

# Optimal replacement and overhaul decisions with imperfect maintenance and warranty contracts

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## Abstract

In this article, we develop a model to help a maintenance decision making situation of a given equipment. We propose a novel model to determine optimal life-cycle duration and intervals between overhauls by minimizing global maintenance costs. We consider a situation where the costumer, which owns the equipment, may negotiate a better warranty contract by offering an improved preventive maintenance program for the equipment. The equipment receives three kind of actions: repairs, overhauls, and replacement. An overhaul represents an imperfect maintenance action, that is, the failure rate is improved but not a point that the equipment is as good as new. Corrective maintenance actions are minimal, in the sense that the failure rate after each repair is the same as before the failure. The proposed strategy surpasses others seen in the literature since it considers at the same time the warranty negotiation situation and the optimal life-cycle duration under imperfect preventive actions. We also propose a simplified approach that facilitates the task of implementing the method in standard solvers.

*Keywords:* Imperfect maintenance; Warranty; Management; Life-cycle cost

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## 1. Introduction

Many systems are sold with a warranty that offers protection to the buyers against early failures during the infancy of the equipment and as a medium of promotion to the vendor. When the warranty period tends to be large, degradation also appears during it and in this case preventive maintenance plays an important role in order to reduce the failure rate, and its evolution in time.

Offering periods of warranty implies extra costs for the vendor. There exist repair costs (corrective maintenance) and possibly penalty costs to be paid, which are associated to downtime. A preventive maintenance program may reduce such corrective costs. Given that the buyer does not pay repairs during the warranty period, there exist no incentives

for him to expense in preventive actions (in the case that the warranty also covers for the downtime costs). For the vendor, it is convenient to perform preventive actions only if they cost less than the expected corrective costs.

From the buyers' point of view, investing in preventive maintenance during and after the warranty period may have significant effects on the life-cycle costs. Consequently, it may be convenient to define preventive policies all along the life-cycle  $T_1$ . The aim of this work is to present a general framework for this situation from the point of view of the buyer.

We consider three kind of maintenance actions: minimal repair (as good as before the failure), imperfect overhaul (between as good as after the previous overhaul and as good as before the overhaul) or replaced (as good as new). Each action has its own costs and may depend on variables such as age and/or quality. We will consider a single component analysis, neglecting scale economies that may appear from negotiating for multi-component systems. We also discard revenue-sharing contracts as studied in Cachon and Lariviere [1]. A cost analysis for systems in series, in parallel and a combination of both may be found in Bai and Pham [2].

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Modelling of the system improvement due to imperfect maintenance is crucial to establish the cost model to be optimized. Malik [3] introduced the concept of virtual age, which essentially says that the system is younger than before the action by some interval  $T_y$ . A similar formulation is offered by Kijima [4]. A limitation of the virtual age model is that it does not alter the reference failure rate function. Nakagawa [5] assumed that the action is minimal with probability  $a$  and perfect with probability  $(1 - a)$ . As it is referred to minimal and perfect actions, this model implies that as time passes the quality of the overhauls must be improved in order to keep  $a$  constant. This has some consequences: if the original failure rate without overhauls of the system is a power function of time, the failure rate is always bounded. Zhang and Jardine [6] propose a failure rate model where after an overhaul, it is between as good as before and as good as after previous overhaul. This model does not bound the failure rate as Nakagawa's, but due to the discontinuities in the failure rate function, it may be difficult to evaluate the number of expected failures during the warranty period, which is variable in our model. Djameludin et al. [7], propose to use a continuous failure rate function  $\lambda$  whose time-dependency parameter is associated to the quality (cost)  $\sigma$  of the maintenance policy, that is

$$\lambda(t, \sigma) = \frac{\beta}{\eta_\sigma} \left( \frac{t}{\eta_\sigma} \right)^{\beta-1}$$

with

$$\eta_\sigma = \eta_0 \left( \frac{1}{1 - \sigma} \right)^\kappa$$

for  $t \in [0, T_1]$  and  $\sigma \in [0, 1)$ . We observe that they use a time non-homogeneous Poisson process. For maximum quality, the failure rate tends to be constant, for minimum quality, the failure rate corresponds to the original failure rate when minimal repairs are performed. A serious limitation of this approach is how to model the relationship between the quality of the preventive policy and the failure rate. A summary of research on imperfect maintenance is offered in Pham and Wang [8].

