Double-spike solutions for a critical inhomogeneous elliptic problem in domains with small holes

Salomón Alarcón
Departamento de Ingeniería Matemática, Universidad de Chile,
Casilla 170 Correo 3, Santiago, Chile (salarcon@dim.uchile.cl)

In this paper we construct solutions which develop two negative spikes as $\varepsilon \to 0^+$ for the problem $-\Delta u = |u|^{4/(N-2)}u + \varepsilon f(x)$ in Ω, $u = 0$ on $\partial\Omega$, where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain exhibiting a small hole, with $f \geqslant 0$, $f \not\equiv 0$. This result extends a recent work of Clapp *et al.* in the sense that no symmetry assumptions on the domain are required.

1. Introduction

This paper deals with the construction of solutions of the problem

$$\left.\begin{aligned} -\Delta u &= |u|^{p-1}u + \varepsilon f(x) \quad &&\text{in } \Omega, \\ u &= 0 &&\text{on } \partial\Omega, \end{aligned}\right\} \tag{1.1}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$, $N \geqslant 3$, which has a small hole, $p = (N+2)/(N-2)$ is the critical Sobolev exponent, $f(x)$ is an inhomogeneous perturbation, $f \geqslant 0$, $f \not\equiv 0$ and $\varepsilon > 0$ is a small parameter.

In the case when $1 < p < (N+2)/(N-2)$, it is well known that if $f = 0$, the associated energy functional to problem (1.1) is even and satisfies the Palais–Smale (PS) condition in $H_0^1(\Omega)$, which implies the existence of infinitely many non-trivial solutions by standard Lyusternik–Schnirelman theory. Also known are many results on existence and multiplicity of sign-changing solutions for small and large inhomogeneous perturbations (see [2, 5, 18, 19, 23, 25]), whereas in [16] it was proved that (1.1) does not admit any positive solution if $\varepsilon > 0$ is too large.

In the critical case, $p = (N+2)/(N-2)$, the embedding $H_0^1(\Omega) \subset L^{p+1}(\Omega)$ is continuous but not compact, so that the (PS) condition does not hold, and serious difficulties in facing the existence question arise. In fact, Pohozaev [17] proved that (1.1) has no solution if $f = 0$ and Ω is strictly star-shaped. In contrast, Brezis and Nirenberg [7] showed that this situation can be reverted by introducing suitable additive perturbations. Rey [20] pointed out that the result in [6] implies that if $f \geqslant 0$, $f \neq 0$ and $f \in H^{-1}(\Omega)$, then at least two positive solutions exist provided that $\varepsilon > 0$ is sufficiently small. Moreover, in [20] it was proven that if $f \geqslant 0$, $f \not\equiv 0$, is sufficiently regular, then at least $\operatorname{cat}(\Omega) + 1$ positive solutions exist for $\varepsilon > 0$ sufficiently small, one of them converging uniformly to 0 while the others concentrate at some special points in Ω, depending on f and the regular part of Green's function of the Laplacian on Ω, as $\varepsilon \to 0$. In parallel to Rey's result in [20], but with a different approach, Tarantello [26] proved that (1.1) admits at least two solutions for $f \not\equiv 0$ satisfying $\|\varepsilon f\|_{H^{-1}(\Omega)} < C_N$, where C_N is an explicit constant; such solutions are positive if $f \geqslant 0$. The effect of the symmetries in further multiplicity of solutions has been considered in some works. Ali and Castro [1] proved that the existence result in [7] is optimal for positive solutions in a ball: if Ω is a ball and $f \equiv 1$, problem (1.1) has exactly two positive solutions for all sufficiently small $\varepsilon > 0$. More recently, Clapp *et al.* [9] proved that if Ω is symmetric with respect to 0, $0 \notin \Omega$, and f is even, then at least $\mathrm{cat}(\Omega) + 2$ positive solutions exist provided that $\|\varepsilon f\|_{H^{-1}}$ is sufficiently small. The results in [1, 7, 9, 20, 26] deal with the existence of positive solutions to problem (1.1), provided that $f \geqslant 0$ and $f \neq 0$, where $\varepsilon > 0$ is a small parameter.

Concerning solutions which are not necessarily positive, Clapp *et al.* [10] showed the existence of solutions of (1.1) under certain symmetry assumptions in the domain Ω and the function f. Such solutions develop k negative spikes, for any $k \geqslant k_0(\Omega)$, where $k_0(\Omega)$ is a sufficiently large number depending on Ω.

In this paper we leave aside any symmetry assumptions on the domain Ω and the perturbation f, and we find solutions to problem (1.1) developing a negative double-spike shape. Additionally, we give precise information about the asymptotic profile of the blow-up of these solutions as $\varepsilon \to 0$ and we indicate a clearly delimited region where the spikes are formed.

More precisely, our setting in problem (1.1) is as follows: let us consider the domain

$$\Omega = \mathcal{D} \setminus \overline{B(P, \mu)}, \tag{1.2}$$

where $\mathcal{D}$ is a bounded smooth domain in $\mathbb{R}^N$, $N \geqslant 3$, $P \in \mathcal{D}$ and $\mu > 0$ is a small number. Let us consider $f \in C^{0,\gamma}(\bar{\Omega})$, for some $0 < \gamma < 1$, such that $\inf_{x\in\Omega} f(x) > 0$ and, by simplicity, we fix $P = 0$. Then our main result is as follows.

THEOREM 1.1. *There exists a constant $\mu_0 = \mu_0(f, \mathcal{D}) > 0$, such that for each $0 < \mu < \mu_0$ fixed, there exists a number $\varepsilon_0 > 0$ and a family of solutions u_ε of (1.1), for $0 < \varepsilon = \varepsilon_n < \varepsilon_0$, with the following property: u_ε has exactly a pair of local minimum points $(\xi_1^\varepsilon, \xi_2^\varepsilon) \in \Omega^2$ with $k_*\mu < |\xi_i^\varepsilon| < k^*\mu$, $i = 1, 2$, for certain constants k_*, k^* independent of μ and such that, for each small $\delta > 0$,*

$$\inf_{\{|x-\xi_i^\varepsilon|>\delta,\ i=1,2\}} u_\varepsilon(x) \to 0 \quad \text{and} \quad \inf_{\{|x-\xi_i^\varepsilon|<\delta\}} u_\varepsilon(x) \to -\infty, \quad i = 1, 2,$$

as $\varepsilon \to 0$.

Indeed, we will find that u_ε is a non-trivial solution of (1.1) of the form

$$u_\varepsilon(x) = -\alpha_N \sum_{i=1}^{2} \left\{ \frac{\varepsilon^{2/(N-2)}\lambda_{i\varepsilon}}{\varepsilon^{4/(N-2)}\lambda_{i\varepsilon}^2 + |x - \xi_i^\varepsilon|^2} \right\}^{(N-2)/2} + \varepsilon^{-1}\hat{\phi}(x) + \theta_\varepsilon(x),$$

where $\theta_\varepsilon(x) \to 0$ uniformly as $\varepsilon \to 0$, $\hat{\phi}$ is the unique solution of the problem

$$\begin{aligned} -\Delta\hat{\phi}(x) &= \varepsilon^2 f(x) \quad \text{in } \Omega, \\ \hat{\phi} &= 0 \qquad\quad\ \text{on } \partial\Omega, \end{aligned}$$

$\alpha_N = (N(N-2))^{(N-2)/4}$ and the points $\xi_i^\varepsilon \to \xi_i$, up to subsequences, where (ξ_1, ξ_2) is a critical point of the functional

$$\Phi(x,y) = \frac{1}{2}\left\{\frac{H(x,x)w^2(y) + 2G(x,y)w(x)w(y) + H(y,y)w^2(x)}{G^2(x,y) - H(x,x)H(y,y)}\right\}$$

defined in the region $\{(x,y) \in \Omega^2 : G(x,y) - H^{1/2}(x,x)H^{1/2}(y,y) > 0,\ x \neq y\}$. Here G and H are, respectively, Green's function of the Laplacian on Ω and its regular part, and w is the unique solution of the problem

$$\begin{aligned} -\Delta w &= f \quad \text{in } \Omega, \\ w &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Additionally, one can identify the limits λ_i of $\lambda_{i\varepsilon}$ as

$$\lambda_i = \left(a_N^{-1}\frac{H(\xi_j,\xi_j)w(\xi_i) + G(\xi_i,\xi_j)w(\xi_j)}{G^2(\xi_i,\xi_j) - H(\xi_i,\xi_i)H(\xi_j,\xi_j)}\right)^{2/(N-2)}, \quad i \neq j,\ i,j = 1,2,$$

where a_N is an explicit constant, and consider the constants k_*, k^* as follows: k_* is the unique solution in $]1,+\infty[$ of the equation

$$\frac{2^{2-N}}{s^{N-2}} = \frac{(s^2+1)^{N-2} + (s^2-1)^{N-2}}{(s^4-1)^{N-2}}$$

and $K \leqslant k^* = k*(\Omega, f) < \infty$, where K is the unique solution in $]1,+\infty[$ of the equation

$$\frac{2^{1-N}}{s^N} = \frac{(s^2-1)^{N-1} + (s^2+1)^{N-1}}{(s^4-1)^{N-1}}.$$

In particular, if f is a constant and Ω is an annulus, then $k^* = K$.

On the other hand, it will be clear from the proof that the small excised domain does not need to be exactly a ball, and we consider this case just for notational simplicity.

The proof of theorem 1.1 follows a Lyapunov–Schmidt reduction procedure, related with this problem. This method has been used for solving problem (1.1) in the critical case (see [10, 20]) and in the slightly supercritical case with $f = 0$ (see [12, 13], and also [21, 22] for related results).

In the next section we derive some basic estimates for the *reduced energy* associated with this problem. Sections 3 and 4 will be devoted to discussion of the finite-dimensional reduction scheme which we use for the construction of solutions of (1.1). In § 5 we introduce an auxiliary function which will be the key in our min–max scheme, which we develop in § 6 to finally establish theorem 1.1.

