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A probabilistic interpretation and stochastic particle approximations of the 3-dimensional Navier-Stokes equations

Abstract. We develop a probabilistic interpretation of local mild solutions of the three dimensional Navier-Stokes equation in the L^p spaces, when the initial vorticity field is integrable. This is done by associating a generalized nonlinear diffusion of the McKean-Vlasov type with the solution of the corresponding vortex equation. We then construct trajectorial (chaotic) stochastic particle approximations of this nonlinear process. These results provide the first complete proof of convergence of a stochastic vortex method for the Navier-Stokes equation in three dimensions, and rectify the algorithm conjectured by Esposito and Pulvirenti in 1989. Our techniques rely on a fine regularity study of the vortex equation in the supercritical L^p spaces, and on an extension of the classic McKean-Vlasov model, which incorporates the derivative of the stochastic flow of the nonlinear process to explain the vortex stretching phenomenon proper to dimension three.

1. Introduction

The Navier-Stokes equation for an homogeneous and incompressible fluid in the whole space or plane, is given by

$$\begin{aligned} &\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = \nu\Delta\mathbf{u} - \nabla\mathbf{p}; \\ &div\,\mathbf{u}(t,x) = 0; \quad \mathbf{u}(t,x)\to 0 \text{ as } |x|\to\infty, \end{aligned} \tag{1}$$

where $\mathbf{u}$ is the velocity field, $\mathbf{p}$ is the pressure function and $\nu > 0$ is the viscosity coefficient assumed to be constant.

In this work we develop a probabilistic interpretation of the Navier-Stokes equation (1) in three dimensions. More precisely, we will consider the vortex equation satisfied by the vorticity field $curl\mathbf{u}$, and we will show in a general functional

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framework that it can be viewed as a generalized McKean-Vlasov equation associated with a nonlinear diffusion process. As a consequence, we will construct and prove the convergence of a stochastic particle method for the solution of (1) in that functional setting.

Thirty years ago, Chorin [10] proposed an heuristical probabilistic algorithm to numerically simulate the solution of the Navier-Stokes equation in two dimensions, by approximating the (scalar) vorticity function by random interacting "point vortices". The convergence of Chorin's vortex method was first mathematically proved in 1982 by Marchioro and Pulvirenti [23], who interpreted the vortex equation in two dimensions with bounded and integrable initial condition as a generalized McKean-Vlasov equation (with a singular interacting kernel) associated with a nonlinear diffusion. (For general expositions on the McKean-Vlasov model and nonlinear processes, we refer the reader to Sznitman [33] or Méléard [25].) Following the pioneering ideas of McKean [22], Marchioro and Pulvirenti defined then some stochastic systems of particles interacting weakly through cutoffed kernels, and for which the empirical measure converges at each time (when the number of particles tends to ∞) to the solution of the vortex equation. The results of [23] were later improved by Méléard [26], [27], who showed the convergence in the path space of the empirical measures of the interacting particle systems or, equivalently, the propagation of chaos for the system of particles. (Propagation of chaos for a system of particles without cutoff has been proved by Osada [29], but only for large viscosities and vorticities which indeed are bounded probability densities).

A rigorous probabilistic interpretation and a stochastic vortex method for the Navier-Stokes equation in three dimensions have been open problems since the paper [23] appeared. An attempt to extend those results to the three dimensional case was done by Esposito and Pulvirenti [13], but they did not give rigorous mathematical proofs of crucial facts.

In three dimensions, the vorticity field $\mathbf{w} = curl\mathbf{u}$ is a solution of the nonlinear equation

$$\begin{aligned} &\frac{\partial \mathbf{w}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{w} = (\mathbf{w} \cdot \nabla)\mathbf{u} + \nu \Delta \mathbf{w}, \\ &div\ w_0 = 0, \end{aligned} \tag{2}$$

where, thanks to the condition of incompressibility, $div\mathbf{u} = 0$, and by the Biot-Savart law, the velocity field $\mathbf{u}$ is equal to

$$\mathbf{u}(t, x) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{(x - y)}{|x - y|^3} \wedge \mathbf{w}(t, y) dy. \tag{3}$$

Here, $\wedge$ stands for the vectorial product in $\mathbb{R}^3$ and, with the notation $K(x) := -\frac{1}{4\pi} \frac{x}{|x|^3}$, the vectorial kernel $K(x) \wedge \cdot$ is the so-called *Biot-Savart kernel* in three dimensions. We refer to Chorin and Marsden [11] Ch. 1, Chorin [10] Ch.1, Marchioro and Pulvirenti [24], or the recent book of Bertozzi and Majda [3] for this facts and for background on vorticity.

The vectorial equation (2) is not conservative due to the *vortex stretching* term $(\mathbf{w} \cdot \nabla)\mathbf{u} = \sum_j \mathbf{w}_j \frac{\partial \mathbf{u}}{\partial x_j}$. In fact, vortex stretching lies in the heart of complex three dimensional phenomena such as transfer of energy and turbulence (see [10] Ch. 5), and is also related to the emergence of singularities (see Beale, Kato, Majda [1] or Bertozzi and Majda [3]).

In this work, we will consider the vortex equation (2) with integrable initial conditions w_0, which belong moreover to a suitable $[L^p(\mathbb{R}^3)]^3$ space (this will be $\frac{3}{2} < p$). Define a probability law ρ_0 on $\mathbb{R}^3$ and a function $h_0 : \mathbb{R}^3 \to \mathbb{R}^3$ (a "vectorial weight") by

$$\rho_0 = \frac{|w_0(x)|}{\|w_0\|_1} dx, \quad \text{and} \quad h_0(x) = \frac{w_0}{\rho_0}(x).$$

Let us further denote by $\mathcal{M}_{3\times 3}$ the space of real 3×3 matrices, and by I the identity matrix. We consider the following stochastic differential equation with values in $\mathbb{R}^3 \times \mathcal{M}_{3\times 3}$, nonlinear in the sense of McKean:

$$\begin{aligned} X_t &= X_0 + \sqrt{2\nu} B_t + \textstyle\int_0^t \mathbf{u}^P(s, X_s) ds, \\ \Phi_t &= I + \textstyle\int_0^t \nabla \mathbf{u}^P(s, X_s) \Phi_s \, ds, \end{aligned} \qquad t \in [0, T], \text{ with} \tag{4}$$

$$\begin{aligned} law(X, \Phi) = P, \quad law(X_0) = \rho_0(x)dx \quad \text{and} \\ \mathbf{u}^P(s, x) = E^P \left[K(x - X_s) \wedge \Phi_s h_0(X_0) \right]. \end{aligned} \tag{5}$$

Our first goal is to establish an equivalence between weak solutions P of (4)–(5) in certain class of probability measures on $C([0, T], \mathbb{R}^3 \times \mathcal{M}_{3\times 3})$, and "mild" solutions $\mathbf{w}$ of equation (2) in the space $[L^p(\mathbb{R}^3)]^3 \cap [L^1(\mathbb{R}^3)]^3$. This correspondence will be given by the relation

$$E^P(\mathbf{f}(X_t)\Phi_t h_0(X_0)) = \int_{\mathbb{R}^3} \mathbf{f}(y)\mathbf{w}(t, y)dy \tag{6}$$

for functions $\mathbf{f} : \mathbb{R}^3 \to \mathbb{R}^3$. According to (6) and to the Biot-Savart law (3), the function $\mathbf{u}^P(t) = \mathbf{u}(t)$ is the velocity field associated with the vorticity field $\mathbf{w}(t)$. To study well-posedness of the nonlinear martingale problem associated with (4)–(5), we will first of all prove (local) existence and (global) uniqueness of a mild solution of (2) in the space $[L^p(\mathbb{R}^3)]^3$, for $\frac{3}{2} < p < 3$. We adapt to this end techniques developed for the study of equation (1) in the so-called "supercritical" spaces (see e.g. Cannone [8] Ch.1). Secondly, we shall prove continuity and boundedness properties of the functions $\mathbf{u}(t)$ and $\nabla\mathbf{u}(t)$ associated with this mild solution. We point out that, although similar analytical statements could be found elsewhere, it will be fundamental for our probabilistic results to give proofs and estimates that explicitly depend on the continuity properties of the Biot-Savart operator.

Our next goal is to construct stochastic particle approximations of the mild solution $\mathbf{w}$ and of $\mathbf{u}$. We will follow a trajectorial approach, in the line of Bossy and Talay [4], Méléard [26] and [27], or Fontbona [14]. However, the present case is much harder. A crucial fact will be that the "vortex stretching process" Φ_t associated to $\mathbf{w}$ turns to be *a priori* bounded independently of randomness. We will define a system of particles $(X^{i,n,\varepsilon,R}, \Phi^{i,n,\varepsilon,R})_{i=1}^{n}$, $n \in \mathbb{N}$, with $\varepsilon > 0$ and $R > 0$ respectively a mollifying parameter of the kernel K and a cutoff threshold of the approximating vortex stretching processes $\Phi^{i,n,\varepsilon,R}$, and we will prove a propagation of chaos result for fixed ε and R. Then, under conditions of smallness of w_0 and $T > 0$ that ensure us existence of a solution $\mathbf{w}$ of (2) on $[0, T]$, we will prove that for $R > 0$ large enough the system $(X^{i,n,\varepsilon_n,R}, \Phi^{i,n,\varepsilon_n,R})$ is chaotic when $\varepsilon_n \to 0$ sufficiently slowly, and has the limit P equal to the solution of (4)–(5). From this we will deduce the convergence to $\mathbf{w}$ of some "weighted empirical process" of the system (with time dependent vectorial weights), and the convergence of an "approximate velocity field" to $\mathbf{u}$. This is the first complete mathematical proof of convergence of a stochastic vortex method for the Navier-Stokes equation in three dimensions, and rectifies a method conjectured by Esposito and Pulvirenti in [13].

The fact that we only impose integrability conditions on w_0 yields singularities of the functions $\mathbf{u}$ and $\nabla \mathbf{u}$ at $t = 0$. To overcome this problem, we extend some techniques of Méléard [27] and Fontbona [14] to construct and suitably approximate nonlinear processes with singular drift terms. As in those works, loss of regularity at $t = 0$ prevents us from obtaining an explicit convergence rate. This could be done under additional assumptions, but the speed we can expect to obtain is certainly not optimal. (This problem is mainly owed to the probabilistic techniques employed; even in the 2d case, improvements in that direction have not yet been made).

Under the assumption that $w_0 \in L_3^p$ with $\frac{3}{2} < p < 3$, we will also show that the SDE

$$\xi_t(x) = x + \sqrt{2\nu} B_t + \int_0^t \mathbf{u}(s, \xi_s(x)) ds, \tag{7}$$

with $\mathbf{u} = \mathbf{u}^P$ given (5), will define a C^1 stochastic flow $\xi : [0, T] \times \mathbb{R}^3 \to \mathbb{R}^3$, and we will have the pointwise identity $(X, \Phi) = (\xi(X_0), \nabla_x \xi(X_0))$. Formula (6) is thus the fact that the vorticity is "transported" by the stochastic flow ξ and "stretched" by its gradient, and generalizes the representation of the vorticity field in the inviscid case ($\nu = 0$) in terms of the (deterministic) flow of the solution of the Euler equation (see [11] Ch. 1). A representation formula for mild solutions equivalent to (6) was partially established in [13] (under the more restrictive assumption that w_0 and its Fourier transform are L^1 functions). Our interpretation in terms of the weight function h_0 is much simpler, and inspired on the approach of Méléard [26] and [27] in two dimensions, where vorticity was represented using a scalar weight being simply "transported" by a nonlinear diffusion process. We also extend in this way ideas of Jourdain [18] for dealing with signed measures in the McKean-Vlasov context. A representation formula in terms of stochastic flow was also proved in Esposito, Marra, Pulvirenti and Sciaretta [12], but these authors needed to restrict themselves to equation (1) on the torus in order to define the underlying probability space, and also to impose additional regularity.

A different probabilistic interpretation is developed in Giet [17], in the case of a bounded domain and non-slip boundary condition. This author extends the ideas of Benachour, Roynette and Vallois [2] in two dimensions, by using a diffusion process with jumps to interpret the coupled system (2) with zero-order term, and a branching process to treat the boundary condition. At an advanced stage of this work, we also became aware of the work of Busnello, Frandoli and Romito [7], who also interpret the vorticity in terms of a stochastic flow and its gradient. These authors use a Bismut-Elworthy formula to recover the velocity field (extending the approach of Busnello in two dimensions [6]) and provide a local existence statement. All these works present an approach "dual" to ours, by using Feynman-Kac type formulae for the vorticity (in terms of the *linear* stochastic flow (7) reversed in time) and aim to represent classical solutions of (2) by mean of probabilistic objects. Due to this fact, they need to assume more regularity of the initial conditions. Furthermore, non of the aforementioned works [12], [7] or [17] relate the nonlinearity to a mean field interaction limit, and they do not lead to implementable stochastic approximations methods for solutions of the Navier-Stokes or the vortex equation.

Finally, let us point out that a probabilistic interpretation and vortex method for the 2d-Navier-Stokes equation with an external force field has recently been developed by the author and Mé léard [15], in terms of particle systems with random space-time births. The three dimensional model with external force field will be treated in a forthcoming paper.

1.1. Notation

- By $\mathcal{M}eas^T$ we denote the space of measurable real valued functions on $[0, T] \times \mathbb{R}^3$.
- $C^{1,2}$ is the set of real valued functions on $[0, T] \times \mathbb{R}^3$ with continuous derivatives up to the first order in $t \in [0, T]$ and up to the second order in $x \in \mathbb{R}^3$. $C_b^{1,2}$ is the subspace of bounded functions in $C^{1,2}$ with bounded derivatives.
- $\mathcal{D}$ is the space of compactly supported functions on $\mathbb{R}^3$ with infinitely many derivatives.
- For all $1 \leq p \leq \infty$ we denote by L^p the space $L^p(\mathbb{R}^3)$ of real valued functions on $\mathbb{R}^3$. By $\|\cdot\|_p$ we denote the corresponding norm and p^* stands for the Hölder conjugate of p. We write $W^{1,p} = W^{1,p}(\mathbb{R}^3)$ for the Sobolev space of functions in L^p with partial derivatives of first order in L^p.
- If E is a space of real valued functions (defined on $\mathbb{R}^3$ or on $[0, T] \times \mathbb{R}^3$), then the notation E_3 is used for the space of $\mathbb{R}^3$-valued functions with scalar components in E. If E has a norm, the norm in E_3 is denoted in the same way.
- For simplicity, if $\mathbf{f}, \mathbf{g} : \mathbb{R}^3 \to \mathbb{R}^3$ are vector fields and $Z : \mathbb{R}^3 \to \mathcal{M}_{3\times 3}$ is a matrix function, we will write $\mathbf{f}\mathbf{g} = \sum_i^3 \mathbf{f}_i \mathbf{g}_i$ and $\mathbf{f}Z$ for the row-vector $(\mathbf{f}^t Z)_i = \sum_{j=1}^3 \mathbf{f}_j Z_{j,i}$. By $\nabla\mathbf{f}$ we denote the gradient of $\mathbf{f}$, that is the matrix $(\nabla\mathbf{f})_{i,j} = \frac{\partial \mathbf{f}_i}{\partial x_j}$. We will simply write $(\nabla\mathbf{f})\mathbf{g}$ for the column-vector $(\sum_j \frac{\partial \mathbf{f}_i}{\partial x_j} \mathbf{g}_j)_i$ (instead of the usual "$(\mathbf{g} \cdot \nabla)\mathbf{f}$").
- C and $C(T)$ are finite positive constants that may change from line to line.

2. Preliminaries

Let $G : \mathbb{R}^3 \backslash \{0\} \to \mathbb{R}$ be the fundamental solution of the Laplace operator. We will denote by K its gradient $\nabla G : \mathbb{R}^3 \backslash \{0\} \to \mathbb{R}^3$, that is, the singular kernel

$$K(x) := -\frac{1}{4\pi} \frac{x}{|x|^3}.$$

For functions $w : \mathbb{R}^3 \to \mathbb{R}^3$, we (formally) define the *Biot-Savart operator* $\mathbf{K}$ by

$$\mathbf{K}(w)(x) := \int_{\mathbb{R}^3} K(x-y) \wedge w(y)dy = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{(x-y)}{|x-y|^3} \wedge w(y)dy. \quad (8)$$

A vector field $w : \mathbb{R}^3 \to \mathbb{R}^3$ with components in $\mathcal{D}'$, and such that

$$\int_{\mathbb{R}^3} \nabla f(x) w(x) dx = 0$$

for all $f \in \mathcal{D}$, is said to have *null divergence in the distribution sense*. We write it $div\ w = 0$.

Remark 2.1. Let $u, w : \mathbb{R}^3 \to \mathbb{R}^3$ be vector fields and distributions. We have $div(curl\ u) = 0$, and if $\mathbf{K}(w)$ is a distribution, then $div\ \mathbf{K}(w) = 0$.

For each $\nu > 0$, we denote by $G^\nu : \mathbb{R}_+ \times \mathbb{R}^3 \to \mathbb{R}$ the heat kernel

$$G_t^\nu(x) = (4\pi\nu t)^{-\frac{3}{2}} \exp\left(-\frac{|x|^2}{4\nu t}\right). \quad (9)$$

We will consider L^p-solutions of (2) defined in two different senses. The first will be useful for analytical purposes. The second will be natural from the probabilistic point of view.

Definition 2.1. *Let $w_0 \in L_3^p$ with $div\ w_0 = 0$. We say that $\mathbf{w} \in L^\infty([0,T], L_3^p)$ is a mild solution of with initial condition w_0, ("mild solution" for short) if the following hold:*

mV1: *The functions $\mathbf{K}(\mathbf{w})_i(t,x) := \mathbf{K}(\mathbf{w}(t,\cdot))_i(x)$, $i = 1,2,3$ are defined a.e. on $[0,T] \times \mathbb{R}^3$*

mV2: *For a.e. $t \in [0,T]$ the following identity holds in $L_3^p(\mathbb{R}^3)$:*

$$\begin{aligned}\mathbf{w}(t,x) = &\int_{\mathbb{R}^3} G_t^\nu(x-y) w_0(y) dy \\ &+ \sum_{j=1}^3 \int_0^t \int_{\mathbb{R}^3} \frac{\partial G_{t-s}^\nu}{\partial y_j}(x-y) \big[\mathbf{K}(\mathbf{w})_j(s,y)\mathbf{w}(s,y) \\ &- \mathbf{w}_j(s,y)\mathbf{K}(\mathbf{w})(s,y)\big] dy\, ds \end{aligned} \quad (10)$$

Definition 2.2. *Let $w_0 \in L_3^p$ and $div\ w_0 = 0$. A function $\mathbf{w} \in L^\infty([0,T], L_3^p)$ is a weak solution on $[0,T]$ of the vortex equation with initial condition w_0 (or "weak solution") if*

wV1: *The functions $\mathbf{K}(\mathbf{w})$ and $\nabla\mathbf{K}(\mathbf{w})$ are defined a.e on $[0,T] \times \mathbb{R}^3$.*

wV2: *The products* $\mathbf{w}_i \mathbf{K}(\mathbf{w})_j$ *and* $\mathbf{w}_i \frac{\partial \mathbf{K}(\mathbf{w})_j}{\partial x_k}$, $i, j = 1, 2, 3$, *belong to* $L^1_{loc}([0,T] \times \mathbb{R}^3)$, *and for every smooth compactly supported function* $\mathbf{f} : [0,T] \times \mathbb{R}^3 \to \mathbb{R}^3$,

$$\begin{aligned}\int_{\mathbb{R}^3} \mathbf{f}(t,y)\mathbf{w}(t,y)dy = &\int_{\mathbb{R}^3} \mathbf{f}(0,y) w_0(y) dy \\ &+ \int_0^t \int_{\mathbb{R}^3} \left[\frac{\partial \mathbf{f}}{\partial s}(s,y) + \nu \triangle \mathbf{f}(s,y) \right. \\ &\left. + \nabla \mathbf{f}(s,y) \mathbf{K}(\mathbf{w})(s,y) + \mathbf{f}(s,y) \nabla \mathbf{K}(\mathbf{w})(s,y) \right] \\ &\times \mathbf{w}(s,y)\, dy\, ds. \end{aligned} \tag{11}$$

We respectively refer to (10) and (11) as the *mild* and the *weak equations*. The two forms are not in general equivalent. Going from weak to mild forms will be important to identify objects of analytic and probabilistic nature. The following partial argument will be useful.

Lemma 2.1. *Let* $\mathbf{w}$ *be a weak solution such that for all* $i, j = 1, 2, 3$ *and* $\psi \in \mathcal{D}_3$,

$$\int_0^t \int_{(\mathbb{R}^3)^2} \sum_{i,j=1}^3 \left| \frac{\partial G^\nu_{t-s}}{\partial y_j}(x-y) \right| |\mathbf{K}(\mathbf{w})_j(s,y)| |\psi_i(x)| |\mathbf{w}_i(s,y)| dx\, dy\, ds < \infty \text{ and}$$

$$\int_0^t \int_{(\mathbb{R}^3)^2} \sum_{i,j=1}^3 G^\nu_{t-s}(x-y) \left| \frac{\partial \mathbf{K}(\mathbf{w})_i}{\partial y_j}(s,y) \right| |\psi_i(x)| |\mathbf{w}_j(s,y)| dx\, dy\, ds < \infty.$$

Then, $\mathbf{w}$ *satisfies the "intermediate" form*

$$\begin{aligned}\mathbf{w}(t,x) = &G^\nu_t * w_0(x) \\ &+ \int_0^t \sum_{j=1}^3 \int_{\mathbb{R}^3} \left[\frac{\partial G^\nu_{t-s}}{\partial y_j}(x-y) [\mathbf{K}(\mathbf{w})_j(s,y) \mathbf{w}(s,y)] \right. \\ &\left. + G^\nu_{t-s}(x-y) [\mathbf{w}_j(s,y) \frac{\partial \mathbf{K}(\mathbf{w})}{\partial y_j}(s,y)] \right] dy\, ds. \end{aligned} \tag{12}$$

Proof. Take fixed $\psi \in \mathcal{D}_3$ and $t \in [0,T]$ and define $\mathbf{f}_t : [0,t] \times \mathbb{R}^3 \to \mathbb{R}^3$ by $\mathbf{f}_t(s,y) = G^\nu_{t-s} * \psi(y)$; this function is of class $(C^{1,2}_b)_3$ and solves the backward heat equation on $[0,t] \times \mathbb{R}^3$ with final condition $\mathbf{f}(t,y) = \psi(y)$. If $\mathbf{w}$ satisfies the hypothesis, by a density argument it also satisfies (11) with the function $\mathbf{f}_t(s,y)$ just defined. Using Fubini's theorem, and since $\psi \in \mathcal{D}_3$ is arbitrary, we deduce that (12) holds. □

Remark 2.2. An inspection of the r.h.s. of (10) shows that, by definition, every mild solution satisfies $div\ \mathbf{w}(t) = 0$. To check that a weak solution satisfies this condition, we need more regularity than what is assumed in its definition. On the other hand, if a weak solution $\mathbf{w}(t)$ as in Lemma 2.1 is known to satisfy $div\ \mathbf{w}(t) = 0$, it could be showed to be a mild solution by integrating by parts the last term in the r.h.s. of (12). However, that argument still requires the knowledge that $\mathbf{w}$ and $\mathbf{K}(\mathbf{w})$ belong to suitable functional spaces.

2.1. Continuity of the Biot-Savart operator

In order to obtain existence and uniqueness results for the vortex equation in Lebesgue and Sobolev spaces, as well as regularity estimates for the associated velocity field, we next establish fundamental continuity properties of the operator $\mathbf{K}$.

Lemma 2.2. *Let $1 < p < 3$ and $\frac{1}{q} = \frac{1}{p} - \frac{1}{3}$ (notice that $q \in]\frac{3}{2}, \infty[$).*

i) For every $w \in L^p_3$, the integral (8) is absolutely convergent for almost every x and one has $\mathbf{K}(w) \in L^q_3$. There exists further a positive constant $\widetilde{C}_{p,q}$ such that

$$\|\mathbf{K}(w)\|_q \leq \widetilde{C}_{p,q} \|w\|_p \tag{13}$$

for all $w \in L^p_3$.

ii) If moreover $w \in W^{1,p}$, then we have $\mathbf{K}(w) \in W^{1,q}_3$, with $\frac{\partial}{\partial x_k}\mathbf{K}(w) = \mathbf{K}\left(\frac{\partial w}{\partial x_k}\right)$, and

$$\left\| \frac{\partial \mathbf{K}(w)}{\partial x_k} \right\|_q \leq \widetilde{C}_{p,q} \left\| \frac{\partial w}{\partial x_k} \right\|_p \tag{14}$$

for all $k = 1, 2, 3$.

Proof. Denote by $K_j(x)$ the j-th component of the vector $K(x)$, and consider the operators

$$f(x) \mapsto \mathcal{K}_j(f)(x) := \int_{\mathbb{R}^3} K_j(x-y) f(y) dy$$

acting on real valued functions f. It is enough to prove the analogue statements for the operators $\mathcal{K}_j$ acting in the spaces $L^p(\mathbb{R}^3)$ and $W^{1,p}(\mathbb{R}^3)$ respectively.

The continuity of $\mathcal{K}_j : L^p(\mathbb{R}^3) \to L^q(\mathbb{R}^3)$, and the absolute convergence of the integral $\mathcal{K}_j * f$ when $f \in L^p(\mathbb{R}^3)$, are easily deduced from the fact that the Riesz transform

$$f(x) \mapsto \int_{\mathbb{R}^3} \frac{f(y)}{|x-y|^2} dy,$$

satisfies precisely those properties (see Stein [31], Ch. 5.). The fact that $\mathcal{K}_j : W^{1,p}(\mathbb{R}^3) \to W^{1,q}(\mathbb{R}^3)$ is continuous will follow from the latter and a density argument, if we show that for all $f \in \mathcal{D}$ the identity

$$\frac{\partial}{\partial x_k}\mathcal{K}_j(f) = \mathcal{K}_j\left(\frac{\partial f}{\partial x_k}\right)$$

hods. Since the integral $\mathcal{K}_j\left(\frac{\partial f}{\partial x_k}\right)(x)$ is absolutely convergent and f has compact support, this is simply derivation under the integral sign using Lebesgue's theorem. □

Part *ii)* in previous lemma ensures us that the "velocity" $\mathbf{K}(w)$ associated with $w \in W_3^{1,p}$, $p \in]1,3[$, belongs to $W_3^{1,q}$ for $q = \frac{3p}{3-p}$. One could however expect that the gradient of the velocity field has similar regularity as vorticity. This will be consequence of next lemma.

