

# Some multipoint boundary value problems of Neumann–Dirichlet type involving a multipoint $p$ -Laplace like operator <sup>☆</sup>

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Dedicated to William Ames

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## Abstract

Let  $\phi$  and  $\theta$  be two increasing homeomorphisms from  $\mathbb{R}$  onto  $\mathbb{R}$  with  $\phi(0) = 0$ ,  $\theta(0) = 0$ . Let  $f : [0, 1] \times \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$  be a function satisfying Carathéodory's conditions, and for each  $i$ ,  $i = 1, 2, \dots, m - 2$ , let  $a_i : \mathbb{R} \mapsto \mathbb{R}$ , be a continuous function, with  $\sum_{i=1}^{m-2} a_i(0) = 1$ ,  $\xi_i \in (0, 1)$ ,  $0 < \xi_1 < \xi_2 < \dots < \xi_{m-2} < 1$ .

In this paper we first prove a suitable continuation lemma of Leray–Schauder type which we use to obtain several existence results for the  $m$ -point boundary value problem:

$$\begin{aligned}(\phi(u'))' &= f(t, u, u'), \quad t \in (0, 1), \\ u'(0) &= 0, \quad \theta(u(1)) = \sum_{i=1}^{m-2} \theta(u(\xi_i)) a_i(u'(\xi_i)).\end{aligned}$$

We note that this problem is at *resonance*, in the sense that the associated  $m$ -point boundary value problem

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$$\begin{aligned}
 &(\phi(u'(t)))' = 0, \quad t \in (0, 1), \\
 &u'(0) = 0, \quad \theta(u(1)) = \sum_{i=1}^{m-2} \theta(u(\xi_i))a_i(u'(\xi_i))
 \end{aligned}$$

has the non-trivial solution  $u(t) = \rho$ , where  $\rho \in \mathbb{R}$  is an arbitrary constant vector, in view of the assumption  $\sum_{i=1}^{m-2} a_i(0) = 1$ .  
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## 1. Introduction

Let  $\phi$  and  $\theta$  be two increasing homeomorphisms from  $\mathbb{R}$  onto  $\mathbb{R}$  with  $\phi(0) = 0$ ,  $\theta(0) = 0$ . Let  $f : [0, 1] \times \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$  be a function satisfying Carathéodory’s conditions, and for each  $i$ ,  $i = 1, 2, \dots, m - 2$ , let  $a_i : \mathbb{R} \mapsto \mathbb{R}$  be a continuous function such that  $\sum_{i=1}^{m-2} a_i(0) = 1$ . We are interested in the problem of existence of solutions for the  $m$ -point boundary value problem:

$$(P) \quad \begin{cases} (\phi(u'))' = f(t, u, u'), & t \in (0, 1), \\ u'(0) = 0, & \theta(u(1)) = \sum_{i=1}^{m-2} \theta(u(\xi_i))a_i(u'(\xi_i)), \end{cases} \quad (1.1)$$

where  $0 < \xi_1 < \xi_2 < \dots < \xi_{m-2} < 1$  are given.

We note the associated  $m$ -point boundary value problem

$$\begin{aligned}
 &(\phi(u'(t)))' = 0, \quad 0 < t < 1, \\
 &u'(0) = 0, \quad \theta(u(1)) = \sum_{i=1}^{m-2} \theta(u(\xi_i))a_i(u'(\xi_i))
 \end{aligned} \quad (1.2)$$

has the non-trivial solution  $u(t) = \rho$ , where  $\rho \in \mathbb{R}$  is an arbitrary constant. Because of this we say that problem (1.1) is at *resonance*.

Problems of this kind have been recently dealt with in the literature. Thus in the case  $\phi(u) = \theta(u) \equiv u$ ,  $u \in \mathbb{R}$ ,  $a_i(s) = \text{constant}$ ,  $i = 1, \dots, m - 2$ , we refer to [1], for  $m = 3$ , and for general  $m$  in [2,7–9]. For general  $\phi$  and  $\theta(u) \equiv u$  the boundary value problem (1.1) has been studied in [3,4] for the boundary conditions  $u'(0) = 0$ ,  $u(1) = u(\eta)$ .

The authors continue in this paper the work initiated in [3,5,6] by generalizing the main results in [2,3] in several respects. To illustrate this fact we state next a particular and simple case that cannot be obtained from any of the results in [2,3].

**Theorem 1.1.** *Let  $f : [0, 1] \times \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$  be a continuous function in the boundary value problem*

$$u'' = f(t, u, u'), \quad t \in (0, 1), \quad (1.3)$$

$$u'(0) = 0, \quad \phi_p(u(1)) = \frac{1}{m-2} \sum_{i=1}^{m-2} \phi_p(u(\xi_i))e^{-(u'(\xi_i))^2}, \quad (1.4)$$

where  $\xi_i \in (0, 1)$ ,  $i = 1, \dots, m - 2$ ,  $0 < \xi_1 < \xi_2 < \dots < \xi_{m-2} < 1$ , and  $\phi_p(s) = |s|^{p-2}s$ ,  $p > 1$ . Suppose  $f$  satisfy the following conditions:

- there exists  $M > 0$  such that for all  $|u| > M$ , and all  $t \in [0, 1]$ , one has

$$uf(t, u, 0) > 0,$$

- there exist a function  $\psi : [0, \infty) \mapsto [0, \infty)$ ,  $\psi$  non decreasing with  $\psi(s) \rightarrow \infty$  as  $s \rightarrow \infty$ , and continuous functions  $d_1, d_2 : [0, 1] \times \mathbb{R} \mapsto [0, \infty)$ , such that for all  $u \in [-M, M]$ ,  $v \in \mathbb{R}$ , and for all  $t \in [0, 1]$ , it holds that

$$|f(t, u, v)| \leq d_1(t, u) + d_2(t, u)\psi(|v|),$$

- let  $C_2 = \max\{d_2(t, u), t \in [0, 1], |u| \leq M\}$ , and assume that

$$0 < \limsup_{s \rightarrow \infty} \frac{\psi(s)}{s} < \frac{1}{2C_2}.$$

Then problem (1.3)–(1.4) has at least one solution  $u \in C^1[0, 1]$ .

The proof of this theorem is a direct consequence of Theorem 3.2 below and is left to the reader. Theorem 3.2 in turn follows from our main existence result in Section 3.

This paper is organized as follows. In Section 2 we state and prove a rather general continuation lemma for the solvability of problem (1.1). Section 3 begins with our main existence result, Theorem 3.1. This theorem as well as rest of the results in this section is consequence of the continuation lemma derived in Section 2.

We shall denote by  $C[0, 1]$  (respectively  $C^1[0, 1]$ ) the classical space of continuous (respectively continuously differentiable) functions with values in  $\mathbb{R}$  defined on the interval  $[0, 1]$ . The norm in  $C[0, 1]$  is denoted by  $|\cdot|_\infty$ . Also, we shall denote by  $L^1(0, 1)$  the space of functions (equivalence classes of) that are integrable on  $(0, 1)$ . The Brouwer and Leray–Schauder degree shall be respectively denoted by  $\deg_B$  and  $\deg_{LS}$ .

