High-frequency chaotic solutions for a slowly varying dynamical system

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Abstract. In this article we study the asymptotic dynamics of highly oscillatory solutions for the unbalanced Allen–Cahn equation with a slowly varying coefficient. We describe the underlying structure of these solutions through a function we call the adiabatic profile, which accounts for the asymptotic area covered by the solutions in the phase space. In finite intervals, we construct solutions given any adiabatic profile. In the case of a periodic coefficient we show that the system has chaotic behavior by constructing high-frequency complex solutions which can be characterized by a bi-infinite sequence of real numbers in $[c_1, c_2] \cup \{0\}$ $(0 < c_1 < c_2)$.

1. *Introduction*

Slowly varying plane Hamiltonian systems appear as models for an ample variety of problems in the sciences and applied sciences: particle mechanics, genetic evolution, physics of alloys, water waves and many more. These three-dimensional systems often present a very complicated behavior giving rise to intricate dynamics, which can be interpreted as spatial or temporal chaos, depending of the problem under study. The general problem is

$$\frac{dz}{dt} = J\nabla_z H(z, \varepsilon t), \quad z(t) \in \mathbb{R}^2,$$

where J is the standard 2×2 symplectic matrix, $\varepsilon > 0$ is a small parameter and H is the Hamiltonian.

Even though we believe that our results could be extended to a general class of Hamiltonian systems, we concentrate our study in one particular second-order system known as the unbalanced Allen–Cahn equation. This problem possesses solutions exhibiting phase transitions and two types of spikes, providing a rich behavior that gives a good idea of a general second-order system. The unbalanced Allen–Cahn equation can be written as

$$w'' - (w^2 - 1)(w - \phi(\varepsilon t)) = 0 \quad \text{in } \mathbb{R}, \tag{1.1}$$

and is also known as the Fisher equation.

Equation (1.1) has been recently studied by Ai, Chen and Hastings [**4**], showing that it possesses transition layers and two types of spike layers. They construct solutions having multiple transition layers at points where $\phi(x) = 0$ and $\phi'(x) \neq 0$, and multiple spike layers at non-zero critical points of ϕ. In all cases, the solutions found in [**4**] have an ε-independent number of transitions or spikes in any bounded interval. They even construct complex solutions indexed by a bi-infinite sequence of integers in $\{0, 1, 2, \ldots, m\}$, $m \in \mathbb{N}$, proving that the system is chaotic.

This article goes a step further in the understanding of the dynamics of system (1.1) by finding the underlying structure governing highly oscillatory solutions. This structure goes far beyond the set of zeros and critical points of ϕ; it is richer and much more interesting. We show that all solutions of equation (1.1) are associated asymptotically to an area-like function, which we call *adiabatic profile*, that accounts for the asymptotic area described by each oscillation in the phase space.

But our main contribution is the converse. In the case of a bounded interval, we show that given any adiabatic profile there exists a family of solutions to (1.1) associated asymptotically to this profile. Assuming some extra global conditions on ϕ, like periodicity, we can also construct solutions in all $\mathbb{R}$, associated to a certain class $\mathcal{C}$ of adiabatic profiles, proving that the system is chaotic. This class $\mathcal{C}$ is indexed by $\mathcal{I} = ([c_1, c_2] \cup \{0\})^{\mathbb{Z}}$, where $0 < c_1 < c_2$.

The solutions that we construct are characterized by their highly oscillatory behavior packed in the form of homoclinic and heteroclinic clusters. Each of these clusters oscillates a number of times asymptotically equal to ω/ε, where $\omega \in [c_1, c_2]$.

Instead of working with (1.1), we prefer to consider the following equivalent form:

$$-\varepsilon^2 u'' + f(x, u) = 0 \quad \text{in } \mathbb{R}, \tag{1.2}$$

where the nonlinearity f is defined as $f(x, u) = (u^2 - 1)(u - \phi(x))$.

In order to describe our results in a more precise manner, we first introduce our hypotheses on the function ϕ:

(ϕ1) $\phi : \mathbb{R} \to (-1, 1)$ is of class C^1.
(ϕ2) All critical points of ϕ are isolated and they are local maxima or local minima.
(ϕ3) If $\phi(x) = 0$ then $\phi'(x) \neq 0$.

We consider the primitive of f given by $F(x, u) = \int_0^u f(x, s)\, ds$, and we define the function $\phi_+^* : \mathbb{R} \to [-1, 1]$ as the unique solution of $F(x, y) = F(x, -1)$ if $\phi(x) < 0$ and $\phi_+^*(x) = -1$ otherwise. Similarly we define $\phi_-^* : \mathbb{R} \to [-1, 1]$ as the unique solution of $F(x, y) = F(x, 1)$ if $\phi(x) > 0$ and $\phi_+^*(x) = 1$ otherwise. These functions satisfy $-1 \leq \phi_-^*(x) < \phi(x) < \phi_+^*(x) \leq 1$ for all $x \in \mathbb{R}$. For a given $x \in \mathbb{R}$ and $e \in (\phi_-^*(x), \phi_+^*(x))$ we denote by $v(x, e; s)$ the solution of

$$v''(s) - f(x, v(s)) = 0, \quad v'(0) = 0, \quad v(0) = e, \tag{1.3}$$

which is periodic and non-constant if $e \neq \phi(x)$. If $\phi(x) > 0$ we let $v_-^*(s)$ be the solution of equation (1.3) with $e = \phi_-^*(x)$, which corresponds to the homoclinic orbit at $(1, 0)$. In an analogous way, when $\phi(x) < 0$ we let $v_+^*(s)$ be the homoclinic orbit at $(-1, 0)$. When $\phi(x) = 0$, we denote by $v_0^*(s)$ the heteroclinic solution of (1.3) with $v_0^*(-\infty) = -1$, $v_0^*(\infty) = 1$ and $v_0^*(0) = 0$.

We define the *trivial adiabatic profile* (or *trivial action*) as the function

$$\mathcal{A}_0(x) = \begin{cases} \displaystyle\int_{-\infty}^{\infty} v_-^{*\,\prime}(s)^2\, ds & \text{if } \phi(x) > 0, \\ \displaystyle\int_{-\infty}^{\infty} v_+^{*\,\prime}(s)^2\, ds & \text{if } \phi(x) < 0, \\ \displaystyle 2\int_{-\infty}^{\infty} v_0^{*\prime}(s)^2\, ds & \text{if } \phi(x) = 0. \end{cases} \tag{1.4}$$

The function $\mathcal{A}_0(x)$ corresponds to the area enclosed by the homoclinic (or heteroclinic) orbit in the phase plane of equation (1.3).

Definition 1.1. We say that the function $\mathcal{A} : \mathbb{R} \to (0, \infty)$ is an *adiabatic profile* (or *action*) if $\mathcal{A}$ is continuous, $\mathcal{A}(x) \leq \mathcal{A}_0(x)$ for all $x \in \mathbb{R}$ and, whenever $\mathcal{A}(x) \neq \mathcal{A}_0(x)$, we have $\mathcal{A}'(x) = 0$.

We define the *support* of an adiabatic profile as

$$\operatorname{supp}(\mathcal{A}) = \{x \in \mathbb{R} \mid \mathcal{A}(x) \neq \mathcal{A}_0(x)\}.$$

Our first theorem describes the asymptotic behavior of a given family of solutions of (1.1) in terms of adiabatic profiles. Given a family $\{u_\varepsilon\}$ of solutions to (1.2) we define an *approximate action* $\mathcal{A}_\varepsilon$ as follows. Consider $v_\varepsilon = v_\varepsilon(x; \cdot)$, the solution to the initial value problem

$$v''(s) - f(x, v(s)) = 0, \tag{1.5}$$

$$v(0) = u_\varepsilon(x), \quad v'(0) = \varepsilon u_\varepsilon'(x). \tag{1.6}$$

If $u_\varepsilon'(x) \geq 0$ we define

$$T_\varepsilon^0(x) = \inf\{s \mid v_\varepsilon'(t) \geq 0,\ -1 \leq v_\varepsilon(t) \leq 1,\ \text{for all } t \in (s, 0)\}$$

and

$$T_\varepsilon^1(x) = \sup\{s \mid v_\varepsilon'(t) \geq 0,\ -1 \leq v_\varepsilon(t) \leq 1,\ \text{for all } t \in (0, s)\}.$$

In the case $u_\varepsilon'(x) \leq 0$ we proceed in an analogous way. Then we define the *approximate action* associated to u_ε as

$$\mathcal{A}_\varepsilon(x) = 2\int_{T_\varepsilon^0(x)}^{T_\varepsilon^1(x)} (v_\varepsilon'(s))^2\, ds. \tag{1.7}$$

We prove the following theorem.

THEOREM 1.1. *There is a sequence $\varepsilon_n \to 0$ such that $\mathcal{A}_{\varepsilon_n}$ converges locally uniformly in $\mathbb{R}$ to an adiabatic profile $\mathcal{A}$. Moreover, if the set of accumulation points of the maxima of u_{ε_n} contains a given interval $[a, b]$ then the* $\operatorname{supp}(\mathcal{A})$ *contains (a, b).*

We should point out that, in an interval where the oscillations of u_ε stay away from the homoclinic (or heteroclinic) orbits, this theorem simply states that the area function is an *adiabatic invariant*, a well-known concept in the theory of averaging. See for example the book by Arnold [**6**, §52]. The main point of our result is its global character. Moreover, in its second part it establishes that if u_ε oscillates in a set which is dense in $[a, b]$, then the adiabatic profile is non-trivial in (a, b). Particularly, this implies that if $x_\varepsilon \to a$ with $u_\varepsilon(x_\varepsilon) \to \pm 1$ then for each $y \in (a, b)$ we have that $\mathcal{A}_\varepsilon(y) \to \mathcal{A}(y)$ with $\mathcal{A}(y) < \mathcal{A}_0(y)$, that is, the solution u_ε *separates* from the homoclinic (or heteroclinic) solution at a. The proof of this fact relies on Proposition 3.3, which happens to be crucial in the analysis leading to Theorem 1.2.

Our main theorem asserts that, on the other hand, given any adiabatic profile $\mathcal{A}$ we can construct a family of solutions having $\mathcal{A}$ as its asymptotic profile.

THEOREM 1.2. *Given a bounded interval I and a non-trivial adiabatic profile $\mathcal{A}$, there exists a family u_ε of solutions to (1.2) in I, with Neumann boundary conditions on ∂I, such that the approximate action $\mathcal{A}_\varepsilon$ associated to u_ε converges to $\mathcal{A}$ in I.*

At this point we mention the earlier work by Kurland [**18**], where the author constructs highly oscillatory *local* solutions for (1.2) for which the oscillations stay away from homoclinic or heteroclinic orbits. This allows a change of variables, transforming the system to action-angle variables. In this context, our results prove the existence of *global* solutions, crossing homoclinic or heteroclinic orbits.

We can also obtain results on the existence of solutions in all $\mathbb{R}$. Actually, as a consequence of Theorem 1.2 and the uniform control of the estimates that can be obtained when ϕ is periodic, we can prove the existence of solutions for (1.1) in $\mathbb{R}$, that exhibit chaotic behavior. We prefer to postpone the precise description of this result to §7.

Oscillatory solutions of slowly varying systems have been studied by many authors. In particular, we mention the work by Hastings and McLeod [**15**] and Gedeon *et al* [**14**]. Earlier results by Hale and Sakamoto [**16**], Angenent *et al* [**8**] and Alikakos *et al* [**5**] are also contributions that motivated our work.

Highly oscillatory solutions are very natural in the context of slowly varying systems, as shown by Kurland [**18**], but as far as we know not much is known in the literature about the construction of solutions that cross homoclinic or heteroclinic orbits. Ai [**1, 2**], Ai and Hastings [**3**], and Ali, Chen and Hastings [**4**] use uses a shooting method to construct solutions for certain equations, somehow related to (1.1), having a number of oscillations of order ε^{-1}. On the other hand, for a one-dimensional Schrödinger equation Felmer and Torres [**11**] obtained such highly oscillatory solutions, describing the associated envelope equation. However, in [**11**] only single-cluster solutions were constructed. In another work, Felmer and Martínez [**12**] obtained single-cluster solutions for the balanced inhomogeneous Allen–Cahn equation in an interval. In [**12**] important technical simplifications were obtained in the existence mechanism.

In [**11**] and [**12**] an important question was left open, which is the construction of multi-cluster solutions. This problem is undertaken here for the unbalanced Allen–Cahn equation.

The problem of gluing concentrating solutions has received an enormous amount of attention during the last 15 years. We mention the pioneering work of Séré [**23**] and

Coti-Zelati and Rabinowitz [**9**], and subsequent papers of many others. Particularly interesting to our analysis is the work of Alessio and Montecchiari [**7**] and Kang and Wei [**17**]. In all these works a good understanding of the properties of the objects to be glued is needed, like uniqueness or non-degeneracy which is expressed in analytical or topological terms. In the problem under study such information seems more elusive, since the clusters we have in mind are solutions that do not survive in a reasonable manner the limit procedure, as in the case of a single- or multi-peak or transition layer.

However, our problem is three-dimensional, two equations plus time, and we can take advantage of that. In a recent work, Nakashima and Tanaka [**21**] (see also Nakashima [**19, 20**]) obtained several existence results on multiple transition layers for the balanced Allen–Cahn equation. Their method is variational and well suited to treat these slowly varying systems. This method was also used to find multiple spikes in the nonlinear Schrödinger equation by del Pino *et al* [**10**].

