Resonant frequency shifts induced by a large spherical object in an air-filled acoustic cavity

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Abstract: Acoustic resonances are modified when objects are introduced into a chamber. The magnitude of these changes depends on the object position, size, and shape, as well as on its acoustic properties. Here, an experimental study concerning the resonant frequency shifts induced by a solid spherical object in a quasi-one-dimensional air-filled acoustic cavity is reported. It is shown that Leung's theory does not account quantitatively for the observations. A novel and simple approach is proposed, based on the wave equation in a cavity of variable cross section. The results fit more accurately the measured frequency shifts.

1. Introduction

The acoustic resonances of a cavity are modified when objects are introduced into the chamber. It is well understood that these changes are more pronounced when either the object size increases or the object itself is a very efficient scatterer. Thus, large solid spheres in air-filled cavities, as in this study, or gas bubbles in a liquid, can have considerable effects on the acoustic properties of resonance chambers. These observations indicate that both volumetric and scattering effects are important.

The inclusion of an object in a resonant chamber is analogous to the one-dimensional problem of a mass attached to a string studied a long time ago by Rayleigh.[1] More recently, the two-dimensional version has also been studied by Laura *et al.*[2] The interest in the use of acoustic levitation for space applications, the detection of blockage in nuclear reactors, and the measurements of properties and volumes of objects, in particular rocks, has drawn attention to the effect of introducing an object in acoustic resonant cavities.[3–9] Most of these studies are theoretical, dealing with more or less elaborate techniques to predict the frequency resonance shifts for various object shapes and sizes, as well for different boundary conditions. Experimental results are scant, exceptions being the rather complete studies performed by Barmatz *et al.*[5] and Chen *et al.*[9]

In this letter, we present an experimental study of longitudinal mode resonant frequency shifts induced by a solid spherical object in a quasi-one-dimensional air-filled acoustic cavity. We compare our results with Leung's theory, which does not account quantitatively for the observations. We propose a novel and simple approach based on the wave equation in a cavity of variable cross section. The results fit more accurately the measured frequency shifts.

2. Experimental setup

The experimental setup is composed of a square section quasi-one-dimensional cavity, of dimensions $L_x \equiv L = 100$ mm and $L_y = L_z = 6.8$ mm. Two duraluminum walls allow rigidity and hold two other static dissipative acrylic walls which in turn allow visualization (see Fig. 1). Another two duraluminum end walls close the cavity on each side. One of them is attached to an

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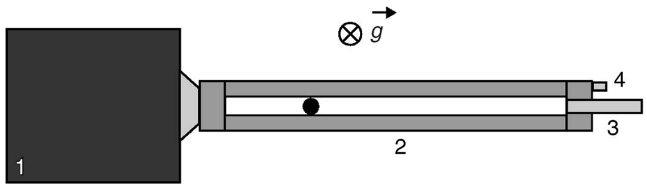

Fig. 1. Experimental setup. (1) Electromechanical vibrator, (2) quasi-one-dimensional square section cavity, (3) microphone, and (4) accelerometer. A metallic magnetic sphere is placed inside and held fixed by another magnetic sphere from outside. The origin is chosen at the wall close to the vibrator.

electromechanical vibrator (Bruel & Kjaer mini-shaker 4810), which provides a maximum force of 10 N in a large frequency range, typically between 100 Hz and 18 kHz, but provides a constant acceleration amplitude in the 100–5000 Hz range. The cavity is placed such that both acrylic walls are normal to gravity, hence the system is visualized from above. It oscillates entirely in the direction of the vibrator's axis, which has been shown to be an efficient way to amplify resonant acoustic modes.[10]

A microphone and an accelerometer (PCB 130D20 and 340A65, respectively) allow measurements of the cavity's end side acoustic pressure and acceleration of the whole system. The microphone is placed inside the cavity, flush with the end wall, and it has a 6.35-mm-diam active surface. The accelerometer is placed at the external end side with its axis parallel to the cavity's axis. The electromechanical shaker is powered by an amplifier with a signal generated by a spectrum analyzer (SR780). Experiments are performed in the analyzer's swept sine mode. The analyzer measures both the pressure and acceleration amplitude values.