Lemma 2.3. *Let* $1 < r < \infty$.

i) For all $w \in L_3^r$, *we have* $\frac{\partial}{\partial x_k}\mathbf{K}(w) \in L_3^r$ *for* $k = 1, 2, 3$. *There exists further a positive constant* $\widetilde{C}_p$ *depending only on r such that*

$$\left\| \frac{\partial \mathbf{K}(w)_j}{\partial x_k} \right\|_r \leq \widetilde{C}_r \|w\|_r \tag{15}$$

for all $j = 1, 2, 3$, *where* $\mathbf{K}(w)_j$ *is the j-th component of* $\mathbf{K}(w)$.

ii) If moreover $w \in W_3^{1,r}$, *then we have* $\frac{\partial}{\partial x_k}\mathbf{K}(w) \in W_3^{1,r}$, *and*

$$\left\| \frac{\partial^2 \mathbf{K}(w)_j}{\partial x_l \partial x_k} \right\|_r \leq \widetilde{C}_r \left\| \frac{\partial w}{\partial x_l} \right\|_r \tag{16}$$

for all $l = 1, 2, 3$.

Proof. Let $\mathcal{K}_j(x)$ be defined as in the previous lemma. Again, it is enough to prove the analogue results for $\mathcal{K}_j$. We will use the following fact (proved for instance in Bertozzi and Majda [3] Ch.2): for any $r \in]1, \infty[$ and $f \in L^r(\mathbb{R}^3)$, each first order derivative of $\mathcal{K}_j(f)$ is the results of a singular integral operator acting on the function f. Thus, by classic results of the singular integrals theory (see Stein [31] Ch. 1.), for each i, j the mapping $f \mapsto \frac{\partial}{\partial x_i}\mathcal{K}_j(f)$ is a continuous operator $L^r \to L^r$ for all $r \in]1, \infty[$, and there exists moreover a homogeneous function $m_{i,j} : \mathbb{R}^3 \to \mathbb{R}$ of degree 0, such that for all $f \in L^2(\mathbb{R}^3)$

$$\mathcal{F}\left(\frac{\partial}{\partial x_i}\mathcal{K}_j(f)\right)(\xi) = \mathcal{F}(m_{i,j})\mathcal{F}(f)(\xi),$$

where $\mathcal{F}(f)(\xi) := \int_{\mathbb{R}^3} e^{-2\pi i \xi \cdot x} g(x)dx$ is the Fourier transform of f. Observe that $\mathcal{F}(m_{i,j})$ is bounded. Using this fact, the previous identity and the inverse transform, we deduce that

$$\frac{\partial}{\partial x_k}\left(\frac{\partial}{\partial x_i}\mathcal{K}_j(f)\right) = \frac{\partial}{\partial x_i}\mathcal{K}_j\left(\frac{\partial f}{\partial x_k}\right)$$

for all $f \in W^{1,2}$. By continuity of $\frac{\partial}{\partial x_i}\mathcal{K}_j$ in L^r and a density argument, this also holds for all $f \in W^{1,r}$ and $r \in]1, \infty[$. The continuity of $\frac{\partial}{\partial x_i}\mathcal{K}_j : W^{1,r}(\mathbb{R}^3) \to W^{1,r}(\mathbb{R}^3)$ follows. □

3. The vortex equation in the supercritical L^p spaces

In this section, we will prove all analytic results that are needed in the sequel. First, we will establish an L^p framework where local existence and global uniqueness for the mild vortex equation (10) will hold. We adapt to this end general techniques for the mild Navier-Stokes (velocity) equation in the so-called super-critical spaces (see Cannone [8], Ch.1). To obtain our probabilistic results, we will also need to prove, both for the velocity field and its gradient, precise Hölder and L^∞ estimates depending only on the L^p norm of the vorticity. Although the solutions we obtain are in correspondence with those in [8], Ch.1, the regularity estimates we need seem not to be available in the analytical literature. This is why we will give complete proofs. Moreover, we will make the role played by K explicit, since our stochastic particle approximations result will require existence and regularity statements that hold "uniformly" for (10) and for a family of approximating equations involving mollified kernels. Let us begin by recalling some well-known facts:

Lemma 3.1. *Let G^ν the heat kernel (9) and $m \in [1,\infty]$. There exist positive constants $c(m)$ and $c'(m)$ such that for all $t > 0$,*

$$\|G^\nu_t\|_m \leq c(m)(\nu t)^{-\frac{3}{2}+\frac{3}{2m}} \quad \textit{and} \tag{17}$$

$$\|\nabla G^\nu_t\|_m \leq c'(m)(\nu t)^{-2+\frac{3}{2m}}. \tag{18}$$

We shall frequently use Young's inequality: if $f \in L^m$ and $g \in L^k$, with $1 \leq m, k \leq \infty$ such that $\frac{1}{r} := \frac{1}{m} + \frac{1}{k} - 1 \geq 0$. Then,

$$f * g \in L^r \text{ and } \|f * g\|_r \leq \|f\|_m \|g\|_k. \tag{19}$$

We easily deduce the following estimates.

Lemma 3.2. *Let $p \in [1,\infty]$, $r \geq p$ and $w_0 \in L^p_3$. There exist positive constants $\overline{C}_1(p)$, $\overline{C}_0(p;r)$ and $\overline{C}_1(p;r)$ such that for all $t > 0$,*

i) $\|G^\nu_t * w_0\|_p \leq \|w_0\|_p$,
ii) $\|\nabla G^\nu_t * w_0\|_p \leq \overline{C}_1(p) t^{-\frac{1}{2}} \|w_0\|_p$,
iii) $\|G^\nu_t * w_0\|_r \leq \overline{C}_0(p;r) t^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{r})} \|w_0\|_p$,
iv) $\|\nabla G^\nu_t * w_0\|_r \leq \overline{C}_1(p;r) t^{-\frac{1}{2}-\frac{3}{2}(\frac{1}{p}-\frac{1}{r})} \|w_0\|_p$.

According to Lemma 3.2, we define for $\mathbf{w} \in \mathcal{M}eas^T_3$, $p \in [1,\infty]$ and $r \geq p$ the norms:

- $|||\mathbf{w}|||_{0,r,(T;p)} := \sup_{0 \leq t \leq T} t^{\frac{3}{2}(\frac{1}{p}-\frac{1}{r})} \|\mathbf{w}(t)\|_r$,
- $|||\mathbf{w}|||_{1,r,(T;p)} := \sup_{0 \leq t \leq T} \left\{ t^{\frac{3}{2}(\frac{1}{p}-\frac{1}{r})} \|\mathbf{w}(t)\|_r + t^{\frac{1}{2}+\frac{3}{2}(\frac{1}{p}-\frac{1}{r})} \sum_{k=1}^{3} \left\| \frac{\partial \mathbf{w}(t)}{\partial x_k} \right\|_r \right\}$,
- $|||\mathbf{w}|||_{0,p,T} := |||\mathbf{w}|||_{0,p,(T;p)}$, and
- $|||\mathbf{w}|||_{1,p,T} := |||\mathbf{w}|||_{1,p,(T;p)}$.

The associated (Banach) spaces will be respectively denoted by

$$\mathbf{F}_{0,r,(T;p)},\ \mathbf{F}_{1,r,(T;p)},\ \mathbf{F}_{0,p,T} \text{ and } \mathbf{F}_{1,p,T}.$$

We shall prove existence and uniqueness results for the mild equation in $\mathbf{F}_{0,p,T}$ for $p \in]\frac{3}{2}, 3[$. We notice that for these values of p, the L_3^p-spaces are in correspondence (*via* the operator $\mathbf{K}$) with the so-called "supercritical L^q spaces" for the velocity field (that is, L_3^q with $q = \frac{3p}{3-p} \in]3, \infty[$.) Then, we will show that the solution belongs to $\mathbf{F}_{1,r,(T;p)}$ for suitable $r \in [3, \infty[$, and we will deduce the regularity properties of $\mathbf{K}(\mathbf{w})$ we need using classic embbeding results of Sobolev spaces.

Given functions $\mathbf{w}, \mathbf{v} \in \mathcal{M}eas_3^T$, we (formally) define

$$\mathbf{B}(\mathbf{w},\mathbf{v})(t,x) = \int_0^t \sum_{j=1}^3 \int_{\mathbb{R}^3} \frac{\partial G_{t-s}^\nu}{\partial y_j}(x-y) \left[\mathbf{K}(\mathbf{w})_j(s,y)\mathbf{v}(s,y) - \mathbf{v}_j(s,y)\mathbf{K}(\mathbf{w})(s,y)\right] dy\, ds. \quad (20)$$

The continuity of $\mathbf{B}$, in some of the spaces we have previously defined will be crucial.

Proposition 3.1. *The bilinear operator* $\mathbf{B} : \mathbf{F}^2 \to \mathbf{F}'$ *is well defined and continuous if*

i) $\frac{3}{2} \le p < 3,\ \frac{3p}{6-p} \le p' < \frac{3p}{6-2p},\quad \mathbf{F} = \mathbf{F}_{0,p,T}$ *and* $\mathbf{F}' = \mathbf{F}_{0,p',T}$
ii) $\frac{3}{2} \le p < 3,\ p \le l < 3,\ \frac{3l}{6-l} \le l' < \frac{3l}{6-2l},\quad \mathbf{F} = \mathbf{F}_{0,l,(T;p)}$ *and* $\mathbf{F}' = \mathbf{F}_{0,l',(T;p)}$
iii) $\frac{3}{2} \le p < 3,\ \frac{3p}{6-p} \le p' < \frac{3p}{6-2p},\quad \mathbf{F} = \mathbf{F}_{1,p,T}$ *and* $\mathbf{F}' = \mathbf{F}_{1,p',T}$
iv) $\frac{3}{2} \le p < 3,\ p \le l < \min\{\frac{6p}{6-p}, 3\},\ \frac{3l}{6-l} \le l' < \frac{3l}{6-2l},\quad \mathbf{F} = \mathbf{F}_{1,l,(T;p)}$ *and* $\mathbf{F}' = \mathbf{F}_{1,l',(T;p)}$.

Proof. The following formula will be used: if $\varepsilon, \theta > -1$, then for all $t > 0$

$$\int_0^t s^\varepsilon (t-s)^\theta\, ds = t^{\varepsilon+\theta+1}\beta(\varepsilon+1, \theta+1),$$

where $\beta(\alpha,\beta) = \int_0^1 s^{\alpha-1}(1-s)^{\beta-1} ds < \infty$ is the Beta function of parameters $\alpha, \beta > 0$.

i) We take the $L^{p'}$ norm to the $i-$th component of (20), and apply inequality (19), with $r = p'$, $m = (\frac{4}{3} + \frac{1}{p'} - \frac{2}{p})^{-1}$ and $k = \frac{3p}{6-p}$. (Notice that $1 \le m < \frac{3}{2}$ and $1 \le k < \infty$). We get

$$\begin{aligned}\|\mathbf{B}(\mathbf{w},\mathbf{v})_i(t)\|_{p'} &\le C \sum_{j=1}^3 \int_0^t \|\nabla G_{t-s}^\nu\|_m \\ &\quad \times \left(\|\mathbf{v}_j(s)\ \mathbf{K}(\mathbf{w})_i(s)\|_k + \|\mathbf{v}_i(s)\ \mathbf{K}(\mathbf{w})_j(s)\|_k\right)\, ds \\ &\le C \int_0^t (t-s)^{\frac{3}{2m}-2} \|\mathbf{w}(s)\|_p\ \|\mathbf{v}(s)\|_p\, ds\end{aligned}$$

by using also estimate (18), Hölder's inequality and inequality (13). Therefore,

$$\|\mathbf{B}(\mathbf{w},\mathbf{v})_i(t)\|_{p'} \le C t^{1-3(\frac{1}{p}-\frac{1}{2p'})} \|\!|\mathbf{w}|\!\|_{0,p,T} \|\!|\mathbf{v}|\!\|_{0,p,T}, \ \forall t \in [0,T], \tag{21}$$

and we conclude that $\|\!|\mathbf{B}(\mathbf{w},\mathbf{v})|\!\|_{0,p',T} \le C_0(p,p')T^{1-3(\frac{1}{p}-\frac{1}{2p'})}\|\!|\mathbf{w}|\!\|_{0,p,T}$ $\|\!|\mathbf{v}|\!\|_{0,p,T}$, with a constant $C_0(p,p')>0$ independent of T, and $1-3(\frac{1}{p}-\frac{1}{2p'})>0$.

ii) We proceed as in *i)*, taking now in Young's inequality (19) $r=l'$, $m=(\frac{4}{3}+\frac{1}{l'}-\frac{2}{l})^{-1}$ and $k=\frac{3l}{6-l}$ (again, we have $1\le m<\frac{3}{2}$ and $1\le k<\infty$). By similar steps we obtain

$$\begin{aligned}\|\mathbf{B}(\mathbf{w},\mathbf{v})_i(t)\|_{l'} &\le C\int_0^t (t-s)^{\frac{3}{2l'}-\frac{3}{l}}\|\mathbf{w}(s)\|_l\ \|\mathbf{v}(s)\|_l\,ds\\ &\le C\int_0^t (t-s)^{\frac{3}{2l'}-\frac{3}{l}}s^{\frac{3}{l}-\frac{3}{p}}ds\,\|\!|\mathbf{w}|\!\|_{0,l,(T;p)}\|\!|\mathbf{v}|\!\|_{0,l,(T;p)}\\ &\le Ct^{1+\frac{3}{2l'}-\frac{3}{p}}\|\!|\mathbf{w}|\!\|_{0,l,(T;p)}\|\!|\mathbf{v}|\!\|_{0,l,(T;p)},\end{aligned}\tag{22}$$

since $\frac{3}{2m}-2=\frac{3}{2l'}-\frac{3}{l}>-1$ and $\frac{3}{l}-\frac{3}{p}>-1$ (the latter because $\frac{3p}{3-p}\ge 3>l$). We conclude that the norm of $\mathbf{B}:(\mathbf{F}_{0,l,(T;p)})^2\to\mathbf{F}_{0,l',(T;p)}$ is bounded by a constant times $T^{1-\frac{3}{2p}}$.

iii) If $\mathbf{w},\mathbf{v}\in\mathbf{F}_{1,p,T}$, we have $\mathbf{K}(\mathbf{w})_j(t)\mathbf{v}(t)_i\in W^{1,\frac{3p}{6-p}}$. Since $G_t^\nu\in W^{1,\frac{3p}{4p-6}}$ we can integrate by parts for each $t\in]0,T]$ and see that $\mathbf{B}(\mathbf{w},\mathbf{v})(t,x)$ is equal to

$$-\sum_{j=1}^{3}\int_0^t\!\!\int_{\mathbb{R}^3}G_{t-s}^\nu(x-y)\frac{\partial}{\partial y_j}\big[\mathbf{K}(\mathbf{w})_j(s,y)\mathbf{v}(s,y)-\mathbf{v}_j(s,y)\mathbf{K}(\mathbf{w})(s,y)\big]\,dy\,ds.$$

Take $f\in\mathcal{D}$ and write

$$\begin{aligned}\overline{\mathbf{B}}_i^f(\mathbf{w},\mathbf{v})(t) := \int_0^t\int\int G_{t-s}^\nu(x-y)|f(x)|\\ \left|\frac{\partial}{\partial y_j}\big[\mathbf{K}(\mathbf{w})_j(s,y)\mathbf{v}_i(s,y)\big]\right|dx\,dy\,ds.\end{aligned}$$

By Hölder and Young's inequalities applied as in *i)*, we deduce, with $\frac{1}{q}=\frac{1}{p}-\frac{1}{3}$, that

$$\begin{aligned}\overline{\mathbf{B}}_i^f(\mathbf{w},\mathbf{v})(t) &\le C\|f\|_{(p')^*}\sum_{j=1}^{3}\int_0^t (t-s)^{\frac{3}{2m}-\frac{3}{2}}\\ &\qquad\left[\left\|\frac{\partial\mathbf{K}(\mathbf{w})_j(s)}{\partial y_j}\right\|_q\|\mathbf{v}_i(s)\|_p+\left\|\frac{\partial\mathbf{v}_i(s)}{\partial y_j}\right\|_p\|\mathbf{K}(\mathbf{w})_j\|_q\right]ds\\ &\le C\|f\|_{(p')^*}\int_0^t(t-s)^{\frac{3}{2m}-\frac{3}{2}}s^{-\frac{1}{2}}ds\,\|\!|\mathbf{w}|\!\|_{1,p,T}\|\!|\mathbf{v}|\!\|_{1,p,T}\\ &\le C\|f\|_{(p')^*}T^{1-3(\frac{1}{p}-\frac{1}{2p'})}\|\!|\mathbf{w}|\!\|_{1,p,T}\|\!|\mathbf{v}|\!\|_{1,p,T}<\infty.\end{aligned}\tag{23}$$

We have used here (17), (13) and the definition of $\|\!|\cdot|\!\|_{1,p,T}$. A similar estimate holds for the term involving the product $\mathbf{v}_j(s)\mathbf{K}_i(\mathbf{w})(s)$. By Fubini and integration by parts, we get

$$\int_{\mathbb{R}^3} \mathbf{B}(\mathbf{w},\mathbf{v})_i(t,x)\frac{\partial f(x)}{\partial x_k}dx = \sum_{j=1}^{3}\int_0^t\int_{\mathbb{R}^3}\left(\int_{\mathbb{R}^3}\frac{\partial G^{\nu}_{t-s}}{\partial x_k}(x-y)f(x)dx\right) \times\frac{\partial}{\partial y_j}\big[\mathbf{K}(\mathbf{w})_j(s,y)\mathbf{v}_i(s,y) -\mathbf{v}_j(s,y)\mathbf{K}(\mathbf{w})_i(s,y)\big]\,dy\,ds \tag{24}$$

for all $f\in\mathcal{D}$. Proceeding as before in (23), we deduce now from (24) that

$$\left|\int_{\mathbb{R}^3}\mathbf{B}(\mathbf{w},\mathbf{v})_i(t,x)\frac{\partial f(x)}{\partial x_k}dx\right| \leq C\|f\|_{(p')^*}\sum_{j=1}^{3}\int_0^t (t-s)^{\frac{3}{2m}-2}s^{-\frac{1}{2}}ds\,\|\!|\mathbf{w}|\!\|_{1,p,T}\|\!|\mathbf{v}|\!\|_{1,p,T} \leq C\|f\|_{(p')^*}t^{\frac{1}{2}-3(\frac{1}{p}-\frac{1}{2p'})}\|\!|\mathbf{w}|\!\|_{1,p,T}\|\!|\mathbf{v}|\!\|_{1,p,T}<\infty.$$

From this and (21), we conclude that $\|\!|\mathbf{B}(\mathbf{w},\mathbf{v})|\!\|_{1,p',T}\leq C_1(p,p')T^{1-3(\frac{1}{p}-\frac{1}{2p'})}$ $\|\!|\mathbf{w}|\!\|_{1,p,T}\ \|\!|\mathbf{v}|\!\|_{1,p,T}$.

iv) Consider $\overline{\mathbf{B}}^f_i(\mathbf{w},\mathbf{v})$ defined as in *iii)*. Using Young's inequality as in *ii)* we get

$$\overline{\mathbf{B}}^f_i(\mathbf{w},\mathbf{v})(t)\leq C\|f\|_{(l')^*}\int_0^t (t-s)^{\frac{3}{2l'}-\frac{3}{l}+\frac{1}{2}}s^{\frac{3}{l}-\frac{3}{p}-\frac{1}{2}}ds\,\|\!|\mathbf{w}|\!\|_{1,l,(T;p)}\|\!|\mathbf{v}|\!\|_{1,l,(T;p)},$$

where the r.h.s. is finite because $\frac{3}{l}-\frac{3}{p}-\frac{1}{2}>-1$ (as follows from $l<\frac{6p}{6-p}$). We deduce that

$$\|\nabla\mathbf{B}(\mathbf{w},\mathbf{v})_i(t)\|_{l'}\leq Ct^{\frac{1}{2}+\frac{3}{2l'}-\frac{3}{p}}\|\!|\mathbf{w}|\!\|_{1,l,(T;p)}\|\!|\mathbf{v}|\!\|_{1,l,(T;p)},$$

whence, the asserted continuity of $\mathbf{B}$ (the norm depends on T in the same way as in *ii)*). □

Remark 3.1. For $p\in]\frac{3}{2},3[$ we have

$$\frac{3p}{6-p}<p<\frac{3p}{6-2p},$$

and so by Lemma 3.2 and Proposition 3.1 *i)* and *iii)* the mild equation (10) makes sense in the spaces $\mathbf{F}_{0,p,T}$ and $\mathbf{F}_{1,p,T}$. Writing

$$\mathbf{w}_0(t,x):=G^{\nu}_t * w_0(x),$$

equation (10) in $\mathbf{F}_{0,p,T}$ or in $\mathbf{F}_{1,p,T}$ is thus equivalent to the abstract equation

$$\mathbf{w}=\mathbf{w}_0+\mathbf{B}(\mathbf{w},\mathbf{w}). \tag{25}$$

3.1. Local existence and global uniqueness

Theorem 3.1. *Let $\frac{3}{2} < p < 3$ and $w_0 \in L^p_3$.*

a) For all $T > 0$, equation (10) has at most one solution in $\mathbf{F}_{0,p,T}$.
b) There is a positive constant $\Gamma_0(p)$ such that equation (10) has a solution in $\mathbf{F}_{0,p,T}$, for all $T > 0$ and $w_0 \in L^p_3$ satisfying

$$T^{1-\frac{3}{2p}} \|w_0\|_p < \Gamma_0(p).$$

To prove global uniqueness, we shall proceed in a similar way as in [14] using next lemma.

Lemma 3.3. *Let $g : [0, T] \to]0, \infty[$ be a bounded measurable function, and suppose there exist constants $C \geq 0$ and $\theta > 0$ such that $g(t) \leq C \int_0^t (t-s)^{\theta-1} g(s)\, ds$ for all $t \in [0, T]$. Then,*

$$g(t) \leq C^2 \beta(\theta, \theta) \int_0^t (t-s)^{2\theta-1} g(s)\, ds.$$

The proof of local existence will rely on a standard contraction argument for the abstract equation (25), based on Banach's fixed point theorem (see for instance Cannone [8]):

Lemma 3.4. *Let $(\mathbf{F}, |||\cdot|||)$ be a Banach space, $\mathbf{B} : \mathbf{F}\times\mathbf{F} :\to \mathbf{F}$ a bilinear application and $\mathbf{y} \in \mathbf{F}$. Suppose there exists a positive constant Λ such that*

$$|||\mathbf{B}(\mathbf{x}_1, \mathbf{x}_2)||| \leq \Lambda |||\mathbf{x}_1|||\, |||\mathbf{x}_2|||$$

for all $\mathbf{x}_1, \mathbf{x}_2 \in \mathbf{F}$. If $4\Lambda |||\mathbf{y}||| < 1$, then for all $\gamma \in [|||\mathbf{y}|||, \frac{1}{4\Lambda}[$ there exists a unique solution of

$$\mathbf{x} = \mathbf{y} + \mathbf{B}(\mathbf{x}, \mathbf{x})$$

in the ball $\mathcal{B}_{R_\gamma} = \{\mathbf{x} \in \mathbf{F} : |||\mathbf{x}||| \leq R_\gamma\}$, $R_\gamma = \frac{1-\sqrt{1-4\Lambda\gamma}}{2\Lambda}$. The solution $\mathbf{x}$ satisfies $|||\mathbf{x}||| \leq 2\gamma$.

Proof of Theorem 3.1: a) Let $\mathbf{w}, \mathbf{v} \in \mathbf{F}_{0,p,T}$ be two solutions. Proceeding as in Proposition 3.1 *i)* (with $r = p$) we obtain

$$\begin{aligned}\|\mathbf{w}(t) - \mathbf{v}(t)\|_p \leq C &\int_0^t (t-s)^{-\frac{3}{2p}} \|\mathbf{w}(s)\|_p\, \|\mathbf{w}(s) - \mathbf{v}(s)\|_p\, ds \\ &+C \int_0^t (t-s)^{-\frac{3}{2p}} \|\mathbf{v}(s)\|_p\, \|\mathbf{w}(s) - \mathbf{v}(s)\|_p\, ds \\ &\leq C \left(|||\mathbf{w}|||_{0,p,T} + |||\mathbf{v}|||_{0,p,T}\right) \int_0^t (t-s)^{-\frac{3}{2p}} \|\mathbf{w}(s) - \mathbf{v}(s)\|_p\, ds.\end{aligned}$$

Let $\theta_N := 2^N(1-\frac{3}{2p}) > 0$ and $N(p)$ be the first integer for which $\theta_N - 1 > 0$. Then, by applying $N(p)$ times Lemma 3.3 , it follows that

$$\|\mathbf{w}(t)-\mathbf{v}(t)\|_p \leq C(T)\left(\|\!|\mathbf{w}|\!\|_{0,p,T} + \|\!|\mathbf{v}|\!\|_{0,p,T}\right)\int_0^t \|\mathbf{w}(s)-\mathbf{v}(s)\|_p\, ds,$$

for some $C(T) > 0$. We conclude by Gronwall's lemma.

b) From Proposition 3.1 *i)*, one has for all $T > 0$ and $\mathbf{w}, \mathbf{v} \in \mathbf{F}_{0,p,T}$ that

$$\|\!|\mathbf{B}(\mathbf{w},\mathbf{v})|\!\|_{0,p,T} \leq C_0(p,p)T^{1-\frac{3}{2p}}\|\!|\mathbf{w}|\!\|_{0,p,T}\ \|\!|\mathbf{v}|\!\|_{0,p,T} \tag{26}$$

where $C_0(p,p) > 0$ does not depend on T. On the other hand, by Lemma 3.2 *i)* we have that $\|\!|\mathbf{w}_0|\!\|_{0,p,T} \leq \|w_0\|_p$. Therefore, by Lemma 3.4, a solution $\mathbf{w} \in \mathbf{F}_{0,p,T}$ to the abstract equation (25) exists if

$$4C_0(p,p)T^{1-\frac{3}{2p}}\|w_0\|_p < 1. \tag{27}$$

□

3.2. Regularity estimates

We need two technical facts:

Remark 3.2. Let $\mathbf{w} \in \mathbf{F}_{0,p,T}$ be a solution of (25) with $\frac{3}{2} < p < 3$. If for each $\tau \in]0,T]$ we write

$$\mathbf{w}_{0,\tau}(t) := G_t^\nu * \mathbf{w}(\tau),\ \text{and}\ \mathbf{w}_\tau(t) := \mathbf{w}(\tau+t),$$

then the function $\mathbf{w}_\tau$ is a solution in $\mathbf{F}_{0,p,T-\tau}$ of the equation

$$\mathbf{v}(t,x) = \mathbf{w}_{0,\tau}(t,x) + \mathbf{B}(\mathbf{v},\mathbf{v})(t,x). \tag{28}$$

(This follows from the semigroup property of G^ν and Fubini's theorem, using similar estimates as in the proof of Proposition 3.1.)