Considering warranty, Djameludin et al. [7] develop a framework to study preventive policies when the vendor offers an initial period of warranty  $T_w$  where he pays labor, materials and downtime costs if a failure occurs. Under this premise, the buyer is not necessarily committed with preventive maintenance. With that agreement, the costs to the buyer (if he decides to perform preventive maintenance during  $[0, T_1]$  are

$$C_r + c_p T_1 + C_m \int_{T_w}^{T_1} \lambda(t, \sigma) dt$$

where  $C_r$  corresponds to the overall cost of a replacement (investment, labor, material, downtime costs);  $c_p$  is the expected overall cost of preventive maintenance per

unit time and  $C_m$  is the overall cost per failure. In this model,  $T_1$  and  $T_w$  are considered as fixed parameters.

Jack and Dagpunar [9] use a virtual age model to determine the quality and the period between overhauls. In their model, the optimal solution is complete renewal at each overhaul (age 0). Jung and Park [10] study the optimal periodic preventive policies following the expiration of warranty. They use the expected maintenance cost rate per unit time from the buyer's perspective. They also use a virtual age model for the failure rate and consider that preventive activities start just after the end of the warranty. This limits the optimality of the preventive maintenance since early preventive actions may reduce even further the life-cycle costs [7]. Warranties may also be defined by several criteria and not only age. Concerning warranties that are limited by age and usage, Chen and Popova [11] propose a simulation-based approach to determine minimal repair and replacement policies. Product design and warranty interaction has been studied, i.e. in Kimura et al. [12]. They derive optimal release policies when the designer (which also acts as the vendor) has to pay the cost for fixing any faults detected during the warranty period. More recently, Kim et al. [13], following Kijima's virtual age model [4], develop a strategy to determine maintenance policies in a similar way as in the article by Djameludin et al. [7], but considering discrete overhaul actions, as we do in this work. In the model by Kim et al., life-cycle duration as well as warranty interval are known a priori. Huang and Zhuo [14] study the selection of the type of warranty to be offered by the vendor. The criterion is the maximization of the vendors profit. They use a fixed continuous failure rate function and, therefore, the model does not reflect explicitly the dependency of the warranty strategy on the maintenance policy. Research dealing on warranty cost analysis has been summarized in Blischke and Murthy [15] and Murthy and Djameludin [16].

The model that is presented here considers that preventive policies are taken from the beginning of the life-cycle. In this way, the failure rate is reduced and costs associated to corrective maintenance are reduced.

We shall propose a failure rate model that takes the advantages of both the model of Zhang and Jardine and the one by Djameludin et al. Based on the proposed model we minimize the expected cost per unit-time. It allows an optimal decision on life-cycle duration and the number of periodic overhauls to perform during it. It also permits a negotiation of the warranty period with the vendor. We show results for the case when the reference failure rate function is growing exponentially with time. We illustrate the methodology through a numerical example and we obtain optimal values for the number of overhauls and the life-cycle duration.

The improvements made over the reference models are:

- Life-cycle  $T_1$  is a decision variable;
- We obtain an optimal value for the warranty period  $T_w$  to be negotiated with the vendor;

- We consider explicitly the downtime cost  $C_{fm}$  which are not paid in general by the vendors;
- We consider the cost of an overhaul  $C_o$  as a function of its quality parameter  $p$ ;
- We propose a continuous failure-rate model that relates the model of Djameludin and Murthy to the model of Zhang and Jardine and permits a long trend analysis to evaluate the life-cycle duration.

## 2. General model

### 2.1. Statement of the problem

Equipment breaks down from time to time, requiring repairs. Also, while the equipment is being repaired, there is a loss in production output. In order to reduce the number of failures, we can periodically overhaul the equipment and perform preventive actions. After some time it may be economically convenient to replace the equipment by another new one.

Our purpose is to determine the optimal life-cycle duration and the interval between overhauls that minimize the global cost per unit time.