## 2. Abstract formulation and deformation lemmas

In this section  $f^* : [0, 1] \times \mathbb{R} \times \mathbb{R} \times [0, 1] \mapsto \mathbb{R}$  will denote a function satisfying Carathéodory's conditions, i.e. (i) for all  $(s, r, \lambda) \in \mathbb{R} \times \mathbb{R} \times [0, 1]$  the function  $f^*(\cdot, s, r, \lambda)$  is measurable on  $[0, 1]$ , (ii) for a.e.  $t \in [0, 1]$  the function  $f^*(t, \cdot, \cdot, \cdot)$  is continuous on  $\mathbb{R} \times \mathbb{R} \times [0, 1]$ , and (iii) for each  $R > 0$  there exists a Lebesgue integrable function  $\rho_R : [0, 1] \mapsto \mathbb{R}$  such that  $|f^*(t, s, r, \lambda)| \leq \rho_R(t)$  for a.e.  $t \in [0, 1]$  and all  $(s, r, \lambda) \in \mathbb{R} \times \mathbb{R} \times [0, 1]$  with  $|s| \leq R$ , and  $|r| \leq R$ . We suppose, further, that  $f^*(t, s, r, 1) = f(t, s, r)$  for all  $(t, s, r) \in [0, 1] \times \mathbb{R} \times \mathbb{R}$ .

Next we establish and prove a continuation lemma for the solvability of problem (1.1). To this end we first introduce an operator  $\mathcal{B} : C^1[0, 1] \times [0, 1] \mapsto \mathbb{R}$  defined by

$$\begin{aligned} \mathcal{B}(u, \lambda) = & \theta \left( \lambda u(1) + (1 - \lambda) \int_0^1 \phi^{-1} \left( \int_0^s f^*(\tau, u(\tau), u'(\tau), \lambda) d\tau \right) ds \right) \\ & - \sum_{i=1}^{m-2} a_i(u'(\xi_i)) \theta \left( \lambda u(\xi_i) + (1 - \lambda) \int_0^{\xi_i} \phi^{-1} \left( \int_0^s f^*(\tau, u(\tau), u'(\tau), \lambda) d\tau \right) ds \right), \end{aligned} \tag{2.1}$$

and then for  $\lambda \in (0, 1]$  we define the family of boundary value problems:

$$(P_\lambda) \quad \begin{cases} \left( \phi \left( \frac{u'}{\lambda} \right) \right)' = f^*(t, u, u', \lambda), \\ u'(0) = 0, \quad \mathcal{B}(u, \lambda) = 0. \end{cases} \quad (2.2)$$

We note that for  $\lambda = 1$  problem (2.2) coincides with problem (1.1). We also define  $F : \mathbb{R} \mapsto \mathbb{R}$  by

$$F(\rho) = \theta \left( \int_0^1 \phi^{-1} \left( \int_0^s f^*(\tau, \rho, 0, 0) d\tau \right) ds \right) - \sum_{i=1}^{m-2} a_i(0) \theta \left( \int_0^{\xi_i} \phi^{-1} \left( \int_0^s f^*(\tau, \rho, 0, 0) d\tau \right) ds \right) \quad (2.3)$$

and note that

$$\mathcal{B}(\rho, 0) = F(\rho). \quad (2.4)$$

Let  $\Omega \subset C^1[0, 1]$  be a bounded open set, we have the following continuation lemma.

**Lemma 2.1.** *Assume that*

- (i) *there is no solution  $u$  to  $(P_\lambda)$ ,  $0 < \lambda < 1$ , such that  $u \in \partial\Omega$ ,*
- (ii) *the equation*

$$F(\rho) = 0 \quad (2.5)$$

*has no solution on  $\partial\Omega \cap \mathbb{R}$ ,*

- (iii) *the Brouwer degree*

$$\deg_B[F, \Omega \cap \mathbb{R}, 0] \neq 0. \quad (2.6)$$

*Then, problem (P) has a solution in  $\overline{\Omega}$ .*

**Proof.** If (1.1) has a solution in  $\partial\Omega$ , then there is nothing to prove, hence we suppose that (1.1) has no solutions belonging to  $\partial\Omega$ . This assumption combined with (i) implies that there are no solutions to  $(P_\lambda)$  in  $\partial\Omega$  for  $0 < \lambda \leq 1$ .

It can be easily proved that problem  $(P_\lambda)$ , for  $\lambda \in (0, 1]$  is equivalent to an abstract equation of the form

$$u = \Psi^*(u, \lambda), \quad (2.7)$$

where  $\Psi^* : C^1[0, 1] \times [0, 1] \mapsto C^1[0, 1]$  is defined by

$$\Psi^*(u, \lambda)(t) := u(0) + \lambda \int_0^t \phi^{-1} \left( \int_0^s f^*(\tau, u(\tau), u'(\tau), \lambda) d\tau \right) ds + \mathcal{B}(u, \lambda). \quad (2.8)$$

Standard arguments show that  $\Psi^*$  is a completely continuous operator. Furthermore setting  $\Psi(u) := \Psi^*(u, 1)$ , we observe that  $u$  is a solution of (P) if and only if it is a fixed point of  $\Psi$ .

At this point it is clear that assumption (i) can now be restated as

$$u \neq \Psi^*(u, \lambda) \quad \text{for all } u \in \partial\Omega, \text{ and for all } \lambda \in (0, 1].$$

We show next that this is also true for  $\lambda = 0$ . We note from (2.8) that

$$\Psi^*(u, 0)(t) = u(0) + \mathcal{B}(u, 0), \quad t \in [0, 1],$$

is a real constant for each  $u \in C^1[0, 1]$ . Thus, if for some  $u \in \partial\Omega$ ,

$$u = \Psi^*(u, 0), \tag{2.9}$$

then, for all  $t \in [0, 1]$ , we have that  $u(t) = s \in \mathbb{R}$  and hence, from (2.9), and (2.4), we obtain

$$s = s + \mathcal{B}(s, 0) = s + F(s).$$

But this implies that  $F(s) = 0$ , for  $s \in \mathbb{R} \cap \partial\Omega$ , contradicting (ii). In this manner we have verified that

$$u \neq \Psi^*(u, \lambda) \quad \text{for all } u \in \partial\Omega, \text{ and for all } \lambda \in [0, 1].$$

Then, from the homotopy invariance property of the Leray–Schauder degree, it follows that

$$\begin{aligned} \deg_{LS}(I - \Psi^*(\cdot, 1), \Omega, 0) &= \deg_{LS}(I - \Psi^*(\cdot, 0), \Omega, 0) = \deg_B(I - \Psi^*(\cdot, 0)|_{\mathbb{R}}, \Omega_0, 0) \\ &= -\deg_B(F, \Omega_0, 0) \neq 0, \end{aligned}$$

where  $\Omega_0 = \Omega \cap \mathbb{R}$ . Hence the mapping  $\Psi = \Psi^*(\cdot, 1)$  has at least one fixed point in  $\Omega$  and therefore problem (1.1) has at least one solution in  $\Omega$ .  $\square$

