s))ds > \int_m^M \omega(V(s), \tilde{e}(s))ds, \tag{8.3}$$

for $\lambda \in [0, 1]$. Hence, there exists a unique solution $\tilde{e}_\lambda$ of (3.1), considered with V_λ, defined in $(m-\ell+1, M+\ell-1)$ for which

$$\int_{m-\ell+1}^{M+\ell-1} \omega(V_\lambda(s), \tilde{e}_\lambda(s))ds = \int_m^M \omega(V(s), \tilde{e}(s))ds. \tag{8.4}$$

We observe that there exists $0 < \bar{\lambda} < 1$ such that

$$\operatorname{supp}(\tilde{e}_\lambda) \subset [m-\ell+2, M+\ell-2] \quad \text{for } \bar{\lambda} \leqq \lambda \leqq 1. \tag{8.5}$$

At this point we fix $t_0 > 0$, so that Lemmas 6.1 and 6.2 hold. We see that for $0 < \varepsilon < \varepsilon_0$ small there exists $\eta > 0$ such that the unique increasing solution of

$$\varepsilon^2 u'' - V_\lambda(x)u + |u|^{p-1}u = 0, \tag{8.6}$$

$$u'(m-\ell) = u'(m-\ell+1+\varepsilon t_0) = 0, \tag{8.7}$$

satisfies

$$u(m-\ell+1+\varepsilon t_0) < \left(\frac{p+1}{2} V_\lambda(m-\ell+1+\varepsilon t_0)\right)^{\frac{1}{p-1}} - 2\eta, \tag{8.8}$$

for $0 \leqq \lambda \leqq \bar{\lambda}$. Also, if we choose ℓ large enough, then for every $\lambda \in [0,1]$,

$$\tilde{e}_\lambda(m-\ell+1) > \left(\frac{p+1}{2} V_\lambda(m-\ell+1)\right)^{\frac{1}{p-1}} - \eta. \tag{8.9}$$

For $x, y \in (m-\ell, M+\ell)$ we define the distance

$$d_\lambda(x,y) = \frac{1}{\varepsilon}\int_x^y \omega(V_\lambda(s), \tilde{e}_\lambda(s))ds,$$

and we let $x_0 = m-\ell$ and $x_{N_\varepsilon+1} = M+\ell$. We introduce the domain in $I\!R^{N_\varepsilon}$

$$\begin{aligned}\Delta_\varepsilon^\lambda = \{&(x_1, x_2, \dots, x_{N_\varepsilon}) \,/\, d_\lambda(x_i, x_{i+1}) > 1, i = 1, \dots, N_\varepsilon - 1\\ &x_1 > m-\ell+1+\varepsilon t_0,\ x_{N_\varepsilon} < M+\ell-1-\varepsilon t_0\}.\end{aligned} \tag{8.10}$$

Theorems 5.1 and 6.1 allow us to define the C^1 functional $g_\varepsilon^\lambda : \Delta_\varepsilon^\lambda \to I\!R$, using the potential V_λ instead of V in (5.6), for $0 < \varepsilon < \varepsilon_0$ and $\lambda \in [0,1]$.

Lemma 8.1. *Additionally assume that there is $c \in (m, M)$ such that $V'(x) > 0$ in $[m, c)$ and $V'(x) < 0$ in $(c, M]$. Then there exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$*

$$\deg(\nabla g_\varepsilon^1, \Delta_\varepsilon^1, 0) = -1.$$

Proof. We consider ℓ large and V_λ as before, and note that the extension of V can be taken strictly increasing in $[m-1, c)$ and strictly decreasing in $(c, M+1]$. Using the homotopy invariance of the degree and Proposition 7.1, we see that we just need to prove that the functional g_ε^λ does not have critical points in $\partial\Delta_\varepsilon^\lambda$, for all $\lambda \in [0,1]$.

We proceed by contradiction. Suppose that there exist sequences $\varepsilon_n \to 0$, $\lambda_n \to \lambda^*$ as $n \to \infty$ such that $g_{\varepsilon_n}^{\lambda_n}$ has a critical point $X^n = (x_1^n, \dots, x_{N_{\varepsilon_n}}^n)$ in $\partial\Delta_{\varepsilon_n}^{\lambda_n}$. To simplify notation, we set $g_n \equiv g_{\varepsilon_n}^{\lambda_n}$ and $\Delta_n = \Delta_{\varepsilon_n}^{\lambda_n}$. Since $\nabla g_n(X^n) = 0$, using (5.7) we can define u_n, a solution of (8.1) in $[m-\ell, M+\ell]$ considered with the potential V_{λ_n}, satisfying $u_n'(m-\ell) = u_n'(M+\ell) = 0$.