### 2.2. Construction of model

Let us consider the following conditions:

- Equipment is subjected to three types of actions: minimal repair, imperfect overhaul, and replacement; each action has its own associated costs;
- Equipment is repaired when it fails;
- Equipment is periodically replaced;
- Equipment receives  $n - 1$  overhauls during its life,

$$n \geq 1, \text{ integer, positive;} \quad (1)$$

- The interval between overhauls  $T_s$  is constant. The life-cycle is given by

$$T_1 = nT_s \quad (2)$$

- An overhaul improves the equipment in term of its failure rate  $\lambda$ ;
- All repairs are minimal, that is, they only return the equipment to production but they don't improve the failure rate;
- The quality of an overhaul (as well as its cost) is dependent on the improvement factor  $p$ ;

$$0 < p < 1 \quad (3)$$

- Material and tradesmen to perform a repair cost  $C_{im}$ ;
- Downtime cost of a repair is  $C_{fm}$ ;
- Overall cost of an overhaul is  $C_o$  (it includes: material, tradesmen, downtime costs);
- Overall cost of a replacement is  $C_r$  (investment, labor, material, downtime costs);
- Failure rate with periodic overhauls is  $\lambda(t)$ ;

- Failure rate if no overhauls are performed is  $\bar{\lambda}(t)$ ;
- The vendor only pays labour and material costs during the warranty period  $T_w$ ; downtime costs are assumed by the costumer;
- The costumer and the vendor agree to negotiate an enlarged warranty period if the costumer performs overhauls during the duration of the contract; we have:

$$T_w < T_1 \quad (4)$$

- Repairs beyond the warranty period are paid by the costumer;
- Recovery value of the equipment is negligible;
- The quality  $p$  is considered constant, as well as its cost,
- The mean time to repair is negligible in front of the mean time between failures;
- The vendor offers a baseline warranty period  $\bar{T}_w$  where the costumer is not obliged to perform overhauls; the costumer is in position to negotiate an extension to the warranty so,

$$\bar{T}_w \leq T_w \quad (5)$$

Our aim is to determine the number of overhauls, their interval  $T_s$  (or equivalently the life-cycle duration  $T_1$ ), and the warranty interval  $T_w$ , that minimize the total expected cost per unit time  $c_g$ .

The expected number of failures during the warranty period is given by

$$\int_0^{T_w} \lambda(t) dt,$$

and during the rest of the life-cycle by

$$\int_{T_w}^{T_1} \lambda(t) dt.$$

The life-cycle cost for the costumer is given by

$$\begin{aligned} C_g(n, T_1, p, T_w) &= C_r + C_o(p)(n - 1) + C_{fm} \int_0^{T_1} \lambda(t) dt + C_{im} \int_{T_w}^{T_1} \lambda(t) dt \end{aligned} \quad (6)$$

and for the vendor by

$$C_{im} \int_0^{T_w} \lambda(t) dt,$$

thus, the life-cycle cost per unit time for the costumer is given by

$$c_g(n, T_1, p, T_w) = \frac{C_g(n, T_1, p, T_w)}{T_1}. \quad (7)$$

If no negotiation occurs, the vendor agrees to pay the labor and material required to minimally repair

the equipment during the reference warranty period:

$$C_{\text{im}} \int_0^{\bar{T}_w} \bar{\lambda}(t) dt$$

The costumer, by his side, is tempted to perform overhauls during the life of the equipment in order to reduce the number of failures. This would reduce the expected number of failures also during the warranty period and the expected cost for the vendor. Performing overhauls imply costs to be paid by the costumer, so the vendor could compensate his efforts by extending the warranty period. The vendor does not want to increase his expected cost, so he would agree to extend the warranty if the expected cost during the extended warranty does no surpass the expected cost for the baseline warranty, that is:

$$C_{\text{im}} \int_0^{T_w} \lambda(t) dt \leq C_{\text{im}} \int_0^{\bar{T}_w} \bar{\lambda}(t) dt \quad (8)$$