At low frequencies, where the cavity is considered as quasi-one-dimensional, the empty cavity resonant frequencies differ very little from those predicted theoretically. The predicted fundamental frequency is given by $\hat{f}_0 = c/2L$, where c is the sound speed in air. c depends on temperature,[11]

$$c = 331.5\sqrt{1 + \frac{T_C}{273}}\ \text{m/s}, \tag{1}$$

where T_C is in degrees Celsius. Hence, $\hat{f}_0$ also has a temperature dependence. However, a temperature variation of $\pm 1\ ^\circ\text{C}$ only induces a $\pm 0.2\%$ change in $\hat{f}_0$. Care was taken in order to avoid larger temperature variations.

At the operating temperature $T_C = 20.75 \pm 0.5\ ^\circ\text{C}$, we have $c = 343.9 \pm 0.3$ m/s, and thus the predicted resonant frequency is $\hat{f}_0 = 1719.3$ Hz. However, the measured value is $f_0 = 1702.7$ Hz, 1% lower than $\hat{f}_0$. We assume this difference is due to a slightly larger effective length $L_{\text{eff}} = 0.101$ m, which is only 1 mm longer. Considering that the pressure sensor front—active—surface is rather soft, it is reasonable to consider the real stiff element to be slightly behind it.

In order to study the resonant frequency shifts induced by a spherical object, a 6.35-mm-diam metallic, magnetic sphere is placed inside and held fixed by means of a similar magnetic sphere placed outside the cavity. Once the intruder is fixed and its position determined, a pressure spectrum is obtained between 1 and 10 kHz, with a roughly constant dimensionless peak acceleration (≈ 0.5 g), resulting in a maximum pressure of ≈ 10 Pa. This was performed for 90 different positions, separated by 1 mm, from $X_0 = 4$ mm to $X_0 = 94$ mm. Because of the magnetic nature of the spheres, no measurements were possible for $X_0 < 4$ mm or $X_0 > 94$ mm.

3. Experimental results and comparison with Leung's theory

The measured resonant frequencies as functions of X_0 are presented in Fig. 2. Results are shown from the first to the fifth longitudinal resonant mode. As previously observed by Leung *et al.*,[3]

