NECESSARY AND SUFFICIENT CONDITIONS TO BE AN EIGENVALUE FOR LINEARLY RECURRENT DYNAMICAL CANTOR SYSTEMS

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Abstract

Necessary and sufficient conditions are given for linearly recurrent Cantor dynamical systems to have measurable and continuous eigenfunctions. Also an example of a linearly recurrent system with a nontrivial Kronecker factor and a trivial maximal equicontinuous factor is constructed explicitly.

1. *Introduction*

Let (X, T) be a topological dynamical system, that is, X is a compact metric space and $T: X \to X$ is a homeomorphism. Let μ be a T-invariant probability measure on X. In the classification of dynamical systems in ergodic theory and topological dynamics, rotation factors play a central role. In the measure-theoretical context this is reflected by the existence of a T-invariant sub-σ-algebra $\mathcal{K}_\mu$ of the Borel σ-algebra of X, $\mathcal{B}_X$, such that

$$L^2(X, \mathcal{K}_\mu, \mu) = \overline{\langle \{f \in L^2(X, \mathcal{B}_X, \mu) \setminus \{0\}; \exists\, \lambda \in \mathbb{C}, f \circ T = \lambda f\} \rangle}.$$

It is the subspace spanned by the eigenfunctions which determines the Kronecker factor. From a purely topological point of view the role of the Kronecker factor is played by the maximal equicontinuous factor. It can be defined in several ways. When (X, T) is minimal (all orbits are dense), it is determined by the continuous eigenfunctions. Thus it is relevant to ask whether there exist continuous eigenfunctions; or even under which conditions measure-theoretical eigenvalues can be associated to continuous eigenfunctions.

In [**6**] these questions are considered for substitutive systems and in [**1**] they are considered for *linearly recurrent systems*. These last systems are characterized by the existence of a nested sequence of clopen (for closed and open) Kakutani–Rokhlin partitions of the system $(\mathcal{P}(n); n \in \mathbb{N})$ verifying some technical conditions we call (KR1), (KR2), …, (KR6) (see below), and such that the height of the towers of each partition increases 'linearly' from one level to the other (see also [**2**, **3**]). A partial answer to the former question is given in terms of the sequence of matrices $(M(n); n \geqslant 1)$ relating towers from different levels in [**1**]. A complete answer to this question is given in the following theorem.

The third author acknowledges financial support from the Programa Iniciativa Científica Milenio P01-005 and FONDECYT 1010447. This project was also partially supported by the international cooperation programme ECOS-CONICYT C03-E03 and FONDECYT for international cooperation agreement 7040166.

We need some extra notations. For each real number x we write $|||x|||$ for the distance of x to the nearest integer. For a vector $V = (v_1, \dots, v_m)^T \in \mathbb{R}^m$, we write

$$\|V\| = \max_{1 \leqslant j \leqslant m} |v_j| \quad \text{and} \quad |||V||| = \max_{1 \leqslant j \leqslant m} |||v_j|||.$$

For $n \geqslant 2$ we put $P(n) = M(n) \dots M(2)$ and$H(1) = M(1)$.

THEOREM 1. *Let (X,T) be a linearly recurrent Cantor system given by an increasing sequence of clopen Kakutani–Rokhlin partitions with associated matrices $(M(n); n \geqslant 1)$, and let μ be the unique invariant measure. Let $\lambda = \exp(2i\pi\alpha)$.*

(1) *λ is an eigenvalue of (X,T) with respect to μ if and only if*

$$\sum_{n \geqslant 2} |||\alpha P(n) H(1)|||^2 < \infty.$$

(2) *λ is a continuous eigenvalue of (X,T) if and only if*

$$\sum_{n \geqslant 2} |||\alpha P(n) H(1)||| < \infty.$$

In [**1**] the authors prove the necessary condition in statement (1) and the sufficient condition in statement (2). One of the most relevant facts is that both conditions do not depend on the order of levels in the towers defining the system but just on the matrices.

2. *Definitions and background*

2.1. *Dynamical systems*

By a *topological dynamical system* we mean a couple (X,T) where X is a compact metric space and $T : X \to X$ is a homeomorphism. We say that it is a *Cantor system* if X is a Cantor space; that is, X has a countable basis of its topology which consists of closed and open sets (clopen sets) and does not have isolated points. We only deal here with minimal Cantor systems.

A complex number λ is a *continuous eigenvalue* of (X,T) if there exists a continuous function $f : X \to \mathbb{C}$, $f \neq 0$, such that $f \circ T = \lambda f$; f is called a *continuous eigenfunction* (associated to λ). Let μ be a T-invariant probability measure, that is $T\mu = \mu$, defined on the Borel σ-algebra $\mathcal{B}_X$ of X. A complex number λ is an *eigenvalue* of the dynamical system (X,T) with respect to μ if there exists $f \in L^2(X, \mathcal{B}_X, \mu)$, $f \neq 0$, such that $f \circ T = \lambda f$; f is called an *eigenfunction* (associated to λ). If the system is ergodic, then every eigenvalue is of modulus 1, and every eigenfunction has a constant modulus. Of course continuous eigenvalues are eigenvalues.

In this paper we mainly consider topological dynamical systems (X,T) which are uniquely ergodic and minimal, that is, systems that admit a unique invariant probability measure which is ergodic, and such that the unique T-invariant sets are X and $\varnothing$ module μ.

2.2. *Partitions and towers*

Sequences of partitions associated to minimal Cantor systems were used in [**5**] to build representations of such systems as adic transformations on ordered Bratteli diagrams. Here we do not introduce the whole formalism of Bratteli diagrams since

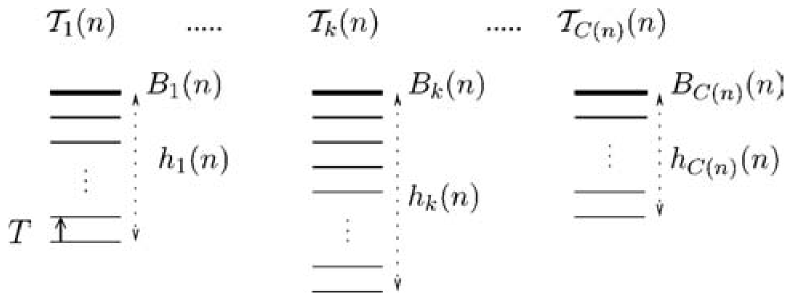

FIGURE 1. *Clopen Kakutani–Rokhlin partition of level* n. (i) X *is partitioned in* $C(n)$ *towers. Each tower* $\mathcal{T}_k(n)$, $1 \leqslant k \leqslant C(n)$, *is composed of* $h_k(n)$ *disjoint sets, called stages of the tower. The top of a tower is the roof* $B_k(n)$. (ii) *The dynamics of* T *consists in going up from one stage to the other of a tower up to the roof. Points in a roof are sent to the bottom of the towers; two points in the same roof can be sent to different towers.*

we will only use the language describing the tower structure. Both languages are very close. We recall some definitions and fix some notations.

Let (X, T) be a minimal Cantor system. A *clopen Kakutani–Rokhlin partition* is a partition $\mathcal{P}$ of X given by

$$\mathcal{P} = \{T^{-j} B_k ; 1 \leqslant k \leqslant C,\ 0 \leqslant j < h_k\}, \tag{2.1}$$

where C is a positive integer, $B_1, \dots, B_C$ are clopen subsets of X, and $h_1, \dots, h_k$ are positive integers. For $1 \leqslant k \leqslant C$, the kth *tower* of $\mathcal{P}$ is

$$\mathcal{T}_k = \bigcup_{j=0}^{h_k - 1} T^{-j} B_k$$

and its height is h_k; the *roof* of $\mathcal{P}$ is the set $B = \bigcup_{1 \leqslant k \leqslant C} B_k$. Let

$$(\mathcal{P}(n) = \{T^{-j} B_k(n); 1 \leqslant k \leqslant C(n),\ 0 \leqslant j < h_k(n)\};\ n \in \mathbb{N}) \tag{2.2}$$

be a sequence of clopen Kakutani–Rokhlin partitions. For every $n \in \mathbb{N}$ and $1 \leqslant k \leqslant C(n)$, $B(n)$ is the roof of $\mathcal{P}(n)$ and $\mathcal{T}_k(n)$ is the kth tower of $\mathcal{P}(n)$ (see Figure 1). We assume that $\mathcal{P}(0)$ is the trivial partition, that is, $B(0) = X$, $C(0) = 1$ and $h_1(0) = 1$.