2. Basic estimates in the reduced energy

Let Ω be a bounded smooth domain in $\mathbb{R}^N$, $N \geqslant 3$, and let us consider the expanded domain

$$\Omega_\varepsilon = \varepsilon^{-2/(N-2)}\Omega, \quad \varepsilon > 0.$$

Using the change of variable

$$v_\varepsilon(x') = -\varepsilon u(\varepsilon^{2/(N-2)}x'), \quad x' \in \Omega_\varepsilon,$$

we note that u solves (1.1) if and only if v_ε solves

$$\left.\begin{aligned} \Delta v + |v|^{p-1}v &= \varepsilon^{p+1}\tilde{f}(x') \quad \text{in } \Omega_\varepsilon, \\ v &= 0 \quad\quad\quad\quad\; \text{on } \partial\Omega_\varepsilon, \end{aligned}\right\} \tag{2.1}$$

where $p = (N+2)/(N-2)$ and $\tilde{f}(x') = f(\varepsilon^{2/(N-2)}x')$. It is well known that all positive solutions of equation $\Delta\vartheta + \vartheta^p = 0$ in $\mathbb{R}^N$ are given by the functions

$$\bar{U}_{\lambda,\xi}(x) = \alpha_N\left(\frac{\lambda}{\lambda^2 + |x-\xi|^2}\right)^{(N-2)/2},$$

with $\lambda > 0$, $\xi \in \mathbb{R}^N$ and $\alpha_N = (N(N-2))^{(N-2)/4}$ [3,7,8,24]. Since Ω_ε is expanding to the whole $\mathbb{R}^N$ as $\varepsilon \to 0$, and $\varepsilon^{p+1}\tilde{f}(x') \to 0$ uniformly as $\varepsilon \to 0$, it is reasonable to assume that, for certain numbers $\lambda_1, \lambda_2 > 0$ and points $\xi_1, \xi_2 \in \Omega$, some solution v_ε of (2.1) becomes

$$v_\varepsilon \sim \bar{U}_{\lambda_1,\xi_1'} + \bar{U}_{\lambda_2,\xi_2'},$$

where $\xi_i' = \varepsilon^{-2/(N-2)}\xi_i \in \Omega_\varepsilon$, and where from now on ξ denotes a point in Ω and ξ' denotes a point in Ω_ε.

From [11], we know that a better approximation to v_ε should be obtained by using the orthogonal projections onto $H_0^1(\Omega_\varepsilon)$ of the functions $\bar{U}_{\lambda,\xi'}$, denoted by $U_{\lambda,\xi'}$, namely the unique solution of the problem

$$\begin{aligned} -\Delta U_{\lambda,\xi'} &= \bar{U}^p_{\lambda,\xi'} \quad \text{in } \Omega_\varepsilon, \\ U_{\lambda,\xi'} &= 0 \qquad\;\; \text{on } \partial\Omega_\varepsilon. \end{aligned}$$

In other words, $U_{\lambda,\xi'} = \bar{U}_{\lambda,\xi'} - \bar{v}_{\lambda,\xi'}$, where $\bar{v}_{\lambda,\xi'}$ solves

$$\begin{aligned} -\Delta \bar{v}_{\lambda,\xi'} &= 0 \qquad\quad \text{in } \Omega_\varepsilon, \\ \bar{v}_{\lambda,\xi'} &= \bar{U}_{\lambda,\xi'} \quad \text{on } \partial\Omega_\varepsilon. \end{aligned}$$

Hence, if we consider $\bar{U} = \bar{U}_{1,0}$, we obtain

$$\bar{v}_{\lambda,\xi'}(x') = \varepsilon^2\lambda^{(N-2)/2}H(\varepsilon^{2/(N-2)}x', \xi)\int_{\mathbb{R}^N}\bar{U}^p + o(\varepsilon^2) \tag{2.2}$$

and, away from $x' = \xi'$,

$$U_{\lambda,\xi'}(x') = \varepsilon^2\lambda^{(N-2)/2}G(\varepsilon^{2/(N-2)}x', \xi)\int_{\mathbb{R}^N}\bar{U}^p + o(\varepsilon^2) \tag{2.3}$$

uniformly for x' on each compact subset of Ω_ε, where G and H are, respectively, Green's function of the Laplacian with the Dirichlet boundary condition on Ω and its regular part. Now, to simplify notation, we consider the function

$$V(x') = U_1(x') + U_2(x'), \quad x' \in \Omega_\varepsilon,$$

where $U_i = U_{\lambda_i,\xi_i'}$, $i = 1, 2$, and we set $\boldsymbol{\xi} = (\xi_1, \xi_2) \in \Omega^2$ and $\boldsymbol{\lambda} = (\lambda_1, \lambda_2) \in \mathbb{R}^2_+$. Then, we look for solutions of problem (2.1) of the form

$$v(x') = V(x') + \tilde{\eta}(x'), \quad x' \in \Omega_\varepsilon, \tag{2.4}$$

which for suitable points ξ and scalars λ will have the remainder term $\tilde{\eta}$ of small order all over Ω_ε. Since solutions of (2.1) correspond to stationary points of its associated energy functional J_ε defined by

$$J_\varepsilon(v) = \frac{1}{2}\int_{\Omega_\varepsilon} |\nabla v|^2 - \frac{1}{p+1}\int_{\Omega_\varepsilon} |v|^{p+1} + \varepsilon^{p+1}\int_{\Omega_\varepsilon} \tilde{f} v, \tag{2.5}$$

we have that if a solution of the form (2.4) exists, then we should have $J_\varepsilon(v) \sim J_\varepsilon(V)$ and the corresponding points $(\boldsymbol{\xi}, \boldsymbol{\lambda})$ in the definition of V also should be 'approximately stationary' for the finite-dimensional functional $(\boldsymbol{\xi}, \boldsymbol{\lambda}) \mapsto J_\varepsilon(V)$. Thus, our first goal is to estimate $J_\varepsilon(V)$. In order to establish the expansion, we consider the function w, which corresponds to the unique solution in $C^{0,\gamma}(\Omega)$ of the problem

$$\left.\begin{aligned} -\Delta w &= f \quad \text{in } \Omega, \\ w &= 0 \quad \text{on } \partial\Omega, \end{aligned}\right\} \tag{2.6}$$

and we make the following choice of the points and parameters: we fix $\delta > 0$ and we define the parameters λ_i as

$$\lambda_i = (a_N^{-1}\Lambda_i)^{2/(N-2)}, \quad i = 1, 2,$$

where $a_N = \int_{\mathbb{R}^N} \bar{U}^p$ and $\Lambda_i \in]\delta, \delta^{-1}[$, for $i = 1, 2$. We also define the set

$$\mathcal{M}_\delta = \{(\boldsymbol{\xi}, \boldsymbol{\Lambda}) : |\xi_1 - \xi_2| > \delta,\ \operatorname{dist}(\xi_i, \partial\Omega) > \delta;\ i = 1, 2\}, \tag{2.7}$$

where $\boldsymbol{\xi} = (\xi_1, \xi_2) \in \Omega^2$ and $\boldsymbol{\Lambda} = (\Lambda_1, \Lambda_2) \in]\delta, \delta^{-1}[$.

LEMMA 2.1. *Let $\delta > 0$ given. The expansion*

$$J_\varepsilon(V) = 2C_N + \varepsilon^2 \Phi(\boldsymbol{\xi}, \boldsymbol{\Lambda}) + o(\varepsilon^2)$$

holds uniformly in the C^1-sense, with respect to $(\boldsymbol{\xi}, \boldsymbol{\Lambda})$ in $\mathcal{M}_\delta$. Here

$$C_N = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla \bar{U}|^2 - \frac{1}{p+1}\int_{\mathbb{R}^N} \bar{U}^{p+1} \tag{2.8}$$

and the function Φ is defined by

$$\Phi(\boldsymbol{\xi}, \boldsymbol{\Lambda}) = \frac{1}{2}\left\{\sum_{i=1}^{2} \Lambda_i^2 H(\xi_i, \xi_i) - 2\Lambda_1\Lambda_2 G(\xi_1, \xi_2)\right\} + \sum_{i=1}^{2} \Lambda_i w(\xi_i). \tag{2.9}$$

The proof of the previous lemma is based on (2.2), (2.3) and some estimates established in [4], and follows a similar procedure to that used to prove [13, lemma 3.2] and [10, proposition 1]; it is therefore omitted here.

3. The finite-dimensional reduction

We first introduce some notation to be used in what follows. For functions u, v defined in Ω_ε we set

$$\langle u, v\rangle = \int_{\Omega_\varepsilon} uv.$$

Let us fix a small number $\delta > 0$ and consider points $(\boldsymbol{\xi}', \boldsymbol{\Lambda})$ in

$$\mathcal{M}_\delta^\varepsilon = \{(\boldsymbol{\xi}', \boldsymbol{\Lambda}) \in \Omega_\varepsilon^2 \times]\delta, \delta^{-1}[^2 \colon |\xi_1' - \xi_2'| > \delta_\varepsilon,\ \operatorname{dist}(\xi_i', \partial\Omega_\varepsilon) > \delta_\varepsilon;\ i = 1, 2\}, \tag{3.1}$$

where $\delta_\varepsilon = \delta\varepsilon^{-2/(N-2)}$, $\boldsymbol{\xi}' = (\xi_1', \xi_2')$ and $\boldsymbol{\Lambda} = (\Lambda_1, \Lambda_2)$. Since all solutions ϑ of the problem $\Delta\vartheta + p\bar{U}_{\Lambda,0}^{p-1}\vartheta = 0$ in $\mathbb{R}^N$ which satisfy $|\vartheta(x)| < C|x|^{2-N}$ belong to

$$\operatorname{span}\left\{\frac{\partial \bar{U}_{\Lambda,0}}{\partial x_j}, \frac{\partial \bar{U}_{\Lambda,0}}{\partial \Lambda}\right\}_{j=1,\dots,N+1}$$

(see [8]), it is convenient to consider, for $i = 1, 2$, the functions

$$\bar{Z}_{ij}(x') = \frac{\partial \bar{U}_i}{\partial \xi_{ij}'}(x'), \quad j = 1, \dots, N, \qquad \bar{Z}_{i(N+1)}(x') = \frac{\partial \bar{U}_i}{\partial \Lambda_i}(x'),$$

and their respective $H_0^1(\Omega_\varepsilon)$-projections Z_{ij}, namely, the unique solutions of

$$\begin{aligned} \Delta Z_{ij} &= \Delta \bar{Z}_{ij} && \text{in } \Omega_\varepsilon, \\ Z_{ij} &= 0 && \text{on } \partial\Omega_\varepsilon. \end{aligned}$$

In order to simplify notation, we will define

$$V = U_1 + U_2 \quad \text{and} \quad \bar{V} = \bar{U}_1 + \bar{U}_2.$$

We start by studying a linear problem which is the basis for the reduction of (2.1): given $h \in L^\infty(\bar{\Omega}_\varepsilon)$, find a function η and constants c_{ij} such that

$$\left.\begin{aligned} \Delta\eta + p|V|^{p-1}\eta &= h + \sum_{i,j} c_{ij} U_i^{p-1} Z_{ij} && \text{in } \Omega_\varepsilon, \\ \eta &= 0 && \text{on } \partial\Omega_\varepsilon, \\ \langle \eta, U_i^{p-1} Z_{ij} \rangle &= 0 && \text{for all } i, j. \end{aligned}\right\} \tag{3.2}$$