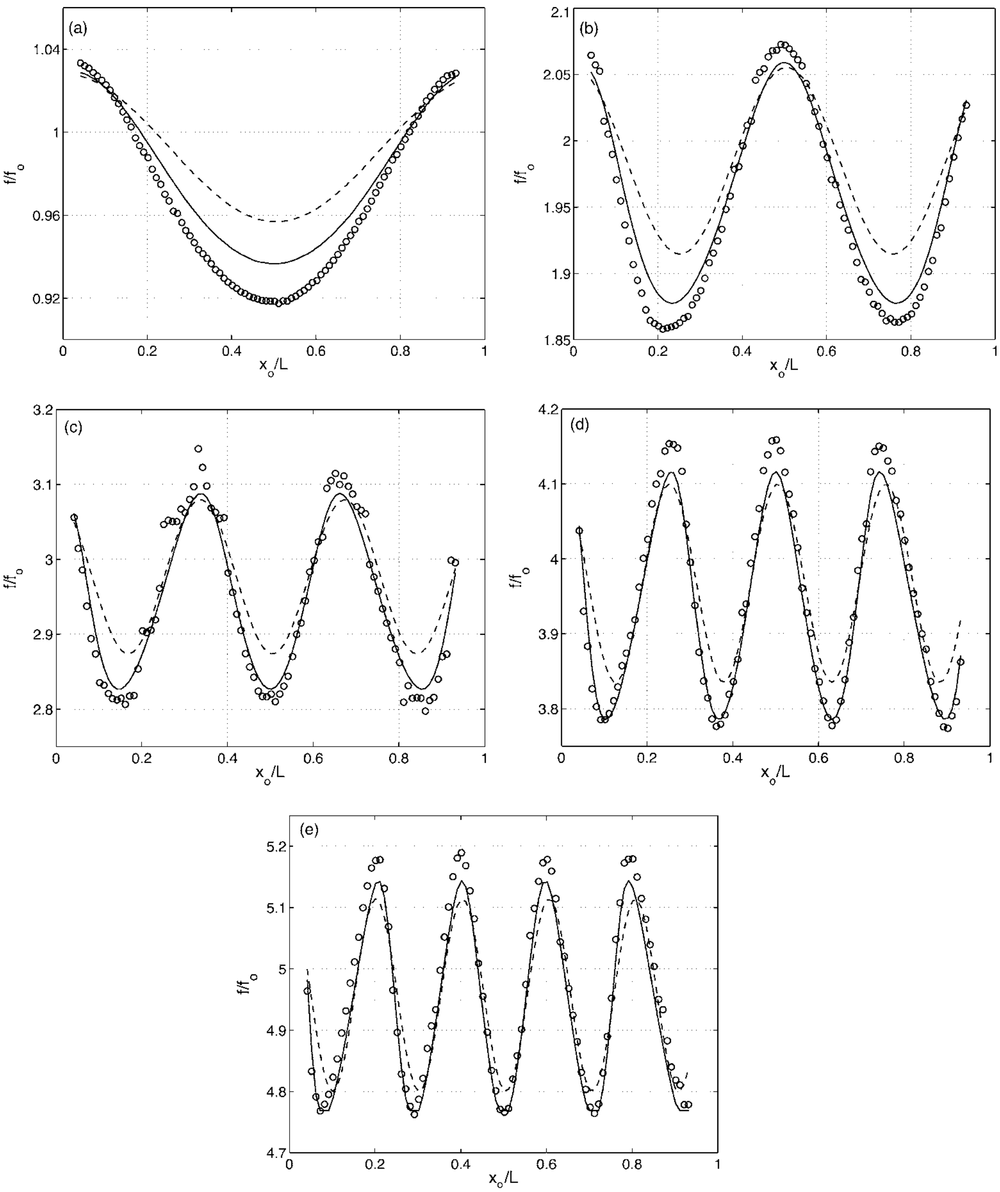

Fig. 2. Normalized resonant frequencies f/f_0 vs X_0 for the first five longitudinal modes (a) $n=1$, (b) $n=2$, (c) $n=3$, (d) $n=4$, and (e) $n=5$: Experiments (○), Leung's prediction (dashed line), and solution of the variable cross-section model (continuous line). Both theoretical predictions are computed using $L_{\text{eff}}=0.101$ m.

the resonant frequencies depend on the sphere's position, varying in an oscillatory way as a function of X_0, such that the number of oscillations is equal to the mode number, $n=1,\ldots,5$. In our case, however, variations are stronger, of the order of 10% peak-to-peak, due to the larger sphere-to-cavity volume ratio. Notice that all except one mode show relatively smooth variations; the third mode indeed presents some noise due to the difficulty in measuring the resonant frequencies from pressure spectra in this case as it was of particularly low amplitude, i.e., with a small signal-to-noise ratio.

We can qualitatively understand the fact that the resonant frequencies vary roughly periodically with X_0 with a very simple argument: By integrating the Helmholtz equation, the wave number is given by[3]

$$k^2 = \frac{\frac{1}{2}\int |\nabla p|^2 dV}{\frac{1}{2}\int p^2 dV} \propto \frac{K}{V}, \tag{2}$$

where integration is performed in the available volume. K and V stand for the kinetic and potential energy, respectively. Thus, when the sphere is located near a pressure maximum, for example at a cavity end wall, which in turn is a velocity node ($\nabla p \approx 0$), the kinetic energy does not change much but the potential energy is reduced. The final result is an increase in the corresponding wave number, hence an increase in the resonant frequency is expected. The inverse argument can be made when the intruder is near a velocity maximum, for example, at $X_0/L \approx 1/2$ for the first mode. In this case k, and thus f, is expected to decrease.

