Short communication
Probabilistic control of materials by a simulation method

Marcelo Elgueta *, Pablo Kittl

Departamento de Ingenierı´a Meca´ nica, Facultad de Ciencias Fıísicas y Matema´ ticas, Universidad de Chile, Beauchef 850, 4 Piso. Casilla 2777, Santiago, Chile

Abstract
This work deals with a method to find the number of required tests so that the failure stress of a material belongs to a given rank of tolerance of cumulative probability. The method is based on the estimation of the dispersion of the Weibull’s parameters (the inferior limit tension \( r_L \), below which there is no failure, the superior limit tension \( r_S \), over which there is always failure, and the fabrication parameters \( m \) and \( K \)) by means of the optimization of a linear correlation between the cumulative probability and the failure tension. Using a property of the Fischer’s matrix and the dispersion of parameters estimated by the simulation method, the number of samples compatible with fixed values of cumulative probability of failure \( F_c \) and tolerance \( dF \) can be found and, consequently, its control. This methodology is applied to the case of a steel SAE 1020, in relation to the yield stress.

1. Introduction
In most materials, the appearance of a failure, the rupture stress in the case of fragile material or the yield stress in a ductile material, is described by the Weibull’s theory [1–3]. The cumulative probability of failure \( F \) is represented by the expression:

\[
F(r) = \frac{1}{\gamma_0} \exp \left( -\frac{V_0 \gamma}{r} \right) \quad [1]
\]

where \( r \) is the tension at which the failure takes place, \( V_0 \) is the volume of the sample, \( V_0 \) is the unit of volume and \( \gamma(\sigma) \) is the specific-risk function or Weibull’s function. In this work, the risk function proposed by Kittl et al. [3] is considered

\[
\gamma(\sigma) = \begin{cases} 
0 & \text{if } 0 \leq \sigma < r_L, \\
\left( K \frac{\sigma - r_L}{r_S - r_L} \right)^m & \text{if } r_L \leq \sigma \leq r_S, \\
\infty & \text{if } r_S < \sigma \leq \infty,
\end{cases} \quad [2]
\]

where \( r_L \) is the inferior limit tension (below which there is no failure), \( r_S \) is the superior limit tension (over which always there is failure) and \( m \) and \( K \) are the fabrication parameters. Replacing (2) in (1) one obtains:

\[
F(\sigma) = 1 - \exp \left\{ -\frac{V_0}{V_0} \left( K \frac{\sigma - r_L}{r_S - r_L} \right)^m \right\} \quad \text{if } r_L \leq \sigma \leq r_S \quad [3]
\]

As it is shown in [2], the determination of \( r_L \) and \( r_S \), in the case of fragile materials allows to know the size the maximal crack to have a null probability of failure (\( r_L \)) and the size of the minimum crack for which the failure probability is 100% (\( r_S \)). The precision whereupon these two tensions can be determined is very important to know those sizes crack. Nevertheless, the procedure to obtain the dispersion by some optimization method and through the Fischer’s matrix [2], implies very troublesome calculations which may be avoided using a simulation method.

2. Simulation method
This method allows to obtain the dispersion of Weibull’s parameters carrying out only one set of experimental

* Corresponding author. Tel.: +56 2 9784543; fax: +56 2 6988453/6896057. E-mail address: melgueta@ing.uchile.cl (M. Elgueta).
measures. On the basis of these measurements, other sets of them can be simulated and be put under a very simple optimization procedure. This allows to find the average value and the dispersion of the Weibull’s parameters.

According to the expression (3), \( F(V, \sigma) \) represents the cumulative probability of failure of a sample of volume \( V \) put under a tension \( \sigma \). Evidently, one assumes that all the specimens have the same volume and are made according to a same fabrication procedure. Supposing by simplicity \( V = V_0 \), \( F \) only depends on \( \sigma \). So, to a fixed \( \sigma_i \) corresponds a cumulative probability \( F(\sigma) \equiv F_i \). The simulation method can be summarized by the following steps:

(a) Take a set of \( N \) random occurrence numbers \( \lambda_i \) with \( 0 \leq \lambda_i \leq 1 \), ordered in increasing order.