Remark 3.3. We recall that if $1 \leq r_1 \leq r \leq r_2 < \infty$, then $L^{r_1} \cap L^{r_2} \subseteq L^r$ with

$$\|f\|_r^r \leq \|f\|_{r_1}^{r_1} + \|f\|_{r_2}^{r_2},\ \forall f \in L^{r_1} \cap L^{r_2}. \tag{29}$$

Thus, if $p \leq r_1 \leq r \leq r_2 < \infty$, we have $\mathbf{F}_{i,(p;r_1),T} \cap \mathbf{F}_{i,(p;r_2),T} \subseteq \mathbf{F}_{i,(p;r),T}$ for $i = 0, 1$, and the following interpolation inequality holds:

$$\|\!|\mathbf{v}|\!\|_{i,(p;r),T}^r \leq \|\!|\mathbf{v}|\!\|_{i,(p;r_1),T}^{r_1} + \|\!|\mathbf{v}|\!\|_{i,(p;r_2),T}^{r_2},\quad \text{for all } \mathbf{v} \in \mathbf{F}_{i,(p;r_1),T} \cap \mathbf{F}_{i,(p;r_2),T}. \tag{30}$$

For $i = 0$ (resp. $i = 1$) this follows by taking in (29) the function $t^{\frac{3}{2p}}\mathbf{v}(t)$ (resp. $t^{\frac{1}{2}+\frac{3}{2p}}\frac{\partial \mathbf{v}(t)}{\partial x_k}$), and then multiplying by $t^{-\frac{3}{2}}$.

Theorem 3.2. *Let* $p \in]\frac{3}{2}, 3[$, $\mathbf{w} \in \mathbf{F}_{0,p,T}$ *be a solution of (10) and* $A > 0$ *an upper bound for* $\|\!|\mathbf{w}|\!\|_{0,p,T}$.

i) *One has* $\mathbf{w} \in \mathbf{F}_{1,p,T}$*, and* $|||\mathbf{w}|||_{1,p,T} \leq C(T, p, A)$*, with* $C(T, p, A)$ *a constant depending on* $\mathbf{w}$ *only through* A*.*

ii) *For all* $p \leq r < \frac{3p}{3-p}$ *one has* $\mathbf{w} \in \mathbf{F}_{1,r,(T;p)}$*. There exists moreover* $\mu_{1,r,p} : \mathbb{R}^3_+ \to \mathbb{R}_+$ *a function which does not depend on* $\mathbf{w}$*, such that*

$$|||\mathbf{w}|||_{1,r,(T;p)} \leq \mu_{1,r,p}(T, \|w_0\|_p, A).$$

iii) *For all* $p \leq r < \infty$ *one has* $\mathbf{w} \in \mathbf{F}_{0,r,(T;p)}$*. Moreover, there exists* $\mu_{0,r,p} : \mathbb{R}^3_+ \to \mathbb{R}_+$ *as in ii) such that*

$$|||\mathbf{w}|||_{0,r,(T;p)} \leq \mu_{0,r,p}(T, \|w_0\|_p, A).$$

Proof. *i)* The proof is similar as in Lemma 4.4 in [14]; we repeat it here since it is an important point for the sequel. Notice that all results for (25) obtained so far apply also to equations (28) with the *same* constants for all $\varepsilon \geq 0$. From Lemma 3.2 one has $|||\mathbf{w}_{0,\tau}|||_{1,p,T'\wedge(T-\tau)} \leq \overline{C}_1(p)|||\mathbf{w}|||_{0,p,T}$ for all $0 < T' < T$. If we choose T' so that

$$(T')^{1-\frac{3}{2p}} A < \Gamma_1(p),$$

where $\Gamma_1(p)^{-1} = 4\overline{C}_1(p) \cdot C_1(p, p)$ with $C_1(p, p)$ as the proof of Proposition 3.1 *iii)*, then

$$|||\mathbf{w}_{0,\tau}|||_{1,p,T'\wedge(T-\tau)} \leq \overline{C}_1(p)A < \frac{1}{4(T')^{1-\frac{3}{2p}} C_1(p, p)}$$

for all $\tau \in [0, T]$. Thus, from Lemma 3.4 we deduce for each $\tau \in [0, T]$ that (28) has a solution in $\mathbf{F}_{1,p,T'}$ (in the ball of radius R_γ defined in Lemma 3.4, with $\gamma = \overline{C}_1(p)|||\mathbf{w}|||_{0,p,T}$). Define $\tau_k := k\frac{T'}{2}$ for $k = 0 \ldots N := [\frac{2T}{T'}]$. Uniqueness for (28) in the space $\mathbf{F}_{0,p,T'\wedge(T-\tau)}$, for each $\tau = \tau_k$, implies that the functions $\mathbf{w}_{(\tau_k)} := \mathbf{w}(\tau_k + \cdot)$ belong to $\mathbf{F}_{1,p,T'\wedge(T-\tau_k)}$ for all $k = 0, \ldots, N$. But one has $\mathbf{w}_{(\tau_k)}(t) = \mathbf{w}_{(\tau_{k-1})}(\frac{T'}{2} + t)$ for all $t \in [0, \frac{T'}{2} \wedge T]$ and $k = 1, \ldots, N$, so we conclude that $\mathbf{w}_{(\tau_k)}, \frac{\partial \mathbf{w}_{(\tau_k)}}{\partial x_j} \in \mathbf{F}_{0,p,T'\wedge(T-\tau_k)}$ for $k = 1, \ldots, N$, implying that $\mathbf{w} \in \mathbf{F}_{1,p,T}$. The estimate for the norm follows from the fact that for all $t \in [0, \frac{T'}{2} \wedge T]$ and $k = 1 \ldots N$, one has

$$\begin{aligned}(\tau_k + t)^{\frac{1}{2}} \|\frac{\partial \mathbf{w}(\tau_k + t)}{\partial x_j}\|_p &\leq C(T')^{-\frac{1}{2}}(\tau_k + t)^{\frac{1}{2}}(t + \frac{T'}{2})^{\frac{1}{2}} \|\frac{\partial \mathbf{w}_{(\tau_{k-1})}(t + \frac{T'}{2})}{\partial x_j}\|_p \\ &\leq C(T, A)|||\mathbf{w}|||_{0,p,T}.\end{aligned}$$

ii) First we notice that $\mathbf{w}_0 \in \mathbf{F}_{1,r,(T;p)}$ for all $r \geq p$, and that $\mathbf{w} \in \mathbf{F}_{1,p,(T;p)} = \mathbf{F}_{1,p,T}$ by *i)*.
In the proof we will repeatedly apply part *iv)* of Proposition 3.1. Observe that at each time we do this we obtain an estimate of the type $|||\mathbf{w}|||_{1,l',(T;p)} \leq c(l, l')\|w_0\|p + \Lambda(T, l, l')A_l^2$, for suitable l and l', and with $\Lambda(T, l, l')$ the norm of the operator $\mathbf{B} : (\mathbf{F}_{1,l,(T;p)})^2 \to \mathbf{F}_{1,l',(T;p)}$ and A_l any upper bound of $|||\mathbf{w}|||_{1,l,(T;p)}$.
We shall distinguishing two cases:

Case a) $2 < p < 3$: We have $3 < \frac{6p}{6-p} < \frac{3p}{6-2p} < \frac{3p}{3-p}$. Therefore, part *iv)* of Proposition 3.1 holds for $l = p$ and any $l' = r \in [p, \frac{3p}{6-2p}[$. Since $\mathbf{w} \in \mathbf{F}_{1,p,(T;p)}$, we deduce that $\mathbf{B}(\mathbf{w}, \mathbf{w}) \in \mathbf{F}_{1,r,(T;p)}$ for all $r \in [p, \frac{3p}{6-2p}[$ from where $\mathbf{w} \in \mathbf{F}_{1,r,(T;p)}$ for those r.
Now we show that further, $\mathbf{w} \in \mathbf{F}_{1,r,(T;p)}$ for all $r \in [\frac{3p}{6-2p}, \infty[$. Let g be the strictly increasing function $g(s) := \frac{3s}{6-2s}$ defined on the interval $]\frac{3}{2}, 3[$. We have $g([p, 3[) = [\frac{3p}{6-2p}, \infty[$. Let $r \in [\frac{3p}{6-2p}, \infty[$ be given. Then $2r \in]\frac{3p}{6-2p}, \infty[$ and we have

$$p < g^{-1}(r) < g^{-1}(2r) < \frac{g^{-1}(2r) + 3}{2} < 3$$

Notice that if we set $l := \frac{g^{-1}(2r)+3}{2}$, then $2r < g(l)$. We can therefore apply Proposition 3.1 *iv)* to this choice of $l \in]p, 3[$ and to $l' = 2r$, which yields $w \in F_{1,2r,(T;p)}$, from where $w \in F_{1,r,(T;p)}$ by the interpolation inequality (30).

Case b) $\frac{3}{2} < p \leq 2$: Now we have $\frac{3p}{6-2p} \leq 3$ and $2 < \frac{6p}{6-p} \leq 3 < \frac{3p}{3-p}$. First we will show that $\mathbf{w} \in F_{1,r,(T;p)}$ for all $r \in]2, \frac{6p}{6-p}[$. Consider the function $g(s)$ on $]\frac{3}{2}, 3[$ as before, and define a sequence l_n by $l_0 = p$, $l_{n+1} = g(l_n)$. Since $g'(s) > 2$, for all $\frac{3}{2} < s < t < 3$ one has $g(t) - g(s) > 2(t - s)$ and consequently there exists $N \in \mathbb{N}$ such that $l_N \leq 2$ and $l_{N+1} > 2$. For all $n = 0, \ldots, N-1$, we have $g([l_n, l_{n+1}[) \subseteq]l_n, \frac{3l_n}{6-2l_n}[$. We apply Proposition 3.1 *iv)* to $l = l_0 = p$ and $l' = \frac{l_0+l_1}{2}$ and get that $\mathbf{w} \in \mathbf{F}_{1,\frac{l_0+l_1}{2},(T;p)}$. Taking then $l = \frac{l_0+l_1}{2}$ and $l' = l_1$ we deduce that $\mathbf{w} \in \mathbf{F}_{1,l_1,(T;p)}$. Applying now this two-step argument starting from l_1 we deduce that $\mathbf{w} \in \mathbf{F}_{1,l_2,(T;p)}$, and iterating we deduce that $\mathbf{w} \in \mathbf{F}_{1,l_N,(T;p)}$. Since $2 < l_{N+1}$ we can apply again Proposition 3.1 *iv)* to $l = l_N$ and some $l' = r_0 \in [2, l_{N+1}[$. If $r_0 \geq \frac{6p}{6-p}$ we conclude by interpolation that $\mathbf{w} \in \mathbf{F}_{1,r,(T;p)}$ for all $r \in]2, \frac{6p}{6-p}[$. Otherwise, we continue the procedure taking $l = r_0$ and $l' = \frac{6p}{6-p} \leq 3 < l_{N+2}$, and we conclude the same fact again by interpolation.

Observe now that $g([2, \frac{6p}{6-p}[) = [3, \frac{p}{2-p}[$ and that $3 < \frac{3p}{3-p} < \frac{p}{2-p}$. To conclude Case b) we will show that indeed $\mathbf{w} \in \mathbf{F}_{1,r,(T;p)}$ for all $r \in [3, \frac{p}{2-p}[$. Take $r_1 \in]r, \frac{p}{2-p}[$, then

$$2 < g^{-1}(r) < g^{-1}(r_1) < \frac{1}{2}(g^{-1}(r_1) + \frac{6p}{6-p}) < \frac{6p}{6-p}$$

Taking $l = \frac{1}{2}(g^{-1}(r_1) + \frac{6p}{6-p})$, we have $r_1 < g(l)$ and then for $l' = r_1$ we have $\mathbf{w} \in \mathbf{F}_{1,l',(T;p)}$. By interpolation we conclude that $\mathbf{w} \in \mathbf{F}_{1,r,(T;p)}$.
Finally, if $A > 0$ is an upper bound for $\| w \|_{0,p,T}$, it is clear that *i)* and the procedures used in Cases a) and b) allow us to exhibit an upper bound for $\| w \|_{1,r,(T;p)}$,

in terms of A, T, $\|w_0\|_p$, and the norms of the operators $\mathbf{B} : (\mathbf{F}_{1,l,(T;p)})^2 \to \mathbf{F}_{1,l',(T;p)}$, for some finite set of indexes (l, l') such that $l \in [p, \min\{3, \frac{6p}{6-p}\}[$ and $l' \in [l, \frac{3l}{6-l}[$. The statement follows.

iii) This is similar to *ii)*, but the proof is easier now, since we do not have the restriction "$l < \frac{6p}{6-p}$" owed to part *vi)* of Proposition 3.1. We just sketch the proof: for any $p \in]\frac{3}{2}, 3[$ we consider the sequence l_n as in Case b) in *ii)*, and chose $N \in \mathbb{N}$ such that $l_N \leq 3$ and $l_N > 3$. By an inductive argument (and interpolation) we deduce that $\rho \in F_{0,r,(T;p)}$ for all $r \in [p, 3[$, and since $g([2, 3[) = [3, \infty[$, we can pass to arbitrary $r \in [p, \infty[$ by choosing a suitable $l \in [2, 3[$ such that $r < g(l)$. □

Denote by $\mathcal{C}^\alpha$ the space of functions $\mathbb{R}^3 \to \mathbb{R}^3$ that are Hölder continuous of index $\alpha \in]0, 1[$. We recall the following standard embbeding of Sobolev spaces (see e.g. [5]):

Lemma 3.5. *For all $m > 3$, the space $W_3^{1,m}$ is continuously embedded into $L_3^\infty \cap \mathcal{C}^{1-\frac{3}{m}}$.*

From this and Theorem 3.2 we finally deduce

Corollary 3.1. *Let $p \in]\frac{3}{2}, 3[$ and $\mathbf{w} \in F_{0,p,T}$ be a solution of the mild equation (10). Write $\mathbf{u}(s, x) := \mathbf{K}(\mathbf{w})(s, x)$. Then, the following holds:*

i)

$$\sup_{t\in[0,T]} t^{\frac{1}{2}} \left\{ \|\mathbf{u}(t)\|_\infty + \|\mathbf{u}(t)\|_{\mathcal{C}^{\frac{2p-3}{p}}} \right\} < \hat{C}(T, p, A) \tag{31}$$

for a constant $\hat{C}(T, p, A) > 0$ depending on $\mathbf{w}$ only through A.

ii) For all $r \in]3, \frac{3p}{3-p}[$, $i = 1, 2, 3$, and any upper bound $A > 0$ of $|||\mathbf{w}|||_{0,p,T}$, we have

$$\sup_{t\in[0,T]} t^{\frac{1}{2}+\frac{3}{2}(\frac{1}{p}-\frac{1}{r})} \left\{ \left\| \frac{\partial \mathbf{u}(t)}{\partial x_i} \right\|_\infty + \left\| \frac{\partial \mathbf{u}(t)}{\partial x_i} \right\|_{\mathcal{C}^{1-\frac{3}{r}}} \right\} < \hat{C}(T, p, r, A) \tag{32}$$

with $\hat{C}(T, p, r, A) > 0$ a constant depending on $\mathbf{w}$ only through $\|w_0\|_p$ and A. In particular, the functions

$$t \mapsto \|\mathbf{u}(t)\|_\infty \text{ and } t \mapsto \|\frac{\partial \mathbf{u}(t)}{\partial x_i}\|_\infty,\ i = 1, 2, 3$$

belong to $L^1([0, T], \mathbb{R})$.

Proof. By Lemma 2.2, one has $\mathbf{u} \in \mathbf{F}_{1,q,T}$, with $q = \frac{3p}{3-p}$, and by Lemma 3.5, we deduce that for $t \in [0, \min\{T, 1\}]$

$$t^{\frac{1}{2}} \left(\|\mathbf{u}(t)\|_\infty + \|\mathbf{u}(t)\|_{\mathcal{C}^{\frac{2p-3}{p}}} \right) \leq Ct^{\frac{1}{2}} \|\mathbf{u}(t)\|_{1,q} \leq C\|\mathbf{u}(t)\|_q + t^{\frac{1}{2}} \|\nabla\mathbf{u}(t)\|_q.$$

On the other hand, if $t \in [\min\{T,1\},T]$, one has

$$t^{\frac{1}{2}}\left(\|\mathbf{u}(t)\|_\infty + \|\mathbf{u}(t)\|_{C^{\frac{2p-3}{p}}}\right) \le Ct^{\frac{1}{2}}\|\mathbf{u}(t)\|_{1,q} \le CT^{\frac{1}{2}}\left(\|\mathbf{u}(t)\|_q + t^{\frac{1}{2}}\|\nabla\mathbf{u}(t)\|_q\right).$$

Since $\|\mathbf{u}(t)\|_q + t^{\frac{1}{2}}\|\nabla\mathbf{u}(t)\|_q \le C|||\mathbf{w}(t)|||_{1,p,T}$ for all $t \in [0,T]$, the statement *i)* follows from Theorem 3.2 *i)*. Statement *ii)* is proved in a similarly way, noting that $\frac{\partial \mathbf{u}}{\partial x_i} \in \mathbf{F}_{1,r,(T;p)}$ by Lemma 2.3 and using Theorem 3.2 *ii)*. The last assertion is straightforward. □

4. The nonlinear martingale problem

In this section, under the additional probabilistic assumption that w_0 is integrable, we will identify the solution $\mathbf{w} \in \mathbf{F}_{0,p,T}$ of the mild vortex equation with a flow of $\mathbb{R}^3$-valued vector measures associated with a generalized nonlinear diffusion of the McKean-Vlasov type. Let us establish some notation required in the sequel:

- We denote by $\mathcal{P}(\mathcal{C}_T)$ the space of probability measures on $\mathcal{C}_T = C([0,T],\mathbb{R}^3 \times \mathcal{M}_{3\times 3})$.
- For any element $P \in \mathcal{P}(\mathcal{C}_T)$, we will write P° for the first marginal $P^\circ = P|_{C([0,T],\mathbb{R}^3)}$, and P' for the second marginal $P' = P|_{C([0,T],\mathcal{M}_{3\times 3})}$.
- The canonical process in $C([0,T],\mathbb{R}^3 \times \mathcal{M}_{3\times 3})$ will be denoted by (X,Φ).
- We use the notation $\mathcal{P}_b(\mathcal{C}_T)$ for the subspace of $\mathcal{P}(\mathcal{C}_T)$ of probability measures Q such that the support of Q' is bounded. (Equivalently, under each law $Q \in \mathcal{P}_b(\mathcal{C}_T)$, the process Φ is bounded independently of t and of the randomness.)
- By $F_{0,p,T}$, $F_{1,p,T}$, $F_{0,r,(T;p)}$ and $F_{1,r,(T;p)}$ we denote the subspaces of $\mathcal{M}eas^T$ that are the real-valued analogues of the spaces $\mathbf{F}$ defined in Section **3**. We use the same notation as therein for the norms.

We define now a "vectorial weight function" in terms of the initial condition w_0, by setting

$$h_0(x) := w_0(x)\frac{\|w_0\|_1}{|w_0(x)|} \tag{33}$$

(with the convention "$\frac{0}{0} = 0$"). Observe that h_0 takes values in the sphere $\|w_0\|_1 \cdot S^2$ or 0.

With each $Q \in \mathcal{P}_b(\mathcal{C}_T)$ we associate a family of $\mathbb{R}^3$-valued vector measures $(\tilde{Q}_t)_{t\in[0,T]}$ on $\mathbb{R}^3$, defined by

$$\tilde{Q}_t(\mathbf{f}) = E^Q(\mathbf{f}(X_t)\Phi_t h_0(X_0)), \tag{34}$$

for all $\mathbf{f} \in \mathcal{D}_3$. Since Φ is bounded, $\tilde{Q}_t$ is absolutely continuous with respect to Q_t°, with

$$h_t^Q(x) := \frac{d\tilde{Q}_t}{dQ_t^\circ}(x) = E^Q(\Phi_t h_0(X_0)|X_t = x), \tag{35}$$

and its total mass is bounded by $\|w_0\|_1 (\sup_{\phi \in supp(Q')} \sup_{t\in[0,T]} |\phi_t|)$.

Notice that $(t,x) \mapsto h_t^Q(x)$ is measurable. With notation (35), we can rewrite (34) as

$$\tilde{Q}_t(\mathbf{f}) = E^Q(\mathbf{f}(X_t) h_t^Q(X_t)). \tag{36}$$

Thus, we can think of $h_t^Q(x)$ as a bounded vectorial weight found at position x at time t.

If now $P \in \mathcal{P}_b(\mathcal{C}_T)$ is such that for each t the probability measure P_t° is absolutely continuous with respect to Lebesgue's measure, then the same holds for the vector measure $\tilde{P}_t$. In that case, and if $\rho : [0,T] \times \mathbb{R}^3 \to \mathbb{R}$ is the family of densities of P_t°, we will denote by

$$\tilde{\rho} : [0,T] \times \mathbb{R}^3 \to \mathbb{R}^3$$

the family of densities of $\tilde{P}_t$ (taking always bi-measurable versions of both of them if they exist). We stress the fact that $\tilde{\rho}_t$ is defined in terms of the joint law of (X_0, X_t, Φ_t).

We will study the following nonlinear martingale problem: to find $P \in \mathcal{P}_b(\mathcal{C}_T)$ such that

- $P^\circ|_{t=0}(dx) = \frac{|w_0(x)|}{\|w_0\|_1} dx$ and for all $0 \le t \le T$, $P_t^\circ(dx) = \rho_t(x)dx$.

- $$f(t,X_t) - f(0,X_0) - \int_0^t \Big[\frac{\partial f}{\partial s}(s,X_s) + \nu \triangle f(s,X_s)ds + \mathbf{K}(\tilde{\rho})(s,X_s)\nabla f(s,X_s) \Big] ds, \tag{37}$$

  $0 \le t \le T$, is a continuous P°-martingale for all $f \in \mathcal{C}_b^{1,2}$;

- $\Phi_t = Id + \int_0^t \nabla \mathbf{K}(\tilde{\rho})(s,X_s)\Phi_s \, ds$, for all $0 \le t \le T$, P-almost surely.

To state our main result on the probabilistic interpretation of the vortex equation, we need

Definition 4.1. $\mathcal{P}^T_{b,\frac{3}{2}}$ *is the space of probability measures* $P \in \mathcal{P}_b(\mathcal{C}_T)$ *satisfying the following conditions:*

- *For each* $t \in [0,T]$*, the time marginal* P_t° *is absolutely continuous with respect to Lebesgue's measure, and the family of densities* $(t,x) \mapsto \rho(t,x)$ *is bi-measurable and belongs to* $F_{0,p,T}$ *for some* $\frac{3}{2} < p$.
- $div\ \tilde{\rho}_t = 0$ *for all* $t \in [0,T]$.

Theorem 4.1. *Assume that* $w_0 \in L^1_3 \cap L^p_3$ *for some* $p \in]\frac{3}{2}, 3[$*. For every* $T > 0$*, the nonlinear martingale problem (37) has at most one solution* P *in the class* $\mathcal{P}^T_{b,\frac{3}{2}}$.

Further, there exists a solution P *in* $\mathcal{P}^T_{b,\frac{3}{2}}$ *such that* P° *has a density family* $\rho \in F_{0,p,T}$, $p \in]\frac{3}{2}, 3[$*, if and only if there exists in* $\mathbf{F}_{0,p,T}$ *a solution* $\mathbf{w}$ *of the mild* *equation (10) with initial condition* w_0. *In that case, for all* $t \in [0, T]$ *one has the relations*

$$\mathbf{w}(t,x) = \tilde{\rho}(t,x), \quad \rho(t,x)\left| E^P(\Phi_t h_0(X_0)|X_t = x)\right| = |\mathbf{w}(t,x)|,$$

and for all $1 \leq p' \leq p$ *and* $p \leq r < \frac{3p}{3-p}$, *it holds that* $\rho \in F_{0,p',T} \cap F_{1,r,(T;p)}$.

Corollary 4.1. *If* $w_0 \in L^1_3 \cap L^p_3$ *for some* $p \in]\frac{3}{2}, 3[$ *and* $T^{1-\frac{3}{2p}} \|w_0\|_p < \Gamma_0(p)$ *(the constant of Theorem 3.1), then (37) has a unique solution in* $\mathcal{P}^T_{b,\frac{3}{2}}$.