In this article we extend this variational approach to the construction of multiple heteroclinic and homoclinic clusters. We combine the basic ideas in [**21**] with the analysis developed in [**12**] in order to gain understanding in this more difficult problem. We refer also to the recent work of the present authors [**13**] where a simpler case is studied using a different point of view.

The organization of this paper is the following. In §2 we study the adiabatic profiles and their relation with envelopes, an alternative way of describing the problem. In §3 we prove Theorem 1.1. We use some ideas from [**12**] for the study of the asymptotic behavior of a family of solutions. This result, which is interesting on its own, is needed for the existence theory. In §4 we study the basic solutions upon which we base our variational method. In §5 we construct one cluster by maximizing a finite-dimensional functional of Nehari type. In §6 we extend the previous construction to the case of finitely many clusters in a finite interval. In §7 we present our results on chaotic solutions. We describe the class of solutions, and we prove the existence result.

2. *Adiabatic profiles and envelopes*

In this section we analyze in more detail the adiabatic profiles, as defined in §1. We also introduce the notion of envelope, which appears to be very useful in the analysis and proof of our theorems. These functions account for the asymptotic amplitude of the solutions we are studying.

We start by discussing the behavior of the trivial adiabatic profile $\mathcal{A}_0$. For this purpose it is convenient to define another two primitives of f,

$$F_+(x,u) = \int_{-1}^{u} f(x,s)\,ds \quad \text{and} \quad F_-(x,u) = \int_{1}^{u} f(x,s)\,ds, \tag{2.1}$$

and we notice that

$$F_+(x,u) - F_-(x,u) = \int_{-1}^{1} f(x,s)\,ds = \frac{4}{3}\phi(x).$$

We define

$$N_\pm^*(x) = \int_{-\infty}^{+\infty} \left(v_\pm^*(s) - \frac{(v_\pm^*(s))^3}{3} \pm \frac{2}{3} \right) ds,$$

where v_+^* and v_-^* were defined in the introduction. These functions satisfy $N_-^*(x) < 0$ and $N_+^*(x) > 0$, where they are defined. The following proposition gives us the behavior of $\mathcal{A}_0$.

PROPOSITION 2.1. *The function $\mathcal{A}_0$ is differentiable at x so that $\phi(x) \neq 0$ and*
(1) *if $\phi(x) > 0$ then $\mathcal{A}_0'(x) = \phi'(x)N_-^*(x)$;*
(2) *if $\phi(x) < 0$ then $\mathcal{A}_0'(x) = \phi'(x)N_+^*(x)$;*
(3) *the function $\mathcal{A}_0$ has its global maximum at points where $\phi(x) = 0$.*

Remark 2.1. The local maximum points of the function $\mathcal{A}_0$ are the positive minima of ϕ, the negative maxima of ϕ and the points where ϕ vanishes. In the latter case the maximum point is a global maximum and $\mathcal{A}_0$ is not differentiable there.

Having the graph of the function ϕ we can draw the qualitative graph of $\mathcal{A}_0$ and then we easily identify all possible adiabatic profiles.

Now we consider the notion of envelope. For $x \in \mathbb{R}$ and $e \in (\phi_-^*(x), \phi_+^*(x)) \setminus \{\phi(x)\}$ we denote by $T(x, e)$ half the period of the solution v of equation (1.3) and we set $T(x, \phi_+^*(x)) = T(x, \phi_-^*(x)) = \infty$ and $T(x, \phi(x)) = 2\pi/\sqrt{1 - \phi^2(x)}$. We define the function $A(x, e)$ as

$$A(x, e) = 2\int_0^{T(x,e)} |v'(x, e; s)|^2\, ds,$$

when $e \in (\phi_-^*(x), \phi_+^*(x))$, and we extend it as

$$A(x, \phi_+^*(x)) = \mathcal{A}_0(x) \quad \text{and} \quad A(x, \phi_-^*(x)) = \mathcal{A}_0(x).$$

We observe that $A(x, \cdot)$ is strictly increasing in $[\phi(x), \phi_+^*(x)]$ and strictly decreasing in $[\phi_-^*(x), \phi(x)]$. Through this function we define the envelope function associated to an adiabatic profile.

Definition 2.1. A function $e : \mathbb{R} \to [-1, 1]$ is said to be an *envelope function* if it is continuous and it satisfies

$$\mathcal{A}(x) = A(x, e(x)), \quad x \in \mathbb{R}, \tag{2.2}$$

for a given adiabatic profile $\mathcal{A}$.

If $\mathcal{A} = \mathcal{A}_0$ then $e = \phi_+^*$ or $e = \phi_-^*$. Thus, ϕ_+^* and ϕ_-^* are envelopes, which we refer to as *trivial* envelopes.

Given an adiabatic profile, an envelope e satisfies either $e(x) \in [\phi(x), \phi_+^*(x)]$ for all x or $e(x) \in [\phi_-^*(x), \phi(x)]$ for all x. For $e \in [\phi_-^*(x), \phi_+^*(x)]$ we define $R(x, e)$ as the unique solution of

$$F(x, R(x, e)) = F(x, e),$$

satisfying $R(x, e) \in [\phi_-^*(x), \phi(x)]$ if $e \in [\phi(x), \phi_+^*(x)]$ and $R(x, e) \in [\phi(x), \phi_+^*(x)]$ if $e \in [\phi_-^*(x), \phi(x)]$. We note that e is an envelope if and only if $R(x, e(x))$ is an envelope and that $R(x, \phi(x)) = \phi(x)$. The function ϕ is also referred to as a trivial envelope.

We define the *support* of an envelope function as

$$\text{supp}(e) = \{x \in \mathbb{R} \mid e(x) \in (\phi_-^*(x), \phi_+^*(x))\}.$$

We observe that if $\mathcal{A}$ and e are related by (2.2) then $\text{supp}(e) = \text{supp}(\mathcal{A})$.

We may characterize the envelope functions as solutions of a first-order differential equation. This characterization will be useful later in proving Theorem 1.1. Assume $e(x)$ is an envelope function and let $x \in \mathrm{supp}(\mathcal{A})$. Then by direct differentiation of (2.2) and integrating by parts we find that

$$\int_0^{T(x,e(x))} f(x,v)v_x\,ds = 0, \tag{2.3}$$

where v_x denotes the derivative of $v = v(x, e(x); s)$ with respect to x. On the other hand, we may write

$$\mathcal{A}(x) = 4\int_0^{T(x,e(x))} \{F(x,v) - F(x,e(x))\}\,ds.$$

Differentiating this expression, and using (2.3), we find that

$$T(x,e(x))f(x,e(x))e' = \int_0^{T(x,e(x))} \frac{\partial}{\partial x}\{F(x,v) - F(x,e(x))\}\,ds,$$

from which we conclude that

$$e'(x) = H(x,e(x)), \tag{2.4}$$

if we define

$$H(x,e(x)) = \phi'(x)\frac{Q(x,e(x)) - (e(x) - \frac{1}{3}e^3(x))}{f(x,e(x))}$$

and

$$Q(x,e(x)) = \frac{1}{T(x,e(x))}\int_0^{T(x,e(x))}\left(v - \frac{v^3}{3}\right)ds.$$

Conversely, if $e(x)$ satisfies equation (2.4), then $\mathcal{A}$ given by (2.2) is constant.

In order to consider equation (2.4) at points $x \notin \mathrm{supp}(\mathcal{A})$, we extend the definition of the function Q by considering $Q(x,\phi_+^*(x)) = -2/3$ if $\phi(x) < 0$, $Q(x,\phi_-^*(x)) = 2/3$ if $\phi(x) > 0$, and $Q(x,\phi(x)) = \phi(x) - \phi^3(x)/3$. Now we give a precise notion of a solution to (2.4).

Definition 2.2. A continuous function $e : \mathbb{R} \to [-1,1]$ is said to be a *solution* of the envelope equation (2.4) if $e(x)$ satisfies (2.4) at every x for which $e(x) \in (-1,1)$.

With these definitions we can easily prove the following result.

PROPOSITION 2.2. *A function* $e : \mathbb{R} \to [-1,1]$ *is an envelope function if and only if it satisfies equation (2.4).*

Remark 2.2. We consider the sets

$$\mathcal{E}_- = \{(x,y) \in \mathbb{R}^2 \mid \phi_-^*(x) \le y \le \phi(x)\},$$
$$\mathcal{E}_+ = \{(x,y) \in \mathbb{R}^2 \mid \phi(x) \le y \le \phi_+^*(x)\}$$

and $\mathcal{E} = \mathcal{E}_- \cup \mathcal{E}_+$. Then we observe that the solutions of the envelope equation come in pairs, $e(x)$ and $R(x,e(x))$, one contained in $\mathcal{E}_+$ and the other in $\mathcal{E}_-$. The boundary of $\mathcal{E}$ is given by the graphs of $\phi_+^*(x)$ and $\phi_-^*(x)$, both envelopes, and the graph of ϕ is a separatrix of $\mathcal{E}$.

We will see later that an envelope e in $\mathcal{E}_+$ is associated to the maximum points of the solutions of (1.2) and the corresponding envelope $R(x, e(x))$ in $\mathcal{E}_-$ is associated to the minimum points of these solutions.

Remark 2.3. Solutions of equation (2.4) are in a one-to-one correspondence with adiabatic profiles. We observe that these solutions exhibit bifurcations at points where the adiabatic profile leaves the trivial profile. This bifurcation is understood by the fact that $H(x, e)$ is not Lipschitz continuous on the trivial envelopes $(x, \phi_+^*(x))$, $(x, \phi_-^*(x))$, as we can see from the analysis on the period function that follows.

We end this section with an asymptotic estimate on the period function. This estimate is very important in our proof of Proposition 3.3 to follow. We consider

$$T_+(x, e) = \int_{\phi(x)}^{e} \frac{d\tau}{\sqrt{2(F(x, \tau) - F(x, e))}} \quad \text{for } (x, e) \in \mathcal{E}_+$$

and

$$T_-(x, e) = \int_{e}^{\phi(x)} \frac{d\tau}{\sqrt{2(F(x, \tau) - F(x, e))}} \quad \text{for } (x, e) \in \mathcal{E}_-.$$

Naturally we have $T(x, e) = T_+(x, e) + T_-(R(x, e))$.

LEMMA 2.1. *There are continuous functions $\gamma_\pm : \mathcal{E}_\pm \to (0, \infty)$, locally Lipschitz in x, such that*

$$T_\pm(x, e) = -\gamma_\pm(x, e) \ln |e \mp 1|, \quad \textit{for } (x, e) \in \mathcal{E}_\pm, \ |e \mp 1| > 0.$$

Proof. We first consider the case $T_+(x, e)$. We notice that the interesting situation occurs near points $(\bar{x}, 1)$, where $\phi(\bar{x}) \geq 0$. By Taylor expansion we have

$$2(F(x, \tau) - F(x, e)) = f'(x, 1)S(\tau, e) + o((\tau - 1)^2 - (e - 1)^2),$$

where $S(\tau, e) = (e - \tau)(e - \tau + 2(1 - e))$. Then we can write

$$T_+(x, e) = \int_{\phi(x)}^{e} \frac{\sqrt{S(\tau, e)}}{\sqrt{2(F(x, \tau) - F(x, e))}} \frac{d\tau}{\sqrt{S(\tau, e)}}.$$

The first term in the integral is continuous and locally Lipschitz in x. For the second term, after some calculations, we find that

$$\int_{\phi(x)}^{e} \frac{d\tau}{\sqrt{S(\tau, e)}} = s(x, e) \ln(1 - e),$$

with $s(x, e)$ continuous and locally Lipschitz in x. Now it is easy to obtain the desired result. The case $T_-(x, e)$ is analogous. □

3. *Asymptotic behavior of solutions to (1.2)*

In this section we analyze the asymptotic behavior of a given sequence of solutions to equation (1.2). Assume we have functions $u_n : [a_n, b_n] \to \mathbb{R}$, with $\lim_{n\to\infty} a_n = \bar{a}$, $\lim_{n\to\infty} b_n = \bar{b}$ and such that

$$-\varepsilon_n^2 u_n'' + f(x, u_n) = 0, \quad u_n'(a_n) = u_n'(b_n) = 0, \tag{3.1}$$

where $\varepsilon_n \to 0$ as $n \to \infty$. Our purpose is to analyze the behavior of sub-sequences of $\{u_n\}$. In particular we are interested in associating to $\{u_n\}$ an envelope, that is a function satisfying (2.4) describing the local maximum points of u_n asymptotically. For a related analysis for other problems we refer to [**11**] and [**12**]. We start with the following result which is a direct consequence of (3.1).

PROPOSITION 3.1. *Under the conditions described above we have the following:*

(1) *If* $x \in (a_n, b_n)$ *is a local maximum of* u_n *then* $\phi(x) < u_n(x) < 1$*, and if* x *is a local minimum of* u_n *then* $-1 < u_n(x) < \phi(x)$.

(2) *If* $x_1 < x_2$ *are two consecutive maxima (or two consecutive minima) of* u_n*, and* $\phi'(x) > 0$ $(\phi'(x) < 0)$ *in* (x_1, x_2) *then* $u_n(x_1) < u_n(x_2)$ $(u_n(x_1) > u_n(x_2))$.

Next we consider an interval $I_+ = (\ell_-, \ell_+) \subset (\bar{a}, \bar{b})$ having one of the following characteristics: (1) $\phi'(x) < 0$, $\phi(x) > 0$ for $x \in [\ell_-, \ell_+]$, (2) $\phi'(x) > 0$, $\phi(x) < 0$ for $x \in [\ell_-, \ell_+]$.