An analytical expression was obtained by Leung *et al.*[3] From a Green's function scattering calculation they obtained the resonant wave number as a function of three parameters: the sphere to cavity volume ratio V_s/V, the sphere to cavity length ratio R/L, and the sphere's relative position X_0/L. Assuming small spherical scatters, such that $kR \ll 1$ where R is the sphere radius and k is the wave number, they performed their calculation for longitudinal modes in a rectangular cavity of length L. The predicted wave number shift is

$$\frac{\delta k}{k_n} = \frac{V_s}{V}\left[-\left(\frac{1}{4} + \frac{67}{360}(k_n R)^2\right)\right] + \frac{V_s}{V}\left[\left(\frac{5}{4} - \frac{229}{360}(k_n R)^2\right)\cos(2k_n X_0)\right], \tag{3}$$

where $\delta k = k - k_n$. The calculation is done up to order $(k_n R)^2$, $k_n = n\pi/L$ being the nth longitudinal mode wave number for the empty cavity. $V_s(V)$ is the sphere (cavity) volume, and X_0 is the sphere's position. From Eq. (3) we obtain the resonant frequencies

$$f = \frac{c k_n}{2\pi}\left(1 + \frac{\delta k}{k_n}\right).$$

The comparison made in Fig. 2 shows that this expression does follow qualitatively the measured resonant frequencies for all modes although important differences are present. In fact, Eq. (3) considers both volumetric and scattering effects $[\sim (k_n R)^2]$, being the volumetric effects dominant. The discrepancies between Leung's prediction and our measurements are probably due to the breakdown of the single spherical scattering approximation (no wall contributions). It is also important to note that Leung *et al.* did not observe differences in resonant frequencies for scatterers made of different solid materials, as expected when both the density and compressibility differences are so large between solids and air. We also verified this in our setup with plastic spheres.

In order to compare quantitatively the predicted resonant frequencies and the measured ones, we define the difference parameter, or error estimator, for the nth mode as

$$\chi_n^2 = \sum_{i=1}^{N} \frac{(f_{\mathrm{t}}^n(i) - f_{\mathrm{m}}^n(i))^2}{N f_0^2}, \tag{4}$$

where $f_{\mathrm{t}}^n(i)$ and $f_{\mathrm{m}}^n(i)$ are the theoretical and measured resonant frequencies, respectively, for the sphere at the ith position, and $N=90$ is the number of positions (i.e., measured resonant frequencies). The χ_n^2 parameters calculated for $n=1,\ldots,5$ are presented in Table 1.

Table 1. Error estimator χ_n^2 for the first five modes, using Leung's formula and the results obtained considering a variable section cavity. All values are $\times 10^{-3}$. The ratio between both parameters is given in the third row.

	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
Leung	0.68	1.57	2.49	2.21	2.50
Wave Eq. (8)	0.14	0.21	0.62	0.64	1.03
Ratio	5.0	7.6	4.0	3.5	2.4

4. Resonant frequecy shifts obtained from the wave equation in a cavity of variable cross section

We assume that we have an acoustic cavity of cross section $S(x)$, in which a homogeneous fluid sustains acoustic waves. In this case the wave equation is modified. Using the usual linear approximation $\rho=\rho_0+\rho'$ and $P=P_0+p$, in the quasi-one-dimensional limit of interest, one obtains[12]

$$\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2}=\frac{1}{S}\frac{\partial}{\partial x}\left(S\frac{\partial p}{\partial x}\right). \tag{5}$$

We can identify three regions in the cavity: Regions I, II and III, for $x<X_0-R$, $X_0-R<x<X_0+R$, and $x>X_0+R$, respectively. The transverse section then is written

$$S=\begin{cases} S_0 & \text{if } x<X_0-R \\ S_0-\pi(R^2-(x-X_0)^2) & \text{if } X_0-R<x<X_0+R \\ S_0 & \text{if } x>X_0+R, \end{cases} \tag{6}$$

where $S_0=L_y\times L_z$. Notice that the exact sphere position in the (y,z) plane does not matter in this framework.