### 3. Existence theorems

Our main existence result is the following

**Theorem 3.1.** *Let  $f : [0, 1] \times \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$  be a Carathéodory function in the boundary value problem*

$$(\phi(u'))' = f(t, u, u'), \quad t \in (0, 1), \tag{3.1}$$

$$u'(0) = 0, \quad \theta(u(1)) = \sum_{i=1}^{m-2} \theta(u(\xi_i))a_i(u'(\xi_i)), \tag{3.2}$$

where  $a_i : \mathbb{R} \mapsto (0, \infty)$  is a continuous function,  $\xi_i \in (0, 1)$ ,  $i = 1, \dots, m - 2$ ,  $0 < \xi_1 < \xi_2 < \dots < \xi_{m-2} < 1$ . Suppose  $f$  satisfies the following conditions:

- (i) *There exist  $M > 0$ , continuous functions  $g_1, g_2 : [0, 1] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ , with  $g_1(t, u, 0) > 0$  for  $u > M$  and  $g_2(t, u, 0) > 0$  for  $u < -M$ , and functions  $\alpha_1, \alpha_2$  in  $L^1(0, 1)$ ,  $\alpha_1(t) > 0$ ,  $\alpha_2(t) < 0$ , for a.e.  $t \in [0, 1]$ , such that for all  $u > M$ , for all  $v \in \mathbb{R}$ , and for a.e.  $t \in [0, 1]$ , one has*

$$f(t, u, v) \geq \alpha_1(t)g_1(t, u, v),$$

*and for all  $u < -M$ , for all  $v \in \mathbb{R}$ , and for a.e.  $t \in [0, 1]$ , it holds that*

$$f(t, u, v) \leq \alpha_2(t)g_2(t, u, v).$$

- (ii) *There exist a continuous function  $\psi : [0, \infty) \mapsto [0, \infty)$ ,  $\psi$  non decreasing, with  $\psi(s) \rightarrow \infty$  as  $s \rightarrow \infty$ , and functions  $d_1, d_2 \in L^1(0, 1)$ , such that*

$$|f(t, u, v)| \leq d_1(t) + d_2(t)\psi(|v|),$$

*for a.e.  $t \in [0, 1]$ , all  $u \in [-M, M]$ , and all  $v \in \mathbb{R}$ .*

(iii) It holds that

$$0 < \|d_2\|_{L^1(0,1)} \limsup_{s \rightarrow \infty} \frac{s\psi(s)}{\Phi^*(\phi(s))} < 1, \quad (3.3)$$

where  $\Phi^*(s) = \int_0^s \phi^{-1}(\tau) d\tau$ ,  $s \in \mathbb{R}$ .

(iv) Suppose, further, that there exist constants  $b_i \geq 0$ ,  $i = 1, 2, \dots, m-2$ , with  $\sum_{i=1}^{m-2} b_i = 1$ ,

$$0 < a_i(v) \leq b_i, \quad \text{for all } v \in \mathbb{R}, \quad i = 1, \dots, m-2,$$

and

$$\sum_{i=1}^{m-2} a_i(0) = 1.$$

Then problem (3.1) has at least one solution  $u \in C^1[0, 1]$ .

**Proof.** We consider the family of boundary value problems  $(P_\lambda)$  with  $f^*(t, u, v, \lambda) = f(t, u, v)$  for all  $(t, u, v, \lambda) \in [0, 1] \times \mathbb{R} \times \mathbb{R} \times [0, 1]$ , i.e. we consider the family of problems:

$$\begin{aligned} \left( \phi \left( \frac{u'}{\lambda} \right) \right)' &= f(t, u, u'), \quad t \in (0, 1), \quad \lambda \in (0, 1], \\ u'(0) &= 0, \quad \mathcal{B}(u, \lambda) = 0, \end{aligned} \quad (3.4)$$

with  $\mathcal{B}(u, \lambda)$  as in (2.1).

We shall show that the family of problems (3.4) satisfies the conditions of Lemma 2.1 to conclude that problem (3.1) has at least one solution in  $C^1[0, 1]$ . Our first step is to define an open set  $\Omega \subset C^1[0, 1]$  such that (i) of Lemma 2.1 holds. Thus let  $\lambda \in (0, 1]$ , and  $u$  be a solution to (3.4).

We claim that  $|u(t)| \leq M$ , for all  $t \in [0, 1]$ , where  $M$  is as in assumption (i) of the theorem. Let  $t_0 \in [0, 1]$  be such that  $u(t_0)$  is a local maximum for  $u$  and suppose first that  $t_0 \in (0, 1)$ , then  $(t - t_0)u'(t) \leq 0$ , for  $t \in (0, 1)$  close to  $t_0$ . Also since  $u'(t_0) = 0$  by integrating the equation in (3.4), we find that

$$\phi \left( \frac{u'(t)}{\lambda} \right) = \int_{t_0}^t f(\tau, u(\tau), u'(\tau)) d\tau \leq 0, \quad (3.5)$$

for all  $t > t_0$  and close to  $t_0$ . Let us assume that  $u(t_0) > M$ , and set  $C_0 = g_1(t_0, u(t_0), 0) > 0$ . Let  $\varepsilon > 0$  be such that  $\varepsilon < C_0/2$ , then there is  $\delta > 0$  such that for all  $(t, u, v)$  such that  $|t - t_0| < \delta$ ,  $|u - u(t_0)| < \delta$ ,  $|v| < \delta$ , it holds that

$$|g_1(t, u, v) - g_1(t_0, u(t_0), 0)| < \varepsilon,$$

implying that

$$g_1(t, u, v) > g_1(t_0, u(t_0), 0) - \varepsilon = C_0 - \varepsilon > C_0/2.$$

Now for the  $\delta > 0$ , where  $\delta$  is as above, there is a  $\mu > 0$ ,  $\mu < \delta$ , such that for all  $|t - t_0| < \mu$ , it holds that

$$|u(t) - u(t_0)| < \delta \quad \text{and} \quad |u'(t)| < \delta.$$

Hence for  $t_0 < t < t_0 + \mu$  we have that

$$g_1(t, u(t), u'(t)) > C_0/2,$$

which implies that for all  $t_0 < t < t_0 + \mu$ ,

$$\begin{aligned} \phi\left(\frac{u'(t)}{\lambda}\right) &= \int_{t_0}^t f(\tau, u(\tau), u'(\tau)) d\tau \geq \int_{t_0}^t \alpha_1(\tau) g_1(\tau, u(\tau), u'(\tau)) d\tau \\ &> C_0/2 \int_{t_0}^t \alpha_1(\tau) d\tau > 0, \end{aligned}$$

which contradicts (3.5).

Assume next that  $t_0 = 0$ , and that  $u(0) > M$ . Then, by integrating the equation in (3.4) we first find that

$$\phi\left(\frac{u'(t)}{\lambda}\right) = \int_0^t f(\tau, u(\tau), u'(\tau)) d\tau,$$

for all  $t > 0$ . Then, as in the previous argument, by using the continuity of the function  $g_1$ , we find that

$$\phi\left(\frac{u'(t)}{\lambda}\right) > 0,$$

for  $t > 0$  small. Hence  $u'(t) > 0$  for  $t > 0$  small, implying that  $u$  is strictly increasing near zero, yielding again a contradiction. Thus if  $t_0 \in [0, 1)$  is such that  $u(t_0)$  is a local maximum for  $u$ , then it must be that  $u(t_0) \leq M$ .