We want to prove that this problem is uniquely solvable with uniform bounds in certain appropriate norms. In other words, we want study the linear operator L_ε associated with (3.2), namely

$$L_\varepsilon(\eta) = \Delta\eta + p|V|^{p-1}\eta, \tag{3.3}$$

under the previous orthogonality conditions. In order to achieve this goal, we introduce the following L^∞-norms with weight. Let $\omega_i = (1 + |x' - \xi_i'|^2)^{-(N-2)/2}$, $i = 1, 2$; for a function θ defined in Ω_ε, we consider the norms

$$\|\theta\|_* = \|(\omega_1 + \omega_2)^{-\sigma}\theta(x')\|_\infty + \|(\omega_1 + \omega_2)^{-\sigma-1}\nabla\theta(x')\|_\infty,$$

where $\sigma = \frac{1}{2}$ if $3 \leqslant N \leqslant 6$, $\sigma = 2/(N-2)$ if $N \geqslant 7$ and

$$\|\theta\|_{**} = \|(\omega_1 + \omega_2)^{-\varsigma}\theta(x')\|_\infty,$$

where $\varsigma = \frac{1}{2}p$ if $3 \leqslant N \leqslant 6$ and $\varsigma = 4/(N-2)$ if $N \geqslant 7$. These norms are similar to those defined in [10] for $N \geqslant 7$ but, for $3 \leqslant N \leqslant 6$, we have modified them, something apparently necessary in this case, since $p \geqslant 2$. Now, we study the invertibility of the linear operator L_ε defined in (3.3). Hence, it is also important to understand the differentiability of L_ε in the variables $(\boldsymbol{\xi}', \boldsymbol{\Lambda}) \in \mathcal{M}_\delta^\varepsilon$.

Proposition 3.1. *Assume that* $(\boldsymbol{\xi}', \boldsymbol{\Lambda}) \in \mathcal{M}_\delta^\varepsilon$. *There then exist* $\varepsilon_0 > 0$ *and* $C > 0$ *such that, for all* $0 < \varepsilon < \varepsilon_0$ *and for all* $h \in C^\alpha(\bar{\Omega}_\varepsilon)$, *the problem (3.2) admits a unique solution* $\eta \equiv M_\varepsilon(h)$. *Moreover, the map* $(\boldsymbol{\xi}', \boldsymbol{\Lambda}, h) \mapsto \eta \equiv M_\varepsilon(h)$ *is of class* C^1 *and satisfies*

$$\|\eta\|_* \leqslant C\|h\|_{**} \quad \text{and} \quad \|\nabla_{(\boldsymbol{\xi}', \boldsymbol{\Lambda})}\eta\|_* \leqslant C\|h\|_{**}.$$

The proof of this proposition follows from a slight variation of the arguments in the proof of [13, propositions 4.1 and 4.2] with the necessary modifications in [14], so we omit it here. In what follows, C represents a generic positive constant that is independent of ε and of the particular points $(\boldsymbol{\xi}', \boldsymbol{\Lambda}) \in \mathcal{M}_\delta^\varepsilon$.

Now, we are ready to begin the finite-dimensional reduction. We want to solve the following nonlinear problem: find a function $\tilde{\eta}$ such that, for certain constants c_{ij}, $i = 1, 2$, $j = 1, \ldots, N+1$, one has

$$\left.\begin{aligned} \Delta(V + \tilde{\eta}) + |V + \tilde{\eta}|^{p-1}(V + \tilde{\eta}) - \varepsilon^{p+1}\tilde{f} = \sum_{i,j} c_{ij} U_i^{p-1} Z_{ij} & \quad \text{in } \Omega_\varepsilon, \\ \tilde{\eta} = 0 & \quad \text{on } \partial\Omega_\varepsilon, \\ \langle \tilde{\eta}, U_i^{p-1} Z_{ij} \rangle = -\langle \phi, U_i^{p-1} Z_{ij} \rangle & \quad \text{for all } i, j, \end{aligned}\right\} \tag{3.4}$$

where ϕ solves the problem

$$\left.\begin{aligned} -\Delta\phi = \varepsilon^{p+1}\tilde{f} & \quad \text{in } \Omega_\varepsilon, \\ \phi = 0 & \quad \text{on } \partial\Omega_\varepsilon. \end{aligned}\right\} \tag{3.5}$$

Note that $V + \tilde{\eta}$ is a solution of (2.1) if the scalars c_{ij} in (3.4) are all zero. Also, we note that the partial differential equation in (3.4) is equivalent in Ω_ε to

$$\Delta\eta + p|V|^{p-1}\eta = -N_\varepsilon(\eta) - R_\varepsilon + \sum_{i,j} c_{ij} U_i^{p-1},$$

where $\eta = \tilde{\eta} - \phi$,

$$N_\varepsilon(\eta) = |V + \eta - \phi|^{p-1}(V + \eta - \phi)_+ - |V|^{p-1}V - p|V|^{p-1}(\eta - \phi) \tag{3.6}$$

and

$$R_\varepsilon = |V|^{p-1}V - \bar{U}_1^p - \bar{U}_2^p - p|V|^{p-1}\phi. \tag{3.7}$$

A first step to solve (3.4) consists of dealing with the following nonlinear problem: find a function φ that, for certain constants c_{ij}, $i = 1, 2$, $j = 1, \ldots, N+1$, solves

$$\left.\begin{aligned} \Delta(V + \tilde{\eta}) + |V + \tilde{\eta}|^{p-1}(V + \tilde{\eta})_+ - \varepsilon^{p+1}\tilde{f} = \sum_{i,j} c_{ij} U_i^{p-1} Z_{ij} & \quad \text{in } \Omega_\varepsilon, \\ \varphi = 0 & \quad \text{on } \partial\Omega_\varepsilon, \\ \langle \varphi, U_i^{p-1} Z_{ij} \rangle = 0 & \quad \text{for all } i, j, \end{aligned}\right\} \tag{3.8}$$

where $\tilde{\eta} = \psi + \varphi - \phi$, with ϕ satisfying (3.5), and the function ψ is chosen as

$$\psi = -M_\varepsilon(R_\varepsilon), \tag{3.9}$$

where M_ε is defined as in proposition 3.1 and R_ε is given by (3.7). Actually, it is easy to check that, for points $(\boldsymbol{\xi}', \boldsymbol{\Lambda}) \in \mathcal{M}_\delta^\varepsilon$, one has

$$\|\psi\|_* \leqslant C\varepsilon^2.$$

Now, in (3.8) we rewrite the equation of interest as

$$\Delta\varphi + p|V|^{p-1}\varphi = -N_\varepsilon(\eta) - (\Delta\psi + p|V|^{p-1}\psi + R_\varepsilon) + \sum_{i,j} c_{ij} U_i^{p-1} Z_{ij},$$

where $\eta = \psi + \varphi$.

Lemma 3.2. *Assume that $(\boldsymbol{\xi}', \boldsymbol{\Lambda}) \in \mathcal{M}_\delta^\varepsilon$. There then exists $C > 0$ such that, for all $\varepsilon > 0$ small enough and $\|\varphi\|_* \leqslant \frac{1}{4}$, one has*

$$\|N_\varepsilon(\psi+\varphi)\|_{**} \leqslant \begin{cases} C(\|\varphi\|_*^2 + \varepsilon\|\varphi\|_* + \varepsilon^{p+1}) & \text{if } 3 \leqslant N \leqslant 6, \\ C(\varepsilon^{2(p-2)}\|\varphi\|_*^2 + \varepsilon^{p^2-3p+2}\|\varphi\|_*^p + \varepsilon^{p^2-p+2}) & \text{if } N \geqslant 7. \end{cases}$$

Proof. Note that $\|\phi\|_* \leqslant C\varepsilon^p$ if $3 \leqslant N \leqslant 6$, $\|\phi\|_* \leqslant C\varepsilon^2$ if $N \geqslant 7$ and $\|\psi\|_* \leqslant C\varepsilon^2$. Since $\|\psi+\varphi\|_* \leqslant \|\psi\|_* + \|\varphi\|_*$, for $\eta = \psi + \varphi$ we have that $\|\eta\|_* < 1$. Also we note that

$$N_\varepsilon(\eta) = C|V + \bar{t}(\eta - \phi)|^{p-2}(\eta - \phi)^2, \tag{3.10}$$

with $\bar{t} \in]0,1[$. Hence, if $3 \leqslant N \leqslant 6$, then

$$|(\omega_1+\omega_2)^{-p/2} N_\varepsilon(\eta)| \leqslant C(\omega_1+\omega_2)^{(p-1)/2}\|\eta-\phi\|_*^2 \leqslant C\|\eta-\phi\|_*^2.$$

On the other hand, for $N \geqslant 7$, if $|\eta| \leqslant \frac{1}{2}(\omega_1+\omega_2)$, we again use (3.10) to obtain

$$\begin{aligned} |(\omega_1+\omega_2)^{-4/(N-2)} N_\varepsilon(\eta)| &\leqslant C(\omega_1+\omega_2)^{(6-N)/(N-2)}\|\eta-\phi\|_*^2 \\ &\leqslant C\varepsilon^{(6-N)/(N-2)}\|\eta-\phi\|_*^2. \end{aligned}$$

In another case we obtain directly from (3.6) that

$$\begin{aligned} |(\omega_1+\omega_2)^{-4/(N-2)} N_\varepsilon(\eta)| &\leqslant C|(\omega_1+\omega_2)^{-4/(N-2)}(\eta-\phi)^p| \\ &\leqslant C\varepsilon^{(6-N)/(N-2)(2/(N-2))}\|\eta-\phi\|_*^p. \end{aligned}$$

The result follows on combining previous estimates.

We now deal with the following problem:

$$\left.\begin{aligned} \Delta\varphi + pV^{p-1}\varphi &= -N_\varepsilon(\eta) + \sum_{i,j} c_{ij} U_i^{p-1} Z_{ij} && \text{in } \Omega_\varepsilon, \\ \varphi &= 0 && \text{on } \partial\Omega_\varepsilon, \\ \langle \varphi, U_i^{p-1} Z_{ij} \rangle &= 0 && \text{for all } i,j, \end{aligned}\right\} \tag{3.11}$$

where $\eta = \psi + \varphi$ and ψ is the function defined in (3.9).