Suppose that we are in case (1) and that a_n, b_n are local maxima of u_n. Assume that $\ell_- \leq x_n^1 < x_n^2 < \cdots < x_n^{s_n} \leq \ell_+$ are the local minima of u_n in $[\ell_-, \ell_+]$ for $i = 1, \ldots, s_n$. Furthermore, we assume that $s_n \to \infty$ as $n \to \infty$.

For case (2), we suppose that a_n, b_n are local minima of u_n, and that $\ell_- \leq x_n^1 < x_n^2 < \cdots < x_n^{s_n} \leq \ell_+$ are the local maxima of u_n in $[\ell_-, \ell_+]$ for $i = 1, \cdots, s_n$, and assume that $s_n \to \infty$ as $n \to \infty$. Considering a sub-sequence if necessary we define $\alpha = \lim_{n\to\infty} x_n^1$ and $\beta = \lim_{n\to\infty} x_n^{s_n}$. We also denote by $y_n^1 < \cdots < y_n^{s_n-1}$, the local maxima or minima of u_n in $(x_n^1, x_n^{s_n})$ (if (1) or (2) holds respectively).

We have the following density property for the extreme points of u_n.

PROPOSITION 3.2. *If* $x_1 < x_2$ *are such that* $[x_1, x_2] \subset (\alpha, \beta)$ *then there is* n_0 *so that for every* $n \geq n_0$ *the solution* u_n *has at least one maximum point and one minimum point in* $[x_1, x_2]$.

Proof. We only sketch the proof. Let us assume for definiteness that case (1) holds (the other case is analogous). If there is no critical point in the interval $[x_1, x_2]$ then, up to a sequence, u_n converges to 1 in $[x_1, x_2]$, except possibly for a point. Using comparison, this implies that for a sequence $y_n^+ \in [x_1, x_2]$ we have $1 - u_n(y_n^+), |u_n'(y_n^+)| \leq e^{-\delta/\varepsilon_n}$. On the other hand, we can prove the existence of $y_n^- \in [\alpha, x_1]$ such that $1 - u_n(y_n^-) \leq e^{-\delta/\varepsilon_n}$. Then multiplying the equation by u_n' and integrating between y_n^- and y_n^+ provides a contradiction. □

The next proposition is crucial in proving our main result. This is the starting point for understanding the relation between the oscillatory solutions with the envelope functions, and consequently with the adiabatic profiles. It states that once a solution of (1.2) starts oscillating at a point x, where for example $\phi(x) > 0$ and $\phi'(x) < 0$, then to the right of x it also oscillates. Moreover, its maximum values become different from the maximum associated to the homoclinic orbits, so entering an asymptotically periodic behavior.

PROPOSITION 3.3. *Suppose that (1) holds:*

(i) *If* $\alpha > \ell_-$ *then* $u_n(y_n^{i_n}) \to 1$ *for all sequences* $\{y_n^{i_n}\}$ *such that* $y_n^{i_n} \to \alpha$.

(ii) *If* $y_n^{i_n} \to \bar{x} \in (\alpha, \ell_+]$ *then* $\limsup_{n\to\infty} u_n(y_n^{i_n}) < 1$*, in particular* $\beta = \ell_+$.

If (2) holds then we have:
(i) *If* $\alpha > \ell_-$ *then* $u_n(y_n^{i_n}) \to -1$ *for all sequences* $\{y_n^{i_n}\}$ *such that* $y_n^{i_n} \to \alpha$.
(ii) *If* $y_n^{i_n} \to \bar{x} \in (\alpha, \ell_+]$ *then* $\limsup_{n\to\infty} u_n(y_n^{i_n}) > -1$*: in particular* $\beta = \ell_+$.

Proof. Assume case (1) holds: the other case can be treated in a similar way.

(i) Since $\ell_- < \alpha$, we notice that $u_n(y_n^1) \to 1$ because rescaling u_n around y_n^1 leads to a homoclinic orbit of the limiting equation. Then, integrating (3.1) between y_n^1 and $y_n^{i_n}$ we obtain

$$F_-(y_n^{i_n}, u_n(y_n^{i_n})) - F_-(y_n^1, u_n(y_n^1)) = \int_{y_n^1}^{y_n^{i_n}} \frac{\partial F_-}{\partial x}(x, u_n(x))\, dx, \tag{3.2}$$

from which it follows that $u_n(y_n^{i_n}) \to 1$.

(ii) If $\limsup_{n\to\infty} u_n(y_n^1) < 1$, then Proposition 3.1 implies the result. Thus, we may assume that, up to a sub-sequence, $\lim_{n\to\infty} u_n(y_n^1) = 1$.

Let us assume that the proposition does not hold. Then, up to a sub-sequence, we have that $u_n(y_n^{i_n}) \to 1$ as $n \to \infty$ and so

$$\lim_{n\to\infty} \int_{y_n^1}^{y_n^{i_n}} \frac{\partial F_-}{\partial x}(x, u_n(x))\, dx = 0. \tag{3.3}$$

Our efforts are directed to proving that this is impossible, by providing a contradiction. We start by observing that Proposition 3.1 implies that $u_n(y_n^{l_n}) \to 1$, for all sequences $\{l_n\}$, with $l_n \in \{1, 2, \ldots, i_n\} \equiv K_n$.

CLAIM. We have that, uniformly in the sequence $\{l_n\} \subset K_n$

$$\lim_{n\to\infty} \frac{y_n^{l_n+1} - y_n^{l_n}}{\varepsilon_n} = \infty.$$

Assuming the contrary, there is a sub-sequence of $\{u_n\}$ and $\{l_n\}$, for which by rescaling u_n around $y_n^{l_n}$ we obtain a sequence v_n that converges to a non-trivial solution of

$$v'' - f(\bar{y}, v) = 0,$$

satisfying $v'(\bar{y}) = v'(\bar{y} + y_0) = 0$, for a certain positive constant y_0. But then $v(\bar{y}) < 1$ since v is periodic, providing a contradiction that proves the claim.

Returning to the proof of Proposition 3.3, part (iii), we next let $r_0 > 0$ and v_-^* be the homoclinic solution of (1.3) with $e = \phi_-^*(x)$, for $x \in [\alpha, \bar{x}]$. Then there exists a positive constant A_1 such that

$$\int_{-r_0}^{r_0} \frac{\partial F_-}{\partial x}(x, v_-^*(s))\, ds = \phi'(x) \int_{-r_0}^{r_0} \left(v_-^* - \frac{(v_-^*)^3}{3} - \frac{2}{3} \right) ds \geq A_1 > 0, \tag{3.4}$$

for all $x \in [\alpha, \bar{x}]$. Thus, letting $z_n^k < y_n^k < z_n^{k+1}$ be the minimum points enclosing y_n^k, we see that

$$\liminf_{n\to\infty} \frac{1}{\varepsilon_n} \int_{z_n^{l_n} - r_0\varepsilon_n}^{z_n^{l_n} + r_0\varepsilon_n} \frac{\partial F_-}{\partial x}(x, u_n(x))\, dx \geq A_1, \tag{3.5}$$

uniformly in the sequence $\{l_n\} \subset K_n$. From here we find

$$\int_{y_n^1}^{y_n^{i_n}} \frac{\partial F_-}{\partial x}(x, u_n(x))\,dx \geq \sum_{k=2}^{i_n-1} \int_{z_n^k - r_0\varepsilon_n}^{z_n^k + r_0\varepsilon_n} \frac{\partial F_-}{\partial x}(x, u_n(x))\,dx$$
$$\geq \varepsilon_n(i_n - 2)A_2, \tag{3.6}$$

for a certain constant $A_2 > 0$. Thus, to get a contradiction between (3.6) and (3.3) we just need to prove that the sequence $\{\varepsilon_n i_n\}$ is bounded away from zero.

From (3.2) and (3.6) we find a positive constant A_4 such that

$$F_-(y_n^k, u_n(y_n^k)) \geq \varepsilon_n k A_4 \quad \text{for all } k \in K_n,$$

which implies that $(1 - u_n(y_n^k))^2 \geq \varepsilon_n k A_5$, for some A_5. This, together with Lemma 2.1, shows that there is a constant $\gamma_1 > 0$ such that for all $k \in K_n$,

$$T(y_n^k, u_n(y_n^k)) \leq -\gamma_1 \ln(\varepsilon_n k A_2). \tag{3.7}$$

Next we estimate $z_n^{k+1} - z_n^k$. We let v_n be the solution of the equation

$$\varepsilon_n^2 v_n'' - f(y_n^k, v_n(x)) = 0,$$

with initial conditions $v_n'(y_n^k) = 0$ and $v_n(y_n^k) = u_n(y_n^k)$. Since $\phi(x) \geq \phi(y_n^k)$ for all $x \in [y_n^{k-1}, y_n^k]$ we have $f(x, s) \geq f(y_n^k, s)$ for all $x \in [y_n^{k-1}, y_n^k]$. While u_n and v_n are decreasing we define x_u and x_v as their inverses, respectively. Then we have

$$-\frac{\varepsilon^2}{2}\frac{d}{ds}\left(\frac{1}{(x_u')^2} - \frac{1}{(x_v')^2}\right) = -f(x_u, s) + f(y_n^k, s),$$

and so $(x_v')^2 > (x_u')^2$. Let $\bar{x}_k \in [y_n^k - \varepsilon_n T(y_n^k, u_n(y_n^k)), y_n^k]$ so that $u_n(\bar{x}_k) = v_n(y_n^k - \varepsilon_n T(y_n^k, u_n(y_n^k)))$. We notice that $(y_n^k - \bar{x}_k)/\varepsilon_n \leq T(y_n^k, u_n(y_n^k))$ and since $u_n(y_n^k) \to 1$ we have that $(\bar{x}_k - z_n^k)/\varepsilon_n$ is bounded, and then $(\bar{x}_k - z_n^k)/\varepsilon_n \leq T(y_n^k, u_n(y_n^k))$. Thus

$$(y_n^k - z_n^k) \leq 2\varepsilon_n T(y_n^k, u_n(y_n^k)) \quad \text{for all } k \in K_n.$$

Using similar arguments, we compare $u_n(x)$ with $u_n(2y_n^k - x)$ and we find the same estimate for $z_n^{k+1} - y_n^k$, concluding that

$$\tfrac{1}{2}(z_n^{k+1} - z_n^k) \leq \varepsilon_n T(y_n^k, u_n(y_n^k)) \quad \text{for all } k \in K_n. \tag{3.8}$$

From here and (3.7), we obtain

$$z_n^{i_n} - z_n^1 = \sum_{k=1}^{i_n-1} z_n^{k+1} - z_n^k \leq 2\varepsilon_n \sum_{k=1}^{i_n} T(y_n^k, u_n(y_n^k)) \leq -2\gamma_1\varepsilon_n \sum_{k=1}^{i_n} \ln(\varepsilon_n k A_2).$$

Hence, using that $M! \geq (rM)^M$ for a certain $r > 0$, we find

$$\tfrac{1}{2}(\bar{x} - \alpha) \leq z_n^{i_n} - z_n^1 \leq -2\gamma_1\varepsilon_n i_n \ln(\varepsilon_n i_n r A_2),$$

from which we conclude that $\{\varepsilon_n i_n\}$ must be bounded away from zero, completing the proof of (ii). This completes the proof of Proposition 3.3. $\square$

Now we study the asymptotic behavior of u_n on the interval I_+. We relate it with an envelope by proving that an approximate envelope converges to a solution of the envelope equation. Subsequently we relate the behavior of u_n with the associated adiabatic profile.

We define the approximate envelope function $e_{\varepsilon_n} : [\ell_-, \ell_+] \to \mathbb{R}$ as follows. In the interval $[y_n^1, y_n^{s_n}]$ we consider

$$e_{\varepsilon_n}(x) = u_n(y_n^k) + \frac{u_n(y_n^{k+1}) - u_n(y_n^k)}{y_n^{k+1} - y_n^k}(y_n^{k+1} - x), \quad x \in [y_n^k, y_n^{k+1}], \tag{3.9}$$

for $k = 1, \ldots, s_n - 1$. If $\alpha > \ell_-$ we extend e_{ε_n} as the trivial envelope ϕ_+^* to $[\ell_-, y_n^1 - \varepsilon_n]$ and in $(y_n^1 - \varepsilon_n, y_n^1)$ we extend it linearly. On the other extreme we extend e_{ε_n} linearly to $[y_n^{s_n}, \ell_+]$. In the case $\alpha = \ell_-$ we simply extend e_{ε_n} linearly to $[\ell_-, y_n^1]$. Then we can prove the following proposition.

PROPOSITION 3.4. *The sequence* $\{e_{\varepsilon_n}\}$ *converges uniformly in* $I_+ = [\ell_-, \ell_+]$ *to a solution* e *of (2.4) in* (ℓ_-, ℓ_+). *Moreover,* $\mathrm{supp}(e) = (\alpha, \ell_+]$ *if* $e(\ell_-) = \phi_+^*(\ell_-)$ *and* $\mathrm{supp}(e) = [\ell_-, \ell_+]$ *if* $e(\ell_-) < \phi_+^*(\ell_-)$.