Now we use Proposition 4.2 to obtain some asymptotic information. Unfortunately, the potential V_λ has jumps, so we cannot use Proposition 4.2 in $[m-\ell, M+\ell]$, but we need to consider a subinterval. Up to a subsequence we can assume that

$x_2^n \to \bar{a}$ and $x_{N_{\varepsilon_n}-1}^n \to \bar{b}$ as $n \to \infty$. Then, the approximate envelope e_n associated to u_n, converges uniformly in $\left[\bar{a}, \bar{b}\right]$ to an envelope e_{λ^*} with potential V_{λ^*} satisfying

$$\int_{\bar{a}}^{\bar{b}} \omega(V_{\lambda^*}(s), e_{\lambda^*}(s))ds = \int_m^M \omega(V(s), e(s))ds. \tag{8.11}$$

We claim that

$$\int_{m-\ell+1}^{M+\ell-1} \omega(V_{\lambda^*}(s), e_{\lambda^*}(s))ds = \int_m^M \omega(V(s), e(s))ds, \tag{8.12}$$

and then e_{λ^*} is uniquely determined because V has a unique maximum point in $[m, M]$. By the same reason we also observe that the envelope $\tilde{e}_{\lambda^*}$ is uniquely determined by (8.4).

In order to prove the claim assume that $e_{\lambda^*}(\bar{a}) < (\frac{2}{p+1} V_{\lambda^*}(\bar{a}))^{2/(p-1)}$. Then $x_3^n - x_1^n \to 0$, which implies that we can use Proposition 4.2 (after Remark 4.3) in the interval $\left[m - \ell + 1, x_{N_{\varepsilon_n}-1}^n\right]$ and we may replace the lower limit $\bar{a}$ in (8.11) by $m - \ell + 1$. Of course, we can do the same if $e_{\lambda^*}(\bar{a}) = (\frac{2}{p+1} V_{\lambda^*}(\bar{a}))^{2/(p-1)}$. By repeating the argument on the other extreme of the interval, we complete the proof of the claim.

To continue we distinguish three cases. By extracting a subsequence if necessary, we just have to consider: Case 1: $x_1^n = m - \ell + 1 + \varepsilon_n t_0$ for all n, Case 2: $x_{N_{\varepsilon_n}}^n = M + \ell - 1 - \varepsilon_n t_0$ for all n, and Case 3: there is a sequence i_n, with $1 \leqq i_n < N_{\varepsilon_n} - 1$, such that $d_{\lambda_n}(x_{i_n}^n, x_{i_n+1}^n) = 1$.

Case 1. Suppose further that $\lambda^* \in \left[0, \bar{\lambda}\right]$. Then, taking into account that $d_{\lambda_n}(x_1^n, x_2^n) \geqq 1$ we have that $\lim_{n\to\infty} u_n(x_1^n) \geqq \tilde{e}_{\lambda^*}(m - \ell + 1)$, which is impossible in view of (8.8) and (8.9). On the other hand, if $\lambda^* \in \left[\bar{\lambda}, 1\right]$ then we claim that $|x_1^n - x_3^n| \geqq 1/2$. Assuming the claim for the moment, we can find $\alpha > 0$ such that for $v_n(x) = u_n(m - \ell + 1 + \varepsilon_n x)$ we have

$$\int_0^{(x_1^n - x_2^n)/\varepsilon_n} V'_{\lambda_n}(m - \ell + 1 + \varepsilon_n x) \frac{v_n^2(x)}{2} \leqq \frac{C}{\varepsilon_n} e^{-\frac{\alpha}{\varepsilon_n}}. \tag{8.13}$$

On the other hand, v_n converges locally and uniformly in $\mathbb{R}$ to the homoclinic solution of

$$v'' + V_{\lambda^*}(m + \ell - 1)v + v^p = 0, \;\; v'(0) = 0, v > 0.$$

At this point we observe that even though $V'(m - \ell + 1) = 0$, in the construction of the extension of the potential we could have further assumed that $V'(m - \ell + 1 + h) \geqq h$, for small $h > 0$. Then (8.13) would be impossible. To complete the argument we prove the claim: by (8.4), (8.12) and the choice of $\tilde{e}$, we see that

$$\overline{\text{supp}(e_{\lambda^*})} \subset \text{supp}(\tilde{e}_{\lambda^*}). \tag{8.14}$$

Then, it follows from (8.5) that $x_3^n \geqq m - \ell + 2$, proving the claim.

Case 2. By a completely analogous argument we see that Case 2 does not take place.

Case 3. Since e_n converges to e_{λ^*} in $\left[\bar{a}, \bar{b}\right]$, we see that it is not possible that $d_{\lambda_n}(x^n_{i_n}, x^n_{i_n+1}) = 1$ with $2 \leqq i_n < N_{\varepsilon_n} - 2$ for all n, because that would imply that e_n converges to $\tilde{e}_{\lambda^*}$. Then we may have that $d_{\lambda_n}(x^n_1, x^n_2) = 1$ for all n. If $\tilde{e}_{\lambda^*}(m + \ell - 1) < (\frac{2}{p+1} V_{\lambda^*}(m + \ell - 1))^{1/(p-1)}$ then $d_{\lambda_n}(x^n_2, x^n_3) \to 1$ and we can conclude as above. When $\tilde{e}_{\lambda^*}(m + \ell - 1) = (\frac{2}{p+1} V_{\lambda^*}(m + \ell - 1))^{1/(p-1)}$ we have two cases. First, if $x^n_1 - m + \ell - 1$ stays away from zero we can use Proposition 4.2 with $a_n = m - \ell + 1$, as in Remark 4.3, and get the same contradiction as before. Second, when $x^n_1 - m + \ell - 1 \to 0$ one has $x^n_3 - x^n_1 > r > 0$ and in this situation we can proceed as in Case 1, using (8.13). The case $i_n = x^n_{N_{\varepsilon_n}}$ is analogous. □

In our next lemma we come back to the original interval.