Proposition 3.3. *Assume that $(\boldsymbol{\xi}', \boldsymbol{\Lambda}) \in \mathcal{M}_\delta^\varepsilon$. There then exists $C > 0$ such that, for all $\varepsilon > 0$ small enough, there exists a unique solution $\varphi = \varphi(\boldsymbol{\xi}', \boldsymbol{\Lambda})$ to problem (3.11). Moreover, the map $(\boldsymbol{\xi}', \boldsymbol{\Lambda}) \mapsto \varphi(\boldsymbol{\xi}', \boldsymbol{\Lambda})$ is of class C^1 for the $\|\cdot\|_*$-norm and it satisfies*

$$\|\varphi\|_* \leqslant C\varepsilon^2 \quad \textit{and} \quad \|\nabla_{(\boldsymbol{\xi}', \boldsymbol{\Lambda})}\varphi\|_* \leqslant C\varepsilon^2.$$

Proof. Let us set

$$\mathcal{F}_r = \{\varphi \in H_0^1(\Omega_\varepsilon) : \|\varphi\|_* \leqslant r\varepsilon^2\},$$

with $r > 0$ a constant to be fixed later. We define the map $A_\varepsilon : \mathcal{F}_r \to H_0^1(\Omega_\varepsilon)$ as

$$A_\varepsilon(\varphi) = -M_\varepsilon(N_\varepsilon(\psi + \varphi)),$$

where M_ε is the operator defined in proposition 3.1. Since $\psi = -M_\varepsilon(R_\varepsilon)$, solving (3.11) is equivalent to finding a fixed point φ for A_ε. From proposition 3.1 and lemma 3.2, we deduce that if $\varphi \in \mathcal{F}_r$ and $\varepsilon > 0$ is small enough, then

$$\|A_\varepsilon(\varphi)\|_* \leqslant r\varepsilon^2$$

for a suitable choice of $r = r(N)$ which we consider fixed from now on. Note that, for $\varphi_1, \varphi_2 \in \mathcal{F}_r$, from lemma 3.2 we have

$$\|A_\varepsilon(\varphi_1) - A_\varepsilon(\varphi_2)\|_* \leqslant C\|N_\varepsilon(\psi + \varphi_1) - N_\varepsilon(\psi + \varphi_2)\|_{**} \leqslant C\varepsilon^p\|\varphi_1 - \varphi_2\|_*,$$

for all $N \geqslant 3$. It follows that, for $\varepsilon > 0$ small enough, the map A_ε is a contraction $\|\cdot\|_*$ in $\mathcal{F}_r$. Therefore, A_ε has a fixed point in $\mathcal{F}_r$.

Concerning differentiability properties, let us recall that $\eta = \psi + \varphi$ is defined by the relation

$$B(\boldsymbol{\xi}', \boldsymbol{\Lambda}, \eta) \equiv \eta + M_\varepsilon(N_\varepsilon(\psi + \varphi)) = 0.$$

We see that

$$D_\eta B(\boldsymbol{\xi}', \boldsymbol{\Lambda}, \eta)[\theta] = \theta + M_\varepsilon(\theta D_\eta N_\varepsilon(\psi + \varphi)) \equiv \theta + \tilde{M}(\theta)$$

and check that

$$\|\tilde{M}(\theta)\|_* \leqslant C\varepsilon\|\theta\|_*.$$

This implies that, for ε small, the linear operator $D_\eta B(\boldsymbol{\xi}', \boldsymbol{\Lambda}, \eta)$ is invertible in the space of the continuous functions in Ω_ε with bounded $\|\cdot\|_*$-norm, with a uniformly bounded inverse depending continuously on its parameters.

Now, let us consider the differentiability with respect to the $\boldsymbol{\xi}'$ variable; for simplicity we write

$$\frac{\partial}{\partial \xi'_{ij}} = \partial_{\xi'_{ij}}.$$

Then

$$\partial_{\xi'_{ij}} B(\boldsymbol{\xi}', \boldsymbol{\Lambda}, \eta) = \partial_{\xi'_{ij}} M_\varepsilon(N_\varepsilon(\psi + \varphi)) + M_\varepsilon(\partial_{\xi'_{ij}} N_\varepsilon(\psi + \varphi)) + M_\varepsilon(D_\eta N_\varepsilon(\psi + \varphi)\partial_{\xi'_{ij}}\psi).$$

It is clear that all expressions which define to $\partial_{\xi'_{ij}} B(\boldsymbol{\xi}', \boldsymbol{\Lambda}, \eta)$ depend continuously on their parameters. Applying the implicit function theorem, we find that $\varphi(\boldsymbol{\xi}', \boldsymbol{\Lambda})$ is a C^1-function in L_*^∞. Additionally, we obtain

$$\partial_{\xi'_{ij}}\varphi = -(D_\eta B(\boldsymbol{\xi}', \boldsymbol{\Lambda}, \eta))^{-1}(\partial_{\xi'_{ij}} B(\boldsymbol{\xi}', \boldsymbol{\Lambda}, \eta))$$

and, using the first part of this proposition, the estimates in the previous lemmas, proposition 3.1 and the fact that $(\boldsymbol{\xi}, \boldsymbol{\Lambda}) \in \mathcal{M}_\delta^\varepsilon$, we conclude that

$$\|\partial_{\xi'_{ij}}\varphi\|_* \leqslant C(\|N_\varepsilon(\psi+\varphi)\|_{**} + \|\partial_{\xi'_{ij}} N_\varepsilon(\psi+\varphi)\|_{**} + \|D_\eta N_\varepsilon(\psi+\varphi)\partial_{\xi'_{ij}}\psi\|_{**}) \leqslant C\varepsilon^2.$$

Similarly, we can analyse the differentiability of B with respect to $\boldsymbol{\Lambda}$. This finishes the proof.

4. The reduced functional

Now we are ready to solve the full problem. Let us consider $(\boldsymbol{\xi}', \boldsymbol{\Lambda}) \in \mathcal{M}_\delta^\varepsilon$ with $\mathcal{M}_\delta^\varepsilon$ defined by (3.1). All the estimates obtained below will be uniform on these points. Let $\varphi = \varphi(\boldsymbol{\xi}', \boldsymbol{\Lambda})$ be the unique solution, given by proposition 3.3, of problem (3.8) with $\tilde{\eta} = \psi + \varphi - \phi$, where φ solves (3.9) and ϕ solves (3.5). Note that if $\boldsymbol{\xi} = \varepsilon^{2/(N-2)}\boldsymbol{\xi}' \in \Omega^2$ and $\boldsymbol{\lambda} = (\lambda_1, \lambda_2) \in \mathbb{R}_+^2$ so that $c_{ij} = 0$ for all i, j, then a solution of (1.1) is

$$u(x) = -\varepsilon^{-1} v(\varepsilon^{-2/(N-2)}x), \quad x \in \Omega,$$

where $v = V + \psi + \varphi(\boldsymbol{\xi}', \boldsymbol{\Lambda}) - \phi$. Hence, u will be a critical point of

$$I_\varepsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{p+1}\int_\Omega |u|^{p+1} - \varepsilon \int_\Omega fu,$$

while v will be one of J_ε given by (2.5). Then it is convenient to consider the following functions defined in Ω:

$$\hat{U}_i(x) = \varepsilon^{-1}U_i(\varepsilon^{-2/(N-2)}x) = U_{\lambda_i^\varepsilon, \xi_i}(x), \quad \hat{\psi}(x) = \varepsilon^{-1}\psi(\varepsilon^{-2/(N-2)}x),$$
$$\hat{\varphi}(\boldsymbol{\xi}, \boldsymbol{\Lambda})(x) = \varepsilon^{-1}\varphi(\boldsymbol{\xi}', \boldsymbol{\Lambda})(\varepsilon^{-2/(N-2)}x) \qquad \hat{\phi}(x) = \varepsilon^{-1}\phi(\varepsilon^{-2/(N-2)}x).$$

Note that $\hat{U}_i = U_{\lambda_{i\varepsilon}, \xi_i}$, where $\lambda_{i\varepsilon} = (c_N \Lambda_i^2 \varepsilon)^{2/(N-2)} \in \mathbb{R}_+$ and $\boldsymbol{\xi} = \varepsilon^{2/(N-2)}\boldsymbol{\xi}'$, with $(\boldsymbol{\xi}, \boldsymbol{\Lambda}) \in \mathcal{M}_\delta$ defined by (2.7). Now, let us set $\hat{U} = \hat{U}_1 + \hat{U}_2$. Consider now the functional

$$\mathcal{I}(\boldsymbol{\xi}, \boldsymbol{\Lambda}) \equiv I_\varepsilon(\hat{U} + \hat{\psi} + \hat{\varphi}(\boldsymbol{\xi}, \boldsymbol{\Lambda}) - \hat{\phi}). \tag{4.1}$$

It is easy to check that

$$\mathcal{I}(\boldsymbol{\xi}, \boldsymbol{\lambda}) = J_\varepsilon(V + \psi + \varphi(\boldsymbol{\xi}', \boldsymbol{\Lambda}) - \phi).$$

Then, setting $\tilde{\eta} = \psi + \varphi(\boldsymbol{\xi}', \boldsymbol{\Lambda}) - \phi$, one shows that $DJ_\varepsilon(V + \tilde{\eta})[\vartheta] = 0$ for all $\vartheta \in H_\varepsilon$, where $H_\varepsilon = \{\vartheta \in H_0^1(\Omega_\varepsilon) : \langle \vartheta, \ V_i^{p-1} Z_{ij}\rangle = 0 \text{ for all } i, j\}$. Also one has

$$\frac{\partial V}{\partial \xi'_{lk}} = Z_{lk} + o(1) \quad \text{for all } l, k, \qquad \frac{\partial V}{\partial \Lambda_{l(N+1)}} = Z_{l(N+1)} + o(1) \quad \text{for all } l,$$

with $o(1) \to 0$ in the $\|\cdot\|_*$-norm as $\varepsilon \to 0$. Then from proposition 3.3 we obtain the following basic result.

Lemma 4.1. *The function $u = \hat{U} + \hat{\psi} + \hat{\varphi}(\boldsymbol{\xi}, \boldsymbol{\Lambda}) - \hat{\phi}$ is a solution of problem (1.1) if only if $(\boldsymbol{\xi}, \boldsymbol{\Lambda})$ is a critical point of $\mathcal{I}$.*

Next step is then to give an asymptotic estimate for $\mathcal{I}(\boldsymbol{\xi}, \boldsymbol{\Lambda})$. Set

$$\sigma_f = \int_\Omega f(x) w(x)\, \mathrm{d}x, \tag{4.2}$$

where w is the solution of (2.6). We then have the following proposition.