Proof. Multiplying (3.1) by u_n' and integrating we get

$$F(y_n^{k+1}, u_n(y_n^{k+1})) - F(y_n^k, u_n(y_n^k)) = \int_{y_n^k}^{y_n^{k+1}} \frac{\partial F}{\partial x}(x, u_n(x))\, dx,$$

from which, after writing $u_n^{k+1} = u_n(y_n^{k+1})$ and $u_n^k = u_n(y_n^k)$,

$$\begin{aligned} &\frac{F(y_n^{k+1}, u_n^{k+1}) - F(y_n^{k+1}, u_n^k)}{u_n^{k+1} - u_n^k} \frac{u_n^{k+1} - u_n^k}{y_n^{k+1} - y_n^k} \\ &\quad = \frac{F(y_n^{k+1}, u_n^k) - F(y_n^k, u_n^k)}{y_n^k - y_n^{k+1}} + \frac{1}{y_n^{k+1} - y_n^k} \int_{y_n^k}^{y_n^{k+1}} \frac{\partial F}{\partial x}(x, u_n(x))\, dx. \end{aligned} \tag{3.10}$$

Now let us consider $x \in (\alpha, \beta)$ and $1 \le k_n \le s_n$ so that $y_n^{k_n} \to x$ and $u_n(y_n^{k_n}) \to u$ as $n \to \infty$, for a sub-sequence if necessary. Then, using Proposition 3.3 we obtain that

$$\lim_{n\to\infty} \frac{F(y_n^{k_n+1}, u_n^{k_n+1}) - F(y_n^{k_n+1}, u_n^{k_n})}{u_n^{k_n+1} - u_n^{k_n}} = f(x, u),$$

where $f(x, u) \neq 0$, since $\phi(x) < u < 1$. Then, from (3.10) we find that for a certain constant C

$$\left| \frac{u_n^{k+1} - u_n^k}{y_n^{k+1} - y_n^k} \right| \le C \quad \text{for all } n, k,\ 1 \le k \le s_n.$$

This implies that the sequence $\{e_n\}$ is equicontinuous and then, by the Arzela–Ascoli theorem, after a sub-sequence, there is a function $e : [\ell_-, \ell_+] \to \mathbb{R}$, so that $e_n \to e$, uniformly. Actually, the Arzela–Ascoli theorem guarantees uniform convergence in every closed interval contained in $(\alpha, \ell_+]$. However, the application of Proposition 3.1 allows one to argue the uniform convergence in all $[\ell_-, \ell_+]$. We observe that $\phi(x) < e(x) < \phi_+^*(x)$, for all $x \in (\alpha, \ell_+]$.

Next we look at the right-hand side of (3.10). We see that the function $v_n(y) = u_n(y_n^{k_n} + \varepsilon_n y)$ converges locally uniformly to v, the solution in $\mathbb{R}$ of the equation (1.3) with $e = e(x)$. Then it follows that

$$\lim_{n\to\infty} \frac{1}{y_n^{k_n+1} - y_n^{k_n}} \int_{y_n^{k_n}}^{y_n^{k_n+1}} \frac{\partial F}{\partial x}(z, u_n(z))\, dz = \frac{1}{\tau(x)} \int_0^{\tau(x)} \frac{\partial F}{\partial x}(x, v(s))\, ds$$

where $\tau(x) = T(x, e(x))$. This completes the proof of Proposition 3.4. □

Next we may consider intervals of the form $I_- = (\ell_+, \ell_-) \subset (\bar{a}, \bar{b})$ having one of the following characteristics: (1) $\phi'(x) > 0$, $\phi(x) > 0$ for $x \in I_-$ or (2) $\phi'(x) < 0$, $\phi(x) < 0$ for $x \in I_-$. Then the analogues of Propositions 3.2 and 3.3 can be proved. Defining the corresponding approximate envelope in I_- we can also prove an analogue of Proposition 3.4.

We notice that the interval $I = [\bar{a}, \bar{b}]$ can be written as the union of intervals of type I_+ and I_-, alternatively. Thus, given our solution u_n in the interval $[a_n, b_n]$, we define an approximate envelope e_{ε_n} in $[a_n, b_n]$ and we have proved the following theorem.

THEOREM 3.1. *Under the definitions and conditions given above, up to a sub-sequence,* $\{e_{\varepsilon_n}\}$ *converges uniformly in* I *to a solution* e *of (2.4).*

We complete this section by proving Theorem 1.1. For this purpose we just need to make the connection between the approximate envelope e_{ε_n} with the approximate action as defined in (1.7). We consider $x \in I$ and the function v_{ε_n} defined by (1.5) and (1.6). We define

$$\tilde{e}_{\varepsilon_n}(x) = \max_{s\in\mathbb{R}} |v_{\varepsilon_n}(x; s)|.$$

It is clear that the approximate action satisfies

$$\mathcal{A}_{\varepsilon_n}(x) = A(x, \tilde{e}_{\varepsilon_n}(x)),$$

so that we only need to prove the following lemma.

LEMMA 3.1. *Up to a sub-sequence,*

$$\lim_{n\to\infty} \tilde{e}_{\varepsilon_n}(x) = \lim_{n\to\infty} e_{\varepsilon_n}(x).$$

Proof. Consider

$$E_{\varepsilon_n}(x) = \frac{\varepsilon^2}{2}|u'_{\varepsilon_n}(x)|^2 - F(x, u_{\varepsilon_n}(x)).$$

Then we have

$$\frac{d}{dx} E_{\varepsilon_n}(x) = \phi'(x)\left\{u_\varepsilon(x) - \frac{1}{3}u_\varepsilon^3(x)\right\},$$

so that $E_{\varepsilon_n}(x)$ is bounded in $W^{1,\infty}(I)$ as $n \to \infty$. In particular $E_{\varepsilon_n}(x)$ has a uniformly convergent sub-sequence. We also have that $\tilde{e}_{\varepsilon_n}(x)$ has a uniformly convergent sub-sequence.

Let $x_0 \in \mathrm{int}(I)$ and suppose that, for $\delta > 0$, local maxima of $u_{\varepsilon_n}(x)$ are dense in $(x_0 - \delta, x_0 + \delta)$. Then we can easily see that $e_{\varepsilon_n}(x_0)$ and $\tilde{e}_{\varepsilon_n}(x_0)$ have a common limit. On the other hand, if for $\delta > 0$ local maxima of $u_\varepsilon(x)$ do not appear densely in $(x_0 - \delta, x_0 + \delta)$, by Proposition 3.4 we have $\lim_{n\to\infty} e_{\varepsilon_n}(x_0) = \phi_+^*(x_0)$. We also have $\lim_{n\to\infty} E_{\varepsilon_n}(x_0) = 0$ and thus $\tilde{e}_{\varepsilon_n}(x_0) \to \phi_+^*(x_0)$. □

Remark 3.1. Let us denote by $N_n(x_0, x_1)$ the number of zeros of u_n in $[x_0, x_1] \subset (\bar{a}, \bar{b})$. Then, by a simple argument as in [**12**], we can prove that

$$\lim_{n\to\infty} N_n \varepsilon_n = \int_{x_0}^{x_1} \frac{1}{T(x, e(x))} dx, \tag{3.11}$$

where e is the envelope associated to $\{u_n\}$.

4. *Existence of basic solutions*

In this section we consider the existence of basic solutions for (1.2). Putting together these solutions we construct a finite-dimensional functional in order to find more complicated solutions, resembling earlier work by Nehari [**22**]. We start with the autonomous equation

$$v''(s) - f(x, v(s)) = 0 \quad v(0) = e,\ v'(0) = 0, \tag{4.1}$$

where $x \in [a, b]$ is a fixed parameter and $e \in (-1, 1)$ and we denote its solution by $v(s) = v(x, e; s)$. When $e \in (\phi(x), \phi_+^*(x))$ the solution $v(x, e; s)$ is periodic and we let $T(x, e)$ be half of its period. When $\phi(x)$ is close to 0 and $e \in (-1, 1)$ is close to 1 then $v(x, e; s)$ remains positive in a symmetric bounded interval, whose length is denoted by $T_{\mathrm{p}}(x, e)$. While if $e \in (-1, 1)$ is close to -1 then $v(x, e; s)$ remains negative in a symmetric bounded interval of length $T_{\mathrm{n}}(x, e)$.

For our nonlinearity $f(x, u)$, the following result was proved by Smoller and Wasserman [**24**]

$$\frac{\partial T}{\partial e}(x, e) > 0 \quad \text{for all } x, e \in (\phi(x), \phi_+^*(x)) \tag{4.2}$$

and also for x such that $\phi(x)$ is close to 0

$$\frac{\partial T_{\mathrm{p}}}{\partial e}(x, e) > 0 \quad \text{and} \quad \frac{\partial T_n}{\partial e}(x, e) < 0, \tag{4.3}$$

for e near 1 in the first inequality and e near -1 in the second one.

Based on the monotonicity of the period we obtain a non-degeneracy property of the linearized equation associated to (4.1). We consider first the case of Neumann boundary conditions. We observe that the solution $v(x, e; s)$ of (4.1) satisfies the Neumann boundary condition

$$v'(0) = v'(T(x, e)) = 0,$$

and $v'(t) < 0$ for $t \in (0, T(x, e))$. We have the following lemma.

LEMMA 4.1. *The equation*

$$h'' - f'(x, v(x, e; s))h = 0,\ h'(0) = h'(T(x, e)) = 0, \tag{4.4}$$

has only the trivial solution. Here $f'(x, u)$ denotes the derivative with respect to u.

Proof. The differential equation in (4.4) has two linearly independent solutions that we can write explicitly as

$$h_1(s) = \frac{dv}{ds}(s) \quad \text{and} \quad h_2(s) = \frac{dv}{de}(s).$$

Differentiating we see that h_1 and h_2 do not satisfy the boundary condition in (4.4). In fact,

$$h_1'(0) = v''(0) = f(x, e) \neq 0$$

and by (4.2), and denoting by T' the derivative of T with respect to e,

$$h_2'(T) = -v''(T)T' = -f(x, R(e))T' \neq 0. \qquad \square$$

A similar statement can be made in the case of homoclinic orbits appearing when $\phi(x) \neq 0$. We assume that $\phi(x) < 0$; the other case is analogous. We write $v_*(x, s)$ for the solution of the equation in (4.1) satisfying the boundary condition $v_*'(x, 0) = 0$, $\lim_{s\to\infty} v_*(x, s) = -1$, with $v_*'(x, s) < 0$ for all $s > 0$. Then we have the following lemma.

LEMMA 4.2. *The equation*

$$h'' - f'(x, v_*(x, s))h = 0, \quad h'(0) = \lim_{s\to\infty} h'(s) = 0, \tag{4.5}$$

has only the trivial solution.

Proof. The equation has two independent solutions. One is $v_*'(x, s)$, which does not satisfy the boundary condition at 0 and which is bounded at infinity. The other solution of (4.5) has to be unbounded. $\square$

Before continuing, let us state an energy estimate for the homoclinic orbits $v_*(x, s)$ in terms of x. Still in the case $\phi(x) < 0$ we define

$$E_+(x) = \int_0^\infty \frac{(v_*')^2}{2} + F_+(x, v_*)\, ds.$$

When $\phi(x) > 0$ we denote by $v_*(x, s)$ the solution equation in (4.1) satisfying the boundary condition $v_*'(x, 0) = 0$, $\lim_{s\to\infty} v_*(x, s) = 1$, with $v_*'(x, s) > 0$ for all $s > 0$ and we consider the energy function

$$E_-(x) = \int_0^\infty \frac{(v_*')^2}{2} + F_-(x, v_*)\, ds.$$

Then we have our next lemma.

LEMMA 4.3. $E_+'(x)\phi'(x) > 0$ *if* $\phi(x) < 0$ *and* $\phi'(x)E_-'(x) < 0$ *if* $\phi(x) > 0$.

Proof. We prove this only for E_+. After a change of variables we obtain that

$$E_+(x) = \int_{-1}^{\phi_+^*(x)} \sqrt{2F_+(x, w)}\, dw$$

and then simple differentiation gives $\phi'(x)E_+'(x) > 0$. $\square$

Next we consider the associated non-autonomous linear problem. We need two extreme functions in order to control the size of the period. First we consider a function $\ell_\infty(x)$ such that $\ell_\infty(x) = +\infty$ for all x, except in a small neighborhood of $\{x \mid \phi(x) = 0\}$, where it is continuous. We also define $\ell_0(x) = T(x, \phi(x))$. Given $x_0 \in [\bar{a}, \bar{b}]$ and $\ell > \ell_0(x_0) = T(x_0, \phi(x_0))$, we consider the solution $v_0 = v(x_0, e(\ell); \cdot)$ of (4.1) with $v_0'(t) < 0$ in $(0, \ell)$. Here $e = e(\ell)$ is the unique e satisfying $T(x_0, e) = \ell$.

LEMMA 4.4. *Given $\delta > 0$, there are positive constants C, ε_0 such that for every $\varepsilon \in (0, \varepsilon_0)$, $x_0 \in [\bar{a}, \bar{b}]$,*

$$\ell_0(x_0) + \delta \leq \ell < \ell_\infty(x_0), \tag{4.6}$$

$x_0 + \varepsilon\ell \leq \bar{b}$ and $g \in C([0, \ell])$, the linear equation

$$h'' - f'(x_0 + \varepsilon s, v_0(s))h = g, \quad h'(0) = h'(\ell) = 0, \tag{4.7}$$

has a unique solution h satisfying

$$\|h\|_2 \leq C\|g\|_0. \tag{4.8}$$

Here $\|\cdot\|_2$ and $\|\cdot\|_0$ denote the natural uniform norms in $C^2([0, \ell])$ and $C([0, \ell])$, respectively.