By an entirely similar argument, using this time the continuity of the function  $g_2$ , it follows that if  $t_0 \in [0, 1)$  is such that  $u(t_0)$  is a local minimum for  $u$ , then it must be that  $u(t_0) \geq -M$ .

Thus, in particular, a solution  $u$  to problem (3.4) can have neither a global maximum  $u(t_0) > M$  nor a global minimum  $u(\tilde{t}_0) < -M$ , at points  $t_0, \tilde{t}_0$  with  $t_0, \tilde{t}_0 \in [0, 1)$ .

Suppose next that  $u$  reaches a global maximum at  $t_0 = 1$  and that  $u(1) > M$ . Then  $u(1) \geq u(t)$  for all  $t \in [0, 1]$ . We observe that in this situation it cannot be that  $u(\xi_i) = u(1)$ , for some  $i = 1, \dots, m-2$ , because  $u(t)$  would reach a global maximum at  $t = \xi_i \in (0, 1)$ . Hence  $u(\xi_i) < u(1)$ , for all  $i = 1, \dots, m-2$ . Similarly  $u(0) < u(1)$ . We will show that this implies

$$\mathcal{B}(u, \lambda) > 0 \tag{3.6}$$

yielding again a contradiction. Indeed, from (2.1),

$$\begin{aligned} \mathcal{B}(u, \lambda) &= \theta \left( \lambda u(1) + (1 - \lambda) \int_0^1 \phi^{-1} \left( \int_0^s f(\tau, u(\tau), u'(\tau)) d\tau \right) ds \right) \\ &\quad - \sum_{i=1}^{m-2} a_i (u'(\xi_i)) \theta \left( \lambda u(\xi_i) + (1 - \lambda) \int_0^{\xi_i} \phi^{-1} \left( \int_0^s f(\tau, u(\tau), u'(\tau)) d\tau \right) ds \right). \end{aligned}$$

Now, since  $u$  is a solution to  $(P_\lambda)$ , we have that

$$u(t) = u(0) + \lambda \int_0^t \phi^{-1} \left( \int_0^s f(\tau, u(\tau), u'(\tau)) d\tau \right) ds, \quad (3.7)$$

which together with the fact that  $u(1) > u(\xi_i)$  for all  $i = 1, \dots, m - 2$  implies that

$$\int_{\xi_i}^1 \phi^{-1} \left( \int_0^s f(\tau, u(\tau), u'(\tau)) d\tau \right) ds > 0$$

for all  $i = 1, \dots, m - 2$ . Then, from

$$\begin{aligned} & \lambda u(1) + (1 - \lambda) \int_0^1 \phi^{-1} \left( \int_0^s f(\tau, u(\tau), u'(\tau)) d\tau \right) ds \\ & - \lambda u(\xi_i) - (1 - \lambda) \int_0^{\xi_i} \phi^{-1} \left( \int_0^s f(\tau, u(\tau), u'(\tau)) d\tau \right) ds \\ & = \lambda(u(1) - u(\xi_i)) + (1 - \lambda) \int_{\xi_i}^1 \phi^{-1} \left( \int_0^s f(\tau, u(\tau), u'(\tau)) d\tau \right) ds > 0, \end{aligned}$$

for all  $i = 1, \dots, m - 2$  and the fact that  $\theta$  is an increasing function, we obtain that

$$\begin{aligned} & \sum_{i=1}^{m-2} a_i(u'(\xi_i)) \theta \left( \lambda u(1) + (1 - \lambda) \int_0^1 \phi^{-1} \left( \int_0^s f(\tau, u(\tau), u'(\tau)) d\tau \right) ds \right) \\ & > \sum_{i=1}^{m-2} a_i(u'(\xi_i)) \theta \left( \lambda u(\xi_i) + (1 - \lambda) \int_0^{\xi_i} \phi^{-1} \left( \int_0^s f(\tau, u(\tau), u'(\tau)) d\tau \right) ds \right). \end{aligned} \quad (3.8)$$

Also, since  $u(0) < u(1)$ , we see from (3.7) that

$$\int_0^1 \phi^{-1} \left( \int_0^s f(\tau, u(\tau), u'(\tau)) d\tau \right) ds > 0.$$

This implies that

$$\lambda u(1) + (1 - \lambda) \int_0^1 \phi^{-1} \left( \int_0^s f(\tau, u(\tau), u'(\tau)) d\tau \right) ds > 0$$

which then gives

$$\theta \left( \lambda u(1) + (1 - \lambda) \int_0^1 \phi^{-1} \left( \int_0^s f(\tau, u(\tau), u'(\tau)) d\tau \right) ds \right) > 0.$$

Accordingly, we see that



$$\begin{aligned}
 \mathcal{B}(u, \lambda) &> \left(1 - \sum_{i=1}^{m-2} a_i(u'(\xi_i))\right) \theta \left( \lambda u(1) + (1 - \lambda) \int_0^1 \phi^{-1} \left( \int_0^s f(\tau, u(\tau), u'(\tau)) d\tau \right) ds \right) \\
 &\geq \left(1 - \sum_{i=1}^{m-2} b_i\right) \theta \left( \lambda u(1) + (1 - \lambda) \int_0^1 \phi^{-1} \left( \int_0^s f(\tau, u(\tau), u'(\tau)) d\tau \right) ds \right) \\
 &= 0.
 \end{aligned} \tag{3.9}$$

Thus  $\mathcal{B}(u, \lambda) > 0$ , which is a contradiction.

In conclusion if a solution  $u$  reaches a global maximum  $u(t_0)$  at some  $t_0 \in [0, 1]$ , then it must be that  $u(t_0) \leq M$ .

An entirely similar argument tells us that if a solution  $u$  reaches a global minimum  $u(t_0)$  at some  $t_0 \in [0, 1]$ , then it must be that  $u(t_0) \geq -M$ . Thus, any solution  $u$  of problem (3.4), satisfies  $\|u\|_\infty \leq M$ .

We next show that derivatives of the solutions are also a priori bounded. We have that the function  $\Phi^*$  is even, increasing in  $[0, \infty)$ , and is such that  $\Phi^*(s) \rightarrow \infty$  as  $s \rightarrow \infty$ . Also it is immediate that for any  $v \in C^1$  such that  $\phi(v') \in C^1$ , it holds that

$$\frac{d}{dt} \Phi^*(\phi(v')) = v'(\phi(v'))'. \tag{3.10}$$

By multiplying the first equation in (3.4) by  $\frac{u'(t)}{\lambda}$  we have that

$$\frac{u'(t)}{\lambda} \left( \phi \left( \frac{u'(t)}{\lambda} \right) \right)' = \frac{u'(t)}{\lambda} f(t, u(t), u'(t)),$$

which by (3.10) becomes

$$\frac{d}{dt} \Phi^* \left( \phi \left( \frac{u'(t)}{\lambda} \right) \right) = \frac{u'(t)}{\lambda} f(t, u(t), u'(t)).$$