Lemma 8.2. *Assume that there is $c \in (m, M)$ such that $V'(x) > 0$ in $[m, c)$ and $V'(x) < 0$ in $(c, M]$. Then there exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$*

$$\deg(\nabla g_\varepsilon, \Delta_\varepsilon, 0) = -1.$$

Proof. We will consider the following homotopy. For $\mu \in [0, 1]$ we define $x^\mu_0 = (1 - \mu)(m - \ell) + \mu m$ and $x^\mu_{N_\varepsilon+1} = (1 - \mu)(M + \ell) + \mu M$ and the domain

$$\begin{aligned}\Delta_{\varepsilon,\mu} = \{(x_1, x_2, \ldots, x_{N_\varepsilon}) \,/\, d(x_i, x_{i+1}) > 1, i = 1, \ldots, N_\varepsilon - 1,\\ x_1 > (1 - \mu)(m - \ell + 1) + \mu\tilde{a} + \varepsilon t_0,\\ x_{N_\varepsilon} < (1 - \mu)(M + \ell - 1) + \mu\tilde{b} - \varepsilon t_0\}.\end{aligned}$$

We then define the function $g_{\varepsilon,\mu}$ in $\Delta_{\varepsilon,\mu}$ as in (5.6). By Lemma 8.1 we just need to check that $g_{\varepsilon,\mu}$ does not have critical points in $\partial\Delta_{\varepsilon\mu}$. We will proceed by contradiction, that is we assume that there exist sequences $\varepsilon_n \to 0, \mu_n \to \mu^*$ and $X^n = (x^n_1, \ldots, x^n_{N_{\varepsilon_n}})$ in $\partial\Delta_n$ such that $\nabla g_n(X^n) = 0$, whereas before we wrote $\Delta_n = \Delta_{\varepsilon_n,\mu_n}$ and $g_n = g_{\varepsilon_n,\mu_n}$. We denote by u_n the induced solution of (8.1) in the interval $[(1 - \mu_n)(m - \ell) + \mu_n m, (1 - \mu_n)(M - \ell) + \mu_n M]$ where we replace the boundary conditions by $u'_n((1 - \mu_n)(m - \ell) + \mu_n m) = 0, u'_n((1 - \mu_n)(M - \ell) + \mu_n M) = 0$.

As in the proof of Lemma 8.1, extracting a subsequence if necessary, we have one of the following situations: Case 1: $x^n_1 = (1 - \mu_n)(m - \ell + 1) + \mu_n\tilde{a} + \varepsilon_n t_0$ for all n, Case 2: $x^n_{N_{\varepsilon_n}} = (1 - \mu_n)(M + \ell - 1) + \mu_n\tilde{b} - \varepsilon_n t_0$ for all n, and Case 3: for every n there exists $1 \leqq i_n \leqq N_\varepsilon - 1$ for which $d(x^n_{i_n}, x^n_{i_n+1}) = 1$.

All three cases can be handled as in the proof of Lemma 8.1. However, here the situation is simpler, since we can use Proposition 4.2 taking $a_n = (1 - \mu_n)(m - \ell) + \mu_n m$ and $b_n = (1 - \mu_n)(M + \ell) + \mu_n M$. □

Now we handle more general cases of positive clusters, allowing V to have several critical points in (a, b), the support of the envelope e.

Lemma 8.3. *Suppose that $V'(x) > 0$ for $x \in [m, a]$, $V'(x) < 0$ if $x \in [b, M]$ then there exists $\varepsilon_0 > 0$ such that for every $0 < \varepsilon < \varepsilon_0$ we have*

$$\deg(\nabla g_\varepsilon, \Delta_\varepsilon, 0) = -1.$$

Proof. Note that under the hypothesis of the lemma $V(m), V(M) < \min_{x\in[a,b]} V(x) = V(a) = V(b)$.

Let $l > 0$ be a fixed but large constant. We define a C^1 potential $\bar{V}$ in the interval $[m-l, M+l]$ so that $\bar{V}(x) = V(x)$ if $x \in [m,a] \cup [b, M]$; $\bar{V}'(c) = 0$ for some $c \in (a,b)$; $\bar{V}'(x) > 0$ in $\big[m-l, c)$; $\bar{V}'(x) < 0$ in $(c, M+l]$; $|\bar{V}'(x)| \leqq C/l$ for some constant C independent of l in $[m-l+2, m-1] \cup [M+1, M+l-2]$; and

$$\bar{V}(m-l+1+x) = \bar{V}(M+l-1-x) = \bar{v}_0 x + \bar{v}_1 \text{ for } x \in [-1, 0],$$

where

$$\bar{v}_1 = \frac{1}{2}\min_{x\in[m,M]} V(x), \quad \bar{v}_0 = \frac{1}{4}\min_{x\in[m,M]} V(x).$$

We also extend V to $[m-l, M+l]$ as $V(x) = \bar{V}(x)$ in $[m-l, m] \cup [M, M+l]$.