Proof. By Fredholm alternative, it is enough to prove (4.8) for all possible solutions of (4.7). Suppose we have a sequence $\varepsilon_n \to 0$, $x_0^n \in [\bar{a}, \bar{b}]$, $\ell_n \geq \ell(x_0^n) + \delta$, and $g_n \in C([0, \ell_n])$ such that h_n satisfies (4.7), $\|g_n\|_0 = 1$ and $\|h_n\|_2 \to \infty$.

We define $\hat{h}_n = h_n/\|h_n\|_2$ and $\hat{g}_n = g_n/\|h_n\|_2$, then we have

$$\hat{h}_n'' - f'(x_0^n + \varepsilon_n s, v_0^n(s))\hat{h}_n = \hat{g}_n, \qquad \hat{h}_n'(x_0^n) = \hat{h}_n'(x_0^n + \ell_n) = 0.$$

Let us assume first that for a sub-sequence $\ell_n \to \bar{\ell} < \infty$. Then, for a sub-sequence we have $x_0^n \to \bar{x}$ and $\hat{h}_n \to \bar{h}$ such that

$$\bar{h}'' - f'(\bar{x}, \bar{v}(s))\bar{h} = 0, \qquad \bar{h}'(0) = \bar{h}'(\bar{\ell}) = 0,$$

where $\bar{v} = v(\bar{x}, \bar{e}; \cdot)$ satisfies (4.1) and $\bar{e}$ is so that $T(\bar{x}, \bar{e}) = \bar{\ell}$. But in view of Lemma 4.1 this is impossible, since $\bar{h} \neq 0$.

Assume next that $\ell_n \to \infty$. We notice that thanks to (4.6) this implies that $\phi(\bar{x}) \neq 0$. Let $s_n \in (0, \ell_n)$ be a point where $s \mapsto \max\{|\hat{h}_n(s)|, |\hat{h}_n'(s)|, |\hat{h}_n''(s)|\}$ attains its global maximum value. If s_n is bounded we proceed as above, with the only difference being that we will use Lemma 4.2 to conclude. In the case $s_n \to \infty$ we center the equation at s_n and we take the limit as $n \to \infty$, for an appropriate sub-sequence. Then we obtain a bounded function $\bar{h}$ satisfying

$$\bar{h}'' - f'(\bar{x}, \pm 1)\bar{h} = 0, \quad s \leq 0$$

and $\bar{h}'(0) = 0$. Since $f'(\bar{x}, \pm 1) > 0$ this is impossible. □

Now we can state our theorem on the existence and uniqueness of basic solutions with Neumann boundary condition.

THEOREM 4.1. *Given $\delta > 0$, there exists ε_0 such that, for every $\varepsilon \in (0, \varepsilon_0)$, $x_0 \in [\bar{a}, \bar{b}]$, $x_0 + \varepsilon\ell \leq \bar{b}$,*

$$\ell_0(x_0) + \delta \leq \ell < \ell_\infty(x_0), \tag{4.9}$$

the equation

$$u'' - f(x_0 + \varepsilon s, u(s)) = 0, \quad u'(0) = u'(\ell) = 0, \; u' > 0 \tag{4.10}$$

has a unique solution, which is differentiable in x_0 and ℓ.

Proof. This theorem is a consequence of the implicit function theorem. We consider $w = u - v_0$, where v_0 is defined just before Lemma 4.4. Then equation (4.10) is equivalent to

$$w'' - f'(x_0 + \varepsilon s, v_0(s))w = Q(x_0, \ell, \varepsilon, s)w + E(x_0, \ell, \varepsilon, s) \tag{4.11}$$

$$w'(0) = w'(\ell) = 0, \tag{4.12}$$

where Q and E are given by

$$\begin{aligned} Q(x_0, \ell, \varepsilon, s)w = {} & f(x_0 + \varepsilon s, v_0 + w) - f(x_0 + \varepsilon s, v_0(s)) \\ & - f'(x_0 + \varepsilon s, v_0(s))w \end{aligned}$$

and

$$E(x_0, \ell, \varepsilon, s) = f(x_0 + \varepsilon s, v_0(s)) - f(x_0, v_0(s)).$$

Now, given $\sigma > 0$, we can find ε_1 so that if $\varepsilon \in (0, \varepsilon_1)$ then

$$|E(x_0, \ell, \varepsilon, t)| \leq \frac{\sigma}{2C},$$

where C is the constant appearing in (4.8). On the other hand, by a direct computation we have

$$|Q(x_0, \ell, \varepsilon, s)w(s)| \leq c|w(s)|^2.$$

Now we choose σ so small that $c\sigma^2 \leq \sigma/(2C)$. Then, given w_1 such that $\|w_1\|_0 \leq \sigma$, the equation

$$w'' - f'(x_0 + \varepsilon s, v_0(s))w = Q(x_0, \ell, \varepsilon, s)w_1 + E(x_0, \ell, \varepsilon, s)$$

with boundary condition (4.12) defines a unique w_2 satisfying $\|w_2\|_2 \leq \sigma$, thanks to Lemma 4.4. Thus we can define an operator $\mathcal{F} : B_\sigma \subset C^0([0, \ell]) \to B_\sigma$, where B_σ is the closed ball centered at 0 and with radius σ. In the same way we can prove that $\mathcal{F}$ is a contraction. Therefore $\mathcal{F}$ has a fixed point that is a solution to (4.11) and (4.12), then solving (4.10) in a unique way.

The differentiability properties of this solution require some extra work that we leave to the reader. □

Remark 4.1. If $\phi : \mathbb{R} \to (-1, 1)$ is periodic, then ε_0 may be taken independent of $\bar{a}$ and $\bar{b}$ in Theorem 4.1. More precisely, given $\delta > 0$ there exists $\varepsilon_0 > 0$ such that, for all $\varepsilon < \varepsilon_0$, $x_0 \in \mathbb{R}$ and $\ell(x_0) + \delta \leq \ell \leq \ell_\infty(x_0)$, equation (4.10) has a unique solution.

When $\phi(x_0)$ is close to zero, the Neumann problem may not behave well, and we prefer to use the Dirichlet problem to construct basic solutions. Arguing as in the Neumann case, but now using property (4.3), we can prove the following existence and uniqueness result.

THEOREM 4.2. *Let $\bar{x} \in (\bar{a}, \bar{b})$ such that $\phi(\bar{x}) = 0$. Then, there exists $\bar{\ell} > 0$ and ε_0 such that, for every $x_0 \in [\bar{x} - \varepsilon_0, \bar{x} + \varepsilon_0]$, $\varepsilon \in (0, \varepsilon_0)$ and $\bar{\ell} \leq \ell \leq (\bar{b} - x_0)/\varepsilon$ the equation*

$$u'' - f(x_0 + \varepsilon s, u(s)) = 0, \quad u(0) = u(\ell) = 0, \tag{4.13}$$

has a unique positive solution (negative solution), which is differentiable in x_0 and ℓ.

Remark 4.2. If $\phi : \mathbb{R} \to (-1, 1)$ is periodic, then $\bar{\ell}$ and ε_0 can be taken independent of $\bar{x}$ in Theorem 4.2.

5. *Construction of single clusters*

We start this section with a few definitions.

Definition 5.1. A solution e of (2.4) is *an increasing heteroclinic envelope* if $\mathrm{supp}(e) = (a, b)$ with a, b satisfying $\phi(a) < 0 < \phi(b), \phi'(a), \ \phi'(b) > 0$. In a similar way we define a *decreasing heteroclinic envelope*.

We say that a solution e of (2.4) is a *positive homoclinic envelope* if $\mathrm{supp}(e) = (a, b)$ with a, b satisfying $\phi(a), \phi(b) < 0, \ \phi'(a) > 0, \ \phi'(b) < 0$. Similarly, we say that a solution e of (2.4) is a *negative homoclinic envelope* if $\mathrm{supp}(e) = (a, b)$ with a, b satisfying $\phi(a), \phi(b) > 0$ and $\phi'(a) < 0, \ \phi'(b) > 0$.

This section is devoted to proving that if $e > \phi$ is an increasing (decreasing) heteroclinic envelope in a given bounded interval then for ε small enough there is a solution u_ε of (1.2) such that its approximate upper envelope e_ε converges to e, uniformly.

PROPOSITION 5.1. *Set $\phi < e_1 \leq e_2$ increasing (decreasing) heteroclinic envelopes of (2.4), with* $\mathrm{supp}(e_i) = (a_i, b_i)$ *for $i = 1, 2$ with $\phi'(x) > 0$ (< 0) in $[a_1, a_2] \cup [b_2, b_1]$. For all $\delta > 0$ and $\bar{a}, \bar{b}$ with $\bar{a} < a_1 < b_1 < \bar{b}$ there exists $\varepsilon_\delta > 0$ such that, for $0 < \varepsilon < \varepsilon_\delta$ and $e_1 \leq e \leq e_2$ solution of (2.4), the equation (1.2) admits a solution u_ε defined in $[\bar{a}, \bar{b}]$ satisfying $u'_\varepsilon(\bar{a}) = u'_\varepsilon(\bar{b}) = 0$, and $\|e_\varepsilon - e\|_{L^\infty(\bar{a},\bar{b})} < \delta$, where e_ε is the approximate envelope of u_ε, defined as in (3.9). The family $\{u_\varepsilon\}$ is called an increasing (decreasing) heteroclinic cluster.*

If e is a positive (negative) homoclinic envelope, we can write a similar statement for the existence of homoclinic clusters. The arguments for the proof of such a statement can be directly extended from those of Proposition 5.1.

We shall prove Proposition 5.1 under the extra assumption that there is exactly one $c \in (a_2, b_2)$ such that $\phi(c) = 0$. The general case can be treated similarly. We need two auxiliary envelopes $\tilde{e}, e_c : \mathbb{R} \to \mathcal{E}_+$ solutions of (2.4) satisfying $\tilde{e} \leq e_1 \leq e_2 \leq e_c$, with $\tilde{e}(c) < e_1(c) < e_2(c) < e_c(c) < 1$. We assume that $\mathrm{supp}(\tilde{e}) = (\tilde{a}, \tilde{b}) \subset (\bar{a}, \bar{b})$ and $\phi'(x) > 0$ in $[\tilde{a}, a_1] \cup [b_1, \tilde{b}]$. We also assume that $\mathrm{supp}(e_c) = (a_c, b_c)$, with $b_c - a_c$ suitably small. We define, for $x, y \in [\bar{a}, \bar{b}]$,

$$d(x, y) = \frac{1}{\varepsilon} \int_x^y \frac{ds}{T(s, \tilde{e}(s))}, \ d_c(x, y) = \frac{1}{\varepsilon} \int_x^y \frac{ds}{T(s, e_c(s))} \tag{5.1}$$

and we introduce the domain in $\mathbb{R}^N$

$$\begin{aligned} \Delta_e^\varepsilon = \{(x_1, x_2, \ldots, x_{N_e^\varepsilon}) \mid x_0 = \bar{a}, x_{N_e^\varepsilon+1} = \bar{b}, x_0 \leq x_1 \leq \cdots \leq x_{N_e^\varepsilon+1}, \\ d(x_i, x_{i+1}) \geq 1, d_c(x_i, x_{i+1}) \leq 1, i = 0, 1, \ldots, N_e^\varepsilon\}, \end{aligned} \tag{5.2}$$

where N_e^ε is chosen so that

$$N_e^\varepsilon = \left\lfloor \frac{1}{\varepsilon} \int_{\bar{a}}^{\bar{b}} \frac{1}{T(s, e(s))} ds \right\rfloor, \tag{5.3}$$

with $\lfloor s \rfloor$ denoting the closest even integer to s. Next we introduce our finite-dimensional functional g^ε. It will be notationally convenient to define an energy density as

$$\sigma_\varepsilon^\pm(x, u) = \varepsilon^2 \frac{u'^2(x)}{2} + F_\pm(x, u(x)).$$

We define $g^\varepsilon : \Delta_e^\varepsilon \to \mathbb{R}$ for $X = (x_1, x_2, \dots, x_{N_e^\varepsilon}) \in \Delta_e^\varepsilon$ as

$$g^\varepsilon(X) = \sum_{i=0}^{i_0-1} \int_{x_i}^{x_{i+1}} \sigma_\varepsilon^+(x, u_i)\, dx + \int_{x_{i_0}}^{c} \sigma_\varepsilon^+(x, u_{i_0})\, dx + \int_c^{x_{i_0+1}} \sigma_\varepsilon^-(x, u_{i_0})\, dx + \sum_{i=i_0+1}^{N_e^\varepsilon} \int_{x_i}^{x_{i+1}} \sigma_\varepsilon^-(x, u_i)\, dx, \tag{5.4}$$

where i_0 satisfies $x_1 \le \cdots \le x_{i_0} \le c \le x_{i_0+1} \le \cdots \le x_{N_e^\varepsilon}$, and u_i is defined as the solution of

$$\varepsilon^2 u_i'' - f(x, u_i) = 0, \quad u_i'(x_i) = 0 = u_i'(x_{i+1}), \tag{5.5}$$

with $(-1)^i u_i' > 0$. We remark that this equation has a unique solution thanks to Theorem 4.1 and the constraints $d(x_i, x_{i+1}) \ge 1$ and $d_c(x_i, x_{i+1}) \le 1$ in Δ_e^ε. We can easily check that

$$\frac{\partial g^\varepsilon}{\partial x_j}(X) = -F_+(x_j, u_j(x_j)) + F_+(x_j, u_{j-1}(x_j)), \quad 1 \le j \le i_0$$

and

$$\frac{\partial g^\varepsilon}{\partial x_j}(X) = -F_-(x_j, u_j(x_j)) + F_-(x_j, u_{j-1}(x_j)), \quad i_0 + 1 \le j \le N_e^\varepsilon.$$

Thus, if $\nabla g^\varepsilon(X) = 0$ then the function u_ε, defined as

$$u_\varepsilon(x) = u_i(x), \quad x \in [x_i, x_{i+1}], \quad i = 0, \dots, N_e^\varepsilon, \tag{5.6}$$

is a solution of (1.2) with $u_\varepsilon'(\bar{a}) = u_\varepsilon'(\bar{b}) = 0$. Consequently, in order to prove Proposition 5.1 we just need to prove that g^ε has an interior critical point. Actually we will show that the maximum of g^ε is achieved in $\mathrm{int}(\Delta_e^\varepsilon)$.