(b) Like the cumulative probability has a random character and verifies \( 0 \leq F(\sigma) \leq 1 \), one can establish a direct correspondence between the \( N \) numbers \( \lambda_i \) and the values taken by \( F(\sigma) \) since both have equal probability. So, one can write

\[ 0 \leq \lambda_i \equiv F(\sigma_i) \leq 1 \]

(c) Calculate \( \sigma_i \) from Eq. (3). One obtains:

\[ \sigma_i = \frac{2\ln \left( \frac{1}{1-\lambda_i} \right)^{1/m} + \sigma_L}{\frac{1}{m} \ln \left( \frac{1}{1-\lambda_i} \right)^{1/m} + 1} \]  

(4)

In this expression, the values of \( \sigma_L \), \( \sigma_S \), \( m \) and \( K \) constitute data collected experimentally.

(d) Consider \( N \) values (number of tests) of cumulative probability given by:

\[ F_i = \frac{i - 1/2}{N} \quad \text{with} \quad i = 1, 2, 3 \ldots N \]  

(5)

Notice that these \( N \) values of \( F_i \) are placed in increasing order.

(e) The correlation

\[ \sigma_i \leftrightarrow F_i \]  

(6)

constitutes precisely the simulation. \( F_i \) is calculated by expression (5) and \( \sigma_i \) by expression (4) and both are ranged in increasing order. In a plot \( F_i \) vs. \( \sigma_i \), one obtains a discrete representation (points) of expression (3).

To obtain the Weibull’s parameters using the simulation, formula (3) is rearranged to obtain a straight line in a plot \( \ln(\ln(1/(1-F_i))) \) vs. \( \ln((\sigma - \sigma_L)/\sigma_S - \sigma) \) as it is shown in Fig. 1. It is obtained (analytic expression)

\[ \ln \left( \frac{1}{1-\sigma_L} \right) = m \cdot \ln \left( \frac{\sigma - \sigma_L}{\sigma_S - \sigma} \right) + m \cdot \ln K \]  

(7)

or (discrete expression),

\[ \ln \left( \frac{1}{1-\sigma_i} \right) = m \cdot \ln \left( \frac{\sigma_i - \sigma_L}{\sigma_S - \sigma_L} \right) + m \cdot \ln K \]  

(8)

When the expression (7) is fitted to the points given by (8), for a given pair of values of \( \sigma_L \) and \( \sigma_S \), it is possible to determine \( m \) and \( K \) and the respective correlation coefficient.

Considering a set of inferior and superior tensions

\[ \sigma_{L_p} < \sigma_{L_{p-1}} < \sigma_{L_{p-2}} \ldots < \sigma_{L0} \quad \text{and} \quad \sigma_{S0} < \sigma_{S1} \ldots < \sigma_{Sp} \]

the Weibull’ parameters are those that give to the best linear correlation.

Making a series of \( M \) simulations by means of \( N \) random numbers \( \lambda_i \) each time, the mean values of \( \sigma_L \), \( \sigma_S \), \( m \) and \( \Delta K \) of dispersions \( \Delta \sigma_L \), \( \Delta \sigma_S \), \( \Delta m \) and \( \Delta K \) are determined from only one experimental procedure with \( N \) specimens. This one is not the first attempt to determine the four Weibull’s parameters and their dispersions; a different method has been proposed in [4], but it has not been tested experimentally.

3. Number of specimens

By means of the Fisher’s matrix [2], the dispersions of parameters can be expressed in the form:

\[ \Delta \sigma_L = \frac{1}{\sqrt{m}} f_{\sigma_L}; \quad \Delta \sigma_S = \frac{1}{\sqrt{m}} f_{\sigma_S}; \]

\[ \Delta m = \frac{1}{\sqrt{m}} f_m; \quad \Delta K = \frac{1}{\sqrt{m}} f_K \]  

(9)

where \( N \) is the number of essays to obtain \( \Delta m \), \( \Delta K \), \( \Delta \sigma_L \), \( \Delta \sigma_S \). Functions \( f_m \), \( f_K \), \( f_{\sigma_L} \), \( f_{\sigma_S} \) depend only on \( m \), \( K \), \( \sigma_L \), \( \sigma_S \) and can be calculated from relationships (9), because the parameters dispersions are known from simulations.