Given that  $\lambda(t)$  is non negative, we observe that the last term in (6)

$$C_{\text{im}} \int_{T_w}^{T_1} \lambda(t) dt$$

is decreasing as  $T_w$  increases. The minimization of the total cost of the costumer implies forcing the constraint (8) to the equality

$$\int_0^{T_w} \lambda(t) dt = \int_0^{\bar{T}_w} \bar{\lambda}(t) dt \quad (9)$$

as a consequence,  $T_w$  is not an active decision variable in the sense that it may be obtained once  $\lambda(t)$  is modelled. Of course, the vendor will agree to extend the warranty only if at least one overhaul is performed during  $[0, T_w]$ :

$$T_s \leq T_w.$$

The failure rate is a crucial indicator of the equipment condition, since it permits failure forecasting and establish appropriate preventive measures like overhauls. We will consider that the failure rate after an overhaul fails between as bad as just before and as well as just after the previous overhaul with some improvement factor  $p \in (0, 1)$ .

Let  $\lambda_k(t)$  be the failure rate after the  $k$ th overhaul. We will express the failure rate as

$$\begin{aligned} \lambda_0(t) &= \bar{\lambda}(t) \\ \lambda_k(t) &= p\lambda_{k-1}(t - T_s) + (1 - p)\lambda_{k-1}(t), \quad (10) \\ k &= 1, \dots, n - 1, \end{aligned}$$

which produce discontinuities in  $\lambda(t)$  as observed in Fig. 1. The curve with no discontinuities represents an equivalent for  $\lambda(t)$  in terms of the expected number of failures.

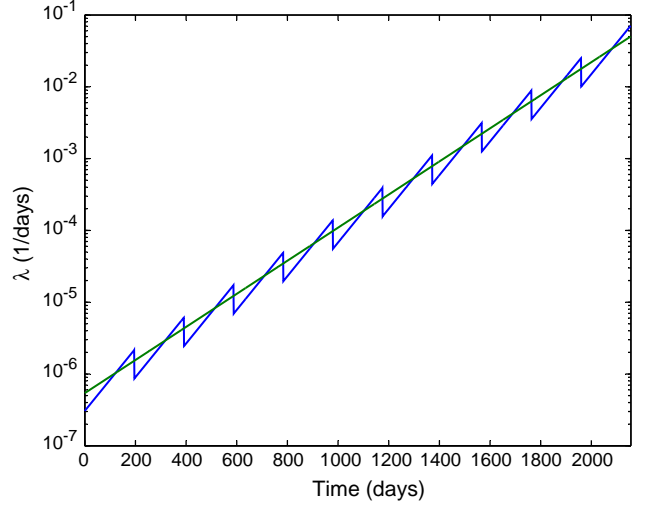


Fig. 1. Model from Zhang and Jardine vs proposed (from the example,  $p=0.7$ ).

In a general situation the aging process shows the pattern observed in Fig. 1. If the improvement factor  $p$  is 0, then

$$\lambda_k(t) = \lambda_{k-1}(t) = \bar{\lambda}(t),$$

in other words, the failure rate is the same as before the overhaul so it may be considered as a minimal repair.

If the improvement factor  $p$  is 1, we have that

$$\lambda_k(t) = \lambda_{k-1}(t - T_s) = \bar{\lambda}(t - kT_s),$$

that is, the overhaul returns the failure rate to the level obtained just after the previous overhaul. Since all overhauls have the same level and their periodicity is constant, overhaul may be considered as a replacement.

It may be proven that [6]

$$\lambda_k(t) = \sum_{i=0}^k \binom{k}{i} p^i (1 - p)^{k-i} \bar{\lambda}(t - iT_s) \quad (11)$$

and since  $T_1 = nT_s$ , we obtain that for any  $n \geq 1$ ,

$$\int_0^{T_1} \lambda(t) dt = \sum_{i=0}^n \binom{n}{i} p^{n-i} (1 - p)^{i-1} \int_0^{iT_s} \bar{\lambda}(t) dt. \quad (12)$$