Proof of Proposition 5.1. We prove that there is an $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ the finite-dimensional functional g^ε achieves its maximum in $\mathrm{int}(\Delta_e^\varepsilon)$. Let us consider sequences $\{\varepsilon_n\}$, $e_n \to e_\infty$, $e_1 \le e_n \le e_2$, and $\{X_n\}$ so that $\varepsilon_n \to 0$, $X_n \in \Delta_{e_n}^{\varepsilon_n}$ and

$$g^{\varepsilon_n}(X_n) \ge g^{\varepsilon_n}(X) \quad \text{for all } X \in \Delta_{e_n}^{\varepsilon_n}.$$

It will be enough to prove that, up to a sub-sequence, for n large we have

$$X_n \in \mathrm{int}(\Delta_{e_n}^{\varepsilon_n}) \quad \text{and} \quad \nabla g^{\varepsilon_n}(X_n) = 0.$$

We write $N_n = N_{e_n}^{\varepsilon_n}$, $g_n = g_{e_n}^{\varepsilon_n}$, $u_n = u_{\varepsilon_n}$ and $\Delta_n = \Delta_{e_n}^{\varepsilon_n}$ for simplicity.

It will be convenient to consider another auxiliary envelope $\hat{e}$ between $\tilde{e}$ and e_1. Let $\hat{e}$ be a solution of (2.4) such that $\mathrm{supp}(\hat{e}) = (\hat{a}, \hat{b})$ and $\bar{a} < \tilde{a} < \hat{a} < a_1$ and $b_1 < \hat{b} < \tilde{b} < \bar{b}$. We define

$$B_n = \{i \mid [x_i^n, x_{i+1}^n] \cap (\hat{a}, \hat{b}) \ne \emptyset\},$$
$$B_n^1 = \{i \mid [x_i^n, x_{i+1}^n] \cap (\hat{a}, a_c) \ne \emptyset\},$$
$$B_n^2 = \{i \mid [x_{i-1}^n, x_i^n] \cap (b_c, \hat{b}) \ne \emptyset\}.$$

The proof of Proposition 5.1 consists of several steps.

Step 1. There exists $j_n \in B_n^1 \cup B_n^2$ such that, up to a sub-sequence,

$$\lim_{n\to\infty} d(x_{j_n}^n, x_{j_n+1}^n) \geq 1 + \kappa, \tag{5.7}$$

with $\kappa > 0$.

Assuming the contrary, we see that $d(x_j^n, x_{j+1}^n)$ approaches 1 uniformly in $j \in B_n^1 \cup B_n^2$. Then we have

$$|B_n^1 \cup B_n^2| = \frac{(1+\gamma_n)}{\varepsilon_n}\left\{\int_{\hat{a}}^{a_c} \frac{1}{T(x, \tilde{e}(x))}\, dx + \int_{b_c}^{\hat{b}} \frac{1}{T(x, \tilde{e}(x))}\, dx\right\},$$

where $\gamma_n \to 0$ as $n \to \infty$. Since $N_n \geq |B_n^1 \cup B_n^2|$, taking the limit we obtain

$$\int_{a_1}^{b_1} \frac{1}{T(x, e_\infty(x))}\, dx \geq \int_{\hat{a}}^{a_c} \frac{1}{T(x, \tilde{e}(x))}\, dx + \int_{b_c}^{\hat{b}} \frac{1}{T(x, \tilde{e}(x))}\, dx,$$

which is a contradiction, if we have chosen $b_c - a_c$ small enough. This proves Step 1.

We may assume without loss of generality that $j_n \in B_n^1$ for all n. We write $B_n^1 = \{i_1, \ldots, i_l\}$, with $i_1 < i_2 < \cdots < i_l$, where we have omitted the index n to simplify notation.

Step 2. For all n, the function u_{ε_n} defined in (5.6) is a solution of (1.2) in $(x_{i_1}^n, x_{i_l+1}^n)$ and satisfies $u'_{\varepsilon_n}(x_{i_1}^n) = u'_{\varepsilon_n}(x_{i_l+1}^n) = 0$.

If not, there is a sequence of integers k_n so that $i_1 < k_n \leq j_n$ for all n (or $j_n + 1 \leq k_n < i_l + 1$ for all n) so that

$$\frac{\partial g_n(X_n)}{\partial x_{k_n}^n} \neq 0$$

and u_n is a solution of (1.2) in $(x_{k_n}^n, x_{j_n+1}^n)$ (or in $(x_{j_n}^n, x_{k_n}^n)$).

Assume that we are in the first case (the other case is completely analogous). From (5.7) and Theorem 3.1 we have that for a certain $\tilde{\kappa} > 0$ it holds that

$$d(x_{k_n}^n, x_{k_n+1}^n) \geq 1 + \tilde{\kappa} \quad \text{for all } n. \tag{5.8}$$

Then we have the following possibilities.

(a) If for a sub-sequence $d(x_{k_n-1}^n, x_{k_n}^n) > 1$, then the point $Y_n = (y_1, \ldots, y_{N_n})$, such that $y_i = x_i$ for all $i \neq k_n$ and y_{k_n} close to $x_{k_n}^n$, also belongs to Δ_n. Choosing y_{k_n} so that

$$\frac{\partial g_n}{\partial x_{k_n}^n}(X_n)(y_{k_n} - x_{k_n}^n) > 0$$

we contradict the maximality of X_n.

(b) For a sub-sequence we have that $d(x_{k_n-1}^n, x_{k_n}^n) = 1$ for all n. Then, up to a sub-sequence, we see that $x_{k_n}^n$ converges to some point $\bar{x}$. By (5.8),

$$\lim_{n\to\infty} F(x_{k_n}, u_{k_n-1}(x_{k_n}^n)) > \lim_{n\to\infty} F(x_{k_n}, u_{k_n}(x_{k_n}^n)),$$

from which we conclude that

$$\frac{\partial g_n}{\partial x_{k_n}^n}(X_n) > 0,$$

for large n. If we define Y_n as before, with y_{k_n} slightly bigger than $x_{k_n}^n$, we see that $Y_n \in \Delta_n$ and X_n is not a maximum point. This completes the proof of Step 2.

Let i_{m_n} be the rightmost index in B_n^2.

Step 3. The function u_{ε_n} defined in (5.6) is a solution of (1.2) in $(x^n_{i_1}, x^n_{i_{m_n}})$ and satisfies $u'_{\varepsilon_n}(x^n_{i_1}) = u'_{\varepsilon_n}(x^n_{i_{m_n}+1}) = 0$.

We see that $x^n_{i_\ell+1} > a_c$. For $j \geq i_\ell + 1$ we need to check $d_c(x^n_j, x^n_{j+1}) < 1$ as well as $d(x^n_j, x^n_{j+1}) > 1$. Proceeding exactly as in Step 2 we prove Step 3 by the maximality of X_n.

Next we prove $i_1 = 0$ and $i_{m_n} = N_n + 1$ to finish the proof. If this is not the case, we may assume without loss of generality that $i_1 \geq 1$ for all n.

Step 4. The approximate envelope associated to $\{u_n\}$, as defined in §3, converges, up to a sub-sequence, to an envelope function e_0. Moreover $\overline{\operatorname{supp}(e_0)} \subset (\hat{a}, \hat{b})$.

We apply Theorem 3.1 to the sequence $\{u_n\}$. We define $a_n = x^n_{i_1}$ if $x^n_{i_1}$ is a minimum point of u_n or $a_n = x^n_{i_1+1}$ if $x^n_{i_1}$ is a maximum point of u_n. We also take $b_n = x^n_{i_{m_n}+1}$ if $x^n_{i_{m_n}+1}$ is a maximum point of u_n, otherwise $b_n = x^n_{i_{m_n}}$. Then Theorem 3.1 implies that the approximate envelope associated to $\{u_n\}$ converges, up to a sub-sequence, to an envelope function e_0 in (a_0, b_0), where $a_0 = \lim_{n\to\infty} a_n$ and $b_0 = \lim_{n\to\infty} b_n$. In view of Remark 3.1 we have that

$$\int_{a_0}^{b_0} \frac{dx}{T(x, e_0(x))} \leq \lim_{n\to\infty} \varepsilon_n N_n = \int_{\bar{a}}^{\bar{b}} \frac{dx}{T(x, e_\infty(x))}.$$

We easily see that

$$\int_{\hat{a}}^{\hat{b}} \frac{dx}{T(x, e_0(x))} = \int_{a_0}^{b_0} \frac{dx}{T(x, e_0(x))}.$$

Then, e_0 and e_∞ being solutions of (2.4) in $[\hat{a}, \hat{b}]$ we have that $e_0(x) \geq e_\infty(x)$ for all $x \in [\hat{a}, \hat{b}]$, and this implies that $\overline{\operatorname{supp}(e_0)} \subset (\hat{a}, \hat{b})$. This proves Step 4.

We observe that this last conclusion implies that $d(x_{i_1}, x_{i_1+1}) > 1$.

Step 5. x_{i_1} is a local minimum of u_n and $x_{i_{m_n}}$ is a local maximum of u_n.

Suppose that x_{i_1} is a local maximum of u_n.

In order to complete our proof we analyze three possible cases:

(1) for a sub-sequence we have $d(x^n_{i_1-1}, x^n_{i_1}) > 1$;
(2) for a sub-sequence we have $d(x^n_{i_1-1}, x^n_{i_1}) = 1$ and $x^n_{i_1-1} \to \bar{x} > \tilde{a}$;
(3) for a sub-sequence we have $d(x^n_{i_1-1}, x^n_{i_1}) = 1$ and $x^n_{i_1-1} \to \tilde{a}$.

For case (1) by the argument in Step 2(a) we can conclude that

$$\frac{\partial g_n}{\partial x^n_{i_1}}(X_n) = 0.$$

Therefore, u_n defined in (5.6) is a solution of (1.2) in $(x^n_{i_1-1}, x^n_{i_{m_n}+1})$ with a maximum in $x^n_{i_1}$ which contradicts Proposition 3.2. To check that case (2) does not hold, we can use an argument as in Step 2(b).

If case (3) holds, we have that there exists a constant $c > 0$ such that $|x^n_{i_1} - x^n_{i_1+1}| > c$. Hence we can define $Y_n = (y^n_1, \ldots, y^n_{N_n}) \in \Delta_n$ as $y^n_i = x^n_i$ if $i \neq i_1$ and $y_{i_1} = x_{i_1} + \zeta$, where $\zeta > 0$ small. We have

$$
\begin{aligned}
g_n(Y_n) - g_n(X_n) = &\int_{x^n_{i_1-1}}^{x^n_{i_1}+\zeta} \sigma^+_{\varepsilon_n}(x, v_{i_1-1})\, dx + \int_{x^n_{i_1}+\zeta}^{x^n_{i_1+1}} \sigma^+_{\varepsilon_n}(x, v_{i_1})\, dx \\
&- \int_{x^n_{i_1-1}}^{x^n_{i_1}} \sigma^+_{\varepsilon_n}(x, u_{i_1-1})\, dx - \int_{x^n_{i_1}}^{x^n_{i_1+1}} \sigma^+_{\varepsilon_n}(x, u_{i_1})\, dx, \qquad (5.9)
\end{aligned}
$$

where v_{i_1-1} and v_{i_1} satisfy (5.5) replacing $x^n_{i_1}$ by $x^n_{i_1} + \zeta$ in both cases. Rescaling these functions as $z_{i_1-1}(t) = v_{i_1-1}(x^n_{i_1} + \zeta - \varepsilon_n t)$ and $z_{i_1}(t) = v_{i_1}(x^n_{i_1} + \zeta + \varepsilon_n t)$, $t \geq 0$, we see both converge to the solution z of the equation

$$
z''(s) - f(\tilde{a} + \zeta, z(s)) = 0, \quad z'(0) = 0,\ z(\infty) = 1,\ z' < 0. \qquad (5.10)
$$

Similarly we define the functions $w_{i_1-1}(t) = u_{i_1-1}(x_{i_1} - \varepsilon_n t)$ and $w_{i_1}(t) = u_{i_1}(x^n_{i_1} + \varepsilon_n t)$, for $t \geq 0$, and we see that they converge to the solution w of (5.10), but replacing $\tilde{a} + \zeta$ by $\tilde{a}$.

Next we rescale the integrals in (5.9) and we obtain

$$
g_n(Y_n) - g_n(X_n) = \varepsilon_n I_n \quad \text{where} \quad \lim_{n\to\infty} I_n = I,
$$

and I is given by

$$
I = 2\int_0^\infty \frac{(z')^2}{2} + F_+(\tilde{a} + \zeta, z)\, ds - 2\int_0^\infty \frac{(w')^2}{2} + F_+(\tilde{a}, w)\, ds.
$$

Then we use Lemma 4.3 to get $I > 0$ and we conclude that $g_n(Y_n) - g_n(X_n) > 0$, which is a contradiction. Therefore, x_{i_1} is a minimum. Similarly we have $x_{i_{m_n}}$ is a maximum.