We say that $(\mathcal{P}(n); n \in \mathbb{N})$ is *nested* if for every $n \in \mathbb{N}$ it satisfies the following.

(KR1) $B(n+1) \subseteq B(n)$.

(KR2) $\mathcal{P}(n+1) \succeq \mathcal{P}(n)$, that is, for all $A \in \mathcal{P}(n+1)$ there exists $A' \in \mathcal{P}(n)$ such that $A \subseteq A'$.

(KR3) $\bigcap_{n \in \mathbb{N}} B(n)$ consists of a unique point.

(KR4) The sequence of partitions spans the topology of X.

In [**5**] it is proved that given a minimal Cantor system (X, T) there exists a nested sequence of clopen Kakutani–Rokhlin partitions fulfilling (KR1)–(KR4) and the following additional technical conditions.

(KR5) For all $n \geqslant 1$, $1 \leqslant k \leqslant C(n-1)$, $1 \leqslant l \leqslant C(n)$, there exists $0 \leqslant j < h_l(n)$ such that $T^{-j} B_l(n) \subseteq B_k(n-1)$.

(KR6) For all $n \geqslant 1$, $B(n) \subseteq B_1(n-1)$.

We associate to $(\mathcal{P}(n); n \in \mathbb{N})$ the sequence of matrices $(M(n); n \geqslant 1)$, where $M(n) = (m_{l,k}(n); 1 \leqslant l \leqslant C(n), 1 \leqslant k \leqslant C(n-1))$ is given by

$$m_{l,k}(n) = \#\{0 \leqslant j < h_l(n); T^{-j} B_l(n) \subseteq B_k(n-1)\}.$$

Notice that (KR5) is equivalent to the following: for all $n \geqslant 1$, $M(n)$ has strictly positive entries. For $n \geqslant 0$, set $H(n) = (h_l(n); 1 \leqslant l \leqslant C(n))^T$. As the sequence of partitions is nested $H(n) = M(n)H(n-1)$ for $n \geqslant 1$. Notice that $H(1) = M(1)$. For $n > m \geqslant 0$ we define

$$P(n,m) = M(n)M(n-1)\dots M(m+1) \quad \text{and} \quad P(n) = P(n,1).$$

Clearly

$$P_{l,k}(n,m) = \#\{0 \leqslant j < h_l(n);\ T^{-j}B_l(n) \subseteq B_k(m)\}$$

for $1 \leqslant l \leqslant C(n),\ 1 \leqslant k \leqslant C(m)$, and

$$P(n,m)H(m) = H(n) = P(n)H(1).$$

2.3. *Linearly recurrent systems*

The notion of linearly recurrent minimal Cantor system (also called linearly recurrent system) in the generality we present below was stated in [**1**]. It is an extension of the concept of linearly recurrent subshift introduced in [**4**].

Definition 2. A minimal Cantor system (X,T) is *linearly recurrent* (with constant L) if there exists a nested sequence of clopen Kakutani–Rokhlin partitions $(\mathcal{P}(n) = \{T^{-j}B_k(n); 1 \leqslant k \leqslant C(n), 0 \leqslant j < h_k(n)\}; n \in \mathbb{N})$ satisfying (KR1)–(KR6) and the following.

(LR) There exists L such that for all $n \geqslant 1$, $l \in \{1,\dots,C(n)\}$ and $k \in \{1,\dots,C(n-1)\}$,

$$h_l(n) \leqslant L\ h_k(n-1).$$

Most of the basic dynamical properties of linearly recurrent minimal Cantor systems are described in [**1**]. In particular, they are uniquely ergodic and the unique invariant measure is never strongly mixing. In addition, $C(n) \leqslant L$ for any $n \in \mathbb{N}$ and the set of matrices $\{M(n); n \geqslant 1\}$ is finite.

To prove Theorem 1 we need to consider the property

(KR5$'$) for all $n \geqslant 2$, $1 \leqslant k \leqslant C(n-1)$, $1 \leqslant l \leqslant C(n)$, there exist $0 \leqslant j < j' < h_l(n)$ such that $T^{-j}B_l(n) \subseteq B_k(n-1)$ and $T^{-j'}B_l(n) \subseteq B_k(n-1)$;

instead of (KR5). This condition is equivalent to saying that the coefficients of $M(n)$ are strictly larger than 1 for $n \geqslant 2$.

Let (X,T) be a linearly recurrent system given by a nested sequence of clopen Kakutani–Rokhlin partitions $(\mathcal{P}(n); n \in \mathbb{N})$ which verifies (KR1)–(KR6) and (LR). Then the sequence of partitions defined by $\mathcal{P}'(0) = \mathcal{P}(0)$ and $\mathcal{P}'(n) = \mathcal{P}(2n-1)$ for $n \geqslant 1$ is a sequence of nested clopen Kakutani–Rokhlin partitions of the system which verifies (KR1)–(KR4), (KR5$'$), (KR6) and (LR) (with another constant). It follows that $M'(1) = M(1)$ and $M'(n) = M(2n-1)M(2n-2)$ for $n \geqslant 2$, where $(M(n); n \geqslant 1)$ and $(M'(n); n \geqslant 1)$ are the sequence of matrices associated to the partitions $(\mathcal{P}(n); n \in \mathbb{N})$ and $(\mathcal{P}'(n); n \in \mathbb{N})$, respectively. Moreover,

$$\sum_{n \geqslant 2} |||\alpha P(n)H(1)|||^p < \infty \ \Leftrightarrow\ \sum_{n \geqslant 2} |||\alpha P'(n)H(1)|||^p < \infty, \tag{2.3}$$

where $\alpha \in \mathbb{R}$ and $p \in \{1,2\}$.

3. *Markov chain associated to a linearly recurrent system*

Let (X,T) be a linearly recurrent system and let μ be its unique invariant measure. Consider a sequence $(\mathcal{P}(n); n \geqslant 0)$ of clopen Kakutani–Rokhlin partitions which satisfies (KR1)–(KR6) and (LR) with constant L and let $(M(n); n \geqslant 1)$ be the sequence of matrices associated. The purpose of this section is to formalize the fact that there exists a Markovian measurable structure behind the tower structure.

The following relation will be of constant use in this paper. For $n \geqslant 1$ put $\mu(n) = (\mu(B_t(n)); 1 \leqslant t \leqslant C(n))$ (the vector of measures of the roofs at level n). It follows directly from the structure of towers that for $1 \leqslant k < n$

$$\mu(n-k) = M^T(n-k+1) \dots M^T(n)\mu(n). \tag{3.1}$$

3.1. *First entrance times and combinatorial structure of the towers*

In this subsection we define several concepts that will be used extensively later. They are illustrated in Figure 2.