We can note that if the failure rate with no overhaul follows an exponential law, that is

$$\bar{\lambda}(t) = e^{\alpha_0 + \alpha_1 t}, \quad \text{with } \alpha_1 > 0, \quad (13)$$

from (12) and (13), we can see that,

$$\begin{aligned} \int_0^{T_1} \lambda(t) dt &= \sum_{i=0}^n \binom{n}{i} p^{n-i} (1 - p)^{i-1} \int_0^{iT_s} \bar{\lambda}(t) dt \\ &= e^{\alpha_0} \frac{[p + (1 - p)e^{\alpha_1 T_s}]^n - 1}{(1 - p)\alpha_1}. \end{aligned}$$

Rai and Singh [17,18] address the problem of estimating the failure rate function using failure data coming from the warranty period. They propose a method to face the incompleteness and inaccuracy of available information, but they do not consider failure rate discontinuities due to overhauls, as it is done in the present study. Oh and Bai [19] consider also the use of after-warranty failure data; again, no explicit consideration of discrete preventive actions are taken into account. Jones and Hayes [20] proposed practical strategies of analyzing large data bases with data from the warranty period.

Concerning the cost of an overhaul, it is logical to impose a dependency between the quality of an overhaul and its cost. We propose

$$C_o(p) = C_{o,\min} e^{\sigma p}, \quad (14)$$

where

$$\sigma = \log \frac{C_{o,\min}}{C_{o,\max}}$$

and  $C_{o,\min}$  and  $C_{o,\max}$  are minimum and maximum (complete renewal) costs for an overhaul.

### 3. Proposed model

The model presented in the previous section shows discontinuities every time an overhaul is performed. Given that the replacement problem considers a long term, instantaneous failure rate values are not important to the optimization problem, and a long term approximation is useful to determine easily the number of failures during the life-cycle and during the warranty period.

#### 3.1. Model parameters

Before computing the model parameters we need a proof to validate the simplified model that we propose. The following result proves the long-trend exponential behavior of the failure rate:

**Theorem 1.** *Let  $\nu \in (0,1)$  and consider the set of points  $P_k = ((\nu + k)T_s, \lambda_k((\nu + k)T_s))$ , for  $k=0, \dots, n-1$ . Then, there exists a constant  $c = c(\nu)$ , such that  $P_k$  belongs to the curve*

$$\tilde{\lambda} = c e^{\hat{\alpha}_1 t},$$

where  $c$  depends only on  $\nu$ .

**Proof.** Note that it is enough to prove that

$$R_k = (t_k, \tilde{y}_k) = ((\nu + k)T_s, \log(\lambda_k((\nu + k)T_s)))$$

belongs to a line of the form  $l\tilde{\lambda} = \tilde{c} + \hat{\alpha}_1 t$ .

First, we can see that

$$\begin{aligned} l_{\tilde{\lambda}_k} &= \log(\lambda_k((\nu + k)T_s)) \\ &= \log\left(\sum_{i=0}^k \binom{k}{i} p^i (1-p)^{k-i} e^{\alpha_0 + \alpha_1((\nu+k)T_s - iT_s)}\right) \\ &= \alpha_0 + \alpha_1 \nu T_s + \log\left(\sum_{i=0}^k \binom{k}{i} p^i ((1-p)e^{\alpha_1 T_s})^{k-i}\right) \\ &= \alpha_0 + \alpha_1 \nu T_s + \log\{(p + (1-p)e^{\alpha_1 T_s})^k\} \\ &= \alpha_0 + \alpha_1 \nu T_s + k \log\{p + (1-p)e^{\alpha_1 T_s}\} \\ &= \alpha_0 + \alpha_1 \nu T_s + k \hat{\alpha}_1 T_s, \end{aligned} \quad (15)$$

thus, for a given  $\nu \in (0,1)$ , we have that for each  $k$ ,  $j \in \{0, \dots, n-1\}$ ,

$$\begin{aligned} \frac{l_{\tilde{\lambda}_j} - l_{\tilde{\lambda}_k}}{t_j - t_k} &= \frac{(\alpha_0 + \alpha_1 \nu T_s + j \hat{\alpha}_1 T_s) - (\alpha_0 + \alpha_1 \nu T_s + k \hat{\alpha}_1 T_s)}{(j-k)T_s} \\ &= \frac{(j \hat{\alpha}_1 T_s) - (k \hat{\alpha}_1 T_s)}{(j-k)T_s} = \hat{\alpha}_1, \end{aligned} \quad (16)$$