Now, let us impose wave solutions of the form $p(x,t)=p(x)e^{-i\omega t}$. We obtain two equations that must be solved. First, the Helmholtz equation

$$k^2p+\frac{d^2p}{dx^2}=0, \tag{7}$$

which must be solved in regions I and III. The second equation, valid for region II, is

$$k^2p+\frac{1}{\hat{S}(x)}\frac{d}{dx}\left[\hat{S}(x)\frac{dp}{dx}\right]=0, \tag{8}$$

where $\hat{S}(x)=S_0-\pi[R^2-(x-X_0)^2]$. In addition to the rigid termination conditions $(dp/dx=0)$ at $x=0$ and $x=L$, the solutions of these equations have to satisfy pressure and acoustic velocity continuity conditions at $x=X_0-R$ and $x=X_0+R$.

The solutions in regions I and III are the usual plane wave solutions

$$p(x)=\begin{cases} P_0\cos kx & \text{(Region I)} \\ P_1\cos k(L-x) & \text{(Region III)}, \end{cases} \tag{9}$$

which satisfy the rigid conditions in the ends of the cavity. Here we put P_0 and P_1 as the (unknown) pressure amplitudes at the left and right sides of the sphere.

In order to solve Eq. (8), we make the change of variable $z=(x-X_0)/R$, obtaining

$$\frac{d^2p}{dz^2} + \frac{2z}{\sigma - 1 + z^2}\frac{dp}{dz} + (kR)^2 p = 0 \tag{10}$$

with $-1 < z < 1$ and $\sigma = s_0 / \pi R^2$. The solution of this equation can be expressed in terms of the confluent Heun function[13] $\mathrm{HC}(\alpha, \beta, \gamma, \delta, \eta; x)$:

$$p(z) = A\mathrm{HC}\left(0, -\frac{1}{2}, 0, -\frac{1}{4}(kR)^2(\sigma-1), \frac{1}{4} + \frac{1}{4}(kR)^2(\sigma-1); -\frac{z^2}{\sigma-1}\right) + B\mathrm{HC}\left(0, \frac{1}{2}, 0, -\frac{1}{4}(kR)^2(\sigma-1), \frac{1}{4} + \frac{1}{4}(kR)^2(\sigma-1); -\frac{z^2}{\sigma-1}\right)z. \tag{11}$$

The boundary conditions (continuity of both pressure and acoustic velocity) can then be written as an homogeneous linear system with four unknowns, P_0, P_1, A, and B. Imposing the determinant of the system to be equal to zero it is possible to find a trascendental equation for k. The results obtained with this procedure are also presented in Fig. 2. We observe that for all modes the comparison is better than the results obtained with Leung's calculation. As before, the parameter χ_n^2 is computed for each mode. Table 1 shows that overall, despite its simplicity, our model performs much better that Leung's calculation.

5. Conclusions

In summary, we have performed an experimental study of the resonance frequency shifts of longitudinal modes in a quasi-one-dimensional air-filled acoustical cavity of rectangular cross section induced by the inclusion of a spherical solid object of diameter comparable to the cross-section length. Measurements were performed for the first five longitudinal modes. Depending on the object position, the measured resonant frequencies vary in an oscillatory way.

Leung's theory, which is valid in the small sphere limit, does account qualitatively for the observations, although important differences are observed. Surprisingly these predictions are not so bad quantitatively, even when the single sphere—no wall—approximation does not hold in our setup. This is probably due to the fact that for a solid sphere in air and for long wavelengths, single scattering effects are small. The question about how the sphere wall interaction (multiple scattering) modifies this remains an open question.

We have developed a simple quasi-one-dimensional model where we consider the cavity with the intruder as a chamber of variable cross section. Hence, this model solely considers volumetric effects. For the well-known wave equation (5) we impose a given form of $S(x)$ from which we can compute the resonant wave numbers, and therefore the resonant frequencies. The global performance of each model is quantified through a difference parameter χ_n^2 for each longitudinal mode. The new predictions agree much better with measurements than Leung's theory.

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