The proof of Theorem 4.1 will be done in several steps. First of all, we shall dwell upon the properties of the evolution equation satisfied by the densities ρ of the marginal P° of a given solution P. The study of this equation will provide *a priori* regularity estimates for the drift term $\mathbf{K}(\tilde{\rho})$ in (37).

4.1. A nonlinear Fokker-Planck equation associated with the vortex equation

Assume for a while that (37) has a solution $P \in \mathcal{P}_b(\mathcal{C}_T)$ which satisfies

$$\int_0^T \int_{\mathbb{R}^3} |\mathbf{K}(\tilde{\rho})(t,x)|\rho(t,x)\, dxdt < \infty. \tag{38}$$

(This is a minimal condition ensuring that $\int_0^t \mathbf{K}(\tilde{\rho})(s, X_s)ds$ has finite variation). By applying Itô's formula to $f(t, X_t)$ for an arbitrary function $f \in C_b^{1,2}$ and taking expectations, we deduce that the couple $(\rho, \tilde{\rho})$ satisfies the following weak evolution equation:

$$\begin{aligned}\int_{\mathbb{R}^3} & f(t,y)\rho(t,y)dy \\ &= \int_{\mathbb{R}^3} f(0,y)\rho_0(y)dy \\ &\quad + \int_0^t \int_{\mathbb{R}^3} \left[\frac{\partial f}{\partial s}(s,y) + \nu \triangle f(s,y) + \mathbf{K}(\tilde{\rho})(s,y)\nabla f(s,y)\right] \\ &\qquad\qquad \times \rho(s,y)\, dy\, ds,\end{aligned} \tag{39}$$

where $\rho_0(x) = \frac{|w_0(x)|}{\|w_0\|_1}dx$. Observe that by (36), one has

$$\tilde{\rho}_t(x) = h_t^P(x)\rho_t(x).$$

If P is fixed, then h^P is also fixed, and then (39) is a nonlinear Fokker-Planck equation for the unknown ρ. To a large extent, we will be able to treat this equation as a scalar analogue of vortex equation. Its mild form is obtained as follows. Fix $\psi \in \mathcal{D}$ and $t \in [0, T]$ and take in (39) the $C_b^{1,2}$-function $f_t : [0,t] \times \mathbb{R}^3 \to \mathbb{R}^3$ given by $f_t(s, y) = G^\nu_{t-s} * \psi(y)$ (which solves the backward heat equation on

$[0,t]\times\mathbb{R}^3$ with final condition $f(t,y)=\psi(y)$). By Lemma 3.1 and condition (38), it is easily checked that

$$\int_0^t \int_{(\mathbb{R}^3)^2} \sum_{j=1}^3 \left|\frac{\partial G^\nu_{t-s}}{\partial y_j}(x-y)\right| |\mathbf{K}(\tilde{\rho})_j(s,y)||\psi(x)|\rho(s,y)dx\,dy\,ds<\infty,$$

and by Fubini's theorem we deduce that

$$\rho(t,x)=G^\nu_t*\rho_0(x) + \int_0^t \sum_{j=1}^3 \int_{\mathbb{R}^3} \frac{\partial G^\nu_{t-s}}{\partial y_j}(x-y)\mathbf{K}(h\rho)_j(s,y)\rho(s,y)\,dy\,ds \tag{40}$$

for all $t\in[0,T]$, where $h=h^P$ and $h\rho(t,x)=h_t(x)\rho(t,x)$.

We will now study some of the analytical properties of equation (40) in a more general situation: assume that $h:[0,T]\times\mathbb{R}^3\to\mathbb{R}^3$ is a fixed arbitrary function of class $L^\infty([0,T],L^\infty_3)$, and define for $\rho,\eta\in\mathcal{M}eas^T$ a function $\mathbf{b}^h(\rho,\eta):[0,T]\times\mathbb{R}^3\to\mathbb{R}^3$ by

$$\mathbf{b}^h(\rho,\eta)(t,x)=\int_0^t \sum_{j=1}^3 \int_{\mathbb{R}^3} \frac{\partial G^\nu_{t-s}}{\partial y_j}(x-y)\mathbf{K}(h\eta)_j(s,y)\rho(s,y)dy\,ds.$$

Remark 4.1. For each $p\in[1,\infty]$ (resp. each $p\in[1,\infty]$ and $r\geq p$), the mapping $\eta\mapsto h\eta$ is continuous from $F_{0,p,T}$ to $\mathbf{F}_{0,p,T}$ (resp. from $F_{0,r,(T;p)}$ to $\mathbf{F}_{0,r,(T;p)}$).

Thus, the following continuity properties of $\mathbf{b}^h$ can be proved in exactly the same way as Proposition 3.1 *i)* and *ii)*:

Lemma 4.1. *The operator* $\mathbf{b}^h:(F,|||\cdot|||)^2\to(F',|||\cdot|||')$ *is well defined and continuous if*

i) $\frac{3}{2}\leq p<3$, $\frac{3p}{6-p}\leq p'<\frac{3p}{6-2p}$, $F=F_{0,p,T}$ *and* $F'=F_{0,p',T}$.
ii) $\frac{3}{2}\leq p<3$, $p\leq r<3$, $\frac{3r}{6-r}\leq r'<\frac{3r}{6-2r}$, $F=F_{0,r,(T;p)}$ *and* $F'=F_{0,r',(T;p)}$

Write now

$$\gamma_0(t,x):=G^\nu_t*\rho_0(x)=G^\nu_t*\frac{|w_0|}{\|w_0\|_1}(x). \tag{41}$$

Since $w_0\in L^p_3$, Lemma 3.1 and Young's inequality imply that $\gamma_0\in F_{1,r,(T;p)}$ for all $r\geq p$. This and the previous lemma give sense to the abstract equation

$$\rho=\gamma_0+\mathbf{b}^h(\rho,\rho) \tag{42}$$

in $F_{0,p,T}$ if $\frac{3}{2}<p<3$, and (40) is equivalent to (42) in that space.

We deduce additional properties of solutions of (40):

Lemma 4.2. *Assume that* $w_0\in L^p_3$*, with* $\frac{3}{2}<p<3$ *and let* $h\in L^\infty([0,T],L^\infty_3)$ *be fixed.*

i) *For all* $T > 0$ *the nonlinear Fokker-Planck equation (40) has at most one solution* $\rho \in F_{0,p,T}$.

ii) *If* $\rho \in F_{0,p,T}$ *is a solution of (40), then* $\rho \in F_{0,r,(T;p)}$ *for all* $p \leq r < \infty$ *with* $|||\rho|||_{0,r,(T;p)} \leq C(T, p, r, |||\rho|||_{0,p,T}) < \infty$.

iii) *We deduce that* $\tilde{\rho} = h\rho$ *satisfies* $\tilde{\rho} \in \mathbf{F}_{0,r,(T;p)}$ *and* $|||\tilde{\rho}|||_{0,r,(T;p)} \leq \tilde{C}(T, h, p, r, |||\tilde{\rho}|||_{0,p,T})$ *for all* $p \leq r < \infty$.

Proof. *i)* is the same as Theorem 3.1 *a)*. To prove *ii)*, we proceed in a similar way as in Theorem 3.2 *iii)*, reasoning now in spaces $F_{0,r;(T,p)}$ instead of $\mathbf{F}_{1,r,(T;p)}$, and using Lemma 4.1 and Remark 4.1). Part *iii)* is immediate from *ii)* and Remark 4.1. □

Now we obtain *a priori* regularity estimates for ρ, $\tilde{\rho}$ and the drift term $\mathbf{K}(\tilde{\rho})$ in (37):

Proposition 4.1. *Let* $\frac{3}{2} < p < 3$ *and* $P \in \mathcal{P}^T_{b,\frac{3}{2}}$ *be solution of (37), with* $\rho \in F_{0,p,T}$. *Then,*

i) $\tilde{\rho} \in \mathbf{F}_{0,r,(T;p)}$ *for all* $r \in [p, \infty[$ *and* $\mathbf{K}(\tilde{\rho}) \in \mathbf{F}_{1,l,(T;\frac{3p}{3-p})}$ *for all* $l \in [\frac{3p}{3-p}, \infty[$.

ii) *We deduce that* $\rho \in F_{1,r,(T;p)}$ *for all* $r \in [p, \frac{3p}{3-p}[$.

Proof. *i)* First notice that ρ belongs to $F_{0,\frac{3}{2},T}$ by inequality (29) since $\rho \in F_{0,1,T} \cap F_{0,p,T}$. Thus, (38) holds by Remark 4.1 and Lemma 2.2 *i)*. We deduce that $\rho \in F_{0,p,T}$ solves the Fokker-Planck equation (40). Therefore, $\tilde{\rho} \in \mathbf{F}_{0,r,(T;p)}$ for all $r \in [p, \infty[$ by Lemma 4.2 *iii)*. Define $q := \frac{3p}{3-p}$. If we take $l \geq q$ and set $r := (\frac{1}{l} + \frac{1}{3})^{-1}$, then one has $r \geq p$, and so Lemma 4.2 *iii)* and Lemma 2.2 *i)* imply that

$$\sup_{t\in[0,T]} t^{\frac{3}{2}(\frac{1}{p}-\frac{1}{r})} \|\mathbf{K}(\tilde{\rho}(t))\|_l < \infty.$$

As $\frac{1}{p} - \frac{1}{r} = \frac{1}{q} - \frac{1}{l}$, this means that $\mathbf{K}(\tilde{\rho}) \in \mathbf{F}_{0,l,(T;q)}$.

We next check that $\mathbf{K}(\tilde{\rho}) \in \mathbf{F}_{1,l,(T;q)}$. From the fact that $\tilde{\rho} \in \mathbf{F}_{0,l,(T;p)}$ holds in particular for all $l \geq q$, we get from Lemma 2.3 *i)* that $\frac{\partial \mathbf{K}(\tilde{\rho})}{\partial x_k} \in \mathbf{F}_{0,l,(T;p)}$ for all $k = 1, 2, 3$. Therefore

$$\sup_{t\in[0,T]} t^{\frac{3}{2}(\frac{1}{p}-\frac{1}{l})} \left\| \frac{\partial \mathbf{K}(\tilde{\rho})}{\partial x_k} \right\|_l < \infty.$$

Since $\frac{3}{2}(\frac{1}{p} - \frac{1}{l}) = \frac{1}{2} + \frac{3}{2}(\frac{1}{q} - \frac{1}{l})$, we conclude that $\mathbf{K}(\tilde{\rho}) \in \mathbf{F}_{1,l,(T;q)}$.

ii) Using the fact that $\mathbf{K}(\tilde{\rho}) \in \mathbf{F}_{1,q,T}$, we prove as in Proposition 3.1 *iii)* that for $p = p'$ the *linear* operator (with ρ fixed) defined by

$$\eta(t,x) \mapsto \mathbf{b}^h(\eta,\rho)(t,x) = \int_0^t \sum_{j=1}^3 \int_{\mathbb{R}^3} \frac{\partial G_{t-s}}{\partial y_j}(x-y)\mathbf{K}(\tilde{\rho})_j(s,y)\eta(s,y)dy\, ds.$$

is continuous from $(F_{1,p,T})^2$ to $F_{1,p,T}$, with norm bounded by a multiple of T to some positive power. Thus, by Banach's fixed point theorem we have a local existence result in $F_{1,p,T'}$ (for some $T' > 0$ possibly smaller than T) for the linear equation

$$\eta = \gamma_0 + \mathbf{b}^h(\rho, \eta). \tag{43}$$

Using the latter, uniqueness for (43) in $F_{0,p,T}$, and the fact that $\mathbf{K}(\tilde{\rho}) \in \mathbf{F}_{1,q,T}$, we can adapt the arguments of Theorem 3.2 *i)* to the linear equation (43) in order to show that the solution $\rho \in F_{0,p,T}$ belongs to $F_{1,p,T}$.

Using further the fact that $\mathbf{K}(\tilde{\rho}) \in \mathbf{F}_{1,r,(T;q)}$ for all $r \geq q$, the operator $\mathbf{b}^h$ can also be shown to be continuous from $(F_{1,m,(T;p)})^2$ to $F_{1,m',(T;p)}$, for $p \leq m < \min\{3, \frac{6p}{6-p}\}$ and $\frac{3m}{6-m} \leq m' < \frac{3m}{6-2m}$ (by the same arguments as in Proposition 3.1 *iv)*). By mimicking the proof of Theorem 3.2 *ii)* we finally conclude that $\rho \in F_{1,r,(T;p)}$ for all $r \in [p, \frac{3p}{3-p}[$. □

Corollary 4.2. *Assume that P is a solution of (37) in the class $\mathcal{P}^T_{b,\frac{3}{2}}$. Then, under the law P, the process Φ is continuous and with finite variation. We deduce that the associated function $\tilde{\rho}$ is a weak solution of the vortex equation.*

Proof. Since condition (38) holds, the process $\mathbf{f}(t, X_t)$ is a semi-martingale under P for any $\mathbf{f} \in C^{1,2}_{b,3}$. On the other hand, $\tilde{\rho} \in \mathbf{F}_{0,\frac{3}{2},T}$, and so Lemma 4.2 *iii)* with $r = 3$ and Lemma 2.3 *i)* yield

$$\int_0^T \int_{\mathbb{R}^3} |\nabla \mathbf{K}(\tilde{\rho})(t,x)| \rho(t,x)\, dxdt < \infty. \tag{44}$$

As Φ is a bounded process under P, the equation defining Φ in (37) and (44) imply that $t \mapsto \Phi_t$ has finite variation. We can thus apply Itô's formula to $\mathbf{f}(t, X_t)\Phi_t$ and see that

$$\mathbf{f}(t, X_t)\Phi_t - \mathbf{f}(0, X_0) - \int_0^t \left[\frac{\partial \mathbf{f}}{\partial s}(s, X_s) + \nu \triangle \mathbf{f}(s, X_s) + \right.$$
$$\left. \nabla \mathbf{f}(s, X_s)\mathbf{K}(\tilde{\rho})(s, X_s) + \mathbf{f}(s, X_s)\nabla \mathbf{K}(\tilde{\rho})(s, X_s) \right] \Phi_s\, ds$$

is a martingale for all $\mathbf{f} \in C^{1,2}_{b,3}$. By multiplying the previous equation by $h_0(X_0)$ and taking expectations, we conclude from the definition of $\tilde{P}_s$ and Fubini's theorem (thanks also to (38) and (44)), that $\tilde{\rho}$ is a solution of the weak vortex equation (11). □

Remark 4.2. We have not used the assumption $div \tilde{\rho}_t = 0$ in the previous results. This condition will be needed later on to prove that $\tilde{\rho}$ is also a mild solution. From this we will deduce additional regularity of $\nabla \mathbf{K}(\tilde{\rho})$, necessary prove that (37) is well posed in $\mathcal{P}^T_{b,\frac{3}{2}}$.

4.2. Existence

We consider now a mild solution $\mathbf{w} \in F_{0,p,T}$ such that $w_0 \in L^1_3$. We will construct a solution $P \in \mathcal{P}^T_{b,\frac{3}{2}}$ of the martingale problem (37), such that $\tilde{\rho}$ defined as in (34) satisfies $\tilde{\rho} = \mathbf{w}$.

By Corollary 3.1, the drift term $\mathbf{K}(\mathbf{w})(t)$ and its gradient $\nabla\mathbf{K}(\mathbf{w})(t)$ are continuous and bounded functions on x for each $t \in]0, T]$, and with singularities in L^∞ and Hölder norm at time $t = 0$. To construct the probability measure P, we will follow a similar strategy as in [14] by an approximation argument by suitable processes involving regularized kernels instead of K. The additional difficulty here is that we have to simultaneously approximate both drift terms $\mathbf{K}(\mathbf{w})(s, X_s)$ and $\nabla\mathbf{K}(\mathbf{w})(s, X_s)\Phi_s$.

Consider $\varphi_\varepsilon : \mathbb{R}^3 \to \mathbb{R}$ a regular approximation of the Dirac mass, that is, $\varphi_\varepsilon(x) = \frac{1}{\varepsilon^3}$ for all $\varepsilon > 0$, with $\varphi : \mathbb{R}^3 \to \mathbb{R}$ a positive, smooth, and rapidly decaying function such that $\int_{\mathbb{R}^3} \varphi(x)dx = 1$.

We define regularized kernels $K_\varepsilon = \varphi_\varepsilon * K$, and the associated mollified operators $\mathbf{K}^\varepsilon$ by

$$\mathbf{K}^\varepsilon(w)(x) := \int_{\mathbb{R}^3} K_\varepsilon(x - y) \wedge w(y)dy$$

Remark 4.3. For all $r \in]1, 3[$ and $m \in]1, \infty[$, and functions $w \in L^r_3$ and $v \in L^m_3$, we have

$$\mathbf{K}^\varepsilon(w) = \mathbf{K}(\varphi_\varepsilon * w) = \varphi_\varepsilon * \mathbf{K}(w) \text{ and } \nabla\mathbf{K}^\varepsilon(v) = \nabla\mathbf{K}(\varphi_\varepsilon * v) = \varphi_\varepsilon * \nabla\mathbf{K}(v).$$

The identities for $\mathbf{K}$ are easily obtained for $w \in \mathcal{D}_3$, and for general w they follow by density and Lemma 2.2 *i)*. The identities for $\nabla\mathbf{K}$ follow for $v \in \mathcal{D}_3$ by considering Fourier transform, and for general v we use again density and Lemma 2.3 *i)*. Consequently, Lemmas 2.2 and 2.3 hold true for all operators $\mathbf{K}^\varepsilon$ with the same constants as $\mathbf{K}$.

We also deduce that $\mathbf{K}^\varepsilon(w)$ converges in L^l_3 to $\mathbf{K}(w)$, for all $r \in]1, 3[$ and $\frac{1}{l} = \frac{1}{r} - \frac{1}{3}$, and that $\frac{\partial \mathbf{K}^\varepsilon(w)}{\partial x_k}$ converges in L^m_3 to $\frac{\partial \mathbf{K}(w)}{\partial x_k}$ for all $m \in]1, \infty[$ and $k = 1, 2, 3$.

Let (ε_n) be a sequence converging to 0, and take in a fixed probability space a standard three dimensional Brownian motion B, and a $\mathbb{R}^3$-valued r.v. X_0 independent of B with law

$$\rho_0(x)dx := \frac{|w_0(x)|}{\|w_0\|_1}dx.$$

Consider the following family of stochastic differential equations indexed by (s, x)

$$\xi^{(n)}_{s,t}(x) = x + \sqrt{2\nu}(B_t - B_s) + \int_s^t \mathbf{K}^{\varepsilon_n}(\mathbf{w})(\theta, \xi^{(n)}_{s,\theta}(x))d\theta, \text{ for all } t \in [s, T] \qquad (45)$$

By Remark 4.3 we have $\mathbf{K}^\varepsilon(\mathbf{w}(t)) = \varphi_\varepsilon * \mathbf{K}(\mathbf{w}(t))$, and so by Young's inequality

$$\|\mathbf{K}^\varepsilon(\mathbf{w}(t))\|_\infty \le \|\varphi_\varepsilon\|_{q^*}\|\mathbf{K}(\mathbf{w}(t))\|_q \le C\|\varphi_\varepsilon\|_{q^*}\|\mathbf{w}\|_{0,p,T}$$

where $q = \frac{3p}{3-p}$. In a similar way, $\|\nabla\mathbf{K}^\varepsilon(\mathbf{w}(t))\|_\infty \le C\|\nabla\varphi_\varepsilon\|_{q^*}\|\mathbf{w}\|_{0,p,T}$ and similar estimates for all derivatives hold. Thus $(s, y) \mapsto \mathbf{K}^\varepsilon(\mathbf{w})(s, y)$ is bounded and continuous in $y \in \mathbb{R}^3$, and has infinitely many derivatives in $y \in \mathbb{R}^3$ that are uniformly bounded in $[0, T] \times \mathbb{R}^3$.

Equations (45) have therefore a unique trajectorial solution for each (s, x). Furthermore, by results of Kunita [21] Ch.2, there is a continuous version of the process $(s, t, x) \mapsto \xi^{(n)}_{s,t}(x)$ which is infinitely many times differentiable in x for all (s, t), and satisfies $\xi^{(n)}_{s,t}(x) \to x$ when $(t - s) \to 0$. The derivative $\nabla\xi^{(n)}_{s,t}(x)$ solves the ordinary differential equation in $\mathcal{M}_{3\times3}$

$$\nabla\xi^{(n)}_{s,t}(x) = Id + \int_s^\tau \nabla\mathbf{K}^{\varepsilon_n}(\mathbf{w})(\theta, \xi^{(n)}_{s,\theta}(x))\nabla\xi^{(n)}_{s,\theta}(x)d\theta. \tag{46}$$

We will denote by $(X^{(n)}, \Phi^{(n)})$ the couple of processes defined on $[0, T]$ by

$$X^{(n)}_t := \xi^{(n)}_{0,t}(X_0), \text{ and } \Phi^{(n)}_t = \nabla\xi^{(n)}_{0,t}(X_0),$$

so that

$$\begin{aligned} X^{(n)}_t &= X_0 + \sqrt{2\nu}B_t + \int_0^t \mathbf{K}^{\varepsilon_n}(\mathbf{w})(s, X^{(n)}_s)ds \\ \Phi^{(n)}_t &= Id + \int_0^t \nabla\mathbf{K}^{\varepsilon_n}(\mathbf{w})(s, X^{(n)}_s)\Phi^{(n)}_s ds. \end{aligned} \tag{47}$$

The law of $(X^{(n)}, \Phi^{(n)})$ clearly belongs to $\mathcal{P}_b(\mathcal{C}_T)$ and will be denoted by $Q^{(n)}$. Moreover, since the drift term in the first equation in (47) is bounded, $(Q^{(n)})^\circ_t$ has a density with respect to Lebesgue's measure. For each $n \in \mathbb{N}$, there exists a bi-measurable version $(t, x) \mapsto \rho^{(n)}(t, x)$ of the densities of $(Q^{(n)})^\circ_t$ (see [28], p. 194), and thus, a bi-measurable version $(t, x) \mapsto \tilde{\rho}^{(n)}(t, x)$ of the densities of $\tilde{Q}^{(n)}_t$. In what follows we shall prove that the sequence $Q^{(n)}$ is uniformly tight, with accumulation points that are solutions of (37). The first step will be to prove the convergence, in a strong enough sense, of the one dimensional time-marginal laws.

We will need a technical result, namely a precise existence and regularity statement for the Cauchy problem associated with the generator of $X^{(n)}$:

$$\begin{aligned} \tfrac{\partial}{\partial s} f(s, y) + \nu\Delta f(s, y) + \mathbf{K}^{\varepsilon_n}(\mathbf{w})(s, y)\nabla f(s, y) = 0, &\quad (s, y) \in [0, \tau[\times\mathbb{R}^3, \\ f(\tau, y) = \phi(y). & \end{aligned} \tag{48}$$

Lemma 4.3. *Let* $\phi \in \mathcal{D}$ *and* $\tau \in]0, T]$. *The backward Cauchy problem (48) has a unique solution* $f \in C_b^{1,2}([0, \tau[\times\mathbb{R}^3) \cap C_b([0, \tau] \times \mathbb{R}^3)$. *Moreover, we have* $f \in C_b^{1,3}([0, \tau] \times \mathbb{R}^3)$.

Proof. Observe that, since for each (s,x) the coefficients in equation (45) are Lipschitz continuous, trajectorial uniqueness (and thus in law) holds for $(\xi^{(n)}_{s,t}(x), t \in [s,\tau])$. Then, if a solution g of (48) exists, it must satisfy $g(s,x) = E(\phi(\xi_{s,\tau}(x))$ by the Feynmann-Kac formula, and thus uniqueness for the Cauchy problem holds.

The previous argument does not provide existence nor regularity. Since standard existence and regularity results for (48) require additional assumptions (e.g. [16]), and do not provide the regularity we need here up to the final time τ, we will give a probabilistic proof of these facts, using regularity properties of the stochastic flow (45) (we are inspired in Theorem 7.1, Ch. 3 in [21]).

From equation (46), it comes that $\nabla\xi_{s,t}(x)$ is bounded and then $\nabla\xi_{s,t}(x) \to Id$ when $s \nearrow t$ for each $t \in [0,\tau]$. Considering the equations satisfied by the higher order derivatives, one can also show that $D^\alpha \xi_{s,t}$ is bounded, and that $D^\alpha \xi_{s,t}(x) \to D^\alpha x$ when $s \nearrow t$, for any multi-index $|\alpha| \leq 3$. It follows that the function $f(s,x) := E(\phi(\xi_{s,\tau}(x))$ has derivatives in x up to the third order, and f and its derivatives are bounded and continuous on $[0,\tau]\times\mathbb{R}^3$.