Step 6. $i_1 = 0$ and $i_{m_n} = N_n + 1$.

Suppose that $i_1 \geq 2$. Since $x^n_{i_1}$ is a minimum we have that there exists a constant $c > 0$ such that $|x^n_{i_1} - x^n_{i_1+1}| > c$. If $d(x^n_{i_1-1}, x^n_{i_1}) > 1$ and $d(x^n_{i_1-2}, x^n_{i_1-1}) > 1$ then proceeding as in Step 5(1) we obtain a contradiction. If $d(x^n_{i_1-1}, x^n_{i_1}) = 1$ and $x_{i_1-1} \to \bar{x} > \tilde{a}$, then we proceed as in Step 5(2) to reach a contradiction. When $d(x^n_{i_1-1}, x^n_{i_1}) = 1$ and $x_{i_1-1} \to \tilde{a}$, then we define $Y_n = (y^n_1, \ldots, y^n_{N_n}) \in \Delta_n$ as $y^n_i = x^n_i$ if $i \neq i_1 - 1, i_1$, $y_{i_1-1} = x_{i_1-1} + \zeta$, $y_{i_1} = x_{i_1} + \zeta$, where $\zeta > 0$ small. Then, following the same reasoning as in Step 5(3) we obtain a contradiction. Similarly, we can show that $i_{m_n} = N_n + 1$.

This ends the proof of Proposition 5.1. □

Remark 5.1. In view of Remark 4.1 we can show that, when $\phi : \mathbb{R} \to (-1, 1)$ is periodic, the number ε_δ in Proposition 5.1 can be chosen independent of $\bar{a}$ and $\bar{b}$. Indeed, if we consider $\bar{a} \leq a_0 < a_1$ and $\bar{b} \geq b_0 > b_1$ then ε_δ depends only on e_1, e_2, a_0, b_0 and δ.

Remark 5.2. We can generalize Proposition 5.1 to solutions having the following degeneracy in the closed interval I: the boundary of supp$(e) \cap$ int(I) contains critical points of ϕ. This occurs if the function e touches a trivial envelope at a critical point of ϕ.

For example let e be a non-trivial envelope with graph in $\mathcal{E}_+$ for all $x \in I$ and for which ϕ has a negative minimum at $\bar{x}$ with $e(\bar{x}) = \phi_+^*(\bar{x})$.

To construct $\{u_\varepsilon\}$ corresponding to e, we argue as in [**13**]. First we approximate e by a sequence of envelopes $\{e_n\}$ such that e_n satisfies the assumptions of Proposition 5.1 and $e_n \to e$. We construct solutions $\{u_i^n\}_{i=1}^\infty$ and use a diagonal argument to conclude.

Remark 5.3. We say that an envelope e on the interval (a, b) is a *boundary envelope* if $e(a)$ (or $e(b)$) does not belong to a trivial envelope. In a similar way to Proposition 5.1, we can construct a family $\{u_\varepsilon\}$ of solutions to equation (1.2) in (a, b) under the Neumann boundary condition, corresponding to any given boundary envelope.

6. *Gluing clusters*

In this section we will prove that it is possible to *glue* an arbitrary finite number of homoclinic or heteroclinic clusters.

THEOREM 6.1. *Set $e_1, e_2 : \mathbb{R} \to \mathcal{E}_+$ solutions of (2.4) with* $\operatorname{supp}(e_i) = \bigcup_{j=1}^k (a_i^j, b_i^j)$ *where $a_1^j < a_2^j < b_2^j < b_1^j$ for all j, $b_1^j < a_1^{j+1}$ for $j = 1, \ldots, k-1$ and $\phi'(x) \neq 0$ in $\bigcup_{j=1}^k [a_1^j, a_2^j] \cup [b_2^j, b_1^j]$. Then for $\bar{a} < a_1^1$ and $\bar{b} > b_1^k$ there exists $\varepsilon_0 > 0$ such that for $e_1 \leq e \leq e_2$ solution of (2.4) there exists u_ε solution of (1.2) satisfying $u_\varepsilon'(\bar{a}) = u_\varepsilon'(\bar{b}) = 0$ with the envelope e_ε converging uniformly to e in $[\bar{a}, \bar{b}]$.*

We notice that from here we directly obtain Theorem 1.2. The proof of Theorem 6.1 is a consequence of the following proposition.

PROPOSITION 6.1. *Under the conditions of Theorem 6.1 with $k = 2$ assume that either, for $i = 1, 2$:*

(a) *$e_i : (a_i^1, b_i^1) \to \mathcal{E}_+$ are increasing heteroclinic envelopes and $e_i : (a_i^2, b_i^2) \to \mathcal{E}_+$ are positive homoclinic envelopes; or*

(b) *$e_i : (a_i^1, b_i^1) \to \mathcal{E}_+$ are positive homoclinic envelopes and $e_i : (a_i^2, b_i^2) \to \mathcal{E}_+$ are increasing heteroclinic envelopes.*

Then there exists $\varepsilon_0 > 0$ such that for any $e_1 \leq e \leq e_2$ solution of (2.4) there exists u_ε solution of (1.2) satisfying $u_\varepsilon'(\bar{a}) = u_\varepsilon'(\bar{b}) = 0$ with envelope e_ε converging uniformly to e in $[\bar{a}, \bar{b}]$.

Proof of Proposition 6.1 part (a). We prove Proposition 6.1 part (a) under the extra assumption that there is exactly one $c \in (a_1^1, b_1^1)$ such that $\phi(c) = 0$ and $\phi < 0$ in (a_1^2, b_1^2). The general case can be treated with minor changes. Following the ideas of §5, we introduce two auxiliary envelopes $\tilde{e}, e_c : \mathbb{R} \to \mathcal{E}_+$ solutions of (2.4), satisfying $\tilde{e} \leq e_1 \leq e_2 \leq e_c$, with $\tilde{e}(c) < e_1(c) \leq e_2(c) < e_c(c) < 1$ and $\operatorname{supp}(e_c) = (a_c, b_c)$, with $a_2^1 < a_c < c < b_c < b_2^1$ and $b_c - a_c$ suitably small. We also assume that $\operatorname{supp}(\tilde{e}) = (\tilde{a}^1, \tilde{b}^1) \cup (\tilde{a}^2, \tilde{b}^2) \subset (\bar{a}, \bar{b})$ and $\phi'(x) > 0$ in $[\tilde{a}^1, a_1^1] \cup [b_1^1, \tilde{b}^1] \cup [\tilde{a}^2, a_1^2]$ and $\phi'(x) < 0$ in $[b_1^2, \tilde{b}^2]$. As in §5, for $x, y \in [\bar{a}, \bar{b}]$, we define the distances $d(x, y)$ and $d_c(x, y)$ as in (5.1). Set $c_* \in (b_1^1, a_1^2)$ such that $\phi(c_*) = 0$ and $\phi'(c_*) < 0$.

We let N_1^ε and N_2^ε be the even integers defined by

$$N_i^\varepsilon = \left\lfloor \frac{1}{\varepsilon} \int_{a_1^i}^{b_1^i} \frac{1}{T(s, e(s))}\, ds \right\rfloor \quad \text{for } i = 1, 2 \tag{6.1}$$

and $N^\varepsilon = N_1^\varepsilon + N_2^\varepsilon + 2$, where we have omitted the dependence on e to keep the notation simpler. Then we introduce the domain in $\mathbb{R}^{N^\varepsilon}$

$$\begin{aligned} \Delta_e^\varepsilon = \{&(x_1, x_2, \ldots, x_{N^\varepsilon}) \mid \bar{a} = x_0 \le x_1 \le \cdots \le x_{N^\varepsilon+1} = \bar{b}, \\ &c_* - \delta \le x_{N_1^\varepsilon+2} \le c_* + \delta,\ |x_{N_1^\varepsilon+j+1} - x_{N_1^\varepsilon+j}| \ge l\varepsilon,\ \text{for } j = 1, 2, \\ &d(x_i, x_{i+1}) \ge 1,\ \ d_c(x_i, x_{i+1}) \le 1, \ \ \text{for } 0 \le i \le N_1^\varepsilon, \\ &N_1^\varepsilon + 3 \le i \le N^\varepsilon\}, \end{aligned} \tag{6.2}$$

where $l > 0$ and $\delta > 0$ are constants to be suitably chosen. In order to define our finite-dimensional functional we consider the function u_i defined by (5.5) with $(-1)^i u_i' \ge 0$ for $i = 0, \ldots, N_1^\varepsilon$ and $(-1)^i u_i' \le 0$ for $i = N_1^\varepsilon + 3, \ldots, N^\varepsilon$. We notice that these solutions are well defined thanks to our constraints $d \ge 1$ and $d_c \le 1$ and Theorem 4.1. For $i = N_1^\varepsilon + 1$ we define u_i to be the solution of

$$\varepsilon^2 u'' - f(x, u) = 0, \tag{6.3}$$

with $u_i'(x_i) = 0$, $u_i(x_{i+1}) = 0$, $u_i' \le 0$. In addition, for $i = N_1^\varepsilon + 2$ the function u_i satisfies (6.3) with $u_i(x_i) = 0$, $u_i'(x_{i+1}) = 0$, $u_i' \ge 0$. If δ is chosen small and l is large, we can properly define these solutions using Theorem 4.2.

Now we define $g^\varepsilon : \Delta_e^\varepsilon \to \mathbb{R}$ for $X \in \Delta_e^\varepsilon$ as

$$\begin{aligned} g^\varepsilon(X) = &\sum_{i=0}^{i_0-1} \int_{x_i}^{x_{i+1}} \sigma_\varepsilon^+(x, u_i)\, dx + \int_{x_{i_0}}^{c} \sigma_\varepsilon^+(x, u_{i_0})\, dx \\ &+ \int_c^{x_{i_0+1}} \sigma_\varepsilon^-(x, u_{i_0})\, dx + \sum_{i=i_0+1}^{j_0-1} \int_{x_i}^{x_{i+1}} \sigma_\varepsilon^-(x, u_i)\, dx + \int_{x_{j_0}}^{c_*} \sigma_\varepsilon^-(x, u_{j_0})\, dx \\ &+ \int_{c_*}^{x_{j_0+1}} \sigma_\varepsilon^-(x, u_{j_0})\, dx + \sum_{i=j_0+1}^{N^\varepsilon} \int_{x_i}^{x_{i+1}} \sigma_\varepsilon^+(x, u_i)\, dx, \end{aligned} \tag{6.4}$$

where i_0, j_0 satisfy $x_1 \le \cdots \le x_{i_0} \le c \le x_{i_0+1} \le \cdots \le x_{j_0} \le c_* \le x_{j_0+1} \le \cdots \le x_{N_e^\varepsilon}$. We observe that by the constraint $d_c(x_i, x_{i+1}) \le 1$ we certainly have $i_0 + 1 \le j_0$.

We can easily check that for $j = N_1^\varepsilon + 2$

$$\frac{\partial g^\varepsilon}{\partial x_j}(X) = \frac{\varepsilon^2}{2}({u_j'}^2(x_j) - {u_{j-1}'}^2(x_j))$$

and for $j \ne N_1^\varepsilon + 2$ we have

$$\begin{aligned} \frac{\partial g^\varepsilon}{\partial x_j}(X) &= F_+(x_j, u_{j-1}(x_j)) - F_+(x_j, u_j(x_j)), \quad j \le i_0 \text{ or } j \ge j_0 + 1, \\ \frac{\partial g^\varepsilon}{\partial x_j}(X) &= F_-(x_j, u_{j-1}(x_j)) - F_-(x_j, u_j(x_j)), \quad i_0 + 1 \le j \le j_0. \end{aligned}$$

Thus, if $\nabla g^{\varepsilon}(X) = 0$ then the function u_ε, defined as

$$u_\varepsilon(x) = u_i(x), \quad x \in [x_i, x_{i+1}], \quad i = 0, \ldots, N^{\varepsilon}, \tag{6.5}$$

is a solution of (1.2) with $u'_\varepsilon(\bar{a}) = u'_\varepsilon(\bar{b}) = 0$.

Hence, to complete the proof of the proposition, it suffices to show that the maximum of g^ε is achieved on $\mathrm{int}(\Delta^\varepsilon_e)$. We proceed by contradiction. Suppose that there exist sequences $\varepsilon_n \to 0$, $e_1 \le e_n \le e_2$ and $X_n = (x_1^n, \ldots, x^n_{N^{\varepsilon_n}}) \in \partial\Delta^{\varepsilon_n}_{e_n}$ such that $g^{\varepsilon_n}(X_n) \ge g^{\varepsilon_n}(X)$ for all $X \in \Delta^{\varepsilon_n}_{e_n}$. For simplicity we denote $x_i = x_i^n$, $\Delta_n = \Delta^{\varepsilon_n}_{e_n}$, $g_n = g^{\varepsilon_n}$ and $N_i^n = N_i^{\varepsilon_n}$, for $i = 1, 2$.

Since $X_n \in \partial\Delta_n$ we have three possible cases:

(1) *Case 1.* Except possibly for a sub-sequence, we have

$$|x_{N_1^n+2} - x_{N_1^n+1}| = \varepsilon_n l \quad \text{or} \quad |x_{N_1^n+3} - x_{N_1^n+2}| = \varepsilon_n l.$$

(2) *Case 2.* Except for a sub-sequence we have that

$$x_{N_1^n+2} = c_* - \delta \quad \text{or} \quad x_{N_1^n+2} = c_* + \delta.$$

(3) *Case 3.* Except for a sub-sequence,

$$d(x_{i_n}, x_{i_n+1}) = 1 \quad \text{or} \quad d_c(x_{i_n}, x_{i_n+1}) = 1$$

for some $0 \le i_n \le N_1^n$ or $N_1^n + 3 \le i_n \le N^n$.