We then define the homotopy potential $V_\lambda = \lambda V + (1-\lambda)\bar{V}$ and we consider $\tilde{e}_\lambda$ the auxiliary envelope associated with V_λ with $\operatorname{supp}(\tilde{e}_\lambda) = (m-l+1, M+l+1)$. It is clear that if l is chosen large enough then

$$\int_{m-\ell+1}^{M+\ell-1} \omega(V_\lambda(s), \tilde{e}_\lambda(s))ds > \int_M^m \omega(V(s), \tilde{e}(s))ds,$$

for all $\lambda \in [0, 1]$. We consider the domain

$$\begin{aligned}\Delta_\varepsilon^\lambda = \{(x_1, x_2, \ldots, x_{N_\varepsilon}) \,/\, d_\lambda(x_i, x_{i+1}) > 1, i = 1, \ldots, N_\varepsilon - 1\\ x_1 > m - l + 1 + \varepsilon t_0,\ x_{N_\varepsilon} < M + l - 1 - \varepsilon t_0\},\end{aligned}$$

where d_λ is defined as before but using the auxiliary envelope $\tilde{e}_\lambda$, and we consider the functional g_ε^λ as before using the potential V_λ. Using arguments as in the proof Lemma 8.1 we can prove that $\nabla g_\varepsilon^\lambda \neq 0$ in $\partial\Delta_\lambda^\varepsilon$ for all $\lambda \in [0, 1]$. Then, using Lemma 8.2 we have

$$\deg(\nabla g_\varepsilon^1, \Delta_\varepsilon^1, 0) = -1. \tag{8.15}$$

To conclude the proof we need to find a homotopy between g_ε^1 and g_ε. For $\mu \in (0, 1)$ we set an envelope $\tilde{e}_\mu$ with support $(\tilde{a}_\mu, \tilde{b}_\mu)$, where $\tilde{a}_\mu = (m-l+1)\mu + \tilde{a}(1-\mu)$ and $V(\tilde{b}_\mu) = V(\tilde{a}_\mu)$. As before we define

$$\begin{aligned}\Delta_\mu^\varepsilon = \{(x_1, x_2, \ldots, x_{N_\varepsilon}) \,/\, d_\mu(x_i, x_{i+1}) > 1, i = 1, \ldots, N_\varepsilon - 1,\\ x_1 > \tilde{a}_\mu + \varepsilon t_0,\ x_{N_\varepsilon} < \tilde{b}_\mu - \varepsilon t_0\},\end{aligned}$$

where the distance d_μ is induced by $\tilde{e}_\mu$ and we consider the corresponding functional g_ε^μ. Using previous arguments we can prove that $\nabla g_\varepsilon^\mu \neq 0$ in $\partial\Delta_\mu^\varepsilon$ for all $\mu \in [0, 1]$. Hence, the result follows using (8.15). □

Proof of Proposition 8.1. After Lemma 8.3 we just need to make a final homotopy to connect our potential V with one potential $\bar{V}$, which coincides with V in some interval $[a-\delta_1, b+\delta_2]$ and satisfies $\bar{V}'(x) > 0$ in $[m, a-\delta_1]$ and $\bar{V}'(x) < 0$ in $[b+\delta_2, M]$. By arguments simpler that those already given above, we can complete the proof. □

Remark 8.1. So far we have considered only the case of a positive cluster associated with a connected component (a, b) of the support of the envelope e. In a completely analogous way we can consider negative clusters. Moreover, under the appropriate changes in the hypothesis for V, and doing some variations of the arguments given already for positive clusters, we can also treat all four possible sign-changing clusters. Actually, considering that $V'(a) < 0$ and $V'(b) > 0$ we can prove

$$\deg(\nabla g_\varepsilon, \Delta_\varepsilon, 0) = 1, \tag{8.16}$$

with the appropriate changes in the definition of N_ε, Δ_ε and g_ε.

The results about the degree of a single cluster that we have obtained in this section allows us to write a first existence theorem

Theorem 8.1. *Assume that* $m < a < b < M$ *and* $V'(a) > 0$ *and* $V'(b) < 0$ *or* $V'(a) < 0$ *and* $V'(b) > 0$. *Let* $e\colon [m, M] \to I\!R$ *be an envelope such that* $\mathrm{supp}(e) = (a, b)$, *then there is* $\varepsilon_0 > 0$ *such that for every* $0 < \varepsilon < \varepsilon_0$ *the equation* (8.1) *admits a solution with* N_ε *critical points in* (m, M). *Moreover, the approximate envelope* e_ε *converges to the envelope* e, *as* $\varepsilon \to 0$.

