Vertex Partitions and Maximum Degenerate Subgraphs

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Abstract: Let G be a graph with maximum degree $d \geq 3$ and $\omega(G) \leq d$, where $\omega(G)$ is the *clique number* of the graph G. Let p_1 and p_2 be two positive integers such that $d = p_1 + p_2$. In this work, we prove that G has a vertex partition $\{S_1, S_2\}$ such that $G[S_1]$ is a maximum order $(p_1 - 1)$-degenerate subgraph of G and $G[S_2]$ is a $(p_2 - 1)$-degenerate subgraph, where $G[S_i]$ denotes the graph induced by the set S_i in G, for $i = 1,2$. On one hand, by using a degree-equilibrating process our result implies a result of Bollobas and Marvel [1]: for every graph G of maximum degree $d \geq 3$ and $\omega(G) \leq d$, and for every p_1 and p_2 positive integers such that $d = p_1 + p_2$, the graph G has a partition $\{S_1, S_2\}$ such that for $i = 1, 2$, $\Delta(G[S_i]) \leq p_i$ and $G[S_i]$ is $(p_i - 1)$-degenerate. On the other hand, our result refines the following result of Catlin in [2]: every graph G of maximum degree $d \geq 3$ has a partition $\{S_1, S_2\}$ such that S_1 is a maximum independent set and $\omega(G[S_2]) \leq d - 1$; it also refines a result of Catlin and Lai [3]: every

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graph G of maximum degree $d \geq 3$ has a partition $\{S_1, S_2\}$ such that S_1 is a maximum size set with $G[S_1]$ acyclic and $\omega(G[S_2]) \leq d-2$. The cases $d = 3$, $(d, p_1) = (4, 1)$ and $(d, p_1) = (4, 2)$ were proved by Catlin and Lai [3].

Keywords: *degenerate subgraphs; vertex partitions*

1. INTRODUCTION

The clique number, $\omega(G)$, of a graph G is the largest integer k such that G contains a complete subgraph of size k. A graph G is k-degenerate if every subgraph of G contains a vertex of degree at most k (Lick and White in [5]). The coloring number $\mathrm{col}(G)$, of a graph G is the smallest integer k such that G is $(k-1)$-degenerate (Erdös and Hajnal in [4]). Then, a graph G has no edge if and only if $\mathrm{col}(G) \leq 1$, and it is acyclic if and only if $\mathrm{col}(G) \leq 2$. Clearly, if G has maximum degree $d = \Delta(G)$, then G is d-degenerate; hence $\mathrm{col}(G) \leq \Delta(G) + 1$. If G is not $(d-1)$-degenerate, then $\mathrm{col}(G) = \Delta(G) + 1$ and G contains a d-regular connected component.

In this work, we prove the following theorem.

Theorem 1.1. *Let G be a graph with maximum degree $d \geq 3$ and $\omega(G) \leq d$. Then, for every p_1 and p_2 such that $d = p_1 + p_2$ the graph G has a partition $\{S_1, S_2\}$ such that $G[S_1]$ is a maximum order $(p_1 - 1)$-degenerate induced subgraph of G and $G[S_2]$ is a $(p_2 - 1)$-degenerate induced subgraph.*

This result generalizes some results relating the maximum degree of the original graph G with the coloring numbers of its parts $G[S_1]$ and $G[S_2]$. On one hand, Catlin in [2] proved that any graph G with maximum degree $d \geq 3$ and $\omega(G) \leq d$, has a partition $\{S_1, S_2\}$ such that S_1 is a maximum independent set and $\Delta(G[S_2]), \omega(G[S_2]) \leq d-1$. The case $p_1 = 1$ of Theorem 1.1 corresponds to a slightly refinement of this result. Later, Catlin and Lai in [3] proved the following.