On the other hand, if in a design the level of failure \( F_c \), with a design tension \( \sigma_c \), and the tolerance \( \delta F \) are adopted, the values of \( \delta m \), \( \delta K \), \( \delta \sigma_L \) and \( \delta \sigma_S \) can be derived from Eq. (3); this fact is outlined in Fig. 2. Explicitly, one obtains:

![Fig. 1. Simulation \( \sigma_i \leftrightarrow F_i \) and its lineal regression.](image-url)
4. Experimental application

To apply the methodology exposed above, 30 specimens of wiredrawing steel SAE 1020 of 0.006 m diameter were tested on an Instron machine [5,6] and 30 values of the yield stress \( \sigma_i \) were determined. In this case, the failure is the plastic deformation. The Weibull’s parameters were found using relation (8), that means

\[
\ln \left\{ \ln \left[ \frac{1}{1 - F_i} \right] \right\} = m \cdot \ln \left( \frac{\sigma_i - \sigma_L}{\sigma_S - \sigma} \right) + m \cdot \ln K
\]

where the values \( \sigma_i \) are the yield stresses measured experimentally. To obtain the Weibull’s parameter, the same methodology explained above was used with \( N = 30 \). Table 1 shows the results obtained for \( \sigma_L \), \( \sigma_S \), \( m \) and \( K \) and constitute the experimental values to be used in the simulations.

This method was applied to 100 simulations of 30 tests each one. The values of \( \sigma_L \) and \( \sigma_S \) giving the best correlation were calculated with module Solver of Microsoft Excel. The results of this simulation are shown in Table 2. It is important outline that the average values of the parameters are practically the same as the ones used for the simulation and, in order to determine the dispersions, the values of the parameters with a cumulative probability smaller than 0.5% and greater than 95% were erased with the aim of eliminating possible computational errors.

With the values of Table 2 and \( N = 30 \), the functions \( f_{\sigma_l} \), \( f_{\sigma_s} \), \( f_m \) and \( f_K \), are evaluated from relations (9). The results are shown in Table 3.

The number of essays required to obtain a cumulative probability of failure compatible between given values of \( F_c \) and \( F_c + \delta F \), can now be found. For example, to the material used in the experiments, Table 4 shows the results obtained for several cases of \( F_c \) and \( F_c + \delta F \). These results are calculated from relationships (9)–(12). Note that in this example, the design stress is the yield stress.

![Diagram of the cumulative probability of failure and its tolerance in relation with the design stress.](image-url)
5. Conclusions

In this work, a simulation method has been implemented to obtain the dispersion and control of the four Weibull’s parameters. The method is based on the optimization of the linear correlation of the cumulative probability of failure and constitutes a simple way to estimate these dispersions, avoiding an excessive number of experimental measurements and troublesome calculations involved when using the Fisher’s matrix. Moreover, using a property of this matrix, the number of samples can be estimated to get fixed values of the cumulative probability of failure and its tolerance.

Eqs. (9)–(12) provide a criterion to expect that approximately 70% of the values of $\sigma_c$ are within a range of acceptance; taking $2\delta F$ instead of $\delta F$, one gets 90%. For the experimental case studied, from the results shown in Table 4, one concludes that the number of essays must be 475 to have a cumulative probability of failure between $10^{-6}$ and $10^{-7}$ and 182 to have a cumulative probability between $10^{-2}$ and $10^{-1}$ (greater cumulative probability of failure requires minor number of samples). The larger value of $N_i$ must be chosen. When a batch of material to be used is available, $N$ samples are assayed and the criterion $\chi^2$ may be used to test whether the batch belongs to the type of material specified.

References