Integrating this expression from 0 to  $t$ , we find

$$\begin{aligned}
 \Phi^* \left( \phi \left( \frac{u'(t)}{\lambda} \right) \right) &\leq \frac{1}{\lambda} \int_0^t |u'(\tau) f(\tau, u(\tau), u'(\tau))| d\tau \\
 &\leq \frac{1}{\lambda} \|u'\|_\infty \int_0^t |f(\tau, u(\tau), u'(\tau))| d\tau,
 \end{aligned}$$

for all  $t \in [0, 1]$ . Next by condition (ii)

$$\begin{aligned}
 \int_0^1 |f(\tau, u(\tau), u'(\tau))| d\tau &\leq \int_0^1 (d_1(\tau) + d_2(\tau) \psi(|u'(\tau)|)) d\tau \\
 &\leq \beta_1 + \beta_2 \psi(\|u'\|_\infty),
 \end{aligned}$$

where  $\beta_1 = \|d_1\|_{L^1(0,1)}$ ,  $\beta_2 = \|d_2\|_{L^1(0,1)}$ . Combining the last two expressions we find that

$$\Phi^* \left( \phi \left( \frac{\|u'\|_\infty}{\lambda} \right) \right) \leq \frac{\|u'\|_\infty}{\lambda} (\beta_1 + \beta_2 \psi(\|u'\|_\infty))$$

and thus

$$\begin{aligned}
 1 &\leq \beta_1 \frac{\frac{\|u'\|_\infty}{\lambda}}{\Phi^*\left(\phi\left(\frac{\|u'\|_\infty}{\lambda}\right)\right)} + \beta_2 \frac{\frac{\|u'\|_\infty}{\lambda} \psi\left(\frac{\|u'\|_\infty}{\lambda}\right)}{\Phi^*\left(\phi\left(\frac{\|u'\|_\infty}{\lambda}\right)\right)} \\
 &\leq \beta_1 \frac{\frac{\|u'\|_\infty}{\lambda}}{\Phi^*\left(\phi\left(\frac{\|u'\|_\infty}{\lambda}\right)\right)} + \beta_2 \frac{\frac{\|u'\|_\infty}{\lambda} \psi\left(\frac{\|u'\|_\infty}{\lambda}\right)}{\Phi^*\left(\phi\left(\frac{\|u'\|_\infty}{\lambda}\right)\right)}.
 \end{aligned} \tag{3.11}$$

Since by (3.3),  $\lim_{s \rightarrow \infty} \frac{s}{\Phi^*(\phi(s))} = 0$ , we find from (3.11) and by using again (3.3), that there must exist an  $R_0 > M$  such that  $\|u'\|_\infty \leq R_0$ . This implies that there exists an  $R_1 > R_0$ , such that for  $\lambda \in (0, 1]$  the family of problems (3.4), or equivalently the equation  $u = \Psi^*(u, \lambda)$ , has no solution on the boundary of the bounded open set  $\Omega = B(0, R_1) \subset C^1[0, 1]$ . Hence condition (i) of Lemma 2.1 is satisfied.

Next, we have that  $F(\rho)$  given by (2.3), for this case, becomes

$$F(\rho) = \theta \left( \int_0^1 \phi^{-1} \left( \int_0^s f(t, \rho, 0) d\rho \right) ds \right) - \sum_{i=1}^{m-2} a_i(0) \theta \left( \int_0^{\xi_i} \phi^{-1} \left( \int_0^s f(t, \rho, 0) d\rho \right) ds \right).$$

Then, since from assumption (i), for all  $\rho > M$  (and hence for  $\rho = R_1$ ), it holds that

$$f(t, \rho, 0) \geq \alpha_1(t) g_1(t, \rho, 0) > 0 \quad \text{for a.e. } t \in [0, 1],$$

and that for all  $\rho < -M$  (and hence for  $\rho = -R_1$ ),

$$f(t, \rho, 0) \leq \alpha_2(t) g_2(t, -\rho, 0) < 0 \quad \text{for a.e. } t \in [0, 1],$$

by using that  $\sum_{i=1}^{m-2} a_i(0) = 1$ , we conclude that  $F(\rho)$  is strictly positive for  $\rho = R_1$  and strictly negative for  $\rho = -R_1$ , implying that condition (ii) of Lemma 2.1 is satisfied. We note that this argument also implies that condition (iii) of Lemma 2.1 is fulfilled (actually in this case, we have  $\deg_B[F, \Omega \cap \mathbb{R}, 0] = 1$ ).  $\square$

The following theorem is a direct consequence of Theorem 3.1 and generalizes Theorem 3.2 in [2]. Also under condition (3.14) below it improves the conditions we used in the particular case of Theorem 3.1 in [3], in the sense that condition (3.2) in that theorem is not needed.

**Theorem 3.2.** *Let  $f : [0, 1] \times \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$  be a continuous function in the boundary value problem*

$$(\phi(u'))' = f(t, u, u'), \quad t \in (0, 1), \tag{3.12}$$

$$u'(0) = 0, \quad \theta(u(1)) = \sum_{i=1}^{m-2} \theta(u(\xi_i)) a_i(u'(\xi_i)), \tag{3.13}$$

where  $a_i : \mathbb{R} \mapsto (0, \infty)$  is a continuous function,  $\xi_i \in (0, 1)$ ,  $i = 1, \dots, m - 2$ ,  $0 < \xi_1 < \xi_2 < \dots < \xi_{m-2} < 1$ . Suppose  $f$  satisfies the following conditions:

- (i) *There exists  $M > 0$  such that for all  $|u| > M$ , and all  $t \in [0, 1]$ , one has*

$$uf(t, u, 0) > 0. \tag{3.14}$$

- (ii) There exist a function  $\psi : [0, \infty) \mapsto [0, \infty)$ ,  $\psi$  non decreasing with  $\psi(s) \rightarrow \infty$  as  $s \rightarrow \infty$ , and continuous functions  $d_1, d_2 : [0, 1] \times \mathbb{R} \mapsto [0, \infty)$ , such that for all  $u \in [-M, M]$ ,  $v \in \mathbb{R}$ , and for all  $t \in [0, 1]$ , it holds that

$$|f(t, u, v)| \leq d_1(t, u) + d_2(t, u)\psi(|v|).$$

- (iii) Let  $C_2 = \max\{d_2(t, u), t \in [0, 1], |u| \leq M\}$ , we assume that

$$0 < C_2 \limsup_{s \rightarrow \infty} \frac{s\psi(s)}{\Phi^*(\phi(s))} < 1, \quad (3.15)$$

where  $\Phi^*(s) = \int_0^s \phi^{-1}(\tau) d\tau$ ,  $s \in \mathbb{R}$ .

- (iv) Suppose, further, that there exist constants  $b_i \geq 0$ ,  $i = 1, 2, \dots, m-2$ , with  $\sum_{i=1}^{m-2} b_i = 1$ ,

$$0 < a_i(v) \leq b_i, \quad \text{for all } v \in \mathbb{R}, i = 1, \dots, m-2,$$

and

$$\sum_{i=1}^{m-2} a_i(0) = 1.$$

Then problem (3.12) has at least one solution  $u \in C^1[0, 1]$ .