Theorem 1.2. *Every graph G with maximum degree $d \geq 3$ and $\omega(G) \leq d$ has a partition $\{S_1, S_2\}$ such that*

1. *For $d = 3$, S_1 is a maximum independent set and $G[S_2]$ is acyclic.*
2. *For $d = 4$, $G[S_1]$ is a maximum acyclic induced subgraph and $G[S_2]$ is acyclic.*
3. *For $d \geq 5$, $G[S_1]$ is a maximum acyclic induced subgraph and $\omega(G[S_2])$, $\Delta(G[S_2]) \leq d-2$.*

The first two cases in Theorem 1.2 correspond to the cases $(d, p_1) = (3, 1)$ and $(d, p_1) = (4, 2)$ in Theorem 1.1, respectively. The case $d \geq 5$, $p_1 = 2$ in Theorem 1.1 is a slightly improvement of the third case in Theorem 1.2. In the best of our knowledge, all the remaining cases are new.

On the other hand, Bollobas and Marvel proved the following [1].

Theorem 1.3. *Let G be a graph of maximum degree $d \geq 3$ and $\omega(G) \leq d$. For every p_1 and p_2 positive integers with $d = p_1 + p_2$, G has a partition $\{S_1, S_2\}$ such that, for $i = 1, 2$, $\mathsf{col}(G[S_i]), \Delta(G[S_i]) \leq p_i$.*

Theorem 1.3 can be derived from Theorem 1.1 by using the following lemma.

Lemma 1.4. *Let G be a graph of maximum degree $d \geq 3$. Let p_1 and p_2 be positive integers with $d = p_1 + p_2$. If G has a partition $\{S_1, S_2\}$ where $\mathsf{col}(G[S_i]) \leq p_i$ for $i = 1, 2$, then G has a partition $\{S_1', S_2'\}$ such that $\mathsf{col}(G[S_i']), \Delta(G[S_i']) \leq p_i$ for $i = 1, 2$.*

Proof. Let $\{S_1, S_2\}$ be a partition of G such that $\mathsf{col}(G[S_i]) \leq p_i$, for $i = 1, 2$ and minimizing $p_2\|S_1\| + p_1\|S_2\|$, where $\|S_i\|$ denotes the number of edges of $G[S_i]$, for $i = 1, 2$. Let us denote by $d_i(v)$ the number of neighbors of a vertex v in S_i, for $i = 1, 2$. If $\Delta(G[S_1]) > p_1$ or $\Delta(G[S_2]) > p_2$, then we can assume that a vertex $v \in S_1$ exists with $d_1(v) > p_1$. Then $d_2(v) < p_2$. Hence $S_1' := S_1 - \{v\}$ and $S_2' := S_2 \cup \{v\}$ satisfy $\mathsf{col}(G[S_i']) \leq p_i$, for $i = 1, 2$. Moreover, $p_2\|S_1\| + p_1\|S_2\| - p_2\|S_1'\| - p_1\|S_2'\| = p_2 d_1(v) - p_1 d_2(v) > p_2 p_1 - p_1 p_2 = 0$ which gives the contradiction. ■

2. PROOF OF THE MAIN RESULT

Let $G = (V, E)$ an undirected graph. Let p be a positive integer. Let us say that a subset S of V is *p-stable* in G if $G[S]$ is $(p-1)$-degenerate. A 1-stable set S in G is precisely an independent set of G. A set $S \subseteq V$ is 2-stable in G if and only if the graph induced by S is acyclic.

Theorem 1.1 will be proved in the following manner.

Theorem 2.1. *Let $G = (V, E)$ be a connected graph with maximum degree $d \geq 3$. Let p_1 and p_2 be positive integers such that $d = p_1 + p_2$. If for every maximum p_1-stable set S_1 in G and every maximum p_2-stable set in $G - S_1$ the union $S_1 \cup S_2$ is not V, then $G = K_{d+1}$.*

Proof. Let H be the set of all tuples $S = (S_1, S_2, v)$ with S_1 a maximum p_1-stable set in G, S_2 a maximum p_2-stable set in $G - S_1$ and $v \notin S_1 \cup S_2$. Then $H \neq \emptyset$. For each $S = (S_1, S_2, v)$ in H and $i = 1, 2$ the following properties hold.