The proof will be achieved by showing that f solves (48). Write $L_\theta\phi(x) := \nu\Delta\phi(x) + \mathbf{K}^{\varepsilon_n}(\mathbf{w})(\theta,x)\nabla\phi(x)$. By the backward Itô formula (Theorem 1.1 in [21], Ch. 3), one has

$$\phi(\xi_{s,t}(x)) = \phi(x) + \sqrt{2\nu}\int_s^t \nabla(\phi\circ\xi_{\theta,t})(x)\widehat{d}B_\theta + \int_s^t L_\theta(\phi\circ\xi_{\theta,t})(x)d\theta \tag{49}$$

where $\int_s^t \cdot\, \widehat{d}B_\theta$ is the backward stochastic integral with respect to B on $[s,t]$ (i.e. the stochastic integral with respect to the standard Brownian motion $(\hat{B}^t_s = B_{t-s} - B_t, s \in [0,t])$ and its natural filtration). Using (49), we check that

$$L_\theta\phi(y) = \lim_{\theta'\to\theta^-} \frac{1}{\theta-\theta'}\left[E(\phi\circ\xi_{\theta',\theta}(y)) - \phi(y)\right],$$

and then the commutation relation $E[L_\theta(\phi\circ\xi_{\theta,t})(x)] = L_\theta E[(\phi\circ\xi_{\theta,t})(x)]$ is obtained, thanks also to the independence of $\xi_{s',s}(x)$ and $\xi_{s,t}(y)$ for $s' < s < t$. It follows then from (49) that

$$f(s,x) - f(s',x) = -\int_{s'}^s L_\theta f(\theta,x)d\theta$$

for all $s, s' \in [0,\tau]$, which ends the proof. □

Lemma 4.4. *For all $t \in [0,T]$ and $n \in \mathbb{N}$, we have $\tilde{\rho}^{(n)}(t) \in L^p_3$, $\rho^{(n)}(t) \in L^p$, and*

$$\sup_{n\in\mathbb{N}} \|\!|\tilde{\rho}^{(n)}|\!\|_{0,p,T} < \infty, \sup_{n\in\mathbb{N}} \|\!|\rho^{(n)}|\!\|_{0,p,T} < \infty. \tag{50}$$

Moreover, $\tilde{\rho}^{(n)}(t)$ converges in L^p_3 for each $t \in [0,T]$ and in $L^1([0,T],L^p_3)$ to $\mathbf{w}(t)$. Similarly, $\rho^{(n)}(t)$ converges in L^p for each $t \in [0,T]$ and in $L^1([0,T],L^p)$, *to the unique solution* $\rho \in F_{0,p,T}$ *of the linear mild equation*

$$\rho(t,x) = G_t^\nu * \rho_0(x) + \int_0^t \sum_{j=1}^3 \int_{\mathbb{R}^3} \frac{\partial G_{t-s}^\nu}{\partial y_j}(x-y)\mathbf{K}(\mathbf{w})_j(s,y)\rho(s,y)\,dy\,ds,\ t \in [0,T]. \quad (51)$$

Proof. By writing It's formula for the product $\mathbf{f}(t, X_t^{(n)})\Phi_t^{(n)}$ and an arbitrary function $\mathbf{f} \in (C_b^{1,2})_3([0,T],\mathbb{R}^3)$, and taking expectations after multiplying by $h_0(X_0)$, we see that $\tilde{\rho}^{(n)}(t)$ is a solution of the following weak equation

$$\int_{\mathbb{R}^3} \mathbf{f}(t,y)\tilde{\rho}^{(n)}(t,y)dy = \int_{\mathbb{R}^3} \mathbf{f}(0,y)w_0(y)dy + \int_0^t \int_{\mathbb{R}^3} \left[\frac{\partial \mathbf{f}}{\partial s}(s,y) + \nu\triangle\mathbf{f}(s,y) + \nabla\mathbf{f}(s,y)\mathbf{K}^{\varepsilon_n}(\mathbf{w})(s,y) + \mathbf{f}(s,y)\nabla\mathbf{K}^{\varepsilon_n}(\mathbf{w})(s,y)\right] \times\tilde{\rho}^{(n)}(s,y)\,dy\,ds. \quad (52)$$

We will prove that $\tilde{\rho}^{(n)}$ solves equation

$$\tilde{\rho}^{(n)}(t,x) = \int_{\mathbb{R}^3} G_t^\nu(x-y)w_0(y)dy + \int_0^t \sum_{j=1}^3 \int_{\mathbb{R}^3} \frac{\partial G_{t-s}^\nu}{\partial y_j}(x-y)\left[\mathbf{K}^{\varepsilon_n}(\mathbf{w})_j(s,y)\tilde{\rho}^{(n)}(s,y) - \tilde{\rho}_j^{(n)}(s,y)\mathbf{K}^{\varepsilon_n}(\mathbf{w})(s,y)\right]dy\,ds. \quad (53)$$

By similar arguments as in Lemma 2.1, the function $\tilde{\rho}^{(n)}$ is seen to solve the linear equation

$$\tilde{\rho}^{(n)}(t,x) = G_t^\nu * w_0(x) + \int_0^t \sum_{j=1}^3 \int_{\mathbb{R}^3} \left[\frac{\partial G_{t-s}^\nu}{\partial y_j}(x-y)[\mathbf{K}^{\varepsilon_n}(\mathbf{w})_j(s,y)\tilde{\rho}^{(n)}(s,y)] + G_{t-s}^\nu(x-y)[\tilde{\rho}_j^{(n)}(s,y)\frac{\partial \mathbf{K}^{\varepsilon_n}(\mathbf{w})}{\partial y_j}(s,y)]\right]dy\,ds. \quad (54)$$

To obtain (53) we must first check that $\tilde{\rho}^{(n)}(s)$ has null divergence in the distribution sense. By the previous lemma, for each $\phi \in \mathcal{D}$ and $t \in]0,T]$, if f is the solution of the Cauchy problem (48), then $\nabla f \in (C_b^{1,2})_3([0,T],\mathbb{R}^3)$. We can therefore plug the function $\mathbf{f} = \nabla f$ in (52), and obtain after simple computations that

$$\int_{\mathbb{R}^3} \nabla\phi(y)\tilde{\rho}^{(n)}(t,y)dy = \int_0^t \int_{\mathbb{R}^3} \nabla\left[\frac{\partial f}{\partial s}(s,y) + \nu\triangle f(s,y) + \mathbf{K}^{\varepsilon_n}(\mathbf{w})(s,y)\nabla f(s,y)\right]\tilde{\rho}^{(n)}(s,y)\,dy\,ds.$$

for all $\phi \in \mathcal{D}$. Thus, $div\ \tilde{\rho}^{(n)}(t) = 0$.

Next, we check that $\tilde{\rho}^{(n)}(t)$ belongs to L_3^p for all $t \in [0, T]$. This is enough to conclude that $\tilde{\rho}^{(n)}$ solves (53) since then, $\int_{\mathbb{R}^3} \nabla\phi(y)\tilde{\rho}^{(n)}(t, y)dy = 0$ for all $\phi \in W^{1,p^*}$, and then we just have to take $\phi = G^{\nu}_{t-s}(x - \cdot)\mathbf{K}^{\varepsilon_n}(\mathbf{w})(s, \cdot) \in W_3^{1,p^*}$.

Fix $m \in [1, \frac{3}{2}[$. Notice that $\|G^{\nu}_{t-s}\|_m$, $\|\nabla G^{\nu}_{t-s}\|_m \in L^1([0, t], ds)$, that $\mathbf{K}^{\varepsilon_n}(\mathbf{w})$ and $\nabla\mathbf{K}^{\varepsilon_n}(\mathbf{w})$ are bounded, and that $\mathbf{w}_0 \in \mathbf{F}_{0,m,T}$. With these facts and Young's inequality we deduce that $\mathbf{w} \in \mathbf{F}_{0,m,T}$. We can chose $m \in [1, \frac{3}{2}[$ such that $p < \frac{3m}{3-m}$, which ensures that r given by $r^{-1} = p^{-1} + 1 - m^{-1}$ belongs to $]1, \frac{3}{2}[$. Using again Young's inequality and that $\mathbf{w} \in \mathbf{F}_{0,m,T}$, we conclude that $\mathbf{w} \in \mathbf{F}_{0,p,T}$.

We next derive an upper bound for $|||\tilde{\rho}^{(n)}|||_{0,p,T}$ independent of n. By standard arguments (cf. Proposition 3.1 *i)*), Lemma 2.2 and Remark 4.3, we obtain

$$\|\tilde{\rho}^{(n)}(t)\|_p \leq |||\mathbf{w}_0|||_{0,p,T} + C|||\mathbf{w}|||_{0,p,T} \int_0^t (t-s)^{-\frac{3}{2p}} \|\tilde{\rho}^{(n)}(s)\|_p \, ds.$$

Iterating this inequality $N(p)$ times, with $N(p)$ the first integer N such that $2^N(1 - \frac{3}{2p}) > 0$, we deduce that

$$\|\tilde{\rho}^{(n)}(t)\|_p \leq C + C' \int_0^t \|\tilde{\rho}^{(n)}(s)\|_p \, ds,$$

with constants that are independent of n, and we conclude by Gronwall's lemma.

Starting now from the fact that $\rho^{(n)}$ solves the linear equation

$$\rho^{(n)}(t, x) = G_t^{\nu} * \rho_0(x) + \int_0^t \sum_{j=1}^{3} \int_{\mathbb{R}^3} \frac{\partial G^{\nu}_{t-s}}{\partial y_j}(x - y)\mathbf{K}^{\varepsilon_n}(\mathbf{w})_j(s, y)\rho^{(n)}(s, y) \, dy \, ds, \quad (55)$$

(which is seen as in Section **4.1**) we establish a uniform L^p bound for $\rho^{(n)}(t)$.

Now we prove the asserted convergence for $\tilde{\rho}^{(n)}$. By taking the L_3^p norm to the difference $\mathbf{w}(t) - \tilde{\rho}^{(n)}(t)$ and proceeding as above, we check that

$$\|\tilde{\rho}^{(n)}(t) - \mathbf{w}(t)\|_p \leq C \int_0^t (t-s)^{-\frac{3}{2p}} \|\mathbf{K}^{\varepsilon_n}(\mathbf{w})(s) - \mathbf{K}(\mathbf{w})(s)\|_q \, ds + C \int_0^t (t-s)^{-\frac{3}{2p}} \|\tilde{\rho}^{(n)}(s) - \mathbf{w}(s)\|_p \, ds.$$

We have also used here the estimates (50). Writing $\theta_0 = 1 - \frac{3}{2p}$ and using induction, we get

$$\|\tilde{\rho}^{(n)}(t) - \mathbf{w}(t)\|_p \leq C \int_0^t \sum_{k=1}^{N} (t-s)^{k\theta_0 - 1} \|\mathbf{K}^{\varepsilon_n}(\mathbf{w})(s) - \mathbf{K}(\mathbf{w})(s)\|_q \, ds + C \int_0^t (t-s)^{N\theta_0 - 1} \|\tilde{\rho}^{(n)}(s) - \mathbf{w}(s)\|_p \, ds$$

Thus, taking a fixed $N = \tilde{N}(p)$ such that $= \tilde{N}(p) > \theta_0^{-1}$, yields

$$\|\tilde{\rho}^{(n)}(t) - \mathbf{w}(t)\|_p \leq C \int_0^t \alpha(t-s)\ \|\mathbf{K}^{\varepsilon_n}(\mathbf{w})(s) - \mathbf{K}(\mathbf{w})(s)\|_q\ ds$$
$$+ C(T) \int_0^t \|\tilde{\rho}^{(n)}(s) - \mathbf{w}(s)\|_p\ ds, \qquad (56)$$

with $\alpha(s) = \sum_{k=1}^{\tilde{N}(p)} s^{k\theta_0 - 1}$. Integrating now between 0 and $\tau \in [0, T]$ gives

$$\int_0^\tau \|\tilde{\rho}^{(n)}(t) - \mathbf{w}(t)\|_p dt \leq C \int_0^T \int_0^t \alpha(t-s)\ \|\mathbf{K}^{\varepsilon_n}(\mathbf{w})(s) - \mathbf{K}(\mathbf{w})(s)\|_q\ ds\ dt$$
$$+ C \int_0^\tau \int_0^t \|\tilde{\rho}^{(n)}(s) - \mathbf{w}(s)\|_p\ ds\ dt,$$

and by Gronwall's lemma,

$$\int_0^\tau \|\tilde{\rho}^{(n)}(t) - \mathbf{w}(t)\|_p dt \leq C \int_0^T \int_0^t \alpha(t-s)\ \|\mathbf{K}^{\varepsilon_n}(\mathbf{w})(s) - \mathbf{K}(\mathbf{w})(s)\|_q\ ds\ dt.$$

Thanks to Remark 4.3, the right hand side converges to 0 by a double application of Lebesgue's theorem. Taking $\tau = T$ gives us convergence in $L^1([0,T], L_3^p)$. Convergence in L_3^p for all $t \in [0, T]$ follows then from (56).

Repeating this reasoning with the difference $\rho^{(m)}(t) - \rho^{(n)}(t)$, $n, m \in \mathbb{N}$, shows us that $\rho^{(n)}$ is Cauchy in $L^1([0,T], L^p)$, and that $\rho^{(n)}(t)$ is also Cauchy in L^p, $\forall t \in [0, T]$. Consequently, there is point-wise convergence of $\rho^{(n)}$ in L^p on the interval $[0, T]$ to a limit $\rho \in L^1([0,T], L^p)$. Estimate (50) implies that $\rho \in F_{0,p,T}$, and using the fact that

$$\left\| \int_0^t \sum_{j=1}^3 \int \frac{\partial G_{t-s}^\nu}{\partial y_j}(x-y)\ \left(\rho^{(n)}(s,y)\, \mathbf{K}^{\varepsilon_n}(\mathbf{w})(s,y) - \rho(s,y)\, \mathbf{K}(\mathbf{w})(s,y)\right)\ dy\ ds \right\|_p$$

is bounded above by $C \int_0^t (t-s)^{-\frac{3}{2p}} \big[\|\rho^{(n)}(s) - \rho(s)\|_p + \|\mathbf{K}^{\varepsilon_n}(\mathbf{w})(s) - \mathbf{K}(\mathbf{w})(s)\|_q\big] ds$ (which goes to 0 as $n \to \infty$), we pass to the limit on n in equation (55) to conclude that ρ solves (51). □

To prove tightness of the sequence $Q^{(n)}$ we will use next version of Gronwall's lemma.

Lemma 4.5. *Let g and k be positive functions on $[0, T]$, such that $\int_0^T k(s)ds < \infty$, g is bounded, and*

$$g(t) \leq C + \int_0^t g(s)k(s)ds \ \ \textit{for all } t \in [0, T].$$

Then, we have

$$g(t) \leq C \exp \int_0^T k(s)ds \ \textit{for all } t \in [0, T].$$

Lemma 4.6. *The sequence $(Q^{(n)}, n \in \mathbb{N})$ is tight.*

Proof. It is enough to prove that each of the two sequences of process $X^{(n)}$ and $\Phi^{(n)}$ have laws that are uniformly tight in n. We will use Aldous' criterion for both of them.

Let R_n, S_n be stopping times in the filtration of $(X^{(n)}, \Phi^{(n)})$ such that $0 \le R_n \le S_n \le T$ and $S_n - R_n \le \Delta$. Thanks to Remark 4.3, Lemma 2.2 *ii)*, Lemma 3.5, and the arguments of Corollary 3.1, we have

$$\begin{aligned}\int_{R_n}^{S_n} |\mathbf{K}^{(\varepsilon_n)}(\mathbf{w})(t, X_t^{(n)})|dt &\le C \int_{R_n}^{S_n} t^{-\frac{1}{2}} |\!|\!|\mathbf{K}^{(\varepsilon_n)}(\mathbf{w})|\!|\!|_{1,q,T} dt \\ &\le C \left(S_n^{\frac{1}{2}} - R_n^{\frac{1}{2}} \right) |\!|\!|\mathbf{w}|\!|\!|_{1,p,T} \le C \Delta^{\frac{1}{2}},\end{aligned}$$

and the criterion applies to $X^{(n)}$. Consider now the processes $\Phi^{(n)}$. Since $\nabla \mathbf{K}^{\varepsilon_n}(\mathbf{w})(t)$ is bounded, each process $\Phi^{(n)}$ is bounded on $[0, T]$ (by a constant depending on ε_n). On the other hand, by Remark 4.3, Lemmas 2.3 *ii)* and Lemma 3.5, we have

$$\left\| \frac{\partial \mathbf{K}^{(\varepsilon_n)}(\mathbf{w})(t)}{\partial x_k} \right\|_\infty \le C t^{-\frac{1}{2} - \frac{3}{2}(\frac{1}{p} - \frac{1}{r})} |\!|\!|\mathbf{w}|\!|\!|_{1,r,(T;p)} \tag{57}$$

for each $r \in]3, \frac{3p}{3-p}[$ and $k = 1, 2, 3$. From this and Lemma 4.5 we deduce that

$$|\Phi_t^{(n)}| \le \exp\left(C T^{\frac{1}{2} - \frac{3}{2}(\frac{1}{p} - \frac{1}{r})} |\!|\!|\mathbf{w}|\!|\!|_{1,r,(T;p)} \right) \tag{58}$$

for all $t \in [0, T]$ and a constant $C > 0$ which does not depend on n. Let now R_n, S_n be stopping times as before, and fix $r \in]3, \frac{3p}{3-p}[$. By using (57) and (58) we establish that

$$\begin{aligned}\int_{R_n}^{S_n} |\nabla \mathbf{K}^{(\varepsilon_n)}(\mathbf{w})(t, X_t^{(n)})||\Phi_t^{(n)}|dt &\le C \left(R_n^{\frac{1}{2} - \frac{3}{2}(\frac{1}{p} - \frac{1}{r})} - S_n^{\frac{1}{2} - \frac{3}{2}(\frac{1}{p} - \frac{1}{r})} \right) \\ &\le C \Delta^{\frac{1}{2} - \frac{3}{2}(\frac{1}{p} - \frac{1}{r})}\end{aligned}$$

for a constant $C > 0$ not depending on n, and the result follows since $\frac{1}{2} - \frac{3}{2}(\frac{1}{p} - \frac{1}{r}) > 0$. □

Remark 4.4. If $\rho \in F_{0,p,T}$ is the unique solution of (51) and P denotes the limit of a convergent subsequence of $Q^{(n)}$, we deduce from Lemma 4.4 that $\int_{\mathbb{R}^3} \psi(x) P_t^\circ(dx) = \int_{\mathbb{R}^3} \psi(x)\rho(t, x)dx$ for all $\psi \in \mathcal{D}$. Consequently, $\rho(t)$ is a probability density.

We can now prove

Proposition 4.2. *Every accumulation point of the sequence $Q^{(n)}$ is a solution of the martingale problem (37) in the class $\mathcal{P}^T_{b,\frac{3}{2}}$.*

Proof. Let P denote the limit of a convergent subsequence renamed $Q^{(n)}$. We take $f \in C_b^{1,2}$, $0 \leq s_1 \leq \cdots \leq s_m \leq s < t \leq T$ and $\lambda : \mathbb{R}^m \to \mathbb{R}$ a continuous bounded function. We will first show that

$$E^P\Bigg[\Bigg(\int_s^t \left\{\frac{\partial f}{\partial \tau}(\tau, X_\tau) + \nu\Delta f(\tau, X_\tau) + \mathbf{K}(\mathbf{w})(\tau, X_\tau)\nabla f(\tau, X_\tau)\right\} d\tau$$

$$+ f(t, X_t) - f(s, X_s)\Bigg) \times \lambda(X_{s_1}, \ldots, X_{s_m})\Bigg] = 0, \tag{59}$$

and that

$$E^P\left[\left|\Phi_t - Id - \int_0^t \nabla\mathbf{K}(\mathbf{w})(\tau, X_\tau)\Phi_\tau d\tau\right|\right] = 0, \tag{60}$$

with (X, Φ) the canonical process and $\mathbf{w} \in \mathbf{F}_{0,p,T}$ the solution of (10) we are given. Notice that the result will follow from (59) and (60) by proving that the density family $\tilde{\rho}$ of $\tilde{P}$ is equal to $\mathbf{w}$.

Define a function $\kappa : C([0, T], \mathbb{R}) \to \mathbb{R}$ by

$$\kappa(\xi) = \Bigg(\int_s^t \left\{\frac{\partial f}{\partial \tau}(\tau, \xi(\tau)) + \nu\Delta f(\tau, \xi(\tau)) + \mathbf{K}(\mathbf{w})(\tau, \xi(\tau))\nabla f(\tau, \xi(\tau))\right\} d\tau$$

$$+ f(t, \xi(t)) - f(s, \xi(s))\Bigg) \times \lambda(\xi(s_1), \ldots, \xi(s_m)) \tag{61}$$

We now check that it is continuous and bounded. From Corollary 3.1 *i)* we see that

$$\int_s^t |\mathbf{K}(\mathbf{w})(\tau, \xi(\tau))\nabla f(\tau, \xi(\tau))|\, d\tau \leq C(T)\|\nabla f\|_\infty |||\mathbf{w}|||_{0,p,T} \text{ and}$$

$$|\mathbf{K}(\mathbf{w})(\tau, x) - \mathbf{K}(\mathbf{w})(\tau, y)| \leq C\tau^{-\frac{1}{2}}|x - y|^{\frac{2p-3}{p}} |||\mathbf{w}|||_{0,p,T}, \ \forall x, y \in \mathbb{R}^3.$$

Thus,

$$\int_s^t |\mathbf{K}(\mathbf{w})(\tau, \xi_1(\tau))\nabla f(\tau, \xi_1(\tau)) - \mathbf{K}(\mathbf{w})(\tau, \xi_2(\tau))\nabla f(\tau, \xi_2(\tau))|\, d\tau$$

$$\leq C(T)\|\nabla f\|_\infty \|\xi_1 - \xi_2\|_\infty^{\frac{2p-3}{p}} |||\mathbf{w}|||_{0,p,T}$$

$$+ C'(T)\|\nabla(\nabla f)\|_\infty \|\xi_1 - \xi_2\|_\infty |||\mathbf{w}|||_{0,p,T},$$

for all $\xi_1, \xi_2 \in C([0, T], \mathbb{R})$. It follows that the mapping $\xi \mapsto \int_s^t \mathbf{K}(\mathbf{w})(\tau, \xi(\tau))\nabla f(\tau, \xi(\tau))\, d\tau$ is continuous and bounded on $C([0, T], \mathbb{R})$, and then the same holds for κ.

Therefore, we have $E^{Q^{(n)}}(\kappa(X)) \to E^P(\kappa(X))$ as $n \to \infty$. Now, from (47) and the definition of $Q^{(n)}$, it follows that

$$E^{Q^{(n)}}\Bigg[\Bigg(\int_s^t \left\{\frac{\partial f}{\partial \tau}(\tau, X_\tau) + \nu\Delta f(\tau, X_\tau) + \mathbf{K}^{\varepsilon_n}(\mathbf{w})(\tau, X_\tau)\nabla f(\tau, X_\tau)\right\} d\tau$$

$$+ f(t, X_t) - f(s, X_s)\Bigg) \times \lambda(X_{s_1}, \ldots, X_{s_m})\Bigg] = 0,$$

and then

$$E^{Q^{(n)}}(\kappa(X)) = E^{Q^{(n)}}\Big[\int_s^t (\mathbf{K}(\mathbf{w})(\tau, X_\tau)\nabla f(\tau, X_\tau) \\ -\mathbf{K}^{(\varepsilon_n)}(\mathbf{w})(\tau, X_\tau)\nabla f(\tau, X_\tau)\Big) d\tau \times \lambda(X_{s_1}, \dots, X_{s_m})\Big].$$

We prove that the latter goes to 0 with n. If $q = \frac{3p}{3-p}$ then we have $q* = \frac{3p}{4p-3} < \frac{3}{2} < p$ and

$$\sup_{k\in\mathbb{N}} \|\!|\rho^{(k)}|\!\|_{0,q*,T} < \infty$$

from estimate (50) and since $\rho^{(n)}(t)$ is a probability density. It follows that

$$\Big|E^{Q^{(n)}}(\kappa(X))\Big| \le CE^{Q^{(n)}}\Big[\int_s^t \Big|\mathbf{K}^{(\varepsilon_n)}(\mathbf{w})(\tau, X_\tau) - \mathbf{K}(\mathbf{w})(\tau, X_\tau)\Big| d\tau\Big] \\ \le C \sup_{k\in\mathbb{N}} \|\!|\rho^{(k)}|\!\|_{0,q*,T} \int_0^T \Big\|\mathbf{K}^{(\varepsilon_n)}(\mathbf{w})(\tau) - \mathbf{K}(\mathbf{w})(\tau)\Big\|_q d\tau,$$

and by Remark 4.3, we conclude that $E^{Q^{(n)}}(\kappa(X)) \to 0$. This proves (59).

We next prove (60). Consider an arbitrary continuous truncation function on matrices $\chi_R : \mathcal{M}_{3\times3} \to \mathcal{M}_{3\times3}$, with $R > 0$, such that $|\chi_R(z)| \le R$ for all $z \in \mathcal{M}_{3\times3}$.

By (58) there exists a constant $R = R_{\mathbf{w}}$ independent of n such that $\sup_{t\in[0,T]} |\Phi^{(n)}| \le R_{\mathbf{w}}$ for all $n \in \mathbb{N}$. We will check that the function $\zeta : C([0,T], \mathbb{R}^3) \times C([0,T], \mathcal{M}_{3\times3}) \to \mathbb{R}$ given by

$$\zeta(\xi, z) := \left|\chi_{R_{\mathbf{w}}}(z_t) - Id - \int_0^t \nabla\mathbf{K}(\mathbf{w})(\tau, \xi(\tau))\chi_{R_{\mathbf{w}}}(z_\tau)d\tau\right| \tag{62}$$

is bounded and continuous. To that end, it is enough to state that the mapping $(\xi, z) \mapsto \int_0^t \nabla\mathbf{K}(\mathbf{w})\,(\tau, \xi(\tau))\chi_{R_{\mathbf{w}}}(z(\tau))d\tau$ is bounded and continuous. Boundedness is consequence of (57). Continuity follows easily from the estimate

$$|\nabla\mathbf{K}(\mathbf{w})(\tau, x) - \nabla\mathbf{K}(\mathbf{w})(\tau, y)| \le C\|\!|\mathbf{w}|\!\|_{1,r,(T;p)}\tau^{-\frac{1}{2}-\frac{3}{2}(\frac{1}{p}-\frac{1}{r})}|x-y|^{1-\frac{3}{r}}, \forall x, y \in \mathbb{R}^3.$$

for any fixed $r \in]3, \frac{3p}{3-p}[$ (see the proof of Corollary 3.1). Thus, proving (60) amounts to check that

$$E^{Q^{(n)}}(\zeta(\xi, z)) \to 0 \tag{63}$$

when $n \to \infty$. Since

$$E^{Q^{(n)}}\left|\chi_{R_{\mathbf{w}}}(z_t) - Id - \int_0^t \nabla\mathbf{K}^{\varepsilon_n}(\mathbf{w})(\tau, \xi(\tau))\chi_{R_{\mathbf{w}}}(z_\tau)d\tau\right| = 0$$

by (47), we have

$$E^{Q^{(n)}}(\zeta(X,\Phi)) \leq R_{\mathbf{w}} E^{Q^{(n)}}\left[\int_s^t \left|\nabla \mathbf{K}^{(\varepsilon_n)}(\mathbf{w})(\tau,X_\tau) - \nabla \mathbf{K}(\mathbf{w})(\tau,X_\tau)\right| d\tau\right]. \tag{64}$$

If $p \geq 2$, the r.h.s. of (64) is bounded above by

$$C \sup_{k\in\mathbb{N}} \|\!|\rho^{(k)}\|\!|_{0,p*,T} \int_0^T \left\|\nabla \mathbf{K}^{(\varepsilon_n)}(\mathbf{w})(\tau) - \nabla \mathbf{K}(\mathbf{w})(\tau)\right\|_p d\tau.$$

The supremum is finite from (50) since $p^* \leq 2$. (63) follows then from Remark 4.3.