which proves that all the points  $(t, l_{\tilde{\lambda}_k})$  belongs to a line of the form

$$l_{\tilde{\lambda}} = \alpha_0 + (\alpha_1 - \hat{\alpha}_1 \nu T_s) + \hat{\alpha}_1 t,$$

and the proof completes.  $\square$

By observation of Fig. 1, let us consider a long term failure rate model defined by

$$\hat{\lambda}(t) = e^{\hat{\alpha}_0 + \hat{\alpha}_1 t},$$

where  $\hat{\alpha}_0$  and  $\hat{\alpha}_1$  are determined by  $p$ . To estimate these parameters, we exploit the formula (10), that is,

$$\lambda_1(T_s) = p \lambda_0(0) + (1-p) \lambda_0(T_s).$$

If we consider the exponential model, we have that

$$\lambda_1(T_s) = p e^{\hat{\alpha}_0} + (1-p) e^{\hat{\alpha}_0 + \hat{\alpha}_1 T_s}. \quad (17)$$

Thus, we may estimate  $\hat{\alpha}_1$  by using the points  $(0, e^{\hat{\alpha}_0})$  and  $(T_s, \lambda_1(T_s))$ , which give us a lower bound for the failure rate at any instant,

$$\hat{\lambda}_{\inf}(t) = e^{\hat{\alpha}_0 + \hat{\alpha}_1 t} \quad (18)$$

with

$$t \in [0, T_1],$$

then substituting (17) into (18)

$$e^{\hat{\alpha}_0 + \hat{\alpha}_1 T_s} = p e^{\hat{\alpha}_0} + (1-p) e^{\hat{\alpha}_0 + \hat{\alpha}_1 T_s}$$

and we obtain,

$$\hat{\alpha}_1 = \frac{\log(p + (1-p)e^{\hat{\alpha}_1 T_s})}{T_s}. \quad (19)$$

In order to estimate  $\hat{\alpha}_0$ , we consider that the continuous model should produce the same number of failures for each

interval  $T_s$  and that the slope in a loglinear diagram is  $\hat{\alpha}_1$ ,

$$\int_0^{T_s} \hat{\lambda}(t, \hat{\alpha}_1) dt = \int_0^{T_s} \bar{\lambda}(t) dt$$

$$\frac{e^{\hat{\alpha}_0}}{\hat{\alpha}_1} (e^{\hat{\alpha}_1 T_s} - 1) = \frac{e^{\alpha_0}}{\alpha_1} (e^{\alpha_1 T_s} - 1)$$

therefore,

$$\begin{aligned} \hat{\alpha}_0 &= \log \left( e^{\alpha_0} \frac{\hat{\alpha}_1}{\alpha_1} \frac{(e^{\alpha_1 T_s} - 1)}{(e^{\hat{\alpha}_1 T_s} - 1)} \right) \\ &= \alpha_0 + \log \left( \frac{\hat{\alpha}_1}{(1-p)\alpha_1} \right). \end{aligned} \quad (20)$$

The use of (18) greatly simplifies the evaluation of constraint (9) and facilitates the optimization process.

### 3.2. Optimal value for the warranty period

If the failure rate with no overhauls follows (13), the expected number of failures during the reference warranty period  $\bar{T}_w$  is

$$\int_0^{\bar{T}_w} \lambda(t) dt = \frac{e^{\alpha_0}}{\alpha_1} (e^{\alpha_1 \bar{T}_w} - 1)$$

then, the non-linear constraint (8) takes the form

$$\frac{e^{\hat{\alpha}_0}}{\hat{\alpha}_1} (e^{\hat{\alpha}_1 \bar{T}_w} - 1) = \frac{e^{\alpha_0}}{\alpha_1} (e^{\alpha_1 \bar{T}_w} - 1) \quad (21)$$

thus, from (19) and (21), we obtain  $T_w$  explicitly:

$$\begin{aligned} T_w &= \frac{\log \left( \frac{e^{\alpha_0}}{e^{\hat{\alpha}_0}} \frac{\hat{\alpha}_1}{\alpha_1} (e^{\alpha_1 \bar{T}_w} - 1) + 1 \right)}{\hat{\alpha}_1} \\ &= \frac{\log(p + (1-p)e^{\alpha_1 \bar{T}_w})}{\log(p + (1-p)e^{\alpha_1 T_s})} T_s. \end{aligned} \quad (22)$$