Proof. By Proposition 8.1 (see also Remark 8.1) there is $\varepsilon_0 > 0$ such that for every $0 < \varepsilon < \varepsilon_0$ we have $\deg(\nabla g_\varepsilon, \Delta_\varepsilon, 0) \neq 0$. Thus (8.1) has a solution u_ε. By Proposition 4.2 such a solution gives rise to an approximate envelope e_ε, which converges to an envelope, as $\varepsilon \to 0$. Because of the choice of N_ε the limiting envelope is precisely the prescribed e. □

Remark 8.2. Theorem 8.1 can be extended, with minor modifications, to include a case where all oscillations concentrate at a point. We assume that V has exactly one critical point in $[m, M]$, say at $c \in (m, M)$. Then, given a sequence of integer numbers N_ε such that $\lim_{\varepsilon\to 0} \varepsilon N_\varepsilon = 0$, there exists a sequence of solutions of (8.1) with N_ε critical points, all of them converging to c, as $\varepsilon \to 0$. Of course, in this case we do not have an associated envelope. In dealing with multi-clusters in the next section, we can also include this situation, but we will not make it explicit.

9. Proof of Theorem 2.2 and extensions

In this section we give the proof of Theorem 2.2 and provide some natural extensions. We start proving the theorem in a particular case to illustrate the technique.

Theorem 9.1. *Assume that $e > e_*$ is an envelope in the interval I such that*

(i) *supp*$(e) = (a_1, b_1) \cup (a_2, b_2), a_1 < b_1 < a_2 < b_2, [a_1, b_2] \subset I$.
(ii) $V'(a_1) < 0$, $V'(b_1) > 0$, $V'(a_2) > 0$, $V'(b_2) < 0$.

Then there exists ε_0 and a family u_ε, $\varepsilon \in (0, \varepsilon_0)$ of solutions to (1.1) *with a Neumann boundary condition in ∂I, such that the approximate envelope e_ε associated to u_ε converges to e.*

Proof. First we remark that there exist numbers m_j, M_j, such that $m_j < a_j < b_j < M_j j = 1, 2$, $M_1 < m_2$, $[m_1, M_2] \subset I$. Moreover, $V'(x) < 0$ in $[m_1, a_1] \cup [b_2, M_2]$ and $V'(x) > 0$ in $[b_1, M_1] \cup [m_2, a_2]$.

We will construct a sign-changing solution in $[m_1, M_1]$ starting and ending with a positive maximum, and a positive solution in $[m_2, M_2]$. The other possible combinations are obtained in a similar way.

We consider two auxiliary envelopes $\tilde{e}_j$, such that $\text{supp}(\tilde{e}_j) = (\tilde{a}_j, \tilde{b}_j)$, with $m_j < \tilde{a}_j < a_j$ and $b_j < \tilde{b}_j < M_j$, for $j = 1, 2$.

We define the numbers

$$N_\varepsilon^j = \left\lfloor \frac{1}{\varepsilon} \int_{a_j}^{b_j} \omega(V(s), e(s)) ds \right\rfloor, \quad j = 1, 2.$$

We define the distance function

$$d^j(x, y) = \frac{1}{\varepsilon} \int_x^y \omega(V(s), \tilde{e}_j(s)) ds,$$

and the subset of $I\!R^{N_\varepsilon^j}$

$$\Delta_\varepsilon^j = \{(x_1^j, x_2^j, \ldots, x_{N_\varepsilon^j}^j) \ / \ d^j(x_i^j, y_{i+1}^j) > 1, \ i = 1, \ldots, N_\varepsilon^j - 1,$$
$$x_1^j > \tilde{a}_j + \varepsilon t_0, \ x_{N_\varepsilon^j}^j < \tilde{b}_j - \varepsilon t_0\}.$$

In each subset Δ_ε^j we define the functionals g_ε^j, as in Section 5. In section 8 we proved that

$$\deg(\nabla g_\varepsilon^j, \Delta_\varepsilon^j, 0) = (-1)^{j+1}, \ j = 1, 2, \tag{9.1}$$

in particular, we proved that

$$\nabla g_\varepsilon^j(X^j) \neq 0, \quad \text{for all } X^j \in \partial \Delta_\varepsilon^j, j = 1, 2. \tag{9.2}$$

We define the function

$$G_\varepsilon^0 : \Delta_\varepsilon^1 \times \Delta_\varepsilon^2 \to I\!R,$$

for $(X^1, X^2) \in \Delta_\varepsilon^1 \times \Delta_\varepsilon^2$ as

$$G_\varepsilon^0(X^1, X^2) = g_\varepsilon^1(X^1) + g_\varepsilon^2(X^2).$$

We also define the functional $G^1_\varepsilon\colon \Delta^1_\varepsilon \times \Delta^2_\varepsilon \to I\!R$ in the following way

$$G^1_\varepsilon(X^1, X^2) = g^1_\varepsilon(X^1) + g^2_\varepsilon(X^2) - \int_{x^1_{N^1_\varepsilon}}^{M_1} \sigma_\varepsilon(x, u^1_{N^1_\varepsilon}) \\ - \int_{m_2}^{x^2_1} \sigma_\varepsilon(x, u^2_0) + \int_{x^1_{N^1_\varepsilon}}^{x^2_1} \sigma_\varepsilon(x, u^{12}),$$

where u^{12} is the positive solution of equation $\varepsilon^2 u'' - V(x)u + u^p = 0$, with a Neumann boundary condition $(u^{12})'(x^1_{N^1_\varepsilon}) = 0 = (u^{12})'(x^2_1)$, and having exactly two local maxima, at $x^1_{N^1_\varepsilon}$ and x^2_1. These solutions and their uniqueness were studied in [12] Lemma 2.1.