Define the first entrance time map to the roof $B(n)$, $r_n : X \to \mathbb{N}$, by

$$r_n(x) = \min\{j \geqslant 0; T^j(x) \in B(n)\}.$$

Since (X,T) is minimal and $B(n)$ is a clopen set, then r_n is finite and continuous. Define the tower of level n map $\tau_n : X \to \mathbb{N}$ by

$$\tau_n(x) = k \text{ if and only if } x \in \mathcal{T}_k(n) \text{ for some } 1 \leqslant k \leqslant C(n).$$

Note that

$$r_n(T(x)) - r_n(x) = \begin{cases} -1 & \text{if } x \notin B(n) \\ h_k(n) - 1 & \text{if } x \in B(n) \text{ and } \tau_n(T(x)) = k. \end{cases} \tag{3.2}$$

Let $n \geqslant 1$ and $1 \leqslant t \leqslant C(n)$. By hypothesis (KR5), several stages in the tower $\mathcal{T}_t(n)$ are included in the roof $B(n-1)$, and in particular stage $B_t(n)$. The number of such stages is

$$m_t(n) = \sum_{k=1}^{C(n-1)} m_{t,k}(n) = \#\{0 \leqslant j < h_t(n); T^{-j} B_t(n) \subseteq B(n-1)\}.$$

Let $\{e_1, e_2, \dots, e_{m_t(n)}\} = \{0 \leqslant j < h_t(n); T^{-j} B_t(n) \subseteq B(n-1)\}$ with $h_t(n) > e_1 > e_2 > \dots > e_{m_t(n)} = 0$. The integers $e_1, \dots, e_{m_t(n)}$ are the first entrance times of points belonging to $\mathcal{T}_t(n) \cap B(n-1)$ into $B_t(n)$. Moreover, for all $1 \leqslant l \leqslant m_t(n)$ there is a unique $k \in \{1, \dots, C(n-1)\}$ such that

$$T^{-e_l} B_t(n) \subseteq B_k(n-1).$$

Denote this k by $\theta_l^t(n-1)$. From (KR6) we have

$$\theta^t_{m_t(n)}(n-1) = 1. \tag{3.3}$$

We set

$$\theta^t(n-1) = \theta_1^t(n-1) \dots \theta^t_{m_t(n)}(n-1) \in \{1, \dots, C(n-1)\}^*. \tag{3.4}$$

Note that $e_l - e_{l+1}$ is the height of the $\theta^t_{l+1}(n-1)$th tower of $\mathcal{P}(n-1)$ for $1 \leqslant l < m_t(n)$. Thus

$$e_l = \sum_{k=l+1}^{m_t(n)} h_{\theta^t_k(n-1)}(n-1).$$

Now, the tower $\mathcal{T}_t(n)$ can be decomposed as a disjoint union of the towers of $\mathcal{P}(n-1)$ it intersects. More precisely, $\mathcal{T}_t(n) = \bigcup_{l=1}^{m_t(n)} \mathcal{E}_{l,t}(n-1)$, where

$$\mathcal{E}_{l,t}(n-1) = \bigcup_{j=e_{l-1}-1}^{e_l} T^{-j} B_t(n) = \bigcup_{j=0}^{h_{\theta^t_l(n-1)}(n-1)-1} T^{-j-e_l} B_t(n).$$

By definition,

$$\mathcal{E}_{l,t}(n-1) \subseteq \bigcup_{j=0}^{h_{\theta^t_l(n-1)}(n-1)-1} T^{-j} B_{\theta^t_l(n-1)}(n-1).$$

For $x \in X$, denote by $l_n(x)$ the unique integer in $\{1, \ldots, m_{\tau_n(x)}(n)\}$ such that $x \in \mathcal{E}_{l_n(x), \tau_n(x)}(n-1)$. The following lemma follows from the construction. The proof is left to the reader.

Lemma 3. *For all $x \in X$ we have*

$$\bigcap_{k=1}^{n} \mathcal{E}_{l_k(x), \tau_k(x)}(k-1) = T^{-r_n(x)} B_{\tau_n(x)}(n), \tag{3.5}$$

$$\{x\} = \bigcap_{n \geqslant 1} \mathcal{E}_{l_n(x), \tau_n(x)}(n-1). \tag{3.6}$$

Moreover, given that

$$(t_n; n \geqslant 0) \in \prod_{n \geqslant 0} \{1, \ldots, C(n)\}, \quad (j_n; n \geqslant 1) \in \prod_{n \geqslant 1} \{1, \ldots, m_{t_n}(n)\}$$

such that $\theta^{t_n}_{j_n}(n-1) = t_{n-1}$ for $n \geqslant 1$, there exists a unique $x \in X$ such that

$$((l_n(x), \tau_n(x)); n \geqslant 1) = ((j_n, t_n); n \geqslant 1). \tag{3.7}$$

Note that the set in (3.5) is the atom of the partition $P(n)$ containing x.

For all $n \geqslant 1$ and $x \in X$ define $s_{n-1}(x) = (s_{n-1,t}(x); 1 \leqslant t \leqslant C(n-1))$ by

$$s_{n-1,t}(x) = \#\{j; r_{n-1}(x) < j \leqslant r_n(x),\ T^j x \in B_t(n-1)\}.$$

It also holds that

$$s_{n-1,t}(x) = \#\{j; l_n(x) < j \leqslant m_{\tau_n(x)}(n),\ \theta^{\tau_n(x)}_j(n-1) = t\}.$$

In other words, the vector $s_{n-1}(x)$ counts, in each coordinate $1 \leqslant t \leqslant C(n-1)$, the number of times the tower $\mathcal{T}_t(n-1)$ is crossed by a point x, after its first return to the roof of level $n-1$, and before reaching the roof of the tower of level n it belongs to. Notice that s_{n-1} does not consider the order in which the towers are visited. In Figure 2 we illustrate the notations introduced previously.

A direct computation yields the following lemma. It will be used extensively in the sequel. Denote by $\langle \cdot, \cdot \rangle$ the usual scalar product.

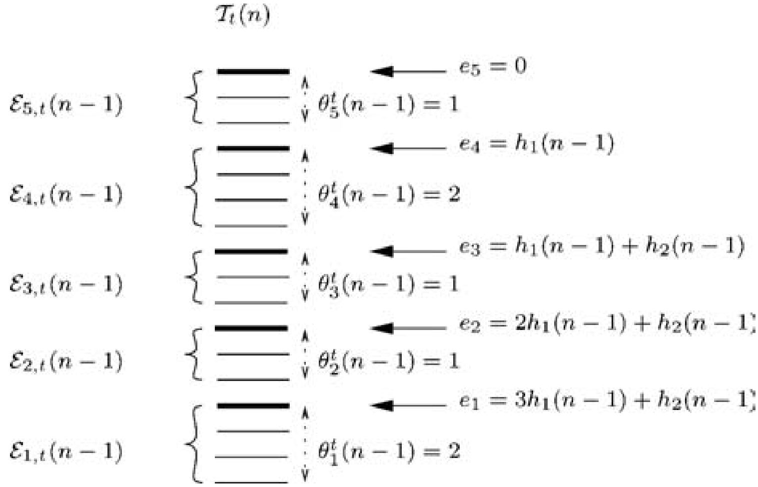

FIGURE 2. *A tower t of $\mathcal{P}(n)$ in an example. We assume that in $\mathcal{P}(n-1)$ there are only two towers and that $m_t(n) = 5$. If $x \in \mathcal{E}_{1,t}(n-1)$, then $s_{n-1}(x) = (3,1)^T$ and $l_n(x) = 1$. If $x \in \mathcal{E}_{4,t}(n-1)$, then $s_{n-1}(x) = (1,0)^T$ and $l_n(x) = 4$.*

LEMMA 4. *For all $x \in X$ and all $n \geqslant 2$ it holds that*

$$r_1(x) = s_0(x), \quad r_n(x) = r_{n-1}(x) + \langle s_{n-1}(x), H(n-1) \rangle,$$
$$r_n(x) = \sum_{j=2}^{n-1} \langle s_j(x), P(j)H(1) \rangle + \langle s_1(x), H(1) \rangle + s_0(x).$$

3.2. *Markov property for the towers*

Now we prove that the sequence of random variables $(\tau_n; n \in \mathbb{N})$ is a non-stationary Markov chain. We need some preliminary computations. Let $n \geqslant 1$. From Lemma 3 we have

$$\mu\big(B_{\tau_n(x)}(n)\big) = \mu\left(\bigcap_{k=1}^{n} \mathcal{E}_{l_k(x), \tau_k(x)}(k-1) \right).$$

Let $(t_i \in \{1, \ldots, C(i)\}; 0 \leqslant i \leqslant n)$. The set $[\tau_n = t_n]$ is the tower $\mathcal{T}_{t_n}(n)$. For $0 \leqslant k < n$, $\tau_k(x)$ is constant on each level of $\mathcal{T}_{t_n}(n)$. By a simple induction, the number of levels of this tower where $\tau_0(x) = t_0, \ldots, \tau_{n-1}(x) = t_{n-1}$ is equal to $m_{t_1,t_0}(1) \ldots m_{t_n,t_{n-1}}(n)$. In other words, the set $[\tau_0 = t_0, \ldots, \tau_n = t_n]$ is the union of $m_{t_1,t_0}(1) \ldots m_{t_n,t_{n-1}}(n)$ levels of the tower $\mathcal{T}_{t_n}(n)$ and

$$\mu[\tau_0 = t_0, \ldots, \tau_n = t_n] = m_{t_1,t_0}(1) \ldots m_{t_n,t_{n-1}}(n) \mu(B_{t_n}(n)). \tag{3.8}$$