### 3.3. Weibull failure rate

A second model of failure rate corresponds to the Weibull failure rate, defined as follows

$$\bar{\lambda}(t) = \left( \frac{\beta}{\eta} \right) \left( \frac{t}{\eta} \right)^{\beta-1}, \quad (23)$$

where  $\beta > 1$  and  $\eta > 0$ .

Note that  $\bar{\lambda}(t)$  can be written in an exponential way, that is,

$$\bar{\lambda}(t) = e^{\log(\beta/\eta^\beta) + (\beta-1)\log(t)},$$

thus, considering

$$\alpha_0 = \log \left( \frac{\beta}{\eta^\beta} \right), \quad (24)$$

and

$$\alpha_1 = (\beta - 1), \quad (25)$$

we can write

$$\bar{\lambda}(t) = e^{\alpha_0 + \alpha_1 \log(t)}. \quad (26)$$

and now, changing the time-scale

$$t_1 = \log(t)$$

we define

$$\lambda_1(t_1) = e^{\alpha_0 + \alpha_1 t_1} \quad (= \bar{\lambda}(t)).$$

But, in this case we have that the size of the intervals on the logarithmic time are not constant, in fact if we define

$$T_{1,i} = \log(iT),$$

then

$$T_{1,i+1} - T_{1,i} = \log((i+1)T) - \log(iT) = \log \left( 1 + \frac{1}{i} \right),$$

that is, the size of the intervals is decreasing to zero. From this remark we may understand the Weibull rate case as an exponential rate case with non constant intervals. In what follows we show results for the cases  $\beta=1, 2$  and  $3$ . Proceeding in the same way as for the exponential case, we impose:

$$\int_0^{nT_s} \hat{\lambda}(t) dt = \sum_{k=1}^n \int_{(k-1)T_s}^{kT_s} \lambda_{k-1}(t) dt, \quad \forall n \in \mathbb{N},$$

and use the expressions for the expected number of failures from reference [6], we obtain the following expression for the continuous failure model:

$$\hat{\lambda}(t) \begin{cases} \frac{1}{\eta} & \beta = 1 \\ \frac{2(1-p)}{\eta^2} t + \frac{pT_s}{\eta^2} & \beta = 2 \\ \frac{3(1-p)^2}{\eta^3} t^2 + \frac{6T_s}{\eta^3} p(1-p)t + \frac{T_s^2}{\eta^3} (2(1-p)^2 - 3(1-p) + 1) & \beta = 3. \end{cases} \quad (27)$$

For these cases, we observe polynomial behavior of order  $\beta-1$ .

## 4. Numerical example

Let us consider the following data, which are expressly similar to those used in reference [6], considering the extra



Table 1  
Optimal solutions for different  $p$

$p$	$n$	$T_1$	$\hat{c}_g$	$T_w$
0.5	3	1391.2	0.1717	775.0
0.6	5	1593.2	0.1654	873.0
0.7	7	1877.8	0.1550	1005.6
0.8	10	24024.7	0.1395	1240.5

parameters needed for our model:  $C_r=200$ ,  $C_{o,\min}=8$ ,  $C_{o,\max}=32$ ,  $C_{fm}=1$ ,  $C_{im}=1$ ,  $\bar{T}_w=730$ . Units are  $10^3$  dollars and days, respectively. The reference failure rate (with no overhaul) follows