PROPOSITION 4.2. *The following expansion holds:*

$$\mathcal{I}(\boldsymbol{\xi},\boldsymbol{\Lambda}) = 2C_N + \varepsilon^2\{\Phi(\boldsymbol{\xi},\boldsymbol{\Lambda}) + \sigma_f\} + o(\varepsilon^2)\theta(\boldsymbol{\xi},\boldsymbol{\Lambda}) \tag{4.3}$$

uniformly in the C^1-sense with respect to $(\boldsymbol{\xi},\boldsymbol{\Lambda}) \in \mathcal{M}_\delta$, where θ is a bounded uniformly function independently of $\varepsilon > 0$. Here C_N is the constant given by (2.8) and Φ is the function given by (2.9).

Proof. The first step to achieve our goal is to prove that

$$\mathcal{I}(\boldsymbol{\xi},\boldsymbol{\Lambda}) - I_\varepsilon(\hat{V} + \hat{\psi} - \hat{\phi}) = o(\varepsilon^2) \tag{4.4}$$

and

$$\nabla_{(\boldsymbol{\xi},\boldsymbol{\Lambda})}(\mathcal{I}(\boldsymbol{\xi},\boldsymbol{\Lambda}) - I_\varepsilon(\hat{V} + \hat{\psi} - \hat{\phi})) = o(\varepsilon^2). \tag{4.5}$$

Let us set $\vartheta = V + \psi - \phi$ and note that

$$\begin{aligned}\mathcal{I}(\boldsymbol{\xi},\boldsymbol{\Lambda}) - I_\varepsilon(\hat{V} + \hat{\psi} - \hat{\phi}) = &-\int_0^1 t\biggl(\int_{\Omega_\varepsilon} N_\varepsilon(\psi+\varphi)\varphi\biggr)\,\mathrm{d}t \\ &+ \int_0^1 t\biggl(\int_{\Omega_\varepsilon} p(|V|^{p-1} - |\vartheta + t\varphi|^{p-1})\varphi^2\biggr)\,\mathrm{d}t.\end{aligned}$$

Now, differentiating with respect to the $\boldsymbol{\xi}$ variable, we obtain

$$\begin{aligned}D_{\boldsymbol{\xi}}(\mathcal{I}(\boldsymbol{\xi},\boldsymbol{\Lambda}) - I_\varepsilon(\hat{\vartheta})) = &-\varepsilon^{-2/(N-2)}\int_0^1 t\int_{\Omega_\varepsilon} p\nabla_{\boldsymbol{\xi}'}[|\vartheta + t\varphi|^{p-1}\varphi^2 - |V|^{p-1}\varphi^2]\,\mathrm{d}t \\ &- \varepsilon^{-2/(N-2)}\int_{\Omega_\varepsilon}\nabla_{\boldsymbol{\xi}'}(N_\varepsilon(\psi+\varphi)\varphi).\end{aligned}$$

Bearing in mind that $\|N_\varepsilon(\psi+\varphi)\|_* + \|\varphi\|_* + \|\psi\|_* + \|\nabla_{\xi_i'}\varphi\|_* + \|\nabla_{\xi_1'}\psi\|_* \leqslant O(\varepsilon^2)$, we see that (4.4) and (4.5) hold.

A second step is to prove that

$$I_\varepsilon(\hat{V} + \hat{\psi} - \hat{\phi}) - I_\varepsilon(\hat{V} - \hat{\phi}) = o(\varepsilon^2) \tag{4.6}$$

and

$$\nabla_{(\boldsymbol{\xi},\boldsymbol{\Lambda})}(I_\varepsilon(\hat{V} + \hat{\psi} - \hat{\phi}) - I_\varepsilon(\hat{V} - \hat{\phi})) = o(\varepsilon^2). \tag{4.7}$$

Put $\eta = V - \phi$ and, by the fundamental calculus theorem, note that

$$\begin{aligned}I_\varepsilon(\hat{\eta} + \hat{\psi}) - I_\varepsilon(\hat{\eta}) = &\int_0^1 (1-t)\biggl(\int_{\Omega_\varepsilon} p|\eta + t\psi|^{p-1}\psi^2 - \int_{\Omega_\varepsilon}|\nabla\psi|^2\biggr)\,\mathrm{d}t \\ &+ \int_{\Omega_\varepsilon}(|V|^p - |\eta|^p - p|V|^{p-1}\phi)\psi + \int_{\Omega_\varepsilon} R_\varepsilon\psi.\end{aligned} \tag{4.8}$$

Now, differentiating with respect to $\boldsymbol{\xi}$ variables, we obtain

$$
\begin{aligned}
D_{\boldsymbol{\xi}}(I_\varepsilon(\hat{\eta}+\hat{\psi})-I_\varepsilon(\hat{\eta})) = {} & \varepsilon^{-2/(N-2)}\int_0^1(1-t)\int_{\Omega_\varepsilon}\nabla_{\boldsymbol{\xi}'}(p|\eta+t\psi|^{p-1}\psi^2-|\nabla\psi|^2)\,\mathrm{d}t \\
& + \varepsilon^{-2/(N-2)}\int_{\Omega_\varepsilon}\nabla_{\boldsymbol{\xi}'}(|V|^p-|\eta|^p-p|V|^{p-1}\phi)\psi \\
& + \varepsilon^{-2/(N-2)}\int_{\Omega_\varepsilon}(|V|^p-|\eta|^p-p|V|^{p-1}\phi)\nabla_{\boldsymbol{\xi}'}\psi \\
& + \varepsilon^{-2/(N-2)}\int_{\Omega_\varepsilon}\nabla_{\boldsymbol{\xi}'}R_\varepsilon\psi+\varepsilon^{-2/(N-2)}\int_{\Omega_\varepsilon}R_\varepsilon\nabla_{\boldsymbol{\xi}'}\psi.
\end{aligned}
$$

Since $\|R_\varepsilon\|_{**}+\|\nabla_{\xi_i'}R_\varepsilon\|_{**}+\|\phi\|_\infty+\|\psi\|_*+\|\nabla_{\xi_i'}\psi\|_*\leqslant O(\varepsilon^2)$ and $\|\phi\|_*\leqslant O(\varepsilon^p)$ if $3\leqslant N\leqslant 6$, $\|\phi\|_*\leqslant O(\varepsilon^2)$ if $N\geqslant 7$, we obtain the result that (4.6) and (4.7) hold.

Finally, we need only the following two estimates to hold:

$$
I_\varepsilon(\hat{V}-\hat{\phi})-I_\varepsilon(\hat{V})=\varepsilon^2\sigma_f+o(\varepsilon^2), \tag{4.9}
$$

where σ_f is given by (4.2), and

$$
D_{(\boldsymbol{\xi},\boldsymbol{\Lambda})}(I_\varepsilon(\hat{V}-\hat{\phi})-I_\varepsilon(\hat{V}))=o(\varepsilon^2). \tag{4.10}
$$

Now, we have

$$
\begin{aligned}
I_\varepsilon(\hat{V}-\hat{\phi})-I_\varepsilon(\hat{V}) = {} & \int_0^1\left(\int_{\Omega_\varepsilon}|\nabla\phi|^2-\int_{\Omega_\varepsilon}p|V-t\phi|^{p-1}\phi^2\right)\mathrm{d}t \\
& + \int_{\Omega_\varepsilon}(\bar{U}_1^p+\bar{U}_2^p-|V-t\phi|^p)\phi. \qquad (4.11)
\end{aligned}
$$

Note that

$$
\int_0^1 t\int_{\Omega_\varepsilon}|\nabla\phi|^2\,\mathrm{d}t=\int_{\Omega_\varepsilon}|\nabla\phi|^2=\varepsilon^{p+1}\int_{\Omega_\varepsilon}\tilde{f}\phi=\varepsilon^2\int_\Omega fw=\varepsilon^2\sigma_f,
$$

and since $\|\phi\|_\infty\leqslant O(\varepsilon^{p+1})$, we have that

$$
\left|\int_{\Omega_\varepsilon}p|V-t\phi|^{p-1}\phi^2\right|\leqslant C\varepsilon^4\int_{\Omega_\varepsilon}(\omega_1+\omega_2)^{p-1}\leqslant o(\varepsilon^2).
$$

On the other hand, it is not difficult to check that

$$
\begin{aligned}
\left|\int_{\Omega_\varepsilon}\left(\sum_{i=1}^2\bar{U}_i^p-|V-t\phi|^p\right)\phi\right| &= \left|\int_{\Omega_\varepsilon}R_\varepsilon\phi+\int_{\Omega_\varepsilon}(|V|^p-|V-t\phi|^p-p|V|^{p-1}\phi)\phi\right| \\
&\leqslant o(\varepsilon^2).
\end{aligned}
$$

The above estimates yield (4.9). Now, from (4.11) we obtain

$$
\begin{aligned}
D_{\boldsymbol{\xi}}(I_\varepsilon(\hat{V}-\hat{\phi})-I_\varepsilon(\hat{V})) = {} & \varepsilon^{-2/(N-2)}\int_0^1 t\int_{\Omega_\varepsilon}p|V-t\phi|^{p-2}\nabla_{\boldsymbol{\xi}'}V\phi^2\,\mathrm{d}t \\
& + \varepsilon^{-2/(N-2)}\int_{\Omega_\varepsilon}\nabla_{\boldsymbol{\xi}'}(\bar{U}_1^p+\bar{U}_2^p-|V-t\phi|^p)\phi, \qquad (4.12)
\end{aligned}
$$

but since $\|\phi\|_\infty \leqslant O(\varepsilon^{p+1})$, it is easy to check that (4.10) holds. Similarly, our results hold for differentiability with respect to $\boldsymbol{\Lambda}$.

REMARK 4.3. Lemma 2.1 and the previous proposition yield

$$\nabla_{(\boldsymbol{\xi},\boldsymbol{\Lambda})}\mathcal{I}(\boldsymbol{\xi},\boldsymbol{\Lambda}) = \varepsilon^2\nabla_{(\boldsymbol{\xi},\boldsymbol{\Lambda})}\Phi(\boldsymbol{\xi},\boldsymbol{\Lambda}) + o(\varepsilon^2)\nabla_{(\boldsymbol{\xi},\boldsymbol{\Lambda})}\theta(\boldsymbol{\xi},\boldsymbol{\Lambda}), \tag{4.13}$$

uniformly with respect to $(\boldsymbol{\xi},\boldsymbol{\Lambda}) \in \mathcal{M}_\delta$, where θ and $\nabla_{(\boldsymbol{\xi},\boldsymbol{\Lambda})}\theta$ are bounded uniformly functions, independently of all $\varepsilon > 0$ small.