1. *Let $L_i(v)$ be the set of all neighbors of v in S_i. Then $|L_i(v)| = p_i$.*

2. *Let $C_i(v)$ the connected component containing v in $G[S_i \cup \{v\}]$. Then, for every $w \in C_i(v)$ the tuple $S_w^v \in H$, where*

$$S_w^v = \begin{cases} ((S_1 \cup \{v\}) - \{w\}, S_2, w) & \text{if } w \in S_1 \\ (S_1, (S_2 \cup \{v\}) - \{w\}, w) & \text{if } w \in S_2 \end{cases}$$

3. *The subgraph $C_i(v)$ is p_i-regular and each vertex in $C_i(v)$ has exactly p_{3-i} neighbors in S_{3-i}.*

Property 1 follows immediately from the definition of H, while Property 2 can be proved by induction on the distance between v and w in $C_i(v)$. Property 3 is obtained by combining Properties 1 and 2.

Let $S = (S_1, S_2, v) \in H$. A subset D of $S_1 \cup S_2$ is a *piece* of S if $G[D]$ is a connected component of $G[S_1]$ or a connected component of $G[S_2]$. Let H^* denote the set of all elements in H with minimum number of pieces. For each $S = (S_1, S_2, v) \in H^*$ and each $i = 1, 2$ the following properties hold.

4. *Let $D_i(v) := V(C_i(v)) - \{v\}$. Then $D_i(v)$ is a piece of S.*

5. *If a vertex in $L_i(v)$ has a neighbor in $L_{3-i}(v)$, then it is adjacent to each vertex in $L_{3-i}(v)$.*

6. *If a vertex in $L_i(v)$ has a neighbor in $L_{3-i}(v)$, then $G[L_{3-i}(v)]$ is a complete subgraph of G.*

Property 4 is a consequence of the definition of H^*. Property 5 is proved as follows. Let $z \in L_i(v)$ be a vertex having a neighbor u in $L_{3-i}(v)$. From Property 2, the tuple $S_z^v \in H$. By applying Property 3 to S_z^v we get that the connected component C' containing z in $G[S_{3-i} \cup \{z\}]$ is p_{3-i}-regular. We know that $L_{3-i}(v)$ is contained in $V(C')$, since by Property 4 we have that $G[D_{3-i}(v)]$ is connected. Therefore, the regularity of C' implies that each vertex in $L_{3-i}(v)$ is adjacent to z.

We now prove Property 6. Let $z \in L_i(v)$ and $u \in L_{3-i}(v)$ such that $zu \in E$. We prove that z is adjacent to every vertex w in $L_i(v)$. From property 5, we know that u and w are adjacent. The same property applied to w implies that w is adjacent to every vertex in $L_{3-i}(v)$. Hence, w has exactly $p_{3-i} - 1$ neighbors in $S_{3-i} - \{u\}$. By using Property 2 twice, we know that $T := S_z^v \in H$ and $T_u^z \in H$. Since the vertex w belongs to the connected component containing u in $G[S_i \cup \{v, u\} - \{z\}]$, Property 3 applied to T_u^z implies that the vertex w has p_{3-i} neighbors in $S_{3-i} \cup \{z\} - \{u\}$. Therefore, w and z are adjacent.

7. *If there exists $S = (S_1, S_2, v) \in H^*$ such that a vertex in $L_i(v)$ has a neighbor in $L_{3-i}(v)$, then G is complete.*

By applying Property 5 first to z and then to any vertex in $L_{3-i}(v)$ we deduce that every vertex in $L_i(v)$ is adjacent to every vertex in $L_{3-i}(v)$. The conclusion follows from Property 6.