If $\frac{3}{2} < p < 2$ we bound the r.h.s. of (64) above by

$$C \sup_{k\in\mathbb{N}} \|\!|\rho^{(k)}\|\!|_{0,p*,(T;p)} \int_0^T t^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{p*})} \left\|\nabla \mathbf{K}^{(\varepsilon_n)}(\mathbf{w})(\tau) - \nabla \mathbf{K}(\mathbf{w})(\tau)\right\|_p d\tau.$$

Since $-\frac{3}{2}(\frac{1}{p} - \frac{1}{p*}) > -1$, using Remark 4.3 we just have to establish that

$$\sup_{k\in\mathbb{N}} \|\!|\rho^{(k)}\|\!|_{0,p*,(T;p)} < \infty$$

to conclude (63). Recall that by Theorem 3.2 *iii)* we have $\mathbf{w} \in F_{0,r,(T;p)}$ for all $r \geq p$. We deduce that the the norm of the linear functional

$$\eta(t,x) \mapsto \int_0^t \sum_{j=1}^3 \int_{\mathbb{R}^3} \frac{\partial G^\nu_{t-s}}{\partial y_j}(x-y)\mathbf{K}^{\varepsilon_n}(\mathbf{w})_j(s,y)\eta(s,y)\,dy\,ds,$$

defined from $F_{0,r,(T;p)}$ to $F_{0,r',(T;p)}$ for $p \leq r < 3$ and $r \leq r' < \frac{3p}{6-2p}$, can be estimated in terms of $\|\!|\mathbf{K}^{\varepsilon_n}(\mathbf{w})\|\!|_{0,l,(T;q)}$ with $l = \frac{3r}{3-r}$, and thus in terms of $\|\!|\mathbf{w}\|\!|_{0,r,(T;p)}$ only (by Lemma 2.2 and Remark 4.3). In particular, there is no dependence on n. Therefore, by an iterative argument as in Theorem 3.2 *iii)*, we obtain an upper bound for $\|\!|\tilde{\rho}^{(n)}\|\!|_{0,p^*,(T;p)}$ not depending on n. We conclude (60).

To finish the proof, we just have to check that for each $t \in [0,T]$, the function $\tilde{\rho}(t)$ given by

$$\int_{\mathbb{R}^3} \mathbf{f}(x)\tilde{\rho}(t,x)dx := E^P(\mathbf{f}(X_t)\Phi_t h_0(X_0)),$$

for all $\mathbf{f} \in \mathcal{D}_3$, is equal to $\mathbf{w}(t)$. Thanks to the convergence $\tilde{\rho}^{(n)} \to \mathbf{w}$ stated in Lemma 4.4, this will hold as soon as the convergence

$$E^{Q^{(n)}}(\mathbf{f}(X_t)\Phi_t h_0(X_0)) \to E^P(\mathbf{f}(X_t)\Phi_t h_0(X_0))$$

is proved. The function h_0 is not necessarily continuous. We will use fact (proved in [18]) that for every $k \in \mathbb{N}$, one can find a continuous bounded function h_0^k such that $\frac{|w_0|}{\|w_0\|_1}(\{h_0^k \neq h_0\}) \leq \frac{1}{k}$, and $|h_0^k| \leq |h_0|$. We also notice that under P, the process

Φ is bounded by the same constant $R_{\mathbf{w}}$ as it is under each law $Q^{(n)}$. Hence, for any $k \in \mathbb{N}$,

$$
\begin{aligned}
&|E^{Q^{(n)}}(\mathbf{f}(X_t)\Phi_t h_0(X_0)) - E^P(\mathbf{f}(X_t)\Phi_t h_0(X_0))| \\
&\quad \leq C(E^{Q^{(n)}}|h_0^k(X_0) - h_0(X_0)| + E^P|h_0^k(X_0) - h_0(X_0)|) \\
&\qquad + |E^{Q^{(n)}}(\mathbf{f}(X_t)\chi_{R_{\mathbf{w}}}(\Phi_t)h_0^k(X_0)) - E^P(\mathbf{f}(X_t)\chi_{R_{\mathbf{w}}}(\Phi_t)h_0^k(X_0))|.
\end{aligned}
$$

We conclude by taking lim sup as $n \to \infty$ and then limit as $k \to \infty$. □

4.3. Uniqueness

Proposition 4.3. *i) If $P \in \mathcal{P}^T_{b,\frac{3}{2}}$ is a solution of (37) with $\rho \in F_{0,p,T}$ and $p \in]\frac{3}{2}, 3[$, then $\mathbf{w} := \tilde{\rho}$ is a solution of the mild vortex equation (10) in the space $\mathbf{F}_{0,p,T}$.*
ii) We deduce that uniqueness holds for (37) in the class $\mathcal{P}^T_{b,\frac{3}{2}}$.

Proof. *i)* Let $P \in \mathcal{P}^T_{b,\frac{3}{2}}$ be a solution of (37). By Proposition 4.1 and Corollary 4.2, $\tilde{\rho}$ is a weak solution in the spaces $\mathbf{F}_{0,p,T} \cap \mathbf{F}_{0,r,(T;p)}$ for all $r \in [p, \infty[$. As in Corollary 4.2, conditions (38) and (44) can be seen to hold, and then it is not hard to check that $\tilde{\rho}$ satisfies the assumptions of Lemma 2.1. Thus, it solves the intermediate mild equation (12). To conclude we need to verify that

$$
\begin{aligned}
\sum_{j=1}^{3} \int_{\mathbb{R}^3} & G^{\nu}_{t-s}(x-y)[\tilde{\rho}_j(s,y)\frac{\partial \mathbf{K}(\tilde{\rho})}{\partial y_j}(s,y)] \\
&+ \frac{\partial G^{\nu}_{t-s}}{\partial y_j}(x-y)[\tilde{\rho}_j(s,y)\mathbf{K}(\tilde{\rho})(s,y)]dy = 0
\end{aligned}
$$

for all $s \in [0,T]$. Since $1 < q* < \frac{3}{2}$ (where $q = \frac{3p}{3-p}$), the function $\tilde{\rho} = h^P \rho$ belongs to $\mathbf{F}_{0,q*,T}$. From the fact that $div\ \tilde{\rho}(s) = 0$ by hypothesis, and that $G^{\nu}_{t-s}(x-\cdot)\mathbf{K}(\tilde{\rho})(s,\cdot) \in W^{1,q}_3$ thanks to Proposition 4.1 *i)*, we obtain the required identity and this proves *i)*.

ii) Assume that P^1 and P^2 are two solutions of (37) in $\mathcal{P}^T_{b,\frac{3}{2}}$, with density families $\rho^1 \in F_{0,p^1,T}$ and $\rho^2 \in F_{0,p^2,T}$ respectively, and such that $p^1, p^2 > \frac{3}{2}$. Then, $\tilde{\rho}^1, \tilde{\rho}^2$ belong to $\mathbf{F}_{0,p,T}$ and are mild solutions of the vortex equation, from where $\tilde{\rho}^1 = \tilde{\rho}^2$ by Theorem 3.1. We write $\mathbf{w} = \tilde{\rho}^1 = \tilde{\rho}^2$.

By arguments in the proof of Proposition 4.1, ρ^1 and ρ^2 solve equation (40) with $h = h^{P^1}$ and $h = h^{P^2}$ respectively. By *i)* we have $h^{P^1}\rho^1 = h^{P^2}\rho^2 = \mathbf{w}$ and so ρ^1 and ρ^2 solve the linear equation

$$
\begin{aligned}
\rho(t,x) &= G^{\nu}_t * \rho_0(x) \\
&\quad + \int_0^t \sum_{j=1}^{3} \int_{\mathbb{R}^3} \frac{\partial G^{\nu}_{t-s}}{\partial y_j}(x-y)\mathbf{K}(\mathbf{w})_j(s,y)\rho(s,y)\,dy\,ds, \qquad (65)
\end{aligned}
$$

in $F_{0,p,T}$. Thus, they are equal, and we write $\rho = \rho^1 = \rho^2$.

We have established that P^1 and P^2 solve a linear martingale problem in $\mathcal{P}^T_{b,\frac{3}{2}}$:

- $Q^\circ|_{t=0}(dx) = \frac{|w_0(x)|}{\|w_0\|_1}dx$ and for all $0 \le t \le T$ and $Q^\circ_t(dx) = \rho_t(x)dx$. is known
- $f(t, X_t) - f(0, X_0) - \int_0^t \Big[\frac{\partial f}{\partial s}(s, X_s) + \nu \triangle f(s, X_s) + \mathbf{K}(\mathbf{w})(s, X_s)\nabla f(s, X_s)\Big]ds, \quad (66)$

  $0 \le t \le T,$ is a continuous Q°-martingale for all $f \in \mathcal{C}_b^{1,2}$;
- $\Phi_t = Id + \int_0^t \nabla\mathbf{K}(\mathbf{w})(s, X_s)\Phi_s\, ds,$ for all $0 \le t \le T,$ Q almost surely.

We can now follow arguments of [27] or [14] to prove the fact that $(P^1)^\circ = (P^2)^\circ$. Indeed, by Corollary 3.1 *i)*, the coefficient $\mathbf{K}(\mathbf{w})$ in (66) satisfies $|\mathbf{K}(\mathbf{w})(t)| \le Ct^{-\frac{1}{2}}$. Consequently, if Q is a solution of (66), and if D_n with $n \in \mathbb{N}$ denotes the shift operator on $C([0, T], \mathbb{R}^3)$ defined by $D_n(\xi) = \xi(\frac{1}{n} + \cdot)$, then the probability measure $Q^\circ \circ D_n^{-1}$ solves a martingale problem with bounded coefficients, and with a fixed initial law given by $\rho(\frac{1}{n}, x)dx$. By classic results of Stroock and Varadhan [32], $Q^\circ \circ D_n^{-1}$ is uniquely determined, and thus $(P^1)^\circ \circ D_n^{-1} = (P^2)^\circ \circ D_n^{-1}$ for all $n \in \mathbb{N}$. By letting $n \to \infty$ we conclude that $(P^1)^\circ = (P^2)^\circ$

It remains to prove that $(P^1)' = (P^2)'$. In virtue of the L^∞ estimates in Corollary 3.1 *ii)*, it is an elementary fact that for each $\xi \in C([0, T], \mathbb{R}^3)$ the O.D.E.

$$z(t) = Id + \int_0^t \nabla\mathbf{K}(\mathbf{w})(s, \xi(s))z(s)ds$$

has a unique continuous solution $t \in [0, T] \mapsto z(t) \in \mathcal{M}_{3\times3}$. Furthermore, using the estimate in Hölder norm of Corollary 3.1 *ii)*, and Gronwall's lemma, it is easily seen that the mapping $\xi \mapsto z$ is continuous. This clearly implies that $(P^1)' = (P^2)'$, and the proof is finished. □

4.4. Strong statements and stochastic flow

Corollary 4.3. *Let $p \in]\frac{3}{2}, 3[$ and $\mathbf{w} \in \mathbf{F}_{0,p,T}$ be a solution of equation (10) with $w_0 \in L^1_3$.*

a) There is strong existence and uniqueness for the stochastic differential equation

$$\begin{aligned} \overline{X}_t &= \overline{X}_0 + \sqrt{2\nu}B_t + \int_0^t \mathbf{K}(\mathbf{w})(s, \overline{X}_s)ds \\ \overline{\Phi}_t &= Id + \int_0^t \nabla\mathbf{K}(\mathbf{w})(s, \overline{X}_s)\overline{\Phi}_s ds \qquad t \in [0, T] \qquad (67) \\ & law(\overline{X}_0) = \frac{|w_0(x)|}{\|w_0\|_1}dx \end{aligned}$$

(linear in the sense of McKean), and one has

$$law((\overline{X}, \overline{\Phi})) = P,$$

the unique solution in $\mathcal{P}^T_{b,\frac{3}{2}}$ *of the nonlinear martingale problem (37) such that* $\tilde{\rho} = \mathbf{w}$.

b) The family of SDE's

$$\xi_{s,t}(x) = x + \sqrt{2\nu}B_t + \int_s^t \mathbf{K}(\mathbf{w})(t, \xi_{s,r}(x))dr, \quad t \in [s, T] \tag{68}$$

with $x \in \mathbb{R}^3$ *and* $s \in [0, T]$*, defines a* C^1*−stochastic flow* ξ*, and one has*

$$(\overline{X}, \overline{\Phi}) = (\xi_{0,\cdot}(X_0), \nabla\xi_{0,\cdot}(X_0)).$$

Proof. *a)* By Theorem 4.1 and Proposition 5.4.11 in [20], there exists in some probability space a weak solution (X, Φ) of the SDE (67). Now, if two solutions $(\overline{X}, \overline{\Phi})$ and $(\overline{Y}, \overline{\Psi})$ of (67) are given in some fixed probability space, then

$$|\overline{X}_t - \overline{Y}_t| \le C \int_0^t s^{\frac{-1}{2} - \frac{3}{2}(\frac{1}{p} - \frac{1}{r})} |\overline{X}_s - \overline{Y}_s| ds$$

for some $r \in]3, \frac{3p}{3-p}[$ and all $t \in [0, T]$ by Corollary 3.1 *ii)*, and we conclude that $\overline{X} = \overline{Y}$ by Lemma 4.5. The fact that $\overline{\Phi} = \overline{\Psi}$ follows as in the last part of Proposition 4.3. Thus, trajectorial uniqueness holds for (67) which yields the result.

b) By Corollary 3.1 and the results in [21], the stochastic flow (68) is well defined for $s \in [0, T]$ and is of class C^1 in x for all $s, t \in]0, T], s < t$. We just have to check that $\xi_{0,t} : \mathbb{R}^3 \to \mathbb{R}^3$ is also C^1. By Lemma 4.5 and similar arguments as in *a)*, the function $\xi_{0,t}$ is globally Lipschitz continuous (independently of the randomness and of $t \in [0, T]$), and the quotients $\delta_t(x, y) := \frac{1}{|x-y|}|\xi_{0,t}(x) - \xi_{0,t}(y)|$ are bounded. With this and the relation

$$\begin{aligned} \xi_{0,t}(y) - \xi_{0,t}(x) = y - x + \int_0^t \int_0^1 \Big[\nabla\mathbf{u}\big(s, \xi_{0,s}(x) + \theta(\xi_{0,s}(y) - \xi_{0,s}(x))\big) \\ -\nabla\mathbf{u}\big(s, \xi_{0,s}(x)\big)\Big] d\theta \cdot \big(\xi_{0,s}(y) - \xi_{0,s}(x)\big) ds \\ + \int_0^t \nabla\mathbf{u}\big(s, \xi_{0,s}(x)\big)\big(\xi_{0,s}(y) - \xi_{0,s}(x)\big) ds \end{aligned}$$

we deduce for a fixed $r \in]3, \frac{3p}{3-p}[$ that

$$\begin{aligned} |\delta_t(x, y) - \delta_t(x, y')| \le C \int_0^t s^{-\frac{1}{2} - \frac{3}{2}(\frac{1}{p} - \frac{1}{r})} \\ \Big[|y - y'|^{1-\frac{3}{r}} + |\delta_s(x, y) - \delta_s(x, y')| \Big] ds \end{aligned}$$

for all $x, y, y' \in \mathbb{R}^3$ thanks also to Corollary 3.1 *ii)*. By Lemma 4.5,

$$|\delta_t(x, y) - \delta_t(x, y')| \le C(T)|y - y'|^{1-\frac{3}{r}}$$

for an absolute constant $C(T) > 0$, and therefore

$$|\delta_t(x, y) - \delta_t(x', y')| \le C\left[|x - x'|^{1-\frac{3}{r}} + |y - y'|^{1-\frac{3}{r}}\right]$$

for all x, x', y, y', which easily yields the conclusion. □

5. A cutoffed and mollified mean field model for the vortex equation

This section provides the theoretical framework to construct pathwise stochastic approximations of the vortex equation (2).

5.1. A generalized McKean-Vlasov equation

Consider a filtered probability space endowed with an adapted standard 3-dimensional Brownian motion B and with a $\mathbb{R}^3$-valued random variable X_0 independent of B. Let $R > 0$ and $\chi_R : \mathcal{M}_{3\times 3} \to \mathcal{M}_{3\times 3}$ be a Lipschitz continuous truncation function such that $|\chi_R(\phi)| \le R$, and let K_ε be the function defined in Section **4.2**. We will study the following system of nonlinear stochastic differential equations of the McKean-Vlasov type:

$$\begin{aligned} X_t^{\varepsilon,R} &= X_0 + \sqrt{2\nu}B_t + \int_0^t \mathbf{u}^{\varepsilon,R}(s, X_s^{\varepsilon,R})ds \\ \Phi_t^{\varepsilon,R} &= Id + \int_0^t \nabla\mathbf{u}^{\varepsilon,R}(s, X_s^{\varepsilon,R})\chi_R(\Phi_s^{\varepsilon,R})ds \end{aligned} \tag{69}$$

with

$$\mathbf{u}^{\varepsilon,R}(s, x) = E\left[K_\varepsilon(x - X_s^{\varepsilon,R}) \wedge \chi_R(\Phi_s^{\varepsilon,R})h_0(X_0)\right] \tag{70}$$

Theorem 5.1. *There is existence and uniqueness (trajectorial and in law) for (69), (70).*

Proof. The proof is adapted from Theorem 1.1 in [33], so we will skip details. Consider the closed subspace $\mathcal{P}(\mathcal{C}_T^0)$ of $\mathcal{P}(\mathcal{C}_T)$ of probability measures Q such that $Q|_{t=0} = law(X_0) \otimes \delta_{Id}$. We define a mapping $\Xi : \mathcal{P}(\mathcal{C}_T^0) \to \mathcal{P}(\mathcal{C}_T^0)$ associating to Q the law $\Xi(Q)$ of the solution of

$$\begin{aligned} X_t^Q &= X_0 + \sqrt{2\nu}B_t + \textstyle\int_0^t \mathbf{u}_Q(s, X_s^Q)ds \\ \Phi_t^Q &= Id + \textstyle\int_0^t \nabla\mathbf{u}_Q(s, X_s^Q)\chi_R(\Phi_s^Q)ds, \end{aligned} \tag{71}$$

where

$$\mathbf{u}_Q(s, x) = E^Q\left[K_\varepsilon(x - X_s) \wedge \chi_R(\Phi_s)h_0(X_0)\right] \tag{72}$$

The coefficients in equation (71) are Lipschitz continuous and bounded functions, and so Ξ is well defined (path-wise). Also by Lipschitz continuity, we just have to prove existence and uniqueness in law for (69), (70), which is equivalent to existence of a unique fixed point for Ξ. The Kantorovitch-Rubinstein (or Vaserstein) distance

$$D_T(Q^1, Q^2) := \inf \left\{ \int_{\mathcal{C}_T^2} \sup_{0\le t\le T} \left[\min\{|x(t)-y(t)|,1\} + \min\{|\phi(t)-\psi(t)|,1\} \right] \Pi(dx,d\phi,dy,d\psi),\ \Pi \text{ has marginals } Q^1 \text{ and } Q^2 \right\}, \tag{73}$$

induces on $\mathcal{P}(\mathcal{C}_T^0)$ the usual weak topology. The required fixed point result can be deduced in a standard way from the following inequality: for all $t \le T$ and $Q^1, Q^2 \in \mathcal{P}(\mathcal{C}_T^0)$,

$$D_t(\Xi(Q^1), \Xi(Q^2)) \le C_T \int_0^t D_s(Q^1, Q^2) ds, \tag{74}$$

with C_T a positive constant, and $D_t(Q^1, Q^2)$ the distance between the projections of Q^1 and Q^2 on $C([0,t], \mathbb{R}^3 \times \mathcal{M}_{3\times 3})$. To prove (74), consider for each $i = 1, 2$ processes (X^i, Φ^i) defined in terms of Q^i as in (71),(72). Take on a different probability space (Ω', P') a coupling $(Y^i, \Psi^i)_{i=1,2}$ of two processes such that $law(Y^i, \Psi^i) = Q^i$. Then,

$$\begin{aligned}
|X_t^1 - X_t^2| &\le \int_0^t |\mathbf{u}_{Q^1}(s, X_s^1) - \mathbf{u}_{Q^1}(s, X_s^2)| + |\mathbf{u}_{Q^1}(s, X_s^2) - \mathbf{u}_{Q^2}(s, X_s^2)| ds \\
&\le \int_0^t \left| E'\left[\left(K_\varepsilon(X_s^1 - Y_s^1) - K_\varepsilon(X_s^2 - Y_s^1) \right) \wedge \chi_R(\Psi_s^1) h_0(X_0) \right] \right| ds \\
&\quad + \int_0^t \left| E'\left[K_\varepsilon(X_s^2 - Y_s^1) \wedge \left(\chi_R(\Psi_s^1) - \chi_R(\Psi_s^2) \right) h_0(X_0) \right] \right| ds \\
&\quad + \int_0^t \left| E'\left[\left(K_\varepsilon(X_s^2 - Y_s^1) - K_\varepsilon(X_s^2 - Y_s^2) \right) \wedge \chi_R(\Psi_s^2) h_0(X_0) \right] \right| ds \\
&\le C \int_0^t \min\{|X_s^1 - X_s^2|, 1\} ds \\
&\quad + C \int_0^t E'\left[\min\{|Y_s^1 - Y_s^2|, 1\} + \min\{\left|\Psi_s^1 - \Psi_s^2\right|, 1\} \right] ds.
\end{aligned}$$

On the other hand, the processes Φ^i, with $i = 1, 2$, are bounded on $[0, T]$:

$$\sup_{t\in[0,T]} \left| \Phi_t^i \right| \le 1 + L_\varepsilon R \|h_0\|_\infty T, \tag{75}$$

with L_ε a Lipschitz constant of K_ε. Thus,

$$
\begin{aligned}
\left|\Phi_t^1 - \Phi_t^2\right| \le & \int_0^t \left|\left(\nabla \mathbf{u}_{Q^1}(s, X_s^1) - \nabla \mathbf{u}_{Q^1}(s, X_s^2)\right) \Phi_s^1\right| + \left|\nabla \mathbf{u}_{Q^2}(s, X_s^2)\left(\Phi_s^1 - \Phi_s^2\right)\right| ds \\
& \le C\left[\int_0^t \left|\nabla \mathbf{u}_{Q_1}(s, X_s^1) - \nabla \mathbf{u}_{Q_1}(s, X_s^2)\right| ds + \int_0^t \min\{\left|\Phi_s^1 - \Phi_s^2\right|, 1\} ds\right] \\
& \le C\Bigg[\int_0^t \min\{|X_s^1 - X_s^2|, 1\} + \min\{\left|\Phi_s^1 - \Phi_s^2\right|, 1\} ds \\
& \quad + \int_0^t E'\left[\min\{|Y_s^1 - Y_s^2|, 1\} + \min\{\left|\Psi_s^1 - \Psi_s^2\right|, 1\}\right] ds\Bigg].
\end{aligned}
$$

The conclusion follows with help of Gronwall's lemma. □

5.2. *Propagation of chaos*

Consider now a probability space endowed with a sequence $(B^i)_{i\in\mathbb{N}}$ of independent 3-dimensional Brownian motions, and a sequence of independent random variables $(X_0^i)_{i\in\mathbb{N}}$ with same law as X_0 and independent of the Brownian motions. For each $n \in \mathbb{N}$ and $R, \varepsilon > 0$, we define the following system of interacting particles:

$$
\begin{aligned}
X_t^{i,\varepsilon,R,n} &= X_0^i + \sqrt{2\nu} B_t^i \\
& \quad + \int_0^t \frac{1}{n} \sum_{j\neq i} K_\varepsilon(X_s^{i,\varepsilon,R,n} - X_s^{j,\varepsilon,R,n}) \wedge \chi_R(\Phi_s^{j,\varepsilon,R,n}) h_0(X_0^j) ds \\
\Phi_t^{i,\varepsilon,R,n} &= Id + \int_0^t \frac{1}{n} \sum_{j\neq i} \left[\nabla K_\varepsilon(X_s^{i,\varepsilon,R,n} - X_s^{j,\varepsilon,R,n}) \wedge \chi_R(\Phi_s^{j,\varepsilon,R,n}) h_0(X_0^j)\right] \\
& \quad \chi_R(\Phi_s^{i,\varepsilon,R,n}) ds,
\end{aligned}
\tag{76}
$$

for $i = 1 \ldots n$, and with $\nabla K(y) \wedge z = \nabla_y (K(y) \wedge z)$ for $y, z \in \mathbb{R}^3$, $y \neq 0$. Notice that the coefficients in the system of SDE's (76) are globally Lipschitz continuous and bounded, so that there is a unique strong solution. We also consider in the same probability space the sequence

$$
\left.
\begin{aligned}
X_t^{i,\varepsilon,R} &= X_0^i + \sqrt{2\nu} B_t^i + \int_0^t \mathbf{u}^{\varepsilon,R}(s, X_s^{i,\varepsilon,R}) ds \\
\Phi_t^{i,\varepsilon,R} &= Id + \int_0^t \nabla \mathbf{u}^{\varepsilon,R}(s, X_s^{i,\varepsilon,R}) \chi_R(\Phi_s^{i,\varepsilon,R}) ds
\end{aligned}
\right., \qquad i \in \mathbb{N}
\tag{77}
$$

of independent copies of (69). Their common law is denoted by $P^{\varepsilon,R}$, and $\bar{h}$, M_ε, L_ε and J_ε denote positive constants such that for all $x, y \in \mathbb{R}$,

- $|h_0(x)| \le \bar{h}$, and
- $|K_\varepsilon(x)| \le M_\varepsilon$, $|K_\varepsilon(x) - K_\varepsilon(y)| \le L_\varepsilon |x - y|$, $|\nabla K_\varepsilon(x) - \nabla K_\varepsilon(y)| \le J_\varepsilon |x - y|$.