5. An auxiliary function on the exterior domain

In this section we consider the domain Ω defined in (1.2) with $P = 0$, $\mu > 0$ small and fixed and we assume that $f \in C^{0,\gamma}(\bar{\Omega})$, for some $0 < \gamma < 1$, with $\min_{x\in\Omega} f(x) = \alpha > 0$. Let w be the unique solution in $C^{2,\gamma}(\bar{\Omega})$ of problem (2.6). It is then easy to check that $w_\mu(x) = \mu^{-2}w(\mu x)$ is the unique $C^{2,\gamma}(\overline{\mu^{-1}\Omega})$ solution of the problem

$$\begin{aligned} -\Delta w_\mu &= \hat{f} \quad \text{in } \mu^{-1}\Omega, \\ w_\mu &= 0 \quad \text{on } \partial(\mu^{-1}\Omega), \end{aligned}$$

where $\hat{f}(x) = f(\mu x)$ for $x \in (\mu^{-1}\Omega)$.

Now, we consider the exterior domain

$$E = \mathbb{R}^N \setminus \overline{B(0,1)}$$

and we denote by G_E and H_E, respectively, Green's function on E and its regular part. By convenience, in the set

$$\boldsymbol{V} = \{(x,y) \in \mathbb{R}^N \times \mathbb{R}^N : G_E(x,y) - H_E^{1/2}(x,x)H_E^{1/2}(y,y) > 0\} \cap (\mu^{-1}\Omega)$$

we define the function

$$\Phi_E(x,y) = \frac{1}{2}\left\{\frac{H_E(x,x)w_\mu^2(y) + 2G_E(x,y)w_\mu(x)w_\mu(y) + H_E(y,y)w_\mu^2(x)}{G_E^2(x,y) - H_E(x,x)H_E(y,y)}\right\}.$$

Then, if x and y are variable vectors whose magnitudes remain constant and we differentiate Φ_E with respect to the angle θ formed between them, we obtain

$$\frac{\partial}{\partial\theta}\Phi_E(x,y) = F(x,y,\theta)\sin\theta$$

for $0 < \theta < \pi$. Since $F(x,y,\theta) > 0$ for all $\theta \in]0,\pi[$, $(x,y) \in \boldsymbol{V}$, we have that for given magnitudes $|x|$ and $|y|$, Φ_E maximizes its value when $\theta = \pi$, is to say when x and y have opposite directions. In the rest of this section we assume that this is the situation.

5.1. A first step to the auxiliary function: a radial case

In this subsection we consider a fixed constant $T > 0$ and the domain

$$\Omega := \mathcal{A}_\mu = \{x \in \mathbb{R}^N : 1 < |x| < \mu^{-1}\} \quad \text{and} \quad f \equiv 1.$$

We write $R := R(\mu, T) = \mu^{-1}T$ so that $w_\mu \in C^{2,\gamma}(\bar{\mathcal{A}}_\mu)$ is defined by

$$w_\mu(x) := W_R(x) = \frac{1}{2N}\left\{\frac{R^2-1}{R^{2-N}-1}|x|^{2-N} - |x|^2 + R^{2-N}\frac{1-R^N}{R^{2-N}-1}\right\}.$$

From the maximum principle we have that W_R is strictly positive in $\mathcal{A}_\mu$. Additionally, it achieves its maximum value in

$$x_\mu^* \in \mathbb{R}^N \quad \text{such that } |x_\mu^*| = R_\mu^* = \left(\frac{(N-2)R^{N-2}(R^2-1)}{2(R^{N-2}-1)}\right)^{1/N}. \tag{5.1}$$

Note that $R_\mu^* \to +\infty$ as $\mu \to 0$. Now we consider an unitary vector $\boldsymbol{e}$ and we set $x = s\boldsymbol{e}$, $y = -t\boldsymbol{e}$ with $s, t > 1$. Then

$$\begin{aligned}
&2\beta_N \Phi_E(x,y)\\
&\quad := 2\beta_N \Phi_R(x,y)\\
&\quad = 2\beta_N \tilde{\Phi}_R(s,t)\\
&\quad = \left(\frac{\tilde{W}_R^2(t)}{(s^2-1)^{N-2}} + 2\left\{\frac{1}{(s+t)^{N-2}} - \frac{1}{(st+1)^{N-2}}\right\}\tilde{W}_R^2(s)\tilde{W}_R^2(t) + \frac{\tilde{W}_R^2(s)}{(t^2-1)^{N-2}}\right)\\
&\qquad \times \left(\left(\frac{1}{(s+t)^{N-2}} - \frac{1}{(st+1)^{N-2}}\right)^2 - \frac{1}{[(s^2-1)(t^2-1)]^{N-2}}\right)^{-1},
\end{aligned}$$

where $\tilde{W}_R(r) = W_R(r\boldsymbol{e})$, for $1 < r < R$.

Remark 5.1. We define in $]1,+\infty[\,\times\,]1,+\infty[$ the following function:

$$\tilde{\Psi}(s,t) = \frac{1}{(s+t)^{N-2}} - \frac{1}{(st+1)^{N-2}} - \frac{1}{[(s^2-1)(t^2-1)]^{(N-2)/2}}. \tag{5.2}$$

From (5.1), it is easy to check that we can choose μ_0 small enough such that for all $0 < \mu < \mu_0$ there exist $1 < k_* < K < R_{\mu_0}^*$ independent of μ, verifying $\tilde{\Psi}(k_*, k_*) = 0$, $\tilde{\Psi}(K,K) = \max_{(x,y)\in E} \tilde{\Psi}(|x|,|y|)$. Moreover, k_* is the unique solution in $]1,+\infty[$ of the equation

$$\frac{2^{2-N}}{s^{N-2}} = \frac{(s^2+1)^{N-2} + (s^2-1)^{N-2}}{(s^4-1)^{N-2}}$$

and K is the unique solution in $]1,+\infty[$ of

$$\frac{2^{1-N}}{s^N} = \frac{(s^2+1)^{N-1} + (s^2-1)^{N-1}}{(s^4-1)^{N-1}}.$$

Now, it is not difficult to prove the following lemma.

Lemma 5.2. *The function $\tilde{\Phi}_R$ achieves only one minimum value at a critical point of the form $(\rho_R, \rho_R) \in\,]k_*, K[$.*

5.2. General case

Let Ω be the domain defined in (1.2), with $P = 0$. In this subsection we consider the values m, M as follows: m is the radius of the biggest ball centred at the origin contained in $\mathcal{D}$ and M is the radius of the smallest ball centred at the origin containing to $\mathcal{D}$. Let w be the unique solution $C^{2,\gamma}(\bar{\Omega})$ of problem (2.6). By the maximum principle, we check that

$$z_m(x) \leqslant w(x) \leqslant z_M(x) \quad \text{for all } \mu < |x| < m,$$

where $z_m(x) = \alpha\mu^2 W_{R_1}(\mu^{-1}x)$ and $z_M(x) = \beta\mu^2 W_{R_2}(\mu^{-1}x)$, with $R_1 = \mu^{-1}m$ and $R_2 = \mu^{-1}M$. Hence,

$$\Phi_{R_1}(\mu^{-1}x, \mu^{-1}y) \leqslant \Phi_E(\mu^{-1}x, \mu^{-1}y) \leqslant \Phi_{R_2}(\mu^{-1}x, \mu^{-1}y) \quad \text{for all } \mu < |x|, |y| < m.$$

Since the function $\tilde{\Psi}(s,s)$ defined in (5.2) is decreasing in its diagonal for values of s greater that K and goes to 0, then is not difficult to show that the system

$$\frac{\tilde{\Phi}_{R_1}(s,s)}{\tilde{\Phi}_{R_2}(K,K)} \geqslant 1, \quad s \geqslant K,$$

possesses a solution, we say k^*, when we have chosen $\mu > 0$ sufficiently small but fixed. Indeed, if we set $\beta = \max_{x\in\Omega} f(x)$ and $(\alpha m^2 - \beta M^2)K^{N-2} + \beta M^2 \neq 0$, then, in the limit for μ, we can choose

$$k^* = \max\left\{K, \left\{\left(\frac{\alpha m^2 K^{N-2}}{(\alpha m^2 - \beta M^2)K^{N-2} + \beta M^2}\right)_+\right\}^{1/(N-2)}\right\}.$$

If $(\alpha m^2 - \beta M^2)K^{N-2} + \beta M^2 = 0$, we change K by a value a few greater than K in the definition of k^*. Then the following lemma is obtained.

Lemma 5.3. *The function $\Phi_E(x,y)$ achieves a relative minimum value in a critical point (x_μ, y_μ) with x_μ and y_μ having opposite directions, and $(|x_\mu|, |y_\mu|) \in]k_*, k^*[$. Moreover, $|x_\mu|$ and $|y_\mu|$ belong to a compact region fully contained in $]k_*, k^*[$, which is independent of all sufficiently small $\mu > 0$.*

Let

$$\mathbb{Q} = \{(x,y) \in \boldsymbol{V} \times \boldsymbol{V} : k_* < |x|, |y| < k^*\}.$$

We then define the following value:

$$c_\mu = \Phi_E(x_\mu, y_\mu) = \min_{(x,y)\in\mathbb{Q}} \Phi_E(x,y). \tag{5.3}$$

Let $\delta_\mu > 0$ be a suitable small value such that the level set

$$\{(x,y) \in \mathbb{Q} : \Phi_E(x,y) = \delta_\mu\}$$

is a closed curve and that $\nabla\Phi_E(x,y)$ does not vanish on it. Let us set

$$\Upsilon_\mu = \{(x,y) \in \mathbb{Q} : \Phi_E(x,y) < \delta_\mu\}. \tag{5.4}$$

Thus, on this region we have that $\Phi_E(x,y) < \delta_\mu$ and if $(x,y) \in \partial\Upsilon_\mu$, then one of the following situations happens: either there is a tangential direction τ to $\partial\Upsilon_\mu$ such that $\nabla\Phi_E(x,y)\cdot\tau \neq 0$, or x and y lie in opposite directions, where $\Phi_E(x,y) = \delta_\mu$ and $\nabla\Phi_E(x,y) \neq 0$ points orthogonally outwards to Υ_μ. Moreover, for fixed sufficiently small $\mu_0 > 0$,

$$\Upsilon_{\hat{\mu}} \subset\subset \Upsilon_\mu \subset\subset \mathbb{Q} \quad \text{for all } 0 < \hat{\mu} < \mu < \mu_0. \tag{5.5}$$