In what follows we prove that there is $\varepsilon_0 > 0$ so that for $0 < \varepsilon < \varepsilon_0$

$$\deg(\nabla G^1_\varepsilon, \Delta^1_\varepsilon \times \Delta^2_\varepsilon, 0) = \deg(\nabla G^0_\varepsilon, \Delta^1_\varepsilon \times \Delta^2_\varepsilon, 0). \tag{9.3}$$

Since the degree at the right is non-zero, we have existence of desired solutions. The asymptotic behavior is already known by the general theory, as discussed in Section 4. Thus, to complete the proof of the theorem, we just need to prove (9.3).

Let us consider $G^\lambda_\varepsilon : \Delta^1_\varepsilon \times \Delta^2_\varepsilon \to I\!R$ defined as $G^\lambda_\varepsilon = \lambda G^1_\varepsilon + (1-\lambda)G^0_\varepsilon$. We use homotopy invariance of the degree in order to prove (9.3). For this we show that in $\partial(\Delta^1_\varepsilon \times \Delta^2_\varepsilon) = (\partial\Delta^1_\varepsilon \times \Delta^2_\varepsilon) \cup (\Delta^1_\varepsilon \times \partial\Delta^2_\varepsilon)$ we have that $\nabla G^\lambda_\varepsilon \neq 0$, for all $\lambda \in [0, 1]$.

Let $(X^1, X^2) \in \partial(\Delta^1_\varepsilon \times \Delta^2_\varepsilon)$ and assume that

$$\frac{\partial g^j_\varepsilon}{\partial x^j_i}(X^j) \neq 0,$$

for some $j = 1, i = 1, 2, \ldots, N^1_\varepsilon - 1$ or $j = 2, i = 2, 3, \ldots, N^2_\varepsilon$ then we clearly have $\nabla G^\lambda_\varepsilon(X^1, X^2) \neq 0$, for all $\lambda \in [0, 1]$. If this is not the case, then we can define solutions u^1 and u^2 to (1.1) in $(m_1, x^1_{N^1_\varepsilon})$ and in (x^2_1, M_2), respectively. We observe that it follows from Proposition 4.2 that the approximate envelope e^2_ε associated with u^2 converges, up to a subsequence, to an envelope $\bar{e}^2$ in the interval $[\bar{x}, M_2]$, where $\bar{x} = \lim x^2_2$ (for notational convenience, here we omit ε). A similar statement can be made on the interval $\left[m_1, x^1_{N^1_\varepsilon}\right]$.

To continue, let us assume that $X^2 \in \partial\Delta^2_\varepsilon$ (if $X^1 \in \partial\Delta^1_\varepsilon$ we proceed similarly). Here we have two possibilities:

(i) If $\bar{e}^2 = \tilde{e}_2$ in $[\bar{x}, M_2]$. In this case we have necessarily $\bar{x} > \tilde{a}_2$ and consequently $u_0(x^2_1) > u_1(x^2_1)$ and $u^{12}(x^2_1) > u_1(x^2_1)$. This implies that

$$\frac{\partial G^0_\varepsilon}{\partial x^2_1} < 0 \quad \text{and} \quad \frac{\partial G^1_\varepsilon}{\partial x^2_1} < 0.$$

It is clear then that $\nabla G^\lambda_\varepsilon(X^1, X^2) \neq 0$, for all $\lambda \in [0, 1]$.

(ii) If $\bar{e}^2 \neq \tilde{e}_2$. Then, for all ε small enough, $d(x_i^2, x_{i+1}^2) > 1$ for all $i = 2, \dots, N_\varepsilon^2 - 1$. Assume further that $d(x_1^2, x_2^2) > 1$ (for a subsequence), then as $X^2 \in \partial\Delta_\varepsilon^2$ we must have $x_1^2 = \tilde{a}_2 + \varepsilon t_0$ and then $x_3^2 - x_1^2 \geqq r > 0$, for some $r > 0$. (The constraint $x_{N_\varepsilon^2 - 1}^2 = \tilde{b}_2 - \varepsilon t_0$ cannot be active, since u^2 is a solution).