Case 1. We may assume that $|x_{N_1^n+2} - x_{N_1^n+1}| = \varepsilon_n l$, for a sub-sequence. Then, after scaling, we see that $u_{N_1^n+1}$ converges to a limiting solution defined in $[0, l]$, while $u_{N_1^n}$ converges to a heteroclinic solution. Then we find that

$$\frac{\partial g_n(X_n)}{\partial x_{N_1^n+1}} = F_\pm(x_{N_1^n+1}, u_{N_1^n}(x_{N_1^n+1})) - F_\pm(x_{N_1^n+1}, u_{N_1^n+1}(x_{N_1^n+1})) < 0.$$

Therefore we can find $Y_n \in \Delta_n$ such that $g_n(Y_n) > g_n(X_n)$, which contradicts our assumption.

Case 2. Since Case 1 does not hold, we may assume that $|x_{N_1^n+2} - x_{N_1^n+1}| > \varepsilon_n l$ and $|x_{N_1^n+3} - x_{N_1^n+2}| > \varepsilon_n l$. We may also assume that $x_{N_1^n+2} = c_* - \delta$. Then, using an argument like in Step 5 in the proof of Proposition 5.1, we can prove that

$$\frac{\partial g_n(X_n)}{\partial x_{N_1^n+2}} = \frac{\varepsilon_n^2}{2}((u'_{N_1^n+2})^2(x_{N_1^n+2}) - (u'_{N_1^n+1})^2(x_{N_1^n+2})) > 0.$$

Hence increasing the value of $x_{N_1^n+2}$ yields a point $Y_n \in \Delta_n$ such that $g_n(Y_n) > g_n(X_n)$, contradicting our assumption again.

Case 3. Here we are in a position of repeating the arguments given in Proposition 5.1, completing the proof of part (a) of Proposition 6.1. □

Proof of Proposition 6.1 part (b). We prove the proposition assuming additionally that there exists a unique $c \in (a_1^2, b_1^2)$ such that $\phi(c) = 0$ and that $\phi < 0$ in $[a_1^1, b_1^1]$. Keeping the notation of part (a), we introduce $\tilde{e}$ and e_c. We notice that now $a_2^2 < a_c < c < b_c < b_2^2$ and $\phi'(x) > 0$ in $[\tilde{a}^1, a_1^1] \cup [\tilde{a}^2, a_1^2] \cup [b_1^2, \tilde{b}^2]$ and $\phi'(x) < 0$ in $[b_1^1, \tilde{b}^1]$. We define the distances d and d_c as before.

With N_i^ε, $i = 1, 2$, as in (6.1), we define $N^\varepsilon = N_1^\varepsilon + N_2^\varepsilon + 1$. Next we introduce the domain

$$\begin{aligned}\Delta_e^\varepsilon = \{&(x_1, x_2, \dots, x_{N^\varepsilon}) \mid \bar a = x_0 \le x_1 \le \cdots \le x_{N^\varepsilon+1} = \bar b,\\ &x_{N_1^\varepsilon} \le \tilde b^1 + \delta,\ x_{N_1^\varepsilon+2} \ge \tilde a^2 - \delta,\\ &d(x_i, x_{i+1}) \ge 1,\ \ d_c(x_i, x_{i+1}) \le 1, \quad \text{for } 0 \le i \le N^\varepsilon\}, \end{aligned} \tag{6.6}$$

where $\delta > 0$ is small and fixed. We set the functional $g^\varepsilon : \Delta_\varepsilon \to \mathbb{R}$ as

$$\begin{aligned} g^\varepsilon(X) = &\sum_{i=0}^{i_0-1} \int_{x_i}^{x_{i+1}} \sigma_\varepsilon^+(x, u_i)\, dx + \sum_{i=i_0+1}^{N^\varepsilon} \int_{x_i}^{x_{i+1}} \sigma_\varepsilon^-(x, u_i)\, dx\\ &+ \int_{x_{i_0}}^{c} \sigma_\varepsilon^+(x, u_{i_0})\, dx + \int_c^{x_{i_0+1}} \sigma_\varepsilon^-(x, u_{i_0})\, dx, \end{aligned}$$

where $x_1 \le \cdots \le x_{i_0} \le c < x_{i_0+1} \le \cdots \le x_{N^\varepsilon+1}$. The function u_i is defined by (5.5).

As before, to complete the proof we just need to show that g^ε achieves its maximum in $\mathrm{int}(\Delta_\varepsilon)$. Assuming the contrary, and with the notational convention of the previous proof, there exist sequences $\{\varepsilon_n\}$, $e_1 \le e_n \le e_2$, $\{X_n\}$ so that $\varepsilon_n \to 0$ as $n \to \infty$ and $X_n \in \partial\Delta_n$ satisfies $g_n(X_n) \ge g_n(X)$ for all $X \in \Delta_n$. We have three possible cases:

(1) *Case 1.* Except possibly for a sub-sequence, we have

$$X_n = (x_1, \dots, x_{N_1^n-1}, \tilde b_1 + \delta, x_{N_1^n+1}, \dots, x_{N_1^n+N_2^n+1}).$$

(2) *Case 2.* Except for a sub-sequence we have that

$$X_n = (x_1, \dots, x_{N_1^n}, x_{N_1^n+1}, \tilde a_2 - \delta, \dots, x_{N_1^n+N_2^n+1}).$$

(3) *Case 3.* Except for a sub-sequence,

$$X_n = (x_1, \dots ., x_{N_1^n+N_2^n+1}),\ x^1_{N_1^n} < \tilde b_1 + \delta,\ x_{N_1^n+2} > \tilde a_2 - \delta$$

and there is i_n such that $d(x_{i_n}, x_{i_n+1}) = 1$ for all $n \in \mathbb{N}$.

Case 1. We assume that $x_{N_1^n-1} > c$. The case $x^n_{N_1^n-1} \le c$ can be handled in a similar way. We consider

$$Y_n = (x_1, \dots, x_{N_1^n-1}, \tilde b_1 + \delta/2, x_{N_1^n+1}, \dots, x_{N_1^n+N_2^n+1}),$$

which belongs to $\partial\Delta_n$ if δ is small. The difference $g_n(Y_n) - g_n(X_n)$ can be written by an expression similar to (5.9). Then, proceeding as in Step 5 in the proof of Proposition 5.1, we can show that

$$g_n(Y_n) - g_n(X_n) = \varepsilon_n J_n, \quad \text{where } \lim_{n\to\infty} J_n = J,$$

and J is given by

$$J = 2\{E_+(\tilde b_1 + \delta/2) - E_+(\tilde b_1 + \delta)\},$$

with

$$E_+(x) = \int_0^\infty \frac{(y')^2}{2} + F_+(x, y)\, ds \quad x \in (c, b + \delta],$$

for y satisfying a suitable limiting equation. Then, in view of Lemma 4.3 we obtain $J > 0$, which leads to a contradiction.

Case 2. This can be handled in a similar way.

Case 3. We proceed following the proof of Proposition 5.1 with minor modifications. □

Proof of Theorem 6.1. In order to glue any pair of clusters, we can proceed as in part (a) or part (b) of Proposition 6.1. The case of a general k can be handled as in Proposition 6.1, with obvious changes of notation, but by means of the same ideas. $\square$

Remark 6.1. Suppose that $\phi : \mathbb{R} \to (-1, 1)$ is 1-periodic and there exist solutions e_1, $e_2 : \mathbb{R} \to [-1, 1]$ of (2.4) with $\mathrm{supp}(e_i) = (a_i, b_i)$ for $i = 1, 2$ with $0 < a_1 < a_2 < b_2 < b_1 < 1$. From the argument given above, for any $\delta > 0$ we can find $\varepsilon_\delta > 0$ such that for a given $k \in \mathbb{N}$ and $\{j_1, j_2, \dots j_k\} \subset \mathbb{Z}$ and for any solution e of (2.4) satisfying

$$e_1(x - j_i) \le e(x) \le e_2(x - j_i) \text{ in } [j_i, j_i + 1],$$

for $i = 1, 2, \dots, k$, $e(x) = \phi_+^*(x)$ in $\mathbb{R} \setminus \bigcup_{i=1}^k [j_i, j_i + 1]$ and $\bar{a} < j_1 < j_k + 1 < \bar{b}$, there exists u_ε $(0 < \varepsilon < \varepsilon_\delta)$ solution of (1.2) with $u_\varepsilon'(\bar{a}) = u_\varepsilon'(\bar{b}) = 0$ satisfying $\|e_\varepsilon(x) - e(x)\|_{L^\infty(\bar{a},\bar{b})} < \delta$.

We remark that ε_δ is independent of k, as can be proved by a contradiction argument in combination with Propositions 3.3 and 3.4.

7. *Existence of chaotic solutions*

In this section we will construct solutions of (1.2) in $\mathbb{R}$ whose envelope will be characterized in terms of a sequence of real numbers.

We assume that $\phi : \mathbb{R} \to (-1, 1)$ is 1-periodic. We fix solutions $e_1, e_2 : \mathbb{R} \to \mathcal{E}_+$ of (2.4), with $\mathrm{supp}(e_i) = (a_i, b_i)$ for $i = 1, 2$ and $0 < a_1 < a_2 < b_2 < b_1 < 1$. Moreover, we suppose that $\phi' \ne 0$ in $[a_1, a_2] \cup [b_2, b_1]$. Set

$$c_i = \int_{a_1}^{b_1} \frac{1}{T(x, e_i(x))}\, dx \quad \text{for } i = 1, 2.$$

Then, for any $\gamma \in [c_1, c_2]$ there exists a unique envelope $e_1 \le e_\gamma \le e_2$ such that

$$\gamma = \int_{a_1}^{b_1} \frac{1}{T(x, e_\gamma(x))}\, dx.$$

For notational convenience we set $e_0(x) = \phi_+^*(x)$.

We will consider sequences $(\gamma_n)_{n\in\mathbb{Z}} \in ([c_1, c_2] \cup \{0\})^{\mathbb{Z}}$.

THEOREM 7.1. *For any $\delta > 0$ there exists $\varepsilon_\delta > 0$ such that for any prescribed sequence $(\gamma_n)_{n\in\mathbb{Z}} \in ([c_1, c_2] \cup \{0\})^{\mathbb{Z}}$ there exists a solution $u_\varepsilon : \mathbb{R} \to [-1, 1]$ of (1.2) such that*

$$\sup_{n\in\mathbb{Z}} \|e_\varepsilon(x + n) - e_{\gamma_n}(x)\|_{L^\infty(0,1)} < \delta,$$

where e_ε is the approximate envelope of u_ε.

Proof. Fix $\delta > 0$ and $(\gamma_n)_{n\in\mathbb{Z}} \in ([c_1, c_2] \cup \{0\})^{\mathbb{Z}}$. By Remark 6.1 there exists $\varepsilon_\delta > 0$ independent of k such that for any $k \in \mathbb{N}$ and $0 < \varepsilon < \varepsilon_\delta$ there exists a solution $u_{k,\varepsilon} : [-k - 1, k + 2] \to [-1, 1]$ of (1.2) with ${u_{k,\varepsilon}}'(-k - 1) = {u_{k,\varepsilon}}'(k + 2) = 0$ satisfying

$$\sup_{|n|\le k} \|e_{k,\varepsilon}(x + n) - e_{\gamma_n}(x)\|_{L^\infty(0,1)} < \delta.$$

Here $e_{k,\varepsilon}$ is the approximate envelope of $u_{k,\varepsilon}$.

For fixed ε we consider the sequence $u_{k,\varepsilon}$. Since the $u'_{k,\varepsilon}$ are bounded independent of k, we can use the Arzela–Ascoli theorem to show that, except for a sub-sequence, $u_{k,\varepsilon} \to u_\varepsilon$ locally uniformly in $\mathbb{R}$ as $k \to \infty$. It can be checked that u_ε is the desired solution. $\square$

Remark 7.1. We can generalize Theorem 7.1 to a more general situation. We assume that $\phi : \mathbb{R} \to (-1, 1)$ is 1-periodic and there exist solutions $e_1^j < e_2^j : \mathbb{R} \to \mathcal{E}_+$ $(j = 1, 2, \ldots, k)$ of (2.4), with $\mathrm{supp}(e_i^j) = (a_i^j, b_i^j)$ for $i = 1, 2$, $j = 1, 2, \ldots, k$ and $0 < a_1^1 < a_2^1 < b_2^1 < b_1^1 < a_1^2 < a_2^2 < b_2^2 < b_1^2 < \cdots < a_1^k < a_2^k < b_2^k < b_1^k < 1$. Set

$$c_i^j = \int_{a_i^j}^{b_i^j} \frac{1}{T(x, e_i^j(x))}\, dx \quad \text{for } i = 1, 2 \text{ and } j = 1, 2, \ldots, k.$$

Then for any given sequence

$$((\gamma_n^1, \gamma_n^2, \ldots, \gamma_n^k))_{n \in \mathbb{N}} \in (([c_1^1, c_2^1] \cup \{0\}) \times ([c_1^2, c_2^2] \cup \{0\}) \times \cdots \times ([c_1^k, c_2^k] \cup \{0\}))^{\mathbb{Z}}$$

we can construct the corresponding solutions u_ε and envelope e_ε as in Theorem 7.1.

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