In particular, from the last equality and the definition of the matrices $(M(n); n \geqslant 1)$ we deduce that

$$\mu[\tau_n = t_n | \tau_{n-1} = t_{n-1}] = \frac{m_{t_n,t_{n-1}}(n) \mu(B_{t_n}(n))}{\mu(B_{t_{n-1}}(n-1))}.$$

Now, given the sequence $(\mathcal{P}(n); n \in \mathbb{N})$ we can prove that $(\tau_n; n \in \mathbb{N})$ is a Markov chain on the probability space $(X, \mathcal{B}_X, \mu)$. Therefore, by (3.1), the matrix $Q(n) = (q_{t,\bar{t}}(n); 1 \leqslant \bar{t} \leqslant C(n), 1 \leqslant t \leqslant C(n-1))$ with

$$q_{t,\bar{t}}(n) = \frac{m_{\bar{t},t}(n)\mu(B_{\bar{t}}(n))}{\mu(B_t(n-1))}$$

is a stochastic matrix.

LEMMA 5. *The sequence of random variables* $(\tau_n; n \in \mathbb{N})$ *is a non-stationary Markov chain with associated stochastic matrices* $(Q(n); n \geqslant 1)$.

Proof. From (3.8) we get

$$\begin{aligned}
\mu[\tau_n &= \bar{t} | \tau_{n-1} = t, \tau_{n-2} = t_{n-2}, \ldots, \tau_0 = t_0] \\
&= \frac{m_{t_1,t_0}(1) \ldots m_{t,t_{n-2}}(n-1) m_{\bar{t},t}(n)\mu(B_{\bar{t}}(n))}{m_{t_1,t_0}(1) \ldots m_{t,t_{n-2}}(n-1)\mu(B_t(n-1))} \\
&= \frac{m_{\bar{t},t}(n)\mu(B_{\bar{t}}(n))}{\mu(B_t(n-1))} \\
&= \mu[\tau_n = \bar{t} | \tau_{n-1} = t] \\
&= q_{t,\bar{t}}(n).
\end{aligned}$$

□

The following lemma provides an exponential mixing property for non-stationary ergodic Markov chains. It is a standard result. The proof can be adapted from that of [**8**, Corollary 2, p. 141]. That is, this corollary can be generalized to the case of a non-stationary Markov chain where the stochastic matrices have not necessarily the same dimension. Alternatively, a direct proof follows from [**8**, inequality (3.3), Theorem 3.1, p. 81] in the case of our particular matrices.

LEMMA 6. *Let* $(\tau_n; n \in \mathbb{N})$ *be the non-stationary Markov chain defined in the previous subsection. There exist* $c \in \mathbb{R}_+$ *and* $\beta \in [0, 1[$ *such that for all* $n, k \in \mathbb{N}$, *with* $k \leqslant n$,

$$\sup_{1 \leqslant t \leqslant C(n-k), 1 \leqslant \bar{t} \leqslant C(n)} |\mu[\tau_n = \bar{t} | \tau_{n-k} = t] - \mu[\tau_n = \bar{t}\,]| \leqslant c\beta^k.$$

4. *Measurable eigenvalues*

The main purpose of this section is to prove statement (1) of Theorem 1 (this is done in Subsection 4.2). In the first subsection we give a general necessary and sufficient condition to be a measurable eigenfunction of a minimal Cantor system.

4.1. *A necessary and sufficient condition to be an eigenvalue*

We give a general necessary and sufficient condition to be an eigenvalue. We do not use it directly to prove our result, but we think it gives an idea of the classical way to tackle the problem and shows that the difficulty relies in understanding the stochastic behaviour of the sequence $(r_n; n \in \mathbb{N})$. We would like to stress the fact that we still do not have a convincing interpretation of the sequence of functions ρ_n which appears in the next theorem.

THEOREM 7. *Let (X, T) be a minimal Cantor system and let μ be an invariant measure. Let $(\mathcal{P}(n); n \in \mathbb{N})$ be a sequence of clopen Kakutani–Rokhlin partitions verifying* (KR1)–(KR4). *A complex number $\lambda = \exp(2i\pi\alpha)$ is an eigenvalue of (X, T) with respect to μ if and only if there exist real functions $\rho_n : \{1, \ldots, C(n)\} \to \mathbb{R}$, $n \in \mathbb{N}$, such that*

$$\alpha(r_n(x) + \rho_n \circ \tau_n(x)) \text{ converges } (\mathrm{mod}\ \mathbb{Z}) \tag{4.1}$$

for μ-almost every $x \in X$ when n tends to infinity.

Proof. Let $\lambda = \exp(2i\pi\alpha)$ be a complex number of modulus 1 such that (4.1) holds and let g be the corresponding limit function. Consider $x \notin \bigcap_{n \in \mathbb{N}} B(n)$ so x does not belong to $B(n)$ for all large enough $n \in \mathbb{N}$. Then from (3.2), we get

$$\frac{\exp(2i\pi g(Tx))}{\exp(2i\pi g(x))} = \lim_{n \to \infty} \lambda^{r_n(Tx) - r_n(x)} = \lambda^{-1}.$$

This implies that λ is an eigenvalue of (X, T) with respect to μ.

Now, assume that λ is an eigenvalue of (X, T) with respect to μ and let $g \in L^2(X, \mathcal{B}_X, \mu)$ be an associated eigenfunction. For all $n \in \mathbb{N}$ let $\phi_n = \lambda^{-r_n}$ and $\psi_n = g/\phi_n$. The map ϕ_n is $\mathcal{P}(n)$-measurable and bounded, so

$$\phi_n \mathbb{E}_\mu(\psi_n | \mathcal{P}(n)) = \mathbb{E}_\mu(\phi_n \psi_n | \mathcal{P}(n)) = \mathbb{E}_\mu(g | \mathcal{P}(n)) \xrightarrow[n \to \infty]{} g$$

μ-almost everywhere. Since $\psi_n \circ T^{-j}/\psi_n = \lambda^{r_n \circ T^{-j} - r_n - j}$, the restriction of ψ_n to each tower of level n is invariant under T. Thus $\mathbb{E}_\mu(\psi_n | \mathcal{P}(n))$ is constant on each of these towers and is therefore equal to the average of ψ_n on each tower.

To finish, for $1 \leqslant i \leqslant C(n)$ we define $\rho_n(i)$ such that

$$\mathrm{Arg}\lambda^{-\rho_n(i)} = \mathrm{Arg}\left(\frac{1}{\mu(B_i(n))} \int_{B_i(n)} \psi_n \, \mathrm{d}\mu\right).$$

This ends the proof. □

REMARK 8. The same proof works if we remove the Cantor and clopen hypotheses.

4.2. *Eigenvalues of linearly recurrent systems*

In this subsection we prove statement (1) of Theorem 1. Recall that (X, T) is linearly recurrent and μ is the unique invariant measure. Let $(\mathcal{P}(n); n \geqslant 0)$ be a sequence of clopen Kakutani–Rokhlin partitions such that (KR1)–(KR6) and (LR) with constant L are satisfied. Let $(M(n); n \geqslant 1)$ be the associated sequence of matrices.

We will need the following lemma. Its proof can be found in [**1**].