Let us now consider the exterior domain

$$E_\mu = \mathbb{R}^N \setminus \overline{B(0,\mu)},$$

and we denote by G_μ and H_μ, respectively, Green's function on E_μ and its regular part. Then

$$G_\mu(x,y) = \mu^{2-N} G_E(\mu^{-1}x, \mu^{-1}y) \quad \text{and} \quad H_\mu(x,y) = \mu^{2-N} H_E(\mu^{-1}x, \mu^{-1}y).$$

In particular, if we set

$$\Sigma_\Omega^\mu = \mu \Upsilon_\mu, \tag{5.6}$$

with Υ_μ defined by (5.4), then Σ_Ω^μ corresponds precisely to the set where

$$\Phi_E(\mu^{-1}x, \mu^{-1}y) < \delta_\mu,$$

with δ_μ defined by (5.4). Moreover, since

$$G(x,y) = G_\mu(x,y) + O(1) \quad \text{for all } (x,y) \in \mu\mathbb{Q},$$

where the quantity $O(1)$ is bounded independently of all small μ, in the C^1-sense, and the same is true for the function H, we have that, in the region $\mu\mathbb{Q}$, the function

$$\Phi_\Omega(x,y) = \frac{1}{2}\left\{\frac{H(x,x)w^2(y) + 2G(x,y)w(x)w(y) + H(y,y)w^2(x)}{G^2(x,y) - H(x,x)H(y,y)}\right\} \tag{5.7}$$

satisfies the following relation:

$$\Phi_\Omega(x,y) = \mu^{N+2}\Phi_E(\mu^{-1}x, \mu^{-1}y) + o(1), \tag{5.8}$$

where the quantity $o(1)$ is bounded independently of all small numbers $\mu > 0$ in the C^1-sense. Additionally, $o(1) \to 0$ as $\mu \to 0$.

6. The min–max scheme and proof of the main result

In this section $\mu > 0$ is a fixed sufficiently small number and Ω is the domain given in (1.2) with $P = 0$. According to the results (4.1) and (4.13), obtained above, our problem reduces to that of finding a critical point for

$$\Phi(\boldsymbol{\xi}, \boldsymbol{\Lambda}) = \frac{1}{2}\left\{\sum_{i=1}^{2} \Lambda_i^2 H(\xi_i, \xi_i) - 2\Lambda_1\Lambda_2 G(\xi_1, \xi_2)\right\} + \sum_{i=1}^{2} \Lambda_i w(\xi_i), \tag{6.1}$$

where $\boldsymbol{\xi} = (\xi_1, \xi_2) \in \Omega^2$ and $\boldsymbol{\Lambda} = (\Lambda_1, \Lambda_2) \in \mathbb{R}_+^2$. Here we consider the function Φ defined over the class $\Sigma_\Omega^\mu \times \mathbb{R}_+^2$, where Σ_Ω^μ is defined by (5.6). Indeed Φ has some singularities on this class which we can avoid by replacing the term $G(\xi_1, \xi_2)$ in (6.1) by

$$G|_M(\xi_1, \xi_2) = \begin{cases} G(\xi_1, \xi_2) & \text{if } G(\xi_1, \xi_2) \leqslant M, \\ M & \text{if } G(\xi_1, \xi_2) > M, \end{cases} \tag{6.2}$$

where M is a big number. Hence, we can work with the modified functional, which, for simplicity, we still denote by Φ.

For every $\boldsymbol{\xi} \in \Sigma_\Omega^\mu$ we choose $d(\boldsymbol{\xi}) = (d_1(\boldsymbol{\xi}), d_2(\boldsymbol{\xi})) \in \mathbb{R}^2$, which is a vector defining the negative direction of the associated quadratic form with Φ. Such a direction exists since $G^2(x,y) - H(x,x)H(y,y) > 0$ over Σ_Ω^μ. More precisely, for fixed $\boldsymbol{\xi}_0 \in \Sigma_\Omega^\mu$, the function

$$\Phi(\boldsymbol{\xi}_0, \boldsymbol{d}) = \frac{1}{2}\left\{ \sum_{i=1}^{2} d_i^2 H(\xi_{0,i}, \xi_{0,i}) - 2d_1 d_2 G(\xi_{0,1}, \xi_{0,2}) \right\} + \sum_{i=1}^{2} d_i w(\xi_{0,i}),$$

regarded as a function of $\boldsymbol{d} = (d_1, d_2)$ only, with $d_1, d_2 > 0$, has a unique critical point $\bar{\boldsymbol{d}}(\boldsymbol{\xi}_0) = (\bar{d}_1(\boldsymbol{\xi}_0), \bar{d}_2(\boldsymbol{\xi}_0))$ given by

$$\bar{d}_i(\boldsymbol{\xi}_0) = \frac{H(\xi_{0,j}, \xi_{0,j}) w(\xi_{0,i}) + G(\xi_{0,i}, \xi_{0,j}) w(\xi_{0,j})}{G^2(\xi_{0,i}, \xi_{0,j}) - H(\xi_{0,i}, \xi_{0,i}) H(\xi_{0,j}, \xi_{0,j})}, \quad i, j = 1, 2, \ i \neq j.$$

In particular,

$$\Phi(\boldsymbol{\xi}_0, \bar{\boldsymbol{d}}(\boldsymbol{\xi}_0)) = \Phi_\Omega(\boldsymbol{\xi}_0), \tag{6.3}$$

where Φ_Ω is the function given by (5.7). Then we simply choose $d(\boldsymbol{\xi}) = \bar{\boldsymbol{d}}(\boldsymbol{\xi})$. Let x_μ and y_μ the points given by (5.3). From now on we consider $\hat{\rho}_\mu = |x_\mu|$ and $\bar{\rho}_\mu = |y_\mu|$. Set

$$\mathbb{S} = \{(x,y) \in \mathbb{Q}^2 : (|x|, |y|) = (\mu\hat{\rho}_\mu, \mu\bar{\rho}_\mu)\}.$$

Let $\mathcal{K}$ be the class of all continuous functions

$$\kappa : \mathbb{S} \times I_0 \times [0,1] \to \Sigma_\Omega^\mu \times \mathbb{R}_+^2$$

such that

(i) $\kappa(\boldsymbol{\xi}, \sigma_0, t) = (\boldsymbol{\xi}, \sigma_0 d(\boldsymbol{\xi}))$ and $\kappa(\boldsymbol{\xi}, \sigma_0^{-1}, t) = (\boldsymbol{\xi}, \sigma_0^{-1} d(\boldsymbol{\xi}))$ for all $\boldsymbol{\xi} \in \mathbb{S}$, $t \in [0,1]$.

(ii) $\kappa(\boldsymbol{\xi}, \sigma, 0) = (\boldsymbol{\xi}, \sigma d(\boldsymbol{\xi}))$ for all $(\boldsymbol{\xi}, \sigma) \in \mathbb{S} \times I_0$, where $I_0 = [\sigma_0, \sigma_0^{-1}]$ and σ_0 is a small number to be chosen later.

Then we define the min–max value as

$$c(\Omega) = \inf_{\kappa \in \mathcal{K}} \sup_{(\boldsymbol{\xi}, \sigma) \in \mathbb{S} \times I_0} \Phi(\kappa(\boldsymbol{\xi}, \sigma, 1)). \tag{6.4}$$

In what follows we prove that $c(\Omega)$ is a critical value of Φ.

Lemma 6.1. *For all sufficiently small $\mu > 0$, the following estimate holds:*

$$c(\Omega) \leqslant \mu^{N+2} c_\mu + o(1),$$

where $o(1) \to 0$ as $\mu \to 0$, and c_μ is the value defined in (5.3).

Proof. For all $t \in [0,1]$, we consider the test path defined as $\kappa(\boldsymbol{\xi}, \sigma, t) = (\boldsymbol{\xi}, \sigma d(\boldsymbol{\xi}))$. Maximizing $\Phi(\boldsymbol{\xi}, \sigma d(\boldsymbol{\xi}))$ in the variable σ, we note that this maximum value is attained at $\sigma = 1$, because of our choice of the vector $d(\boldsymbol{\xi})$. Hence, from (6.3), we have

$$\max_{\sigma \in I_0} \Phi(\boldsymbol{\xi}, \sigma d(\boldsymbol{\xi})) = \Phi(\boldsymbol{\xi}, d(\boldsymbol{\xi})).$$

On the other hand, by the definition of $\mathbb{S}$, we see that

$$\Phi_E(\mu^{-1}\xi_1, \mu^{-1}\xi_2) = c_\mu.$$

Then the conclusion is immediate from (5.8) and the definition of $c(\Omega)$.

In order to prove that $c(\Omega)$ is indeed a critical point of Φ we need an intersection lemma. The idea behind this result is the topological continuation of the set of solutions of an equation (see [15]). For every $(\boldsymbol{\xi}, \sigma, t) \in \mathbb{S} \times I_0 \times [0,1]$ we define

$$\kappa(\boldsymbol{\xi}, \sigma, t) = (\tilde{\xi}(\xi, \sigma, t), \tilde{\Lambda}(\boldsymbol{\xi}, \sigma, t)) \in \Sigma_{\Omega}^{\mu} \times \mathbb{R}_{+}^{2},$$

with $\tilde{\xi} = (\tilde{\xi}_1, \tilde{\xi}_2)$, $\tilde{\Lambda} = (\tilde{\Lambda}_1, \tilde{\Lambda}_2)$, and we define the set

$$\mathbb{M} = \{(\boldsymbol{\xi}, \sigma) \in \mathbb{S} \times I_0 : \tilde{\Lambda}_1(\boldsymbol{\xi}, \sigma, 1) \cdot \tilde{\Lambda}_2(\boldsymbol{\xi}, \sigma, 1) = 1\}.$$

The following lemma has been proved by Del Pino *et al.* in [12, lemma 6.2]. Therefore, we omit the proof here.

Lemma 6.2. *For every open neighbourhood W of $\mathbb{M}$ in $\mathbb{S} \times I_0$, the projection $g : W \to \mathbb{S}$ induces a monomorphism in cohomology, that is*

$$g^* : H^*(\mathbb{S}) \to H^*(W)$$

is injective.