**Proof.** We notice that the function  $f(t, u, v)$  in this theorem satisfies the assumption (i) of Theorem 3.1 with  $g_1(t, u, v) \equiv f(t, u, v)$ ,  $g_2(t, u, v) \equiv -f(t, u, v)$ ,  $\alpha_1(t) \equiv 1$ ,  $\alpha_2(t) \equiv -1$  in view of our assumption that  $f$  is a continuous function and satisfies the condition (i) of this theorem. It satisfies the assumption (ii) of Theorem 3.1 with  $d_1(t) = \max\{d_1(t, u), |u| \leq M\}$ ,  $d_2(t) = \max\{d_2(t, u), |u| \leq M\}$  for all  $t \in [0, 1]$ . Also it is clear that assumption (iii) of Theorem 3.1 is implied by (3.15). Finally noticing that the assumption (iv) of this theorem is the same as the assumption (iv) of Theorem 3.1, we see that the result is an immediate consequence of Theorem 3.1.  $\square$

**Corollary 3.1.** Let  $f : [0, 1] \times \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$  in problem (1.1) be a continuous function that satisfies the following conditions:

- (i) There exist non-negative functions  $\tilde{d}_1(t)$ ,  $\tilde{d}_2(t)$ , and  $r(t)$  in  $L^1(0, 1)$  such that

$$|f(t, u, v)| \leq \tilde{d}_1(t)\phi(|u|) + \tilde{d}_2(t)\phi(|v|) + r(t),$$

for a.e.  $t \in [0, 1]$  and all  $u, v \in \mathbb{R}$ .

- (ii) There exists  $u_0 > 0$  such that  $uf(t, u, v) > 0$  for  $u \in \mathbb{R}$ , with  $|u| > u_0$  and  $v \in \mathbb{R}$ .  
 (iii)  $\tilde{d}_2$  satisfies

$$\|\tilde{d}_2\|_{L^1(0,1)} < \liminf_{s \rightarrow \infty} \frac{\Phi^*(s)}{s\phi^{-1}(s)}, \quad (3.16)$$

where as before  $\Phi^*(s) = \int_0^s \phi^{-1}(\tau) d\tau$ ,  $s \in \mathbb{R}$ .

- (iv) Suppose, further, that there exist constants  $b_i \geq 0$ ,  $i = 1, 2, \dots, m-2$ , with  $\sum_{i=1}^{m-2} b_i = 1$ ,

$$0 < a_i(v) \leq b_i, \quad \text{for all } v \in \mathbb{R}, i = 1, \dots, m-2,$$

and

$$\sum_{i=1}^{m-2} a_i(0) = 1.$$

Then problem (1.1) has at least one solution  $u \in C^1[0, 1]$ .

**Proof.** It is an immediate consequence of Theorem 3.2. Indeed setting in Theorem 3.2,  $d_1(t, u) = \tilde{d}_1(t)\phi(|u|) + r(t)$ ,  $d_2(t, u) = \tilde{d}_2(t)$ , and  $\psi(s) = \phi(s)$ ,  $s \in [0, \infty)$ , it is immediate that conditions (i)–(iv) of this corollary imply that conditions (i)–(iv) of Theorem 3.2 are satisfied.  $\square$

In the following theorem we replace the assumption (ii) of Corollary 3.1 by the assumption “there exists  $u_0 > 0$  such that  $uf(t, u, v) < 0$  for  $t \in [0, 1]$ ,  $u \in \mathbb{R}$ , with  $|u| > u_0$  and  $v \in \mathbb{R}$ .” In this case we assume that the continuous functions  $a_i : \mathbb{R} \mapsto (0, \infty)$ ,  $i = 1, \dots, m - 2$ , in problem (1.1) are constant functions, i.e. there exist constants  $b_i \geq 0$ ,  $i = 1, 2, \dots, m - 2$ , with  $\sum_{i=1}^{m-2} b_i = 1$ , and  $0 < a_i(v) \equiv b_i$ , for all  $v \in \mathbb{R}$ ,  $i = 1, \dots, m - 2$ .

**Theorem 3.3.** Let  $f : [0, 1] \times \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$  in problem (2.2) be a continuous function that satisfies the following conditions:

(i) There exist non-negative functions  $d_1(t)$ ,  $d_2(t)$ , and  $r(t)$  in  $L^1(0, 1)$  such that

$$|f(t, u, v)| \leq d_1(t)\phi(|u|) + d_2(t)\phi(|v|) + r(t),$$

for a.e.  $t \in [0, 1]$  and all  $u, v \in \mathbb{R}$ .

(ii) There exists  $u_0 > 0$  such that  $uf(t, u, v) < 0$  for  $t \in [0, 1]$ ,  $u \in \mathbb{R}$ , with  $|u| > u_0$  and  $v \in \mathbb{R}$ .

(iii) Suppose  $\|d_2\|_{L^1(0,1)} < 1$ , and the function  $\Gamma : [0, \infty) \mapsto [0, \infty)$ , defined for  $z \in \mathbb{R}$  by

$$\Gamma(z) = \phi^{-1}\left(\frac{\|d_1\|_{L^1(0,1)}}{1 - \|d_2\|_{L^1(0,1)}}\phi(z) + \frac{\|r\|_{L^1(0,1)}}{1 - \|d_2\|_{L^1(0,1)}}\right),$$

satisfies the condition

$$\limsup_{z \rightarrow \infty} \frac{\Gamma(z)}{z} < 1. \tag{3.17}$$

(iv) Suppose, further, that there exist constants  $b_i \geq 0$ ,  $i = 1, 2, \dots, m - 2$ , with  $\sum_{i=1}^{m-2} b_i = 1$ .

Then problem

$$(\phi(u'))' = f(t, u, u'), \quad t \in (0, 1),$$

$$u'(0) = 0, \quad \theta(u(1)) = \sum_{i=1}^{m-2} b_i \theta(u(\xi_i)) \tag{3.18}$$

has at least one solution  $u \in C^1[0, 1]$ .

**Proof.** Let us set  $f^*(t, u, v, \lambda) = f(t, u, v)$ , in Theorem 3.1. We shall show that for all  $(t, u, v, \lambda) \in [0, 1] \times \mathbb{R} \times \mathbb{R} \times (0, 1)$ , and  $0 < a_i(v) \equiv b_i$ , for all  $v \in \mathbb{R}$ ;  $i = 1, \dots, m - 2$ , the family of problems

$$\begin{aligned} \left( \phi \left( \frac{u'}{\lambda} \right) \right)' &= f(t, u, u'), \quad t \in (0, 1), \lambda \in (0, 1], \\ u'(0) &= 0, \quad \mathcal{B}(u, \lambda) = 0, \end{aligned} \quad (3.19)$$

with  $\mathcal{B}(u, \lambda)$  as defined in (2.1), satisfies conditions (i)–(iii) of Lemma 2.1.