LEMMA 9. *Let $u \in \mathbb{R}^{C(1)}$ be a real vector such that $\|P(n)u\| \to 0$ as $n \to \infty$. Then there exist $m \geqslant 2$, an integer vector $w \in \mathbb{Z}^{C(m)}$ and a real vector $v \in \mathbb{R}^{C(m)}$ with*

$$P(m)u = v + w \quad \text{and} \quad \|P(n, m)v\| \to 0 \text{ as } n \to \infty.$$

Assume that the following condition holds.

$$\sum_{n\geqslant 2} \|\alpha P(n)H(1)\|^2 < \infty. \tag{4.2}$$

Then $\|P(n)(\alpha H(1))\| \to 0$ as $n \to \infty$. From Lemma 9 there exist an integer $n_0 \geqslant 2$, a real vector $v \in \mathbb{R}^{C(n_0)}$ and an integer vector $w \in \mathbb{Z}^{C(n_0)}$ such that $P(n_0)(\alpha H(1)) = v + w$ and $P(n, n_0)v \to 0$ as $n \to \infty$. By modifying a finite number of towers, if needed, we can assume without loss of generality that $n_0 = 1$ and that $H(1) = (1, \ldots, 1)^T$. Thus condition (4.2) implies that

$$\sum_{n\geqslant 2} \|P(n)v\|^2 < \infty. \tag{4.3}$$

From (2.3), we can also assume without loss of generality that (KR5′) holds. That is, entries of matrices $M(n)$ are larger than 2 for all $n \geqslant 2$.

For $n \geqslant 1$ we define $g_n : X \to \mathbb{R}$ by

$$g_n(x) = s_0(x) + \langle s_1(x), v\rangle + \sum_{j=2}^{n-1} \langle s_j(x), P(j)v\rangle.$$

Since we are assuming that $H(1) = (1, \ldots, 1)^T$, then $s_0 = 0$ and

$$g_n(x) = \sum_{j=1}^{n-1} \langle s_j(x), P(j)v\rangle,$$

where we set $P(1) = \mathrm{Id}$.

Lemma 10. *If (4.2) holds, then the sequence $(f_n = g_n - \mathbb{E}_\mu(g_n); n \geqslant 1)$ converges in $L^2(X, \mathcal{B}_X, \mu)$.*

Proof. Let $n \geqslant 1$. Recall that $\mathcal{P}(n)$ is the partition of level n and let $\mathcal{T}(n)$ be the coarser partition $\{\mathcal{T}_j(n); 1 \leqslant j \leqslant C(n)\}$. As usual we identify the finite partitions with the σ-algebras they span and we use the same notation. Thus $\mathcal{T}(n)$ is the σ-algebra spanned by the random variable τ_n.

Let X_n be the random variable given by

$$X_n = \langle s_n, P(n)v\rangle - \mathbb{E}_\mu(\langle s_n, P(n)v\rangle).$$

We decompose it as $X_n = Y_n + Z_n$, where

$$Y_n = \mathbb{E}_\mu(X_n|\mathcal{P}(n)) \quad \text{and} \quad Z_n = \langle s_n, P(n)v\rangle - \mathbb{E}_\mu(\langle s_n, P(n)v\rangle|\mathcal{P}(n)).$$

We write $\kappa_n = \|P(n)v\|$. Observe that there is a positive constant K such that for all $n \geqslant 1$ we have $|X_n| \leqslant K\kappa_n$, $|Y_n| \leqslant K\kappa_n$ and $|Z_n| \leqslant K\kappa_n$.

First we show that the series $\sum Z_n$ converges. Let m and n be positive integers with $m < n$. The random variable Z_m is measurable with respect to $\mathcal{P}(m+1)$, and thus also with respect to $\mathcal{P}(n)$. Since $\mathbb{E}_\mu(Z_n|\mathcal{P}(n)) = 0$ we get $\mathbb{E}_\mu(Z_m \cdot Z_n) = 0$. As $|Z_n| \leqslant K\kappa_n$ for every $n \geqslant 1$, the series $\sum \mathbb{E}_\mu(Z_n^2)$ converges, and thus the orthogonal series $\sum Z_n$ converges in $L^2(X, \mathcal{B}_X, \mu)$.

Now we prove that the series $\sum Y_n$ converges in $L^2(X, \mathcal{B}_X, \mu)$. Fix $1 \leqslant \bar{t} \leqslant C(n+1)$ and $1 \leqslant j \leqslant m_{\bar{t}}(n+1)$. The set $\mathcal{E}_{j,\bar{t}}(n)$ is included in the tower $\mathcal{T}_t(n)$ where $t = \theta_j^{\bar{t}}(n)$. Moreover, the intersections of all levels of $\mathcal{T}_t(n)$ with $\mathcal{E}_{j,\bar{t}}(n)$ are levels of $\mathcal{T}_{\bar{t}}(n+1)$ (see Figure 2) and thus have the same measure $\mu(B_{\bar{t}}(n+1))$. As each level of the tower $\mathcal{T}_t(n)$ has measure $\mu(B_t(n))$ we have

$$\mu(\mathcal{E}_{j,\bar{t}}(n)|\mathcal{P}(n))(x) = \begin{cases} \dfrac{\mu(B_{\bar{t}}(n+1))}{\mu(B_t(n))} & \text{if } x \in \mathcal{T}_t(n) \\ 0 & \text{otherwise.} \end{cases} \tag{4.4}$$

Observe that this conditional probability is constant in each atom of $\mathcal{T}(n)$ and thus

$$\mu(\mathcal{E}_{j,\bar{t}}(n)|\mathcal{P}(n)) = \mu(\mathcal{E}_{j,\bar{t}}(n)|\mathcal{T}(n)).$$

As s_n is constant on each set $\mathcal{E}_{j,\bar{t}}(n)$, the same property holds for X_n and thus

$$Y_n = \mathbb{E}_\mu(X_n|\mathcal{P}(n)) = \mathbb{E}_\mu(X_n|\mathcal{T}(n)). \tag{4.5}$$

In particular, Y_n is equal to a constant on each set $[\tau_n = \bar{t}\,]$ and we write $y_{\bar{t}}$ for this constant. Fix k with $0 \leqslant k \leqslant n$. If $\tau_{n-k}(x) = t$ then

$$\mathbb{E}_\mu(Y_n|\mathcal{T}(n-k))(x) = \sum_{\bar{t}=1}^{C(n)} \mu[\tau_n = \bar{t}\ |\tau_{n-k} = t]y_{\bar{t}}.$$

From $\sum_{\bar{t}=1}^{C(n)} \mu[\tau_n = \bar{t}\,]y_{\bar{t}} = \mathbb{E}_\mu(Y_n) = 0$ we get

$$|\mathbb{E}_\mu(Y_n|\mathcal{T}(n-k))(x)| \leqslant \sum_{\bar{t}=1}^{C(n)} |\mu[\tau_n = \bar{t}|\tau_{n-k} = t] - \mu[\tau_n = \bar{t}]||y_{\bar{t}}|.$$

From Lemma 6, we deduce the fact that $C(n)$ is bounded independently of n and $|Y_n| \leqslant K\kappa_n$ so that for some positive constant C

$$|\mathbb{E}_\mu(Y_n|\mathcal{T}(n-k))(x)| \leqslant \sum_{\bar{t}=1}^{C(n)} |\mu[\tau_n = \bar{t}|\tau_{n-k} = t] - \mu[\tau_n = \bar{t}]||y_{\bar{t}}| \leqslant C\beta^k\kappa_n.$$

As Y_{n-k} is measurable with respect to $\mathcal{T}(n-k)$ we have

$$|\mathbb{E}_\mu(Y_n \cdot Y_{n-k})| \leqslant C\beta^k\kappa_n\mathbb{E}_\mu(|Y_{n-k}|) \leqslant C\beta^k\kappa_n\kappa_{n-k}.$$

For $1 \leqslant m < n$ we compute

$$\begin{aligned}
\mathbb{E}_\mu\left(\left(\sum_{k=m}^{n} Y_k\right)^2\right) &= \sum_{m\leqslant j,l\leqslant n} \mathbb{E}_\mu(Y_j \cdot Y_l) \leqslant C \sum_{m\leqslant j,l\leqslant n} \beta^{|j-l|}\kappa_j\kappa_l \\
&\leqslant C\sum_{r=0}^{n-m} \beta^r \sum_{l=m}^{n-r} \kappa_l\kappa_{l+r} \leqslant 2C\sum_{r=0}^{n-m} \beta^r \sum_{l=m}^{n} \kappa_l^2 \\
&\leqslant \frac{2C}{1-\beta}\sum_{l=m}^{n} \kappa_l^2.
\end{aligned}$$

Since the series $\sum \kappa_j^2$ converges, the partial sums of the series $\sum Y_n$ form a Cauchy sequence in $L^2(X, \mathcal{B}_X, \mu)$. □

The following lemma completes the proof of statement (1) of Theorem 1.