We have that

$$\frac{\partial G_\varepsilon^0}{\partial x_1^2} = \frac{d}{dx_1^2}\left[\int_{m_2}^{x_1^2} \sigma(x, u_0) + \int_{x_1^2}^{x_2^2} \sigma(x, u_1)\right] = I_1 + I_2.$$

We analyze the second integral (the other is similar). Re-scaling $v(s) = u_1(x_1^2 + \varepsilon s)$ and differentiating directly we obtain

$$I_2 = -\left[V(x_2^2)\frac{v^2(s)}{2} - \frac{v^{p+1}(s)}{p+1}\right]_{s=(x_2^2 - x_1^2)/\varepsilon} + \varepsilon \int_0^{(x_2^2 - x_1^2)/\varepsilon} V'(x_1^2 + \varepsilon s)\frac{v^2(s)}{2}\, ds.$$

Here the first term is exponentially small and the second one is positive. This implies that

$$\frac{\partial G_\varepsilon^0}{\partial x_1^2} > 0.$$

By similar reasons we have this inequality changing G_ε^0 by G_ε^1 and then $\nabla G_\varepsilon^\lambda(X^1, X^2) \neq 0$, for all $\lambda \in [0, 1]$.

In case $d(x_1^2, x_2^2) = 1$ for all $\varepsilon > 0$ small, we have $x_1^2 - \tilde{a}_2 \to 0$, as ε approaches 0, and this implies $x_3^2 - x_1^2 \geqq r$, for some $r > 0$. We then apply the argument give above. This completes the proof. □

Corollary 9.1. *Assume that $e > e_*$ is an envelope in the interval I such that*

(i) *supp*$(e) = (a_1, c) \cup (c, b_2)$, $a_1 < c < b_1$, $[a_1, b_1] \subset I$.
(ii) $V'(a_1) < 0$, $V'(c) = 0$, $V'(b_2) > 0$.

Then there exists ε_0 and a family u_ε, $\varepsilon \in (0, \varepsilon_0)$ of solutions to (1.1) *with a Neumann boundary condition in ∂I, such that the approximate envelope e_ε associated to u_ε converges to e.*

Proof. For a every $k \geqq k_0$ we consider numbers $b_1^k = c - 2/k$, $M_1^k = c - 1/k$, $m_2^k = c + 1/k$ and $a_2^k = c + 2/k$, and an envelope e_k such that $\text{supp}(e_k) = (a_1^k, b_1^k) \cup (a_2^k, b_2^k)$, where a_1^k and b_2^k are chosen appropriately. We see that for $k \geqq k_0$, and k_0 large enough, we can apply Theorem 9.1 to obtain ε_0^k and solutions to (1.1) u_ε^k, $\varepsilon \in (0, \varepsilon_0^k)$ such that their approximate envelope $e_\varepsilon^k \to e_k$. Since $e_k \to e$ as $k \to \infty$, from here we construct the desired family of solutions. □

An extension of Theorem 9.1 to envelopes with support made up of a finite number of intervals and allowing maximal sign generality follows.

Theorem 9.2. *Assume that $e > e_*$ an envelope such that*

(i) *$supp(e) = \cup_{k=1}^{\ell_1}(a_k^p, b_k^p) \cup \cup_{k=1}^{\ell_2}(a_k^c, b_k^c)$ is a disjoint union.*
(ii) *There are numbers m, M, such that $\overline{supp(e)} \subset (m, M) \subset I$.*
(iii) *$V'(a_k^p) > 0$, $V'(b_k^p) < 0$, $V'(a_k^c) < 0$ and $V'(b_k^c) > 0$, for all possible k.*

Given $s^k \in \{+, -\}$ and $(s_1^k, s_2^k) \in \{+, -\}^2$ Then there exists ε_0 and a family u_ε, $\varepsilon \in (0, \varepsilon_0)$ of solutions to (1.1) *with a Neumann boundary condition in ∂I, such that the approximate envelope e_ε associated to u_ε converge to e and so that:*

(i) *u_ε is a positive cluster in (a_k^p, b_k^p) if $s^k = +$ and a negative cluster if $s^k = -$, for all $1 \leqq k \leqq \ell_1$.*
(ii) *u_ε is a sign-changing cluster in (a_k^c, b_k^c) such that the first critical point of u_ε is positive (negative) if $s_1^k = +$ ($s_1^k = -$) and the last one is positive (negative) if $s_2^k = +$ ($s_2^k = -$).*

Proof. The proof of Theorem 9.1 can be adapted to this more general case, without major changes. □

Proof of Theorem 2.2.. Theorem 2.2 is essentially Theorem 9.2, except for the fact that the support of e may be the infinite union of disjoint open intervals. This case can be treated by a limiting process similar to that used in Corollary 9.1. □

Remark 9.1. In all theorems discussed so far we have considered envelopes e with support completely contained in I, that is, $e = e_0$ over ∂I. Our results can be extended to also consider boundary layers, but we do not provide details.

Remark 9.2. At this point we want to mention two extensions to solutions of (1.1) to all $I\!R$. We observe that the envelope equation can be considered naturally in all $I\!R$ and then, in principle, we could try to obtain the analogue of Theorem 2.2 for every envelope in $I\!R$. However, this is not possible unless some control on the potential and the envelope is considered at infinity.

First we consider the case of a periodic potential. With our construction and the use of the Arzela–Ascoli Theorem, we can construct solutions associated with envelopes with support having infinitely many components. In [16], this is done for the unbalanced Allen–Cahn equation. Choosing the envelopes appropriately we can construct in this way various kinds of chaotic solutions to (1.1).