Let  $u(t)$  be a solution of (3.19) for some  $\lambda \in (0, 1)$ . We claim that there exists a  $\tilde{t} \in [0, 1]$  such that

$$-u_0 \leq u(\tilde{t}) \leq u_0. \quad (3.20)$$

Indeed let us assume that  $u(t) > u_0 > 0$  for all  $t \in [0, 1]$  then we see from assumption (ii) that  $f(t, u(t), u'(t)) < 0$  for all  $t \in [0, 1]$ . We then get, for all  $t \in [0, 1]$ ,

$$(\phi(u'))' < 0 \quad (3.21)$$

and since  $u'(0) = 0$  we obtain that  $u(t)$  is strictly decreasing and in particular  $u(1) < u(\xi_i)$  for all  $i$ . Further, we see that

$$\int_0^1 \phi^{-1} \left( \int_0^s f(\tau, u(\tau), u'(\tau)) d\tau \right) ds \leq \int_0^{\xi_i} \phi^{-1} \left( \int_0^s f(\tau, u(\tau), u'(\tau)) d\tau \right) ds$$

for all  $i$ , since  $f(t, u(t), u'(t)) < 0$  for all  $t \in [0, 1]$ . Hence, we obtain

$$\begin{aligned} \lambda u(1) + (1 - \lambda) \int_0^1 \phi^{-1} \left( \int_0^s f(\tau, u(\tau), u'(\tau)) d\tau \right) ds \\ < \lambda u(\xi_i) + (1 - \lambda) \int_0^{\xi_i} \phi^{-1} \left( \int_0^s f(\tau, u(\tau), u'(\tau)) d\tau \right) ds \end{aligned}$$

for all  $i$ . We, next, use  $\mathcal{B}(u, \lambda) = 0$  and the above inequality to obtain first that

$$\begin{aligned} &\theta \left( \lambda u(1) + (1 - \lambda) \int_0^1 \phi^{-1} \left( \int_0^s f(\tau, u(\tau), u'(\tau)) d\tau \right) ds \right) \\ &= \sum_{i=1}^{m-2} b_i \theta \left( \lambda u(\xi_i) + (1 - \lambda) \int_0^{\xi_i} \phi^{-1} \left( \int_0^s f(\tau, u(\tau), u'(\tau)) d\tau \right) ds \right) \\ &> \sum_{i=1}^{m-2} b_i \theta \left( \lambda u(1) + (1 - \lambda) \int_0^1 \phi^{-1} \left( \int_0^s f(\tau, u(\tau), u'(\tau)) d\tau \right) ds \right), \end{aligned}$$

and then use the assumption  $\sum_{i=1}^{m-2} b_i = 1$ , to yield the contradiction

$$\begin{aligned} &\theta \left( \lambda u(1) + (1 - \lambda) \int_0^1 \phi^{-1} \left( \int_0^s f(\tau, u(\tau), u'(\tau)) d\tau \right) ds \right) \\ &> \theta \left( \lambda u(1) + (1 - \lambda) \int_0^1 \phi^{-1} \left( \int_0^s f(\tau, u(\tau), u'(\tau)) d\tau \right) ds \right). \end{aligned}$$

Similarly, if we assume that  $u(t) < -u_0 < 0$  for all  $t \in [0, 1]$  we shall arrive at a contradiction. Accordingly, there exists a  $\tilde{t} \in [0, 1]$  such that  $-u_0 \leq u(\tilde{t}) \leq u_0$ , proving the claim (3.20).

As an immediate consequence of (3.20) we obtain that

$$\|u\|_\infty \leq u_0 + \|u'\|_\infty. \quad (3.22)$$

Next, we see from assumption (i) of the theorem that

$$|f(t, u(t), u'(t))| \leq d_1(t)\phi(|u(t)|) + d_2(t)\phi(|u'(t)|) + r(t),$$

and hence by integrating the equation in (3.19) from 0 to  $t \in [0, 1]$  and using that  $u'(0) = 0$ , we obtain that

$$\phi(\|u'\|_\infty) \leq \phi(\|u\|_\infty)\|d_1\|_{L^1(0,1)} + \phi(\|u'\|_\infty)\|d_2\|_{L^1(0,1)} + \|r\|_{L^1(0,1)}. \quad (3.23)$$

Thus, solving for  $\|u'\|_\infty$ , and then combining with (3.22) we find that

$$\|u'\|_\infty \leq \phi^{-1}\left(\frac{\|d_1\|_{L^1(0,1)}}{1 - \|d_2\|_{L^1(0,1)}}\phi(\|u\|_\infty) + \frac{\|r\|_{L^1(0,1)}}{1 - \|d_2\|_{L^1(0,1)}}\right), \quad (3.24)$$

and then

$$\|u\|_\infty \leq u_0 + \Gamma(\|u\|_\infty).$$

Accordingly, we get using our assumption

$$\limsup_{z \rightarrow \infty} \frac{\Gamma(z)}{z} < 1,$$

that there must exist a  $z_0 > 0$  such that

$$\|u\|_\infty \leq z_0. \quad (3.25)$$

Finally, by (3.24) and (3.25) there is a positive constant  $R_0 > u_0$  (where  $u_0$  is as in assumption (ii)) such that

$$\|u\|_{C^1[0,1]} \leq R_0.$$

Thus for any fixed  $R > R_0$  if  $\Omega = B(0, R) \subset C^1[0, 1]$ , then for each  $0 < \lambda < 1$  the family of problems (3.19) has no solution on  $\partial\Omega$ . Thus (i) of Lemma 2.1 is satisfied.

Next, we observe that for our case

$$F(\rho) = \theta \left( \int_0^1 \phi^{-1} \left( \int_0^s f(\tau, \rho, 0) d\tau \right) ds \right) - \sum_{i=1}^{m-2} b_i \theta \left( \int_0^{\xi_i} \phi^{-1} \left( \int_0^s f(\tau, \rho, 0) d\tau \right) ds \right)$$

and note that

$$B(\rho, 0) = F(\rho).$$

Then assumption (ii) implies that for all  $\rho \in \mathbb{R}$ , with  $|\rho| \geq u_0$ ,

$$\rho F(\rho) < 0. \quad (3.26)$$

Since this holds when  $\rho = R$ , we have that (ii) of Lemma 2.1 is satisfied. That (iii) of that lemma holds is immediate from (3.26). We have thus proved that the family of problems (3.19) satisfies

all the conditions of Lemma 2.1 and hence existence of a solution for problem (3.18) follows from that lemma. This completes the proof of the theorem.  $\square$

Next consider the problem

$$\begin{aligned} (\phi(u'))' + f(t, u, u') &= q(t), \quad t \in (0, 1), \\ u'(0) = 0, \quad \theta(u(1)) &= \sum_{i=1}^{m-2} \theta(u(\xi_i)) a_i (u'(\xi_i)), \end{aligned} \quad (3.27)$$

where  $f : [0, 1] \times \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$  is continuous, and  $q \in L^1(0, 1)$ . We define

$$\bar{q} = \sup \left\{ \frac{1}{s} \int_0^s q(\tau) d\tau \mid s \in (0, 1] \right\},$$

and

$$\underline{q} = \inf \left\{ \frac{1}{s} \int_0^s q(\tau) d\tau \mid s \in (0, 1] \right\}.$$

We have

**Theorem 3.4.** *Let  $q \in L^1(0, 1)$  and  $f : [0, 1] \times \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$  in problem (3.27) be a continuous function satisfying the following conditions:*

- (i) *There exist non-negative functions  $d_1(t)$ ,  $d_2(t)$ , and  $r(t)$  in  $L^1(0, 1)$  such that*

$$|f(t, u, v)| \leq d_1(t)\phi(|u|) + d_2(t)\phi(|v|) + r(t),$$

*for a.e.  $t \in [0, 1]$  and all  $u, v \in \mathbb{R}$ .*

- (ii) *There exists a  $d > 0$  such that*

$$\begin{aligned} f(t, u, v) &< \underline{q} \quad \text{for } u \geq d, \\ f(t, u, v) &> \bar{q} \quad \text{for } u \leq -d, \end{aligned}$$