LEMMA 11. *Let $f \in L^2(X, \mathcal{B}_X, \mu)$ be the limit of the sequence $(f_n; n \geqslant 1)$. The function* $\exp(2i\pi f)$ *is an eigenfunction of (X, T) with respect to μ associated to the eigenvalue* $\exp(2i\pi\alpha)$.

Proof. Remark that $g_n(x) = \alpha r_{n-1}(x) \pmod{\mathbb{Z}}$. From relation (3.2),

$$f_n(Tx) = f_n(x) - \alpha \pmod{\mathbb{Z}}$$

holds outside of the roof $B(n)$ and $\mu(B(n)) \to 0$ as $n \to \infty$. We conclude using Lemma 10. □

5. *Continuous eigenvalues of linearly recurrent systems*

Let (X, T) be a linearly recurrent dynamical system with constant L. The main purpose of this section is to prove the necessary condition in statement (2) of Theorem 1. We recall that the sufficient condition was proved in [**1**].

5.1. *A necessary and sufficient condition to be a continuous eigenvalue*

In this subsection we only assume that $(\mathcal{P}(n); n \in \mathbb{N})$ is a sequence of clopen Kakutani–Rokhlin partitions describing the system (X, T) which satisfies (KR1)–(KR6). We give a general necessary and sufficient condition to be a continuous eigenvalue.

PROPOSITION 12. *Let $\lambda = \exp(2i\pi\alpha)$ be a complex number of modulus* 1*. The following conditions are equivalent.*

(i) *λ is a continuous eigenvalue of the minimal Cantor system (X, T).*

(ii) *$(\lambda^{r_n(x)}; n \geqslant 1)$ converges uniformly in x, that is, the sequence $(\alpha r_n(x); n \geqslant 1)$ converges* $\pmod{\mathbb{Z}}$ *uniformly in x.*

Proof. We start by proving that (1) implies (2). Let g be a continuous eigenfunction associated to λ. For all $n \geqslant 1$ and all $x \in X$ we have $T^{r_n(x)}(x) \in B(n) \subseteq B_1(n-1)$ (the last inclusion is due to (KR6)). Hence, using (KR3), we deduce that $\lim_{n\to\infty} T^{r_n(x)}(x) = u$ uniformly in x, where u is the unique element of $\bigcap_{n\geqslant 0} B(n)$. Because the eigenfunction g is uniformly continuous, $\lambda^{r_n(x)} = g(T^{r_n(x)}(x))/g(x)$ tends to $g(u)/g(x)$ uniformly in x.

Now we prove that (2) implies (1). We set $\phi(x) = \lim_{n\to\infty} \lambda^{r_n(x)}$. Since the convergence is uniform and r_n is continuous, then ϕ is continuous.

Let x be such that $x \notin B(n)$ for infinitely many n. Then, from (3.2), we obtain $\phi(T(x)) = \lambda^{-1}\phi(x)$. Using the minimality of (X, T) and the continuity of ϕ, we obtain $\phi(T(y)) = \lambda^{-1}\phi(y)$ for all $y \in X$. Consequently λ is a continuous eigenvalue. □

COROLLARY 13. *Let λ be a complex number of modulus* 1*.*

(1) *If λ is a continuous eigenvalue of (X, T), then*

$$\lim_{n\to\infty} \lambda^{h_{j_n}(n)} = 1$$

uniformly in $(j_n; n \in \mathbb{N}) \in \prod_{n\in\mathbb{N}}\{1, \ldots, C(n)\}$.

(2) *If*

$$\sum_{m\geqslant 1} \left(\frac{\sup_{k\in\{1,\ldots,C(m+1)\}} h_k(m+1)}{\inf_{k\in\{1,\ldots,C(m)\}} h_k(m)} \right) \sup_{k\in\{1,\ldots,C(m)\}} \left|\lambda^{h_k(m)} - 1\right| < \infty,$$

then λ is a continuous eigenvalue of (X, T).

Proof. Let g be a continuous eigenfunction of λ. Then it is uniformly continuous. Let $\epsilon > 0$. There exists $n_0 \in \mathbb{N}$ such that $|g(y) - g(u)| < \epsilon/2$ for all $y \in B_1(n_0)$, where $\{u\} = \bigcap_{n\in\mathbb{N}} B(n)$.

Let $(j_n; n \in \mathbb{N}) \in \prod_{n\in\mathbb{N}}\{1, \ldots, C(n)\}$. For all $n \in \mathbb{N}$ we take $x(n) \in B_{j_n}(n)$ and we set $y(n) = T^{-h_{j_n}(n)}(x(n)) \in B(n)$. Hence, using (KR6), for all $n \geqslant n_0 + 1$ the points $x(n)$ and $y(n)$ belong to $B(n) \subseteq B_1(n_0)$. Consequently

$$\begin{aligned} \left|\lambda^{h_{j_n}(n)} - 1\right| &= \left|g\left(T^{h_{j_n}(n)} y(n)\right) - g(y(n))\right| \\ &\leqslant |g(x(n)) - g(u)| + |g(u) - g(y(n))| < \epsilon. \end{aligned}$$

Now we prove (2). It suffices to remark, by Lemma 4, that for all $x \in X$ and all $0 < n < m$,

$$\begin{aligned} \left|\lambda^{r_m(x)} - \lambda^{r_n(x)}\right| &= \left|1 - \lambda^{r_m(x) - r_n(x)}\right| \\ &\leqslant \sum_{l=n}^{m-1} \frac{\sup_{k\in\{1,\ldots,C(l+1)\}} h_k(l+1)}{\inf_{k\in\{1,\ldots,C(l)\}} h_k(l)} \sup_{k\in\{1,\ldots,C(l)\}} \left|1 - \lambda^{h_k(l)}\right|. \end{aligned}$$

Hence, from Proposition 12, λ is a continuous eigenvalue. □

Remark that for linearly recurrent systems, statement (2) gives the sufficient condition for λ to be a continuous eigenvalue. This was proved in [**1**].

5.2. *The linearly recurrent case*

Now we assume that (X, T) is linearly recurrent and we prove Theorem 1(2). We also assume without loss of generality that the sequence of partitions verifies (KR5′), that is, entries of $M(n)$ are bigger than 2 for any $n \geqslant 2$ (see the discussion in Subsection 2.3). To prove the result we introduce an intermediate statement which gives a more precise interpretation to the necessary condition.

PROPOSITION 14. *Let $\lambda = \exp(2i\pi\alpha)$ be a complex number of modulus* 1. *The following properties are equivalent.*

(1) *λ is a continuous eigenvalue of the minimal Cantor system (X, T).*

(2) *There exist $n_0 \in \mathbb{N}$, $v \in \mathbb{R}^{C(n_0)}$, $z \in \mathbb{Z}^{C(n_0)}$, such that $\alpha P(n_0)H(1) = v + z$, $P(n, n_0)v \to 0$ as $n \to \infty$ and the series*

$$\sum_{j\geqslant n_0+1} \langle s_j(x), P(j, n_0)v\rangle$$

converges for every $x \in X$.

(3) $\sum_{n\geqslant 2} \|\alpha P(n)H(1)\| < \infty$.

Proof. In [**1**] it is proved that (3) implies (1).

We prove that (1) implies (2). Assume that λ is a continuous eigenvalue of (X, T). We deduce from statement (1) of Corollary 13 that $\|\alpha P(n)H(1)\|$ converges to 0 as n tends to ∞. By Lemma 9, there are $n_0 \in \mathbb{N}$, $v \in \mathbb{R}^{C(n_0)}$ and $z \in \mathbb{Z}^{C(n_0)}$ such that $\alpha P(n_0)H(1) = v + z$ and $P(n, n_0)v \to 0$ as $n \to \infty$. By modifying a finite number of towers we can assume without loss of generality that $n_0 = 1$.