Recall that $|\chi_R(\phi)| \le R$ for all $\phi \in \mathcal{M}_{3\times 3}$, and that . We may and shall assume that χ_R is a Lipschitz-continuous function, with Lipschitz constant equal to 1:

$$
|\chi_R(\phi) - \chi_R(\psi)| \le |\phi - \psi| \text{ for all } \phi, \psi \in \mathcal{M}_{3\times 3}.
$$

Theorem 5.2. *For $\varepsilon > 0$ sufficiently small and all $R > 0$, we have*

$$E\left[\sup_{t\in[0,T]}\left\{|X_t^{i,\varepsilon,R,n} - X_t^{i,\varepsilon,R}| + |\Phi_t^{i,\varepsilon,R,n} - \Phi_t^{i,\varepsilon,R}|\right\}\right] \leq \frac{1}{\sqrt{n}} C(\varepsilon, R, \bar{h}, T) \tag{78}$$

for all $i \leq n$, where

$$C(\varepsilon, R, \bar{h}, T) = C_1\varepsilon(1 + R\bar{h}T)(R\bar{h}T)\exp\{C_2\varepsilon^{-9}\bar{h}T(R+1)(\bar{h}+RT)\}$$

for some positive constants C_1, C_2 independent of R, ε,T and $\bar{h}$. We deduce that the system (76) is chaotic with limiting law $P^{\varepsilon,R} \in \mathcal{P}(\mathcal{C}_T)$. That is, for all $k \in \mathbb{N}$,

$$\begin{aligned} law &\left((X^{1,\varepsilon,R,n}, \Phi^{1,\varepsilon,R,n}), (X^{2,\varepsilon,R,n}, \Phi^{2,\varepsilon,R,n}), \ldots, (X^{k,\varepsilon,R,n}, \Phi^{k,\varepsilon,R,n})\right) \\ &\Longrightarrow (P^{\varepsilon,R})^{\otimes k} \end{aligned} \tag{79}$$

when $n \to \infty$ in the space $\mathcal{P}((\mathcal{C}_T)^k)$.

Proof. Convergence (79) is a simple consequence of (78), which we now prove. The proof is an extension of the arguments of Theorem 1.4 in [33], but we shall make the computations explicit in order to keep trace of the constants. Superscripts ε, R on processes will be dropped to simplify notation. We have

$$\begin{aligned} |X_t^{i,n} - X_t^i| \leq & \int_0^t \Big|\frac{1}{n}\sum_{j=1}^n \left(K_\varepsilon(X_s^{i,n} - X_s^{j,n}) - K_\varepsilon(X_s^i - X_s^{j,n})\right) \wedge \chi_R(\Phi_s^{j,n})h_0(X_0^j)\Big|ds \\ &+ \int_0^t \Big|\frac{1}{n}\sum_{j=1}^n \left(K_\varepsilon(X_s^i - X^{j,n}) - K_\varepsilon(X_s^i - X_s^j)\right) \wedge \chi_R(\Phi_s^{j,n})h_0(X_0^j)\Big|ds \\ &+ \int_0^t \Big|\frac{1}{n}\sum_{j=1}^n K_\varepsilon(X_s^i - X_s^j) \wedge \left(\chi_R(\Phi_s^{j,n}) - \chi_R(\Phi_s^j)\right) h_0(X_0^j)\Big|ds \\ &+ \int_0^t \Big|\frac{1}{n}\sum_{j=1}^n K_\varepsilon(X_s^i - X_s^j) \wedge \chi_R(\Phi_s^j)h_0(X_0^j) \\ &- \int K_\varepsilon(X_s^i - x(s)) \wedge \chi_R(\phi(s))h_0(x(0))P^{\varepsilon,R}(dx, d\phi)\Big|ds \end{aligned}$$

Hence,

$$\begin{aligned} |X_t^{i,n} - X_t^i| \leq & L_\varepsilon R\bar{h}\int_0^t \left\{|X_s^{i,n} - X_s^i| + \frac{1}{n}\sum_{j=1}^n |X_s^{j,n} - X_s^j|\right\}ds \\ &+ L_R M_\varepsilon \bar{h}\int_0^t \frac{1}{n}\sum_{j=1}^n |\Phi_s^{j,n} - \Phi_s^j|ds + \int_0^t I(n, R, \varepsilon, s)ds, \end{aligned}$$

with $I(n,R,\varepsilon,s) = \left|\frac{1}{n}\sum_{j=1}^{n}\mathcal{G}((X^i,\Phi^i),(X^j,\Phi^j))\right|$ and

$$\mathcal{G}((X^i,\Phi^i),(X^j,\Phi^j)) = \Big[K_\varepsilon(X^i_s - X^j_s)\wedge\chi_R(\Phi^j_s)h_0(X^j_0)$$
$$-\int K_\varepsilon(X^i_s - x(s))\wedge\chi_R(\phi(s))h_0(x(0))P^{\varepsilon,R}(dx,d\phi)\Big].$$

Thanks to the exchangeability of the system (76), we obtain

$$E\{\sup_{r\le t}|X^{i,n}_r - X^i_r|\} \le 2L_\varepsilon R\bar{h}\int_0^t E\{\sup_{r\le s}|X^{i,n}_r - X^i_r|\}ds$$
$$+L_R M_\varepsilon\bar{h}\int_0^t E\{\sup_{r\le s}|\Phi^{i,n}_r - \Phi^i_r|\}ds$$
$$+\int_0^t E(I(n,R,\varepsilon,s))ds.$$

Now, each of the n squared terms in the sum $I(n,R,\varepsilon,s)^2$ is bounded by $\frac{1}{n^2}(2M_\varepsilon R\bar{h})^2$. On the other hand, for $j\neq k$ we have, thanks to independence,

$$E\left(\mathcal{G}((X^i,\Phi^i),(X^j,\Phi^j))|(X^i,\Phi^i),(X^k,\Phi^k)\right)$$
$$=\int\mathcal{G}((X^i,\Phi^i),(x,\phi))P^{\varepsilon,R}(dx,d\phi)=0$$

which means that the “crossed terms” in the squared sum $I(n,R,\varepsilon,s)^2$ have null expectation. We deduce that

$$E(I(n,R,\varepsilon,s)^2) \le \frac{1}{n}(2M_\varepsilon R\bar{h})^2.$$

Then,

$$E\{\sup_{r\le t}|X^{i,n}_r - X^i_r|\} \le 2L_\varepsilon R\bar{h}\int_0^t E\{\sup_{r\le s}|X^{i,n}_r - X^i_r|\}ds$$
$$+L_R M_\varepsilon\bar{h}\int_0^t E\{\sup_{r\le s}|\Phi^{i,n}_r - \Phi^i_r|\}ds$$
$$+\frac{1}{\sqrt{n}}(2M_\varepsilon R\bar{h})t. \tag{80}$$

On the other hand, we have

$$|\Phi^{i,n}_t - \Phi^i_t| \le \int_0^t \Big|\Big[\frac{1}{n}\sum_{j=1}^{n}\left(\nabla K_\varepsilon(X^{i,n}_s - X^{j,n}_s) - \nabla K_\varepsilon(X^i_s - X^{j,n}_s)\right)$$
$$\wedge\chi_R(\Phi^{j,n}_s)h_0(X^j_0)\Big]\chi_R(\Phi^{i,n}_s)\Big|ds$$
$$+\int_0^t \Big|\Big[\frac{1}{n}\sum_{j=1}^{n}\left(\nabla K_\varepsilon(X^i_s - X^{j,n}) - \nabla K_\varepsilon(X^i_s - X^j_s)\right)$$
$$\wedge\chi_R(\Phi^{j,n}_s)h_0(X^j_0)\Big]\chi_R(\Phi^{i,n}_s)\Big|ds$$

$$+\int_0^t \big|\big[\frac{1}{n}\sum_{j=1}^n \nabla K_\varepsilon(X_s^i - X_s^j) \wedge \Big(\chi_R(\Phi_s^{j,n}) - \chi_R(\Phi_s^j)\Big)\, h_0(X_0^j)\big]$$

$$\chi_R(\Phi_s^{i,n})\big|ds$$

$$+\int_0^t \big|\big[\frac{1}{n}\sum_{j=1}^n \nabla K_\varepsilon(X_s^i - X_s^j) \wedge \chi_R(\Phi_s^j)h_0(X_0^j)\big]$$

$$\Big(\chi_R(\Phi_s^{i,n}) - \chi_R(\Phi_s^i)\Big)\big|ds$$

$$+\int_0^t \big|\big[\frac{1}{n}\sum_{j=1}^n \nabla K_\varepsilon(X_s^i - X_s^j) \wedge \chi_R(\Phi_s^j)h_0(X_0^j)\big]\chi_R(\Phi_s^i)$$

$$-\big[\int \nabla K_\varepsilon(X_s^i - x(s)) \wedge \chi_R(\phi(s))h_0(x(0))P^{\varepsilon,R}(dx, d\phi)\big]$$

$$\chi_R(\Phi_s^i)\big|ds.$$

Notice that

$$\sup_{t\in[0,T]} |\Phi_t^{i,n}|,\ \sup_{t\in[0,T]} |\Phi_t^i| \le C_{\varepsilon,R,T} := (1 + L_\varepsilon R\bar{h}T) \tag{81}$$

for all $n \in \mathbb{N}$. Thus,

$$|\Phi_t^{i,n} - \Phi_t^i| \le J_\varepsilon R\bar{h}C_{\varepsilon,R,T} \int_0^t \Big\{|X_s^{i,n} - X_s^i| + \frac{1}{n}\sum_{j=1}^n |X_s^{j,n} - X_s^j|\Big\}ds$$

$$+L_\varepsilon L_R\bar{h}C_{\varepsilon,R,T} \int_0^t \frac{1}{n}\sum_{j=1}^n |\Phi_s^{j,n} - \Phi_s^j|\, ds$$

$$+L_\varepsilon R\bar{h} \int_0^t |\Phi_s^{i,n} - \Phi_s^i|\, ds + \int_0^t I'(n, R, \varepsilon, s)\, ds,$$

with

$$I'(n, R, \varepsilon, s) = \Bigg|\frac{1}{n}\sum_{j=1}^n \Bigg[\Big(\nabla K_\varepsilon(X_s^i - X_s^j) \wedge \chi_R(\Phi_s^j)h_0(X_0^j)\Big)\chi_R(\Phi_s^{i,n})$$

$$-\int \nabla K_\varepsilon(X_s^i - x(s)) \wedge \chi_R(\phi(s))h_0(x(0))P^{\varepsilon,R}(dx, d\phi)\chi_R(\Phi_s^i)\Bigg]\Bigg|.$$

We conclude in a similar way as before that

$$E\{\sup_{r\le t}|\Phi_r^{i,n} - \Phi_r^i|\} \le 2J_\varepsilon R\bar{h}C_{\varepsilon,R,T} \int_0^t E\{\sup_{r\le s}|X_r^{i,n} - X_r^i|ds$$

$$+\Big(L_\varepsilon L_R\bar{h}C_{\varepsilon,R,T} + L_\varepsilon R\bar{h}\Big)\int_0^t E\{\sup_{r\le s}|\Phi_r^{i,n} - \Phi_r^i|\}\, ds$$

$$+\frac{1}{\sqrt{n}}(2L_\varepsilon R\bar{h}C_{\varepsilon,R,T})t. \tag{82}$$

Bringing together (80) and (82), we get

$$\begin{aligned}
&E\{\sup_{r\le t}|X_r^{i,n}-X_r^i|+|\Phi_t^{i,n}-\Phi_t^i|\}\\
&\quad\le 2R\bar h\left(L_\varepsilon+J_\varepsilon C_{\varepsilon,R,T}\right)\int_0^t E\{\sup_{r\le s}|X_s^{i,n}-X_s^i|\}ds\\
&\qquad+\bar h\left(L_R M_\varepsilon+L_\varepsilon L_R C_{\varepsilon,R,T}+L_\varepsilon R\right)\int_0^t E\{\sup_{r\le s}|\Phi_r^{i,n}-\Phi_r^i|\}ds\\
&\qquad+\frac{2R\bar h}{\sqrt n}(M_\varepsilon C_{\varepsilon,R,T}+L_\varepsilon)t. \qquad (83)
\end{aligned}$$

Finally, we notice that

$$\begin{aligned}
|K*\varphi_\varepsilon(x)|\le C\sup_{z\in\mathbb{R}^3}\{\varphi_\varepsilon(z)\}&\int_{|x-y|\le 1}|x-y|^{-2}dy\\
&+C\int_{|x-y|\ge 1}\varphi_\varepsilon(y)dy\le\frac{C}{\varepsilon^3}+C
\end{aligned}$$

and then, $M_\varepsilon\le C\varepsilon^{-3}$ for all ε small enough. We deduce in a similar way that $L_\varepsilon\le C\varepsilon^{-4}$ and $J_\varepsilon\le C\varepsilon^{-5}$. As observed in Jourdain and Méléard [19], if $g:\mathbb{R}^+\to\mathbb{R}^+$ is a bounded function such that $g(t)\le c_1\int_0^t g(s)ds+c_2t$ for all $t\in[0,T]$, then $g(t)\le\frac{c_2}{c_1}\exp(c_1T)$. This and (83) provide an upper bound for the l.h.s. of (78) by the constant $\frac{c_2}{c_1}\exp(c_1T)$, where

$$c_1=2R\bar h\left(L_\varepsilon+J_\varepsilon(1+RL_\varepsilon\bar hT)\right)+\bar h(M_\varepsilon+L_\varepsilon(1+RL_\varepsilon\bar hT)+L_\varepsilon R)$$

and $c_2=\frac{2R\bar h}{\sqrt n}(M_\varepsilon(1+RL_\varepsilon\bar hT)+L_\varepsilon)$. The statement follows by noting the existence of universal positive constants $\mathbf{C}$, $\mathbf{C}'$, $\mathbf{C}''$ and ε_0 (in particular independent of R, $\bar h$ and T) such that

$$\mathbf{C}J_\varepsilon L_\varepsilon(R\bar h)^2T\le c_1\le\mathbf{C}'J_\varepsilon L_\varepsilon\bar h(R+1)(\bar h+RT)$$

and that, for all $\varepsilon\in]0,\varepsilon_0[$,

$$c_2\le\mathbf{C}''\frac{L_\varepsilon^2}{\sqrt n}R\bar h(1+R\bar hT).$$

□

We can take for instance χ_R defined by

$$\chi_R(\phi)=\begin{cases}\phi & if\,|\phi|\le R,\\ \frac{R}{|\phi|}\,\phi & \text{if } |\phi|\ge R.\end{cases}$$

(which is the truncation function proposed in [13]).

Remark 5.1. In [13], Esposito and Pulvirenti claimed (without proving) the existence of a nonlinear process satisfying analogous conditions as (69),(70), but without the truncation χ_R on the process Φ *inside the expectation* that we imposed in (70). Indeed, truncating "*outside the expectation*" in (69) is not strictly necessary: the two previous theorems can also be proved in that case, by bounding $|\Phi|$ and $|\Phi^i|$ above by $\exp\{L_\varepsilon R\bar{h}T\}$ (thanks to Gronwall's lemma), instead of the bounds (75) and (81). In turn, it seems not possible to obtain these results in the way conjectured in [13] (truncating Φ *only* outside the expectation). In fact, one cannot then provide a bound for the process Φ by absolute constants as in (75), which is crucial for estimate (74) to hold (or for an analogous to it with a different metric), and therefore to ensure that an iteration (fixed point) procedure will converge.

6. The 3 dimensional stochastic vortex method

We will now state and prove our main result. Recall that $\mathbf{w} \in \mathbf{F}_{0,p,T}$ (with $p \in]\frac{3}{2}, 3[$) is the solution of the mild vortex equation (10) with initial condition $w_0 \in L^p_3 \cap L^1_3$ given by Theorem 3.1. We have $T^{1-\frac{3}{2p}} \|w_0\|_p < \Gamma_0(p)$ for the constant $\Gamma_0(p) > 0$ given therein, and $|||\mathbf{w}|||_{0,p,T} \leq 2\|w_0\|_p$. Let us write

$$\mathbf{u}(t, x) = \mathbf{K}(\mathbf{w})(t, x)$$

for all $(t, x) \in [0, T] \times \mathbb{R}^3$, and fix a real number

$$r_\circ \in]3, \frac{3p}{3-p}[.$$

Remark 6.1. By taking in Corollary 3.1 *ii)* $A = 2\|w_0\|_p$ and $r = r_\circ$, we deduce the existence of a constant $\hat{C}(\|w_0\|_p, T, p)$, depending on $\mathbf{w}$ only through $\|w_0\|_p$, such that

$$\|\nabla \mathbf{u}(t)\|_\infty \leq t^{-\frac{1}{2}-\frac{3}{2}(\frac{1}{p}-\frac{1}{r_\circ})} \hat{C}(\|w_0\|_p, T, p) \text{ for all } t \in [0, T],$$

Consequently, if $P \in \mathcal{P}^T_{b,\frac{3}{2}}$ is the solution of (37) associated with $\mathbf{w}$, under P we have

$$R(w_0, T) := \exp\left\{\hat{C}(\|w_0\|_p, T, p) \int_0^T t^{-\frac{1}{2}-\frac{3}{2}(\frac{1}{p}-\frac{1}{r_\circ})} dt\right\} \geq \sup_{t\in[0,T]} |\Phi_t| \; a.e. \tag{84}$$

Theorem 6.1. *Assume that* $w_0 \in L^p_3 \cap L^1_3$ *with* $p \in]\frac{3}{2}, 3[$ *and* $T^{1-\frac{3}{2p}} \|w_0\|_p < \Gamma_0(p)$. *Fix* $R \geq R(w_0, T)$ *(defined in (84)) and let* (ε_n) *be a sequence converging to* 0 *in such way that*

$$\frac{1}{\sqrt{n}} \exp\left\{C_2 \varepsilon_n^{-9} \|w_0\|_1 T\,(R+1)\,(\|w_0\|_1 + TR)\right\} \to 0,$$

where C_2 *is the absolute constant provided by Theorem 5.2. Furthermore, define for each* $n \in \mathbb{N}$ *a system of interacting particles on* $\mathbb{R}^3 \times \mathcal{M}_{3\times 3}$ *by*

$$Z^{i,n} := (X^{i,\varepsilon_n,R,n}, \Phi^{i,\varepsilon_n,R,n}),$$

and let P be the unique solution in $\mathcal{P}^T_{b,\frac{3}{2}}$ of the nonlinear martingale problem (37). Then, for all $k \in \mathbb{N}$, when $n \to \infty$, we have

$$law(Z^{1,n}, Z^{2,n}, ..., Z^{k,n}) \Longrightarrow P^{\otimes k}$$

in the space $\mathcal{P}(\mathcal{C}^k_T)$.

Remark 6.2. Theorem 6.1 will hold if for instance $\varepsilon_n = (c \ln n)^{-9}$, with

$$0 < c < C_2^{-1} \left((R+1)(\|w_0\|_1 + 1)(T+1)\right)^{-2}.$$

The proof of Theorem 6.1 will use similar techniques as those in [27] or [14] for the equations considered therein. First we will prove that under the conditions ensuring existence of the solution $\mathbf{w}$, P can be approximated by a family of solutions P^n of some nonlinear martingale problems with regular interactions, associated with the solutions $\mathbf{w}^{\varepsilon_n} \in \mathbf{F}_{0,p,T}$ of mollified vortex equations (involving the smoothed kernel K_{ε_n}).

6.1. *The mollified equations*

Consider the operator $\mathbf{K}^\varepsilon$ defined as in Section **4.3**, and for each $\varepsilon > 0$ define

$$\mathbf{B}^\varepsilon(\mathbf{v}', \mathbf{v})(t,x) = \int_0^t \sum_{j=1}^3 \int_{\mathbb{R}^3} \frac{\partial G_{t-s}}{\partial y_j}(x-y) \left[\mathbf{K}^\varepsilon(\mathbf{v}')_j(s,y)\mathbf{v}(s,y) - \mathbf{v}_j(s,y)\mathbf{K}^\varepsilon(\mathbf{v}')(s,y)\right] dy\, ds. \quad (85)$$

Remark 6.3. In virtue of Remark 4.3, the operator $\mathbf{B}^\varepsilon : \mathbf{F}^2 \to \mathbf{F}'$ satisfies the same continuity properties as the operator $\mathbf{B}$ in the spaces $\mathbf{F}$, $\mathbf{F}'$ considered in Proposition 3.1. Moreover, for each such pair $(\mathbf{F}, \mathbf{F}')$, the norm of $\mathbf{B}^\varepsilon$ is bounded by the norm of $\mathbf{B}$.

Therefore, the same existence and regularity results of Theorem 3.1, Theorem 3.2 and Corollary 3.1 hold true **with the same constants**, for the family of mollified equations

$$\mathbf{v}(t,x) = \mathbf{w}_0 + \mathbf{B}^\varepsilon(\mathbf{v}, \mathbf{v}) \quad (86)$$

We denote by $\mathbf{w}^\varepsilon \in \mathbf{F}_{0,p,T}$ the unique solution of the mollified equation (86). By the previous remark, as in Section **4.3**, for each $\varepsilon > 0$ the stochastic differential equations

$$\xi^\varepsilon_{s,t}(x) = x + \sqrt{2\nu}(B_t - B_s) + \int_s^t \mathbf{K}^\varepsilon(\mathbf{w}^\varepsilon)(\theta, \xi^\varepsilon_{s,\theta}(x))d\theta, \text{ for all } t \in [s,T]$$

define a process $(s,t,x) \mapsto \xi^\varepsilon_{s,t}(x)$ which is continuously differentiable in x for all (s,t).

Let (ε_n) be a sequence converging to 0 and define

$$X^n_t := \xi^{\varepsilon_n}_{0,t}(X_0), \text{ and } \Phi^n_t = \nabla\xi^{\varepsilon_n}_{0,t}(X_0) \text{ with } t \in [0,T].$$

The law of (X^n, Φ^n) is denoted by P^n, and we write

$$\rho^n(t,x) \text{ and } \tilde{\rho}^n(t,x)$$

for bi-measurable versions of the densities of $(P^n)_t^\circ$ and $\tilde{P}_t^n$ respectively. By similar arguments as in the proof of Lemma 4.4, it is checked that $\tilde{\rho}^n \in \mathbf{F}_{0,p,T}$, and that

$$\tilde{\rho}^n(t,x) = \int_{\mathbb{R}^3} G_t^\nu(x-y)w_0(y)dy \int_0^t \sum_{j=1}^3 \int_{\mathbb{R}^3} \frac{\partial G_{t-s}^\nu}{\partial y_j}(x-y)\left[\mathbf{K}^{\varepsilon_n}(\mathbf{w}^{\varepsilon_n})_j(s,y)\tilde{\rho}^n(s,y) - \tilde{\rho}_j^n(s,y)\mathbf{K}^{\varepsilon_n}(\mathbf{w}^{\varepsilon_n})(s,y)\right] dy\, ds. \tag{87}$$

Since uniqueness in $\mathbf{F}_{0,p,T}$ holds for equation (87), we deduce that

$$\tilde{\rho}^n = \mathbf{w}^{\varepsilon_n} \tag{88}$$

for all $n \in \mathbb{N}$. Thus, (X^n, Φ^n) solves the nonlinear stochastic differential equation

$$\begin{aligned} X_t^n &= X_0 + \sqrt{2\nu}B_t + \int_0^t \mathbf{u}^{\varepsilon_n}(s, X_s^n)ds \\ \Phi_t^n &= Id + \int_0^t \nabla\mathbf{u}^{\varepsilon_n}(s, X_s^n)\Phi_s^n ds \end{aligned} \tag{89}$$

with

$$\mathbf{u}^{\varepsilon_n}(s,x) = E\left[K_{\varepsilon_n}(x - X_s^n) \wedge \Phi_s^n h_0(X_0)\right]. \tag{90}$$

The reader should compare this process *without truncation* χ_R with that in (69)–(70).

Proposition 6.1. *i) For all $t \in [0,T]$ and $n \in \mathbb{N}$, one has $\tilde{\rho}^n(t) \in L_3^p$, $\rho^n(t) \in L^p$, and*

$$\sup_{n\in\mathbb{N}} |||\tilde{\rho}^n|||_{0,p,T} < \infty, \sup_{n\in\mathbb{N}} |||\rho^n|||_{0,p,T} < \infty. \tag{91}$$

Moreover, $\tilde{\rho}^n(t)$ converges in L_3^p for each $t \in [0,T]$ and in $L^1([0,T], L_3^p)$ to $\mathbf{w}(t)$. Similarly, $\rho^n(t)$ converges in L^p for each $t \in [0,T]$ and in $L^1([0,T], L^p)$ to $\rho(t)$ (the unique solution of the linear equation (51)).

ii) The sequence $(P^n, n \in \mathbb{N})$ is uniformly tight.

iii) When $n \to \infty$, one has $P^n \Longrightarrow P$.