*for a.e.  $t \in [0, 1]$  and all  $v \in \mathbb{R}$ .*

- (iii) *Suppose  $\|d_2\|_{L^1(0,1)} < 1$ , and the function  $\Gamma : [0, \infty) \mapsto [0, \infty)$ , defined by*

$$\Gamma(z) = \phi^{-1} \left( \frac{\|d_1\|_{L^1(0,1)}}{1 - \|d_2\|_{L^1(0,1)}} \phi(z) + \frac{\|\tilde{r}\|_{L^1(0,1)}}{1 - \|d_2\|_{L^1(0,1)}} \right),$$

*where  $\tilde{r}(t) = r(t) + q(t)$ , satisfies the condition*

$$\limsup_{z \rightarrow \infty} \frac{\Gamma(z)}{z} < 1. \quad (3.28)$$

- (iv) *Suppose, further, that there exist constants  $b_i \geq 0$ ,  $i = 1, 2, \dots, m-2$ , with  $\sum_{i=1}^{m-2} b_i = 1$ ,*

$$0 < a_i(v) \leq b_i, \quad \text{for all } v \in \mathbb{R}, \quad i = 1, \dots, m-2,$$

*and*

$$\sum_{i=1}^{m-2} a_i(0) = 1.$$

*Then problem (3.27) has at least one solution  $u \in C^1[0, 1]$ .*

This theorem improves the conditions of Theorem 3.2 in [3], in fact conditions (3.22) and (3.23) in that theorem are no longer needed for its validity.

**Proof.** We shall show that for  $f^*(t, u, v, \lambda) = q(t) - f(t, u, v)$  for all  $(t, u, v, \lambda) \in [0, 1] \times \mathbb{R} \times \mathbb{R} \times (0, 1]$ , the family of problems

$$\begin{aligned} \left(\phi\left(\frac{u'}{\lambda}\right)\right)' &= q(t) - f(t, u, u'), \quad t \in (0, 1), \quad \lambda \in (0, 1], \\ u'(0) &= 0, \quad \mathcal{B}(u, \lambda) = 0, \end{aligned} \quad (3.29)$$

with  $\mathcal{B}(u, \lambda)$  as defined in (2.1), satisfies conditions (i)–(iii) of Lemma 2.1.

Let  $u(t)$  be a solution of (3.29) for some  $\lambda \in (0, 1)$ . Then, by integrating the equation in (3.29) from 0 to  $t \in (0, 1]$  and using that  $u'(0) = 0$ , we obtain that

$$\begin{aligned} \phi\left(\frac{u'(t)}{\lambda}\right) &= \int_0^t (q(\tau) - f(\tau, u(\tau), u'(\tau))) d\tau \\ &> \int_0^t (\underline{q} - f(\tau, u(\tau), u'(\tau))) d\tau > 0, \end{aligned} \quad (3.30)$$

for all  $t \in [0, 1]$ . Similarly

$$\phi\left(\frac{u'(t)}{\lambda}\right) < 0 \quad \text{if } u(t) < -d \text{ for all } t \in [0, 1]. \quad (3.31)$$

Next we claim that there exists a  $\tilde{t} \in [0, 1]$  such that

$$-d \leq u(\tilde{t}) \leq d. \quad (3.32)$$

Indeed assume first that  $u(t) > d$  for all  $t \in [0, 1]$ . Then we see from (3.30) that  $u'(t) > 0$  for all  $t \in [0, 1]$  and hence that  $u(t)$  is strictly increasing. Thus as in the proof of Theorem 3.1, repeating the argument that goes from (3.6) to (3.8), it follows that  $\mathcal{B}(u, \lambda) > 0$ , yielding a contradiction. Similarly, if we assume that  $u(t) < -d$  for all  $t \in [0, 1]$  we arrive to a contradiction. Hence the claim (3.32) is proved.

As an immediate consequence of (3.32) we obtain that

$$\|u\|_\infty \leq d + \|u'\|_\infty. \quad (3.33)$$

Next, we see from assumption (i) of the theorem that

$$|q(t) - f(t, u(t), u'(t))| \leq d_1(t)\phi(|u(t)|) + d_2(t)\phi(|u'(t)|) + \tilde{r}(t),$$

and hence by (3.30), we obtain that

$$\phi(\|u'\|_\infty) \leq \phi(\|u\|_\infty)\|d_1\|_{L^1(0,1)} + \phi(\|u'\|_\infty)\|d_2\|_{L^1(0,1)} + \|\tilde{r}\|_{L^1(0,1)}. \quad (3.34)$$

Thus, solving for  $\|u'\|_\infty$ ,

$$\|u'\|_\infty \leq \phi^{-1}\left(\frac{\|d_1\|_{L^1(0,1)}}{1 - \|d_2\|_{L^1(0,1)}}\phi(\|u\|_\infty) + \frac{\|\tilde{r}\|_{L^1(0,1)}}{1 - \|d_2\|_{L^1(0,1)}}\right), \quad (3.35)$$

and combining with (3.33), we find that

$$\|u\|_\infty \leq d + \Gamma(\|u\|_\infty).$$



Accordingly, using our assumption

$$\limsup_{z \rightarrow \infty} \frac{\Gamma(z)}{z} < 1,$$

we get that there must exist a  $z_0 > 0$  such that

$$\|u\|_\infty \leq z_0. \tag{3.36}$$

Finally, by (3.35) and (3.36) there is a positive constant  $R_0 > d$  (where  $d$  is as in assumption (ii)) such that

$$\|u\|_{C^1[0,1]} \leq R_0.$$

Thus for any fixed  $R > R_0$  if  $\Omega = B(0, R) \subset C^1[0, 1]$ , then for each  $0 < \lambda < 1$  the family of problems (3.29) has no solution on  $\partial\Omega$ . Thus (i) of Lemma 2.1 is satisfied.

Next formula (2.3), applied to our present situation, yields that

$$F(\rho) = \theta \left( \int_0^1 \phi^{-1} \left( \int_0^s (q(\tau) - f(\tau, \rho, 0)) d\tau \right) ds \right) - \sum_{i=1}^{m-2} a_i(0) \theta \left( \int_0^{\xi_i} \phi^{-1} \left( \int_0^s (q(\tau) - f(\tau, \rho, 0)) d\tau \right) ds \right), \tag{3.37}$$

where we note that

$$\mathcal{B}(\rho, 0) = F(\rho). \tag{3.38}$$

Then, for all  $\rho \in \mathbb{R}$ , with  $|\rho| \geq d$ , assumption (ii), together with  $\sum_{i=1}^{m-2} a_i(0) = 1$ , implies that

$$\rho F(\rho) > 0. \tag{3.39}$$

Since this holds when  $\rho = R$ , we have that (ii) of Lemma 2.1 is satisfied. That (iii) of that lemma holds is immediate from (3.39). We have thus proved that the family of problems (3.29) satisfies all the conditions of Lemma 2.1 and hence existence of a solution for problem (3.27) follows from that lemma. This completes the proof of the theorem.  $\square$

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