By Lemma 4, for $n \geqslant 1$ and $x \in X$, $r_n(x) = \sum_{j=1}^{n-1} \langle s_j(x), P(j)H(1)\rangle + s_0(x)$, where we put $P(1) = I$. Then

$$\alpha r_n(x) = \sum_{j=1}^{n-1} \langle s_j(x), P(j)v\rangle + \sum_{j=1}^{n-1} \langle s_j(x), P(j)z\rangle + \alpha s_0(x).$$

From Proposition 12, $\sum_{j=1}^{n-1} \langle s_j(x), P(j)v\rangle + \alpha s_0(x) \to v(x) \pmod{\mathbb{Z}}$ as $n \to \infty$. We distinguish two cases. If $v(x) \in (0,1)$, we write $\sum_{j=1}^{n-1} \langle s_j(x), P(j)v\rangle + \alpha s_0(x) = V_n(x) + v_n(x)$ with $V_n(x) \in \mathbb{Z}$ and $v_n(x) \in [0,1)$; if $v(x) = 0$ we consider $v_n(x) \in [-1/2, 1/2)$. Then, in both cases, $(v_n(x); n \in \mathbb{N})$ converges and *a fortiori* $(v_{n+1}(x) - v_n(x); n \geqslant 1) \to 0$ as $n \to \infty$. Moreover,

$$\begin{aligned} \sum_{j=1}^{n} \langle s_j(x), P(j)v\rangle - \sum_{j=1}^{n-1} \langle s_j(x), P(j)v\rangle \\ &= \langle s_n(x), P(n)v\rangle = V_{n+1}(x) - V_n(x) + v_{n+1}(x) - v_n(x). \end{aligned}$$

Since, for a linearly recurrent system $\{s_n(x); x \in X, n \in \mathbb{N}\}$ is bounded, $P(n)v \to 0$ and $(v_{n+1}(x) - v_n(x)) \to 0$ as $n \to \infty$. We conclude that $V_n(x)$ is a constant integer for all large enough $n \in \mathbb{N}$. Consequently the series $\sum_{j \geqslant 2} \langle s_j(x), P(j)v\rangle$ converges.

Now we prove that (2) implies (3). We assume, without loss of generality, that $n_0 = 1$ and that for any $x \in X$ the series

$$\sum_{j \geqslant 2} \langle s_j(x), P(j)v\rangle \in \mathbb{R}$$

converges. It suffices to prove that $\sum_{j \geqslant 2} \|P(j)v\| < \infty$.

For $n \geqslant 2$, define $i(n) \in \{1, \ldots, C(n)\}$ such that

$$|\langle e_{i(n)}, P(n)v\rangle| = \max_{i \in \{1, \ldots, C(n)\}} |\langle e_i, P(n)v\rangle|,$$

where e_i is the ith canonical vector of $\mathbb{R}^{C(n)}$. Let

$$I^+ = \{n \geqslant 2; \langle e_{i(n)}, P(n)v\rangle \geqslant 0\}, \quad I^- = \{n \geqslant 2; \langle e_{i(n)}, P(n)v\rangle < 0\}.$$

To prove that $\sum_{j \geqslant 2} \|P(j)v\| < \infty$ we only need to show that

$$\sum_{j \in I^+} \langle e_{i(j)}, P(j)v\rangle < \infty \quad \text{and} \quad -\sum_{j \in I^-} \langle e_{i(j)}, P(j)v\rangle < \infty.$$

Since the arguments we will use are similar in both cases we only prove the first fact.

To prove that $\sum_{j \in I^+} \langle e_{i(j)}, P(j)v\rangle < \infty$ we only show that

$$\sum_{j \in I^+ \cap 2\mathbb{N}} \langle e_{i(j)}, P(j)v\rangle < \infty, \tag{5.1}$$

and analogously it can be proved that

$$\sum_{j \in I^+ \cap (2\mathbb{N}+1)} \langle e_{i(j)}, P(j)v\rangle < \infty.$$

We construct two points $x, y \in X$ such that $s_n(x) - s_n(y) = e_{i(n)}$ if $n \in I^+ \cap 2\mathbb{N}$ and $s_n(x) - s_n(y) = 0$ elsewhere. By hypothesis, from this fact we conclude (5.1).

To construct x and y, according to Lemma 3, we only need to produce sequences

$$(t_n; n \in \mathbb{N}) \in \prod_{n\in\mathbb{N}} \{1, \ldots, C(n)\}, \quad (j_n; n \geqslant 1) \in \prod_{n\geqslant 1} \{1, \ldots, m_{t_n}(n)\}$$

and

$$(\bar{t}_n; n \in \mathbb{N}) \in \prod_{n\in\mathbb{N}} \{1, \ldots, C(n)\}, \quad (\bar{j}_n; n \geqslant 1) \in \prod_{n\geqslant 1} \{1, \ldots, m_{\bar{t}_n}(n)\}$$

such that

$$\theta^{t_n}_{j_n}(n-1) = t_{n-1} \quad \text{and} \quad \theta^{\bar{t}_n}_{\bar{j}_n}(n-1) = \bar{t}_{n-1} \text{ for all } n \geqslant 1. \tag{5.2}$$

The point x is the unique one such that $\tau_n(x) = t_n$ and $l_n(x) = j_n$. Point y is defined analogously with respect to $\bar{t}_n$ and $\bar{j}_n$. Given $n \in (I^+ \cap 2\mathbb{N})^c$ put $t_n = \bar{t}_n = 1$.

For $n \in I^+ \cap 2\mathbb{N}$, by property (KR5′), there exists $k \in \{1, \ldots, m_1(n+1) - 1\}$ such that $\theta^1_{k+1}(n) = i(n)$. Put

$$\bar{t}_n = i(n), \quad \bar{j}_n = m_{\bar{t}_n}(n), \quad t_n = \theta^1_k(n) \quad \text{and} \quad j_n = m_{t_n}(n).$$

Using (3.3) we obtain $\theta^{t_n}_{j_n}(n-1) = 1$ and $\theta^{\bar{t}_n}_{\bar{j}_n}(n-1) = 1$. Then we set $\bar{t}_{n+1} = t_{n+1} = 1$, $\bar{j}_{n+1} = k+1$ and $j_{n+1} = k$. Consequently, the relations (5.2) are satisfied for n and $n+1$.

Now we treat the remaining case: $n \in (I^+)^c \cap 2\mathbb{N}$. We recall that $t_n = \bar{t}_n = t_{n+1} = \bar{t}_{n+1} = 1$. It suffices to set

$$j_n = \bar{j}_n = m_1(n) \quad \text{and} \quad j_{n+1} = \bar{j}_{n+1} = m_1(n+1)$$

to fulfil relations (5.2).

For each $n \in I^+ \cap 2\mathbb{N}$, the towers of level n visited by x and y after their first entrance time to $B(n)$ and before their first entrance time to $B(n+1)$ are

$$\mathcal{S}_n(x) = \{\theta^1_{k+1}(n), \ldots, \theta^1_{m_1(n+1)}(n)\} \quad \text{and} \quad \mathcal{S}_n(y) = \{\theta^1_{k+2}(n), \ldots, \theta^1_{m_1(n+1)}(n)\},$$

respectively. Therefore, $s_n(x) - s_n(y) = e_{i(n)}$.

On the other hand, if $n \notin I^+ \cap 2\mathbb{N}$, then $\mathcal{S}_n(x)$ and $\mathcal{S}_n(y)$ are the empty set. Hence $s_n(x) = s_n(y) = 0$. □

6. *Example: measurable and non-continuous eigenvalues*

We construct explicitly a system with a nontrivial Kronecker factor but having a trivial equicontinuous factor. Let us consider the commuting matrices

$$A = \begin{bmatrix} 5 & 2 \\ 2 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

We set $\varphi = (1+\sqrt{5})/2$. Let $e = (\varphi, 1)^T$, $f = (-1, \varphi)^T$, $\alpha_A = 3 + 2\varphi$, $\beta_A = 5 - 2\varphi$, $\alpha_B = 1 + \varphi$ and $\beta_B = 2 - \varphi$. Observe that $\alpha_A > \alpha_B > \beta_A > 1 > \beta_B > 0$ and $\{e, f\}$ is a base of $\mathbb{R}^2$ made of the common eigenvectors associated to eigenvalues α_A, β_A of A and α_B, β_B of B respectively.