Proof. i) The uniform bound for $|||\tilde{\rho}^n|||_{0,p,T}$ is clear from (88) and Remark 6.3, and the bound for $|||\rho^n|||_{0,p,T}$ follows as in Lemma 4.4. The proof of the convergence $\tilde{\rho}^n \to \mathbf{w}$ is also similar as therein, using the estimate

$$\begin{aligned} \|\tilde{\rho}^n(t) - \mathbf{w}(t)\|_p \le C \int_0^t (t-s)^{-\frac{3}{2p}} \|\mathbf{K}^{\varepsilon_n}(\mathbf{w})(s) - \mathbf{K}(\mathbf{w})(s)\|_q\, ds \\ + C \int_0^t (t-s)^{-\frac{3}{2p}} \|\tilde{\rho}^n(s) - \mathbf{w}(s)\|_p\, ds \end{aligned}$$

which can be deduce with Remark 4.3. The convergence of ρ^n is obtained in a similar way.

ii) In virtue of the uniform bounds for $|||\mathbf{w}^{\varepsilon_n}|||_{1,p,T}$ and $|||\mathbf{w}^{\varepsilon_n}|||_{1,r,(T;p)}$ pointed out in Remark 6.3, the proof is done exactly in the same way as Lemma 4.6.

iii) We just have to identify the limiting points in a similar way as in Proposition 4.2. If Q is the limit of a convergent subsequence renamed P^n, we only need to check that $E^Q(\kappa(X)) = 0$ and $E^Q(\zeta(X, \Phi)) = 0$, where $\kappa : C([0, T], \mathbb{R}^3) \to \mathbb{R}$ and $\zeta : C([0, T], \mathbb{R}^3) \times C([0, T], \mathcal{M}_{3\times 3}) \to \mathbb{R}$ are the functions defined in (61) and (62). We know that

$$E^{P^n}\Bigg[\bigg(\int_s^t \left\{\frac{\partial f}{\partial \tau}(\tau, X_\tau) + \nu\Delta f(\tau, X_\tau) + \mathbf{K}^{\varepsilon_n}(\mathbf{w}^{\varepsilon_n})(\tau, X_\tau)\nabla f(\tau, X_\tau)\right\} d\tau$$

$$+ f(t, X_t) - f(s, X_s)\bigg) \times \lambda(X_{s_1}, \ldots, X_{s_m})\Bigg] = 0,$$

and therefore

$$E^{P^n}(\kappa(X)) = E^{P^n}\bigg[\int_s^t \Big(\mathbf{K}(\mathbf{w})(\tau, X_\tau)\nabla f(\tau, X_\tau) - \mathbf{K}^{(\varepsilon_n)}(\mathbf{w}^{(\varepsilon_n)})(\tau, X_\tau)\nabla f(\tau, X_\tau)\Big)d\tau \times \lambda(X_{s_1}, \ldots, X_{s_m})\bigg].$$

We deduce that

$$\left|E^{P^n}(\kappa(X))\right| \leq C \sup_{k\in\mathbb{N}} |||\rho^k|||_{0,q*,T} \int_0^T \left\|\mathbf{K}^{\varepsilon_n}(\mathbf{w}^{\varepsilon_n})(\tau) - \mathbf{K}(\mathbf{w})(\tau)\right\|_q \, d\tau$$

$$\leq C \int_0^T \left(\left\|\mathbf{w}^{\varepsilon_n}(\tau) - \mathbf{w}(\tau)\right\|_p + \left\|\mathbf{K}^{\varepsilon_n}(\mathbf{w})(\tau) - \mathbf{K}(\mathbf{w})(\tau)\right\|_q \right) d\tau$$

thanks to Remark 4.3 (and with C a finite constant), and we conclude with Remark 4.3 that $E^Q(\kappa(X)) = 0$. In a similar way, one can adapt the arguments of Proposition 4.2 to prove that $E^Q(\zeta(X, \Phi)) = 0$. The only point that needs special attention is to establish the uniform bound

$$\sup_{k\in\mathbb{N}} |||\rho^k|||_{0,p*,(T;p)} < \infty,$$

when $p < 2$. This can be justified by similar arguments as in Proposition 4.2 using the fact that the norm of the linear functional

$$\eta(t, x) \mapsto \int_0^t \sum_{j=1}^3 \int_{\mathbb{R}^3} \frac{\partial G^\nu_{t-s}}{\partial y_j}(x - y)\mathbf{K}^{\varepsilon_n}(\mathbf{w}^{\varepsilon_n})_j(s, y)\eta(s, y)\, dy\, ds,$$

defined from $F_{0,r,(T;p)}$ to $F_{0,r',(T;p)}$, can be estimated in terms of $|||\mathbf{w}^{\varepsilon_n}|||_{0,r,(T;p)}$ by Remark 4.3, and the latter is bounded independently of n as asserted in Remark 6.3. □

6.2. Convergence of the particle approximations

By (90), (88) and the definition of $\tilde{\rho}^n$, the drift term $\mathbf{u}^{\varepsilon_n}$ of the nonlinear process (X^n, Φ^n) is given by

$$\mathbf{u}^{\varepsilon_n}(t,x) = \mathbf{K}^{\varepsilon_n}(\mathbf{w}^{\varepsilon_n})(t,x).$$

The following fact is crucial (a similar remark was done by Esposito and Pulvirenti [13] in a more restrictive functional setting):

Remark 6.4. Since $|||\mathbf{w}^{\varepsilon_n}|||_{0,p,T} \leq 2\|w_0\|_p$, we have

$$\|\nabla \mathbf{u}^{\varepsilon_n}(t)\|_\infty \leq t^{-\frac{1}{2}-\frac{3}{2}(\frac{1}{p}-\frac{1}{r})}\hat{C}(\|w_0\|_p, T, p)$$

for all $t \in [0,T]$, $n \in \mathbb{N}$, with $\hat{C}(\|w_0\|_p, T, p)$ the same constant of Remark 6.1. Thus, from (89) and Lemma 4.5 it follows that for all $n \in \mathbb{N}$, almost surely

$$\sup_{t\in[0,T]} |\Phi_t^n| \leq R(w_0, T). \tag{92}$$

Consequently, we have the identity $\chi_R(\Phi^n) = \Phi^n$ for all $R \geq R(w_0,T)$, and then, the nonlinear process (89)–(90) is a weak solution of the nonlinear McKean-Vlasov equation (69)–(70) on the interval $[0,T]$. Uniqueness in law for (69)–(70) implies the following result.

Proposition 6.2. *Let $R \geq R(w_0,T)$ and let the processes $(X^{\varepsilon_n,R}, \Phi^{\varepsilon_n,R})$ and (X^n, Φ^n) be defined on $[0,T]$ respectively by (69)–(70) and (89)–(90). Then, for all $n \in \mathbb{N}$ we have*

$$P^n = law(X^{\varepsilon_n,R}, \Phi^{\varepsilon_n,R}),$$

Proof of Theorem 6.1: Let $k \in \mathbb{N}$ be fixed. Consider the set $\mathcal{P}(\mathcal{C}_T^k)$ of probability measures on the space

$$\mathcal{C}_T^k := C([0,T], (\mathbb{R}^3)^k \times (\mathcal{M}_{3\times 3})^k),$$

and denote by D_T^k the Kantorovich-Rubinstein distance on $\mathcal{P}(\mathcal{C}_T^k)$, associated to the following distance on $\mathcal{C}_T^k$: $\sum_{j=1}^k \left\{ \sup_{0\leq t\leq T} \left[\min\{|x^j(t) - y^j(t)|, 1\} + \min\{|\phi^j(t) - \psi^j(t)|, 1\} \right] \right\}$.

Denote by $\overline{Z}^{i,n} := (X^{i,\varepsilon_n,R}, \Phi^{i,\varepsilon_n,R})$ the nonlinear processes (77) and $\varepsilon = \varepsilon_n$. If P is the solution in $\mathcal{P}_{b,\frac{3}{2}}^T$ of the nonlinear martingale problem (37), we have

$$\begin{aligned}
&D_T^k\left((law(Z^{1,n},\ldots,Z^{k,n}), P^{\otimes k}\right) \\
&\quad \leq D_T^k\left((law(Z^{1,n},\ldots,Z^{k,n}), law(\overline{Z}^{1,n}\ldots,\overline{Z}^{k,n})\right) \\
&\qquad + D_T^k\big(law(\overline{Z}^{1,n}\ldots,\overline{Z}^{k,n}), P^{\otimes k}\big) \\
&\quad \leq C\sum_{i=1}^k E\Bigg[\sup_{t\in[0,T]}\left\{|X_t^{i,\varepsilon_n,R,n} - X_t^{i,\varepsilon_n,R}| + |\Phi_t^{i,\varepsilon_n,R,n} - \Phi_t^{i,\varepsilon_n,R}|\right\}\Bigg] \\
&\qquad + D_T^k\big((P^n)^{\otimes k}, P^{\otimes k}\big),
\end{aligned}$$

thanks also to Proposition 6.2. The term involving the sum is bounded by

$$k\frac{C\varepsilon_n}{\sqrt{n}}\exp\left\{C_2\varepsilon_n^{-9}\|w_0\|_1 T\,(R+1)\,(\|w_0\|_1+RT)\right\}$$

by Theorem 5.2, and goes to 0 by the choice of ε_n. The last term goes to 0 thanks Proposition 6.1 *iii)*. □

Remark 6.5 The distance between P^n and P could estimated in terms of $\|\mathbf{K}^{\varepsilon_n}(\mathbf{w}^{\varepsilon_n}) - \mathbf{K}(\mathbf{w})\|_\infty$ and $\|\nabla\mathbf{K}^{\varepsilon_n}(\mathbf{w}^{\varepsilon_n}) - \nabla\mathbf{K}(\mathbf{w})\|_\infty$, but an explicit dependence on ε_n of these quantities seems hard to obtain without additional regularity assumption. On the other hand, an attempt to improve the propagation of chaos estimates in Section **5** should take into account specific properties of the interaction kernel, which we have not been able to do here.

A first consequence is convergence at the level of empirical processes. Consider the space $\mathcal{M}_3(\mathbb{R}^3)$ of finite $\mathbb{R}^3$-valued measures on $\mathbb{R}^3$, endowed with the weak topology, and the space $C([0,T],\mathcal{M}_3(\mathbb{R}^3))$ with the topology of uniform convergence.

Corollary 6.1 *The family* $(\tilde{\mu}_t^{n,\varepsilon_n,R})_{0\le t\le T}$ *of* $\mathbb{R}^3$*-weighted empirical measures on* $\mathbb{R}^3$

$$\tilde{\mu}_t^{n,\varepsilon_n,R} = \frac{1}{n}\sum_{i=1}^{n}\delta_{X_t^{i,\varepsilon_n,R,n}}\cdot\left(\chi_R(\Phi_t^{i,\varepsilon_n,R,n})h_0(X_0^i)\right)$$

converges in law and in probability to $(\mathbf{w}(t,x)dx)_{0\le t\le T}$ *in the space* $C([0,T],\mathcal{M}_3(\mathbb{R}^3))$.

Proof Since $law(Z^{1,n},...,Z^{n,n})$ is exchangeable, propagation of chaos in Theorem 6.1 is equivalent to the convergence in law (and in probability) of the empirical measure of the system to P, as a probability measure in the path space (see [33]). This implies that

$$E\left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{f}(X_t^{i,\varepsilon_n,R,n})\chi(\Phi_t^{i,\varepsilon_n,R,n})\mathbf{f}_0(X_0^i)\right)\to E^P\left(\mathbf{f}(X_t)\chi(\Phi_t)\mathbf{f}_0(X_0)\right),$$

for all continuous bounded functions $\mathbf{f}_0,\mathbf{f}:\mathbb{R}^3\to\mathbb{R}^3$ and $\chi:\mathcal{M}_{3\times3}\to\mathcal{M}_{3\times3}$. Let $k\in\mathbb{N}$ and h_0^k be a continuous bounded function approximating h_0 as in Proposition 4.2. Since under P we have $\chi_R(\Phi_t)=\Phi_t$, it follows that

$$\begin{aligned}
&\left|E\langle\tilde{\mu}_t^{n,\varepsilon_n,R},\mathbf{f}\rangle-\int_{\mathbb{R}^3}\mathbf{f}(x)\mathbf{w}(t,x)dx\right|\\
&\quad\le E\left|\frac{1}{n}\sum_{i=1}^n\mathbf{f}(X_t^{i,\varepsilon_n,R,n})\chi_R(\Phi_t^{i,\varepsilon_n,R,n})h_0(X_0^i)-E^P(\mathbf{f}(X_t)\chi_R(\Phi_t)h_0(X_0)))\right|\\
&\quad\le E\left|\frac{1}{n}\sum_{i=1}^n\mathbf{f}(X_t^{i,\varepsilon_n,R,n})\chi_R(\Phi_t^{i,\varepsilon_n,R,n})(h_0(X_0^i)-h_0^k(X_0^i))\right|\\
&\qquad+E\left|\frac{1}{n}\sum_{i=1}^n\mathbf{f}(X_t^{i,\varepsilon_n,R,n})\chi_R(\Phi_t^{i,\varepsilon_n,R,n})h_0^k(X_0^i)-E^P(\mathbf{f}(X_t)\chi_R(\Phi_t)h_0^k(X_0)\right|\\
&\qquad+E^P|\mathbf{f}(X_t)\chi_R(\Phi_t)(h_0^k(X_0)-h_0(X_0))|.
\end{aligned}$$

By similar arguments as in the proof of Proposition 4.2 we conclude that

$$\limsup_{n\to\infty}\left|E\langle\tilde{\mu}_t^{n,\varepsilon_n,R},\mathbf{f}\rangle-\int_{\mathbb{R}^3}\mathbf{f}(x)\mathbf{w}(t,x)dx\right|=0.$$

□

6.3. Stochastic approximations of the velocity field

Finally, we prove the convergence of the "approximated velocity field", defined by

$$\mathbf{K}^{\varepsilon_n}(\tilde{\mu}^{n,\varepsilon_n,R})(t,x):=\int_{\mathbb{R}^3}K_{\varepsilon_n}(x-y)\wedge\tilde{\mu}_t^{n,\varepsilon_n,R}(dy),$$

to the local solution $\mathbf{u}(t)=\mathbf{K}(\mathbf{w})(t)$ of the Navier-Stokes equation in $\mathbf{F}_{0,q,T}$. We need a technical lemma:

Lemma 6.1 *Under the assumptions of Theorem 6.3, we have*

$$\|\nabla\mathbf{w}^\varepsilon(t)-\nabla\mathbf{w}(t)\|_p\to 0\ \textit{for all}\ t\in]0,T],\ \textit{and}\ \int_0^T\|\nabla\mathbf{w}^\varepsilon(t)-\nabla\mathbf{w}(t)\|_p dt\to 0$$

when $\varepsilon\to 0$.

Proof It is not hard to check (by similar reasons as in Lemma 4.4) that $div\,\mathbf{w}^\varepsilon(t)=0$. Let us write $\mathbf{w}^{\varepsilon;\tau}:=\mathbf{w}^\varepsilon(\tau+\cdot)$ and $\mathbf{w}^{0;\tau}=\mathbf{w}(\tau+\cdot)$. For each $\varepsilon\ge 0$ and $\tau\in[0,T]$, $\mathbf{w}^{\tau,\varepsilon}\in\mathbf{F}_{1,p,T-\tau}$ solves the "shifted" equation $\mathbf{w}^{\varepsilon;\tau}=G_t^\nu*\mathbf{w}^\varepsilon(\tau)+\mathbf{B}^\varepsilon(\mathbf{w}^{\varepsilon;\tau},\mathbf{w}^{\varepsilon;\tau})$. Taking derivatives in this equation yields, for $k=1,2,3$,

$$\begin{aligned}
\frac{\partial(\mathbf{w}^{\varepsilon;\tau})_k}{\partial x_i}(t,x)&=\int_{\mathbb{R}^3}\frac{\partial G_t^\nu}{\partial x_i}(x-y)(\mathbf{w}^\varepsilon)_k(\tau,y)dy\\
&\quad-\int_0^t\sum_{j=1}^3\int_{\mathbb{R}^3}\frac{\partial G_{t-s}^\nu}{\partial x_i}(x-y)\Bigg[\mathbf{K}^{\varepsilon;\tau}(\mathbf{w}^{\varepsilon;\tau})_j(s,y)\frac{\partial\mathbf{w}_k^{\varepsilon;\tau}(s,y)}{\partial y_j}\\
&\quad-\mathbf{w}_j^{\varepsilon;\tau}(s,y)\frac{\partial\mathbf{K}^\varepsilon(\mathbf{w}^{\varepsilon;\tau})_k(s,y)}{\partial y_j}\Bigg]dy\,ds.
\end{aligned}$$

From this and the uniform bounds for $|||\mathbf{w}^\varepsilon|||_{0,p,T}$ we deduce (with similar arguments as in Theorem 3.1 *a)*) that

$$\begin{aligned}\|\nabla \mathbf{w}^{\varepsilon;\tau}(t) - \nabla \mathbf{w}^{0;\tau}(t)\|_p \le Ct^{-\frac{1}{2}}\|\mathbf{w}^\varepsilon(\tau) - \mathbf{w}(\tau)\|_p \\ +C\int_0^t (t-s)^{-\frac{3}{2p}}(\tau+s)^{-\frac{1}{2}}\big[\|\mathbf{w}^{\varepsilon;\tau}(s) - \mathbf{w}^{0;\tau}(s)\|_p \\ +\|\mathbf{K}^\varepsilon(\mathbf{w}^{0,\tau})(s) - \mathbf{K}(\mathbf{w}^{0,\tau})(s)\|_q\big]ds \\ +C\int_0^t (t-s)^{-\frac{3}{2p}}\big[\|\nabla \mathbf{w}^{\varepsilon;\tau}(s) + \nabla \mathbf{w}^{0;\tau}(s)\|_p \\ +\|\nabla \mathbf{K}^\varepsilon(\mathbf{w}^{0;\tau})(s) - \nabla \mathbf{K}(\mathbf{w}^{0;\tau})(s)\|_q\big]ds.\end{aligned}$$

Now define $\delta^{\varepsilon,\tau}(t) := \|\nabla \mathbf{w}^{\varepsilon;\tau}(t) - \nabla \mathbf{w}^{0;\tau}(t)\|_p$, and

$$\begin{aligned}\Delta^{\varepsilon,\tau}(t) := \tau^{-\frac{1}{2}}\left(\|\mathbf{w}^{\varepsilon;\tau}(t) - \mathbf{w}^{0;\tau}(t)\|_p + \|\mathbf{K}^\varepsilon(\mathbf{w}^{0,\tau})(t) - \mathbf{K}(\mathbf{w}^{0,\tau})(t)\|_q\right) \\ +\|\nabla \mathbf{K}^\varepsilon(\mathbf{w}^{0,\tau})(t) - \nabla \mathbf{K}(\mathbf{w}^{0,\tau})(t)\|_q,\end{aligned}$$

so that for all $t \in]0, T-\tau]$ we have

$$\delta^{\varepsilon,\tau}(t) \le Ct^{-\frac{1}{2}}\|\mathbf{w}^\varepsilon(\tau) - \mathbf{w}(\tau)\|_p + \int_0^t (t-s)^{-\frac{3}{2p}}(\Delta^{\varepsilon,\tau}(s) + \delta^{\varepsilon,\tau}(s))ds.$$

Observe that since $\mathbf{w}^{0,\tau} \in \mathbf{F}_{0,p,T-\tau} \cap \mathbf{F}_{0,q,T-\tau}$, the convergence $\Delta^{\varepsilon,\tau}(t) \to 0$ holds for each $t \in]0, T-\tau[$ when $\varepsilon \to 0$ (cf. Remark 4.3).

As in the proof of Lemma 4.4 (and with the same notation), it follows by induction that

$$\begin{aligned}\delta^{\varepsilon,\tau}(t) \le C\|\mathbf{w}^\varepsilon(\tau) - \mathbf{w}(\tau)\|_p \sum_{k=1}^{\tilde{N}(p)} t^{(k-1)\theta_0 - \frac{1}{2}} \\ +C\int_0^t \alpha(t-s)\Delta^{\varepsilon,\tau}(s)ds + C(T)\int_0^t \delta^{\varepsilon,\tau}(s)ds.\end{aligned} \tag{93}$$

Thus, integrating and using Gronwall's lemma yield, for all $\lambda \in [0, T-\tau]$,

$$\int_0^\lambda \delta^{\varepsilon,\tau}(t)dt \le C(T)\|\mathbf{w}^\varepsilon(\tau) - \mathbf{w}(\tau)\|_p + C'(T)\int_0^{T-\tau}\int_0^t \alpha(t-s)\Delta^{\varepsilon,\tau}(s)ds\, dt.$$

Therefore, $\int_0^\lambda \delta^{\varepsilon,\tau}(t)dt \to 0$ and from this and (93) we deduce that $\delta^{\varepsilon,\tau}(t) \to 0$ for all $t \in]0, T-\tau[$. Consequently, $\nabla \mathbf{w}^\varepsilon(t) \to \nabla \mathbf{w}(t)$ in $(L_3^p)^3$ for all $t \in]0, T]$. Since $\mathbf{w}^\varepsilon$ is bounded in $\mathbf{F}_{1,p,T}$ uniformly in ε, the convergence takes also place in $L^1([0, T], (L_3^p)^3)$. □

Corollary 6.2 *Let* $T^{1-\frac{3}{2p}}\|w_0\|p < \Gamma_0(p)$ *and denote* $\mathbf{u} = \mathbf{K}(\mathbf{w})$. *Let* $\varepsilon_n = (c\ln n)^{-9}$ *be a sequence satisfying the condition of Theorem 6.1. Then, when* $n \to \infty$, *we have*

$$\sup_{x\in\mathbb{R}^3} E\left(\left|\mathbf{K}^{\varepsilon_n}(\tilde{\mu}^{n,\varepsilon_n,R})(t,x) - \mathbf{u}(t,x)\right|\right) \to 0$$

for each $t \in]0,T]$, *and*

$$\sup_{x\in\mathbb{R}^3} E\left(\int_0^T \left|\mathbf{K}^{\varepsilon_n}(\tilde{\mu}^{n,\varepsilon_n,R})(t,x)-\mathbf{u}(t,x)\right| dt\right)\to 0.$$

Proof For all $(t,x)\in[0,T]\times\mathbb{R}^3$, it holds that

$$\begin{aligned}
&\left|\mathbf{K}^{\varepsilon_n}(\tilde{\mu}^{n,\varepsilon_n,R})(t,x)-\mathbf{u}(t,x)\right| \\
&\quad\le \left|\mathbf{K}^{\varepsilon_n}(\tilde{\mu}^{n,\varepsilon_n,R})(t,x)\right. \\
&\qquad \left. -\frac{1}{n}\sum_{i=1}^n K_{\varepsilon_n}(x-X_t^{i,\varepsilon_n,R})\wedge(\chi_R(\Phi_t^{i,\varepsilon_n,R})h_0(X_0^i))\right| \\
&\qquad +\left|\frac{1}{n}\sum_{i=1}^n K_{\varepsilon_n}(x-X_t^{i,\varepsilon_n,R})\wedge(\chi_R(\Phi_t^{i,\varepsilon_n,R})h_0(X_0^i))\right. \\
&\qquad \left. -\int_{\mathcal{C}_T} K_{\varepsilon_n}(x-y(s))\wedge\chi_R(\phi(s))h_0(x(0))P^{\varepsilon_n,R}(dy,d\phi)\right| \\
&\qquad +|\mathbf{K}^{\varepsilon_n}(\mathbf{w}^{\varepsilon_n})(t,x)-\mathbf{u}(t,x)|
\end{aligned} \tag{94}$$

with $P^{\varepsilon_n,R}=P^{\varepsilon_n}=law(X^{i,\varepsilon_n,R},\Phi^{i,\varepsilon_n,R})$. The independence of the processes $(X^{i,\varepsilon_n,R},\Phi^{i,\varepsilon_n,R})$, $i\in\mathbb{N}$, imply that the expectation of the second term is bounded by $\frac{1}{\sqrt{n}}(2M_{\varepsilon_n}R\|w_0\|_1)$. Thus,

$$\begin{aligned}
E\left|\mathbf{K}_{\varepsilon_n}(\tilde{\mu}^{n,\varepsilon_n,R})(t,x)-\mathbf{u}(t,x)\right| &\le (L_{\varepsilon_n}R\|w_0\|_1+M_{\varepsilon_n}\|w_0\|_1)\frac{C\varepsilon(1+R\|w_0\|_1T)}{\sqrt{n}(R\|w_0\|_1T)} \\
&\quad\times\exp\{C_2\varepsilon_n^{-9}\|w_0\|_1T(R+1)(\|w_0\|_1+RT)\} \\
&\quad+\frac{1}{\sqrt{n}}(2M_{\varepsilon_n}R\|w_0\|_1) \\
&\quad+\|\mathbf{K}^{\varepsilon_n}(\mathbf{w}^{\varepsilon_n})(t)-\mathbf{K}^{\varepsilon_n}(\mathbf{w})(t)\|_\infty \\
&\quad+\|\mathbf{K}^{\varepsilon_n}(\mathbf{w})(t)+\mathbf{u}(t)\|_\infty.
\end{aligned} \tag{95}$$

The first term is bounded by $C\frac{(\ln n)^{\alpha_1}}{n^{\alpha_2}}$ for some constants $C,\alpha_1,\alpha_2>0$ and goes to 0 when $n\to\infty$. The same holds for the second term for some other constants. The third is bounded by $C\|\mathbf{w}^{\varepsilon_n}(t)-\mathbf{w}(t)\|_{W^{1,p}}$ by Remark 4.3, and goes to 0 for each $t\in]0,T]$ and in $L^1([0,T],\mathbb{R})$, thanks to Proposition 6.1 *i)* and Lemma 6.1. The convergence of the last term for each $t\in]0,T]$ and in $L^1([0,T],\mathbb{R})$ is obtained by standard arguments. □

Remark 6.6 An improvement of the estimate of the first term in the r.h.s. of (94) (by avoiding the dependence on the divergent constants L_{ε_n} and M_{ε_n} used in the l.h.s. of (95)), could be envisaged by adapting argument of Méléard [27] for the two dimensional vortex equation, relying on uniform L^p-estimates for the densities of the particles. These estimates are based on the results on generators in generalized divergence form of Osada [30], and on a representation formula for the Biot-Savart kernel in 2 dimensions given therein. We have not been able to generalize that formula (and the consequent argument) to the three dimensional case.

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