In the rest of the proof we show the existence of $S = (S_1, S_2, v) \in H^*$ such that a vertex in $L_i(v)$ has a neighbor in $L_{3-i}(v)$ by constructing a maximal structure in G as follows. We say that w *blocks* a piece D of $S = (S_1, S_2, v) \in H$ if $w \notin D$, it has exactly p_i neighbors in S_i and $G[D \cup \{w\}]$ is p_i-regular, where i is the index such that $D \subseteq S_i$.

Let k be the largest integer such that there exist $S = (S_1, S_2, v) \in H^*$ and non empty pairwise disjoint pieces $B_1, \ldots, B_k$ of S such that v blocks B_k and for every $j = 1, \ldots, k-1$, there exists $z \in B_{j+1}$ such that z blocks B_j.

By using the maximality of k, we first prove that there is j such that $B_j = D_2(v)$ and $B_k = D_1(v)$ or $B_j = D_1(v)$ and $B_k = D_2(v)$. Let us assume that $B_k \subseteq S_1$ (the case $B_k \subseteq S_2$ is similar). Since v blocks B_k, from Property 3 we know that $B_k = D_1(v)$. Let us consider $B_{k+1} = D_2(v) \cup \{v\} - \{w\}$, for $w \in D_2(v)$ such that $G[B_{k+1}]$ is connected. This choice of w and Property 2 imply that $S' = (S_1, S_2', w) \in H^*$, where $S_2' := (S_2 \cup \{v\}) - \{w\}$. From Property 1, w has exactly p_2 neighbors in S_2'. Moreover, $B_{k+1} \subseteq S_2'$ is a piece of S' and it is blocked by w. By the maximality of k and since v blocks B_k, there exists $j < k$ such that $B_j \cap B_{k+1} \neq \emptyset$. Since $v \notin S_1 \cup S_2$ and B_j is a piece of S we get that $B_j \cap (D_2(v) - \{w\}) \neq \emptyset$. Finally, B_j and $D_2(v)$ are pieces of S. Therefore, $B_j = D_2(v)$.

We now prove that there is a vertex in $L_2(v)$ having a neighbor in $L_1(v)$. Let $z \in B_{j+1}$ be a vertex such that z blocks B_j. Then z has exactly p_2 neighbors in S_2. Since v also blocks B_j, the set of all neighbors of z in S_2 is $L_2(v)$. To finish the proof we show that $z \in L_1(v)$. Let $w \in L_2(v)$. From Property 2 the tuple $S_w^v \in H$. Since w is adjacent to z, Property 3 applied to S_w^v implies that z has p_2 neighbors in $S_2 \cup \{v\} - \{w\}$. Hence $z \in L_1(v)$ ■

It is worth to mention that the following analogous of Theorem 1.1 can be proved.

Theorem 2.2. *Let G be a graph with maximum degree $d \geq 3$ and $\omega(G) \leq d$. Then, for every p_1 and p_2 such that $d = p_1 + p_2$ the graph G has a partition $\{S_1, S_2\}$ such that $G[S_1]$ is a maximum order $(p_1 - 1)$-colorable induced subgraph of G and $G[S_2]$ is a $(p_2 - 1)$-degenerate induced subgraph.*

The proof follows the same steps as those of Theorem 1.1, when we consider H as the set of all tuples (S_1, S_2, v) with $G[S_1]$ a maximum order $(p_1 - 1)$-colorable induced subgraph of G, $G[S_2]$ a maximum order $(p_2 - 1)$-degenerate induced subgraph of $G - S_1$ and $v \notin S_1 \cup S_2$. Both results can be presented jointly in terms of an induced subgraph hereditary graph property P. The requirement for the property P is the following: a positive integer p exists such that the addition of a new vertex v to a graph $G \in P$ generates a new graph $G' \in P$, whenever the degree of v in G' is less than p. Hence a possible extension of our result is the following. For each property P as above, every non complete graph G has a partition $\{S_1, S_2\}$ where $G[S_1]$ is a maximum order induced subgraph with property P and $G[S_2]$ is a $(n - p - 1)$-degenerate induced subgraph of G.

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