We define recursively the sequence $(v_n; n \geqslant 1)$ of real numbers by $v_1 = 1$ and for all $n > 1$,

$$v_{n+1} = \begin{cases} \beta_A v_n & \text{if } n v_n \leqslant 1 \\ \beta_B v_n & \text{if } n v_n > 1. \end{cases}$$

Notice that the sequence $(nv_n; n \geqslant 1)$ is uniformly bounded and uniformly bounded away from 0. Now let $H(1) = M(1) = (1,1)^T$ and for $n \geqslant 1$,

$$M(n+1) = \begin{cases} A & \text{if } nv_n \leqslant 1 \\ B & \text{if } nv_n > 1. \end{cases}$$

Note that $M(n) = A$ for infinitely many values of n.

Define the words in $\{1,2\}^*$, $\theta^1(A) = 2211111$, $\theta^2(A) = 22211$, $\theta^1(B) = 211$ and $\theta^2(B) = 21$. Let(X,T) be a minimal Cantor system such that there is a sequence of clopen Kakutani–Rokhlin partitions $(\mathcal{P}(n); n \in \mathbb{N})$ verifying (KR1)–(KR6) with associated sequence of matrices $(M(n); n \geqslant 1)$. Moreover, we require that for $n \geqslant 1$ and $t \in \{1,2\}$, $\theta^t(n) = \theta^t(M(n+1))$ holds (see (3.4) for the definition of $\theta^t(n)$). This is possible by [**5**], using Bratteli diagrams. It is clear that (X,T) is linearly recurrent. We call μ its unique ergodic measure.

A symbolic way to see this system is by considering the substitutions $\sigma_A : \{1,2\} \to \{1,2\}^*$, $\sigma_A(1) = 2211111$, $\sigma_A(2) = 22211$, and $\sigma_B : \{1,2\} \to \{1,2\}^*$, $\sigma_B(1) = 211$, $\sigma_B(2) = 21$. Define a sequence of substitutions $(\sigma_n; n \geqslant 1)$ by $\sigma_1 = \mathrm{Id}$ and, for all $n > 1$, $\sigma_{n+1} = \sigma_n \circ \sigma_{M(n)}$. It follows that $\ldots 111\sigma_n(1).\sigma_n(2)222\ldots$ converges to some $\omega \in \{1,2\}^{\mathbb{Z}}$, where the dot indicates the position to the left of 0 coordinate. We set $X = \overline{\{T^n(\omega), n \in \mathbb{Z}\}}$, where T is the shift map.

Before studying the system (X,T) defined by this sequence of matrices, we need a general property. We keep the notations of previous sections.

Lemma 15. *Let $v \in \mathbb{R}^{C(1)}$. If $\lim_{n\to\infty} \|P(n)v\| = 0$, then v is orthogonal to the vector $\mu(1) = (\mu(B_k(1)); 1 \leqslant k \leqslant C(1))^T$.*

Proof. Let $v \in \mathbb{R}^{C(1)}$ be such that $\lim_{n\to\infty} \|P(n)v\| = 0$. Then, for $n > 1$,

$$\begin{aligned} |\langle \mu(1), v\rangle| &= |\langle P^T(n)\mu(n), v\rangle| \\ &= |\langle \mu(n), P(n)v\rangle| \\ &\leqslant \|P(n)v\|, \end{aligned}$$

and the last term converges to 0 as $n \to \infty$. Thus v is orthogonal to $\mu(1)$. $\square$

Proposition 16. *Let (X,T) be the linearly recurrent system defined above. The set of eigenvalues of (X,T) is*

$$E_\mu = \left\{\exp(2i\pi\alpha) \in \mathbb{C}; \alpha = (\varphi - 1, 2 - \varphi)\, A^{-l} w,\ l \geqslant 0, w \in \mathbb{Z}^2\right\}.$$

None of these eigenvalues is continuous except the trivial one.

Proof. Let $v = -(\varphi - 2)f = (\varphi - 2, \varphi - 1)^T$ and $n \geqslant 2$. Hence $P(n)v = P(n)(\varphi H(1)) \pmod{\mathbb{Z}^2}$. Also, since v is an eigenvector of A and B, from the definition of v_n we get

$$P(n)v = \beta_{M(n)} \ldots \beta_{M(2)} v = v_n v.$$

The sequence $(v_n)_{n\geqslant 1}$ was constructed so that nv_n is uniformly bounded and uniformly bounded away from 0. It follows that

$$\sum_{n\geqslant 2} v_n = \infty \quad \text{and} \quad \sum_{n\geqslant 2} v_n^2 < \infty, \tag{6.1}$$

and

$$\sum_{n\geqslant 2}\|P(n)v\| = \infty \quad \text{and} \quad \sum_{n\geqslant 2}\|P(n)v\|^2 < \infty. \tag{6.2}$$

In particular, $\lim_{n\to\infty}\|P(n)v\| = 0$ and, by Lemma 15, v is orthogonal to $\mu(1) = (\mu(B_1(1)), \mu(B_2(1))^T$.

Claim. *Let $\alpha \in \mathbb{R}$ and $\lambda = \exp(2i\pi\alpha)$: $|||P(n)(\alpha H(1))||| \to 0$ as $n \to \infty$ holds if and only if $\lambda \in E_\mu$. Moreover, if $\lambda \in E_\mu$, then $|||P(n)(\alpha H(1))||| = c\|P(n)v\|$, for some positive constant c.*

Proof of the claim. First assume that $|||P(n)(\alpha H(1))||| \to 0$ as $n \to \infty$ holds. By Lemma 9, there exist $m \geqslant 2$, an integer vector $w \in \mathbb{Z}^{C(m)}$ and a real vector $v' \in \mathbb{R}^{C(m)}$ with $P(m)(\alpha H(1)) = v' + w$ and $\|P(n)P(m)^{-1}v'\| \to 0$ as $n \to \infty$. From Lemma 15, vector $P(m)^{-1}v'$ is orthogonal to $\mu(1)$. Hence there exists $k \in \mathbb{R}$ such that $P(m)^{-1}v' = kv$ and

$$P(m)(\alpha H(1)) = kP(m)v + w. \tag{6.3}$$

Suppose that $k=0$. It is not difficult to show by induction that $\gcd(h_1(m), h_2(m))=1$. Then, since w is an integer vector, $\alpha \in \mathbb{Z}$ and $\lambda = 1$ which belongs to E_μ.

Suppose that $k \neq 0$. Then, $k = W_1 - W_2$, where $P(m)^{-1}w = (W_1, W_2)^T$. This gives

$$\alpha H(1) = \begin{pmatrix}\varphi - 1 & 2-\varphi\\ \varphi - 1 & 2-\varphi\end{pmatrix} P(m)^{-1}w. \tag{6.4}$$

The determinants of the matrices A and B are respectively equal to 11 and 1. Therefore, since $P(m) = A^{l_m}B^{k_m}$ for some $l_m, k_m \geqslant 0$,

$$\alpha = (\varphi - 1, 2-\varphi)A^{-l_m}w',$$

with $w' \in \mathbb{Z}^2$. Hence $\lambda \in E_\mu$.

Conversely, let $\lambda \in E_\mu$. Then, since $M(n) = A$ for infinitely many $n \geqslant 2$, for n large enough we get

$$P(n)(\alpha H(1)) = P(n)\begin{bmatrix}\varphi - 2 & -(\varphi-2)\\ \varphi-1 & -(\varphi-1)\end{bmatrix} + w,$$

where $w \in \mathbb{Z}^2$. Therefore, $|||P(n)(\alpha H(1))||| = c\|P(n)v\|$, for some positive constant c, which proves the claim. □

Finally, the proposition follows from Theorem 1 and property (6.2). □

This proposition gives an example of a minimal dynamical system with a nontrivial Kronecker factor and a trivial maximal equicontinuous factor.

Acknowledgements. We would like to thank Claude Dellacherie for the reference [**7**] concerning the convergence of 'almost' Martingale processes that was at the base of the proof of the measure-theoretical statement of the paper, and François Parreau for illuminating discussions.

The first and second authors thank the CMM–CNRS, which made this collaboration possible. Part of this version of the paper was written while the second and third authors visited the Max Planck Institute of Mathematics (Bonn). The support and hospitality of both institutions are very much appreciated.

We also thank the referee of this paper for many valuable comments which improved the proof of Theorem 1(1) and the presentation of the paper.

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