Second, assuming that the potential is bounded away from zero at infinity, we can construct solutions associated with any envelope with a bounded support.

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References

1. AI, S.: Multi-pulse like orbits for a singularly perturbed nearly integrable system. *J. Differential Equations* **179**, 384–432 (2002)
2. AI, S.: Multi-bump solutions to Carrier's problem. *J. Math. Anal. Appl.* **277**, 405–422 (2003)
3. AI, S., CHEN, X., HASTINGS, S.: Layers and spikes in Non homogeneous bistable reaction-diffusion equations. To appear Trans. Amer. Math. Soc
4. AI, S., HASTINGS, S.: A shooting approach to layers and chaos in a forced Duffing equation *J. Differential Equations* **185**, 389–436 (2002)
5. ALESSIO, F., MONTECCHIARI, P.: Multibump solutions for a class of Lagrangian systems slowly oscillating at infinity. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **16**, 107–135 (1999)
6. ALIKAKOS, N., BATES, P.W., FUSCO, G.: Solutions to the nonautonomous bistable equation with specified Morse index. I. Existence. *Trans. Amer. Math. Soc.* **340**, 614–654 (1993)
7. ARNOLD, V.I.: *Mathematical Methods of Classical Mechanics*, 2nd ed. Springer-Verlag, Berlin, 1989
8. BERESTYCKI, H.: Le nombre de solutions de certains problèmes semi-linéaires elliptiques. *J. Funct. Anal.* **40**, 1–29 (1981)
9. BOURLAND, F., HABERMAN, R.: Separatrix crossing: Time potentials with dissipation. *SIAM J. Appl. Math.* **50**, 1716–1744 (1990)
10. CHOW, S.-N., WANG, D.: On the monotonicity of the period function of some second order equations. *Čaposis Přest. Mat.* **111**, 14–25 (1986)
11. COTI ZELATI, V., RABINOWITZ, P.: Homoclinic orbits for second order Hamiltonian systems possessing superquadratic potentials. *J. Amer. Math. Soc.* **4**, 693–727 (1992)
12. DEL PINO, M., FELMER, P., TANAKA, K.: An elementary construction of complex patterns in nonlinear Schrödinger equations. *Nonlinearity* **15**, 1653–1671 (2002)
13. FELMER, P., TORRES, J.J.: Semi classical limits for the nonlinear Schrödinger equation. *Commun. Contemp. Math.* **4**, 481–512 (2002)
14. FELMER, P., MARTÍNEZ, S.: High energy solutions for a phase transition problem. *J. Differential Equations* **194**, 198–220 (2003)
15. FELMER, P., MARTÍNEZ, S., TANAKA, K.: Multi-clustered high energy solutions for a phase transition problem. Proc. Roy. Soc. Edinburgh Sect. A **135**, 731–765 (2005)
16. FELMER, P., MARTÍNEZ, S., TANAKA, K.: High frequency chaotic solutions for a slowly varying dynamical system. To appear in Ergodic Theory Dynam Systems
17. GEDEON, T., KOKUBU, H., MISCHAIKOW, K., OKA, H.: Chaotic solutions in slowly varying perturbations of Hamiltonian systems with applications to shallow water sloshing. *J. Dynam. Differential Equations* **14**, 63–84 (2002)
18. HASTINGS, S., MC LEOD, K.: On the periodic solutions of a forced second order equation. *J. Nonlinear Sci.* **1**, 225–245 (1991)
19. KANG, X., WEI, J.: On interacting bumps of semi-classical states of nonlinear Schrödinger equations. *Adv. Differential Equations* **5**, 899–928 (2000)
20. KURLAND, H.: Monotone and oscillating equilibrium solutions of a problem arising in population genetics. Nonlinear Partial Differential Equations (Durham, N.H., 1982), 323–342, Contemporary Mathematics, Volume 17, Providence, R.I.: American Mathematical Society, 1983
21. NAKASHIMA, K.: Multi-layered stationary solutions for a spatially inhomogeneous Allen-Cahn equation. *J. Differential Equations* **191**, 234–276 (2003)
22. NAKASHIMA, K.: Stable transition layers in a balanced bistable equation. *Differ. Integral Equ. Appl.* **13**(7–9), 1025–1038 (2000)
23. NAKASHIMA, K., TANAKA, K.: Clustering layers and boundary layers in spatially inhomogeneous phase transition problems. *Ann. Inst. H. Poincaré Anal Non Linéaire,* **20**, 107–143 (2003)
24. NEHARI, Z.: Characteristic values associated with a class of nonlinear second-order differential equations. *Acta Math.* **105**, 141–175 (1961)

25. NEISHTADT, A.: Passage through a separatrix in a resonance problem with a slowly-varying parameter. *PPM* **39**, 621–632 (1975)
26. NEISHTADT, A.: About changes of adiabatic invariant at passage through separatrix. *Plasma Physics* **12**, 992–1001 (1986)
27. SÉRÉ, E.: Existence of infinitely many homoclinic orbits in Hamiltonian systems. *Math. Z.* **209**, 27–42 (1991)

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