$$\bar{\lambda}(t) = e^{-15+0.01t},$$

We have

$$\hat{c}_g(n, T_1) = \frac{C_r + C_{o,\min}e^{\beta p}(n-1) + C_{fm} \frac{e^{\hat{\alpha}_0}}{\hat{\alpha}_1}(e^{\hat{\alpha}_1 T_1} - 1) + C_{im} \frac{e^{\hat{\alpha}_0}}{\hat{\alpha}_1}(e^{\hat{\alpha}_1 T_1} - e^{\hat{\alpha}_1 T_w})}{T_1}$$

the following results are obtained:  $n^*=7$ ,  $T_1^*=1877$  days,  $T_w^*=1006$  days,  $c_g(n^*, T_s^*)=0.1550$  USD/day. The system must be replaced every 5.15 years and must be overhauled every  $1877/(7-1)=313$  days. The warranty may be negotiated with the vendor from 2 years to  $1006/365=2.76$  years.

The example was solved using the solver of Excel. Table 1 gives the optimal solutions for various improvement factors. For the given cost structure the larger the improvement factor is, the more overhauls should be performed but also the life-cycle and the extended warranty are larger. A sensitivity analysis for  $p=0.7$  is shown in Figs. 2 and 3, respectively. We observe that the cost per unit

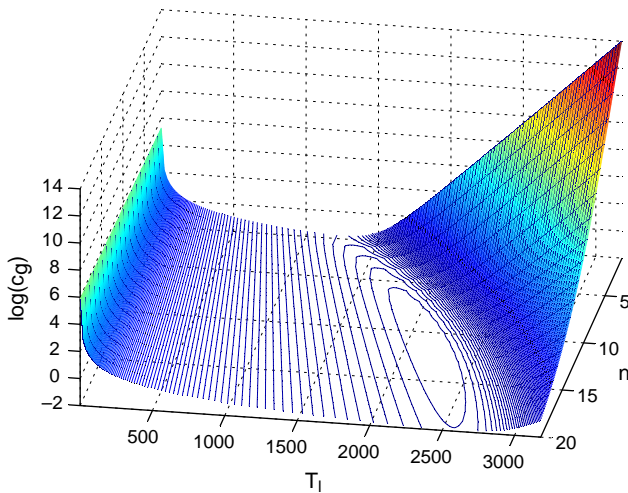


Fig. 2. Study of the topology of the cost function for  $p=0.7$ .

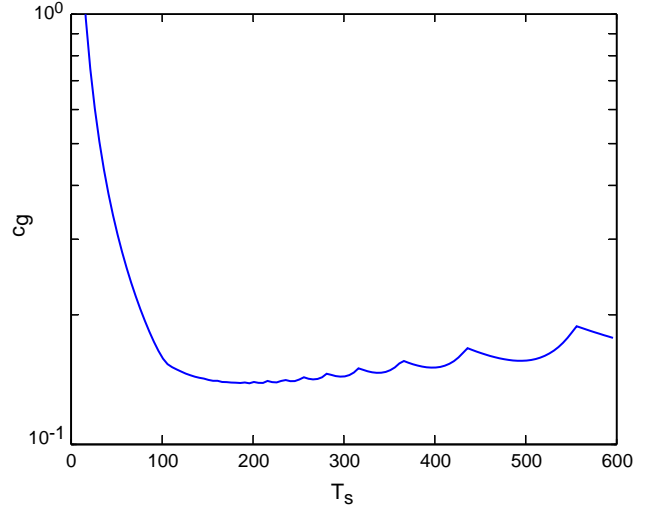


Fig. 3. Minima for varying intervals between overhauls ( $p=0.7$ ).

time is quite insensitive to  $T_1$  in the range (1500, 2500) days. If instead of  $T_1$ , we use  $T_s$  as decision variable (as it is done in Zhang and Jardine), some local minima may perturb seriously the finding of the global minimum as it is observed in Fig. 3.

### 5. Final comments

We have presented a model to relate the long term failure rate with the improvement factor of the overhauls and their interval. We established a cost optimization model to determine optimal levels of preventive maintenance. The model includes easily a negotiation criterion for extending the warranty period. The formulation has been simplified to permit the use of standard spreadsheet solvers to solve the minimization problem. The extension of the model to considerate discounted costs is straightforward. In a forthcoming article we study the effects of considering different interoverhaul time intervals.

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