Proposition 6.3. *There exists a constant $A > 0$ such that*

$$\sup_{(\boldsymbol{\xi}, \sigma) \in \mathbb{S} \times I_0} \Phi(\kappa(\boldsymbol{\xi}, \sigma, 1)) \geqslant -A \quad \textit{for all } \kappa \in \mathcal{K}.$$

Proof. Note that $\boldsymbol{\xi} \in \Sigma_{\Omega}^{\mu}$ implies that $\xi_i \in B(0, \mu k^*) \setminus B(0, \mu k_*)$, for $i = 1, 2$, with $\hat{\rho}_\mu, \bar{\rho}_\mu \in]k_*, k^*[$ for any μ sufficiently small. Thus, we can find a number $\delta_0 > 0$ such that if $|\xi_1 - \xi_2| < \delta_0$, then $\xi_1 \cdot \xi_2 > 0$. Let $A_0 > 0$ be such that $G(x, y) \geqslant A_0$ implies $|x - y| < \delta_0$.

We argue by contradiction. Let us assume that, for certain $\kappa \in \mathcal{K}$, we have

$$\Phi(\kappa(\boldsymbol{\xi}, \sigma, 1)) \leqslant -A_0 \quad \text{for all } (\boldsymbol{\xi}, \sigma) \in \mathbb{S} \times I_0.$$

This implies that, for all $(\boldsymbol{\xi}, \sigma) \in \mathbb{M}$, $(\tilde{\xi}, \tilde{\sigma}) = (\tilde{\xi}(\boldsymbol{\xi}, \sigma, 1), \tilde{\Lambda}(\boldsymbol{\xi}, \sigma, 1))$, we have

$$2G(\tilde{\xi}_1, \tilde{\xi}_2) - (\tilde{\Lambda}_1^2 H(\tilde{\xi}_1, \tilde{\xi}_1) + 2\tilde{\Lambda}_1 w(\tilde{\xi}_1) + \tilde{\Lambda}_2^2 H(\tilde{\xi}_2, \tilde{\xi}_2) + 2\tilde{\Lambda}_2 w(\tilde{\xi}_2)) \geqslant 2A_0$$

and since $H(\tilde{\xi}_i, \tilde{\xi}_i) > 0$ and $w(\tilde{\xi}_i) > 0$, we conclude that if we take a small neighbourhood W of $\mathbb{M}$ in $\mathbb{S} \times I_0$, then for every $(\boldsymbol{\xi}, \sigma) \in W$ we have

$$G(\tilde{\xi}(\xi, \sigma, 1)) \geqslant A_0.$$

Hence, $|\tilde{\xi}_1 - \tilde{\xi}_2| < \delta_0$. Let us fix points $\zeta_i \in \mathbb{R}^N$, $i = 1, 2$, such that $|\zeta_1| = \hat{\rho}_\mu$ and $|\zeta_2| = \bar{\rho}_\mu$. Then $\boldsymbol{\zeta} = (\zeta_1, \zeta_2) \in \mathbb{S}$. Setting $\kappa_1 = \kappa(\cdot, 1)$, we see that, because of the above conclusion, $\kappa_1(W) \subset (\Sigma_{\Omega}^{\mu} \setminus T(\boldsymbol{\zeta})) \times \mathbb{R}_{+}^{2}$, where $T(\boldsymbol{\zeta}) = \{(t_1\zeta_1, t_2\zeta_2) : t_1, t_2 \in]k, K[\}$.

Consider the map $s : \Sigma_{\Omega}^{\mu} \times \mathbb{R}_{+}^{2} \to \mathbb{S}$ defined componentwise as

$$s(\boldsymbol{\xi}, \boldsymbol{\Lambda}) = \mu\left(\frac{\hat{\rho}_\mu \xi_1}{|\xi_1|}, \frac{\bar{\rho}_\mu \xi_2}{|\xi_2|}\right).$$

Then $\kappa_0^* \circ s^* : H^*(\mathbb{S}) \to H^*(\mathbb{S} \times I_0)$, where $\kappa_0 = \kappa(\cdot, 0)$ is an isomorphism. By the homotopy axiom we then deduce that $\kappa_1^* \circ s^*$ is also an isomorphism. We consider the following commutative diagram:

$$\begin{array}{ccccc} H^*(\mathbb{S}\times I_0) & \xleftarrow{\kappa_1^*} & H^*(\Sigma_\Omega^\mu \times \mathbb{R}_+^2) & \xleftarrow{\kappa^*} & H^*(\mathbb{S}) \\ \downarrow i_1^* & & \downarrow i_2^* & & \downarrow i_3^* \\ H^*(W) & \xleftarrow{\tilde{\kappa}_1^*} & H^*(\kappa_1(W)) & \xleftarrow{\tilde{s}^*} & H^*(\mathbb{S}\setminus\{\boldsymbol{\zeta}\}), \end{array}$$

where i_1, i_2 and i_3 are inclusion maps, $\tilde{\kappa}_1 = \kappa_1|_W$ and $\tilde{s} = s|_{\kappa_1(W)}$. From lemma 6.2 we have that i_1^* is a monomorphism, which is a contradiction of the fact that $H^{2N}(\mathbb{S}\setminus\{\boldsymbol{\zeta}\}) = 0$. Thus, the result follows.

In order to prove that the min–max number (6.4) is a critical value of Φ, we need to take into consideration the fact that the domain in which Φ is defined is not necessarily closed for the gradient flow of Φ. The following lemma is given towards this aim.

LEMMA 6.4. *Assume that* $\mu > 0$ *is a sufficiently small number. Let* $(\xi^n, \Lambda^n) \in \Sigma_\Omega^\mu \times \mathbb{R}_+^2$ *be a sequence such that*

$$\nabla_{\boldsymbol{\Lambda}}\Phi(\boldsymbol{\xi}_n, \boldsymbol{\Lambda}_n) \to 0. \tag{6.5}$$

Then each component of $\boldsymbol{\Lambda}_n$ *is bounded above and below by positive constants.*

Proof. Note that $\bar{\Sigma}_\Omega^\mu \subset\subset \Omega$. Hence, $w(\xi_i) > 0$, $i = 1, 2$, for all $\boldsymbol{\xi} \in \bar{\Sigma}_\Omega^\mu$. We set $\boldsymbol{\xi}_n = (\xi_{1,n}, \xi_{2,n})$ and $\boldsymbol{\Lambda}_n = (\Lambda_{1,n}, \Lambda_{2,n})$. Then (6.5) is equivalent to

$$\Lambda_{i,n} H(\xi_{i,n}, \xi_{i,n}) - \Lambda_{j,n} G(\xi_{i,n}, \xi_{j,n}) + w(\xi_{i,n}) \to 0, \quad i, j = 1, 2, \; i \neq j.$$

It is clear that $|\boldsymbol{\Lambda}_n| \to 0$ or $\Lambda_{i,n} \to 0$ and $\Lambda_{j,n} \to C$, with non-zero C and with $i \neq j$, cannot happen. Hence, we can suppose that $|\boldsymbol{\Lambda}_n| \to +\infty$. Since H and G remain uniformly controlled (μ is fixed), we easily see that $\Lambda_{1,n} \to +\infty$ and $\Lambda_{2,n} \to +\infty$. We set $\tilde{\Lambda}_{i,n} = \Lambda_{i,n}/|\boldsymbol{\Lambda}_n|$, for $i = 1, 2$, and passing to a subsequence, if necessary, we may assume that this sequence it approaches a non-zero vector $(\hat{\Lambda}_1, \hat{\Lambda}_2)$ with $\hat{\Lambda}_i \neq 0$ for $i = 1, 2$. It follows that

$$\tilde{\Lambda}_{i,n} H(\xi_{i,n}, \xi_{i,n}) - \tilde{\Lambda}_{j,n} G(\xi_{1,n}, \xi_{2,n}) + \frac{w(\xi_{i,n})}{|\boldsymbol{\Lambda}_n|} \to 0, \quad i, j = 1, 2, \; i \neq j.$$

For a suitable subsequence, for some $(\bar{\xi}_1, \bar{\xi}_2) \in \bar{\Sigma}_\Omega^\mu$, we obtain the system

$$\frac{\hat{\Lambda}_1}{\hat{\Lambda}_2} = \frac{G(\bar{\xi}_1, \bar{\xi}_2)}{H(\bar{\xi}_1, \bar{\xi}_1)} \quad \text{and} \quad \frac{\hat{\Lambda}_2}{\hat{\Lambda}_1} = \frac{G(\bar{\xi}_1, \bar{\xi}_2)}{H(\bar{\xi}_2, \bar{\xi}_2)}.$$

Hence,

$$G^2(\bar{\xi}_1, \bar{\xi}_2) - H(\bar{\xi}_1, \bar{\xi}_1) H(\bar{\xi}_2, \bar{\xi}_2) = 0,$$

which is a contradiction, since the quantity on the left-hand side in the previous equality is strictly positive when $\mu > 0$ is chosen sufficiently small. This finishes the proof.

Proposition 6.5. *Let us assume that $\mu > 0$ is a sufficiently small number. Then the functional Φ satisfies the (PS) condition in the region $\Sigma_{\Omega}^{\mu} \times \mathbb{R}_{+}^{2}$ at the level $c(\Omega)$ given in (6.4).*

Proof. Let us consider a sequence $(\boldsymbol{\xi}_n, \boldsymbol{\Lambda}_n) \in \Sigma_{\Omega}^{\mu} \times \mathbb{R}_{+}^{2}$ such that

$$\nabla_{\boldsymbol{\Lambda}} \Phi(\boldsymbol{\xi}_n, \boldsymbol{\Lambda}_n) \to 0 \quad \text{and} \quad \nabla_{\boldsymbol{\xi}}^{\tau} \Phi(\boldsymbol{\xi}_n, \boldsymbol{\Lambda}_n) \to 0,$$

where $\nabla_{\boldsymbol{\xi}}^{\tau} \Phi$ corresponds to the tangential gradient of Φ to $\partial \Sigma_{\Omega}^{\mu} \times \mathbb{R}_{+}^{2}$ in the case when $\boldsymbol{\xi}_n$ approaches $\partial \Sigma_{\Omega}^{\mu}$ or the full gradient otherwise. From the previous lemma, the components of $\boldsymbol{\Lambda}_n$ are bounded above and below by positive constants, so that we may assume, passing to a subsequence if necessary, that $(\boldsymbol{\xi}_n, \boldsymbol{\Lambda}_n) \to (\boldsymbol{\xi}_0, \boldsymbol{\Lambda}_0) \in \bar{\Sigma}_{\Omega}^{\mu} \times \mathbb{R}_{+}^{2}$ and $\Phi(\boldsymbol{\xi}_n, \boldsymbol{\Lambda}_n) \to c(\Omega)$. Then

$$\nabla_{\boldsymbol{\Lambda}} \Phi(\boldsymbol{\xi}_0, \boldsymbol{\Lambda}_0) = 0.$$

Observe that if $\boldsymbol{\xi}_0 \in \operatorname{Int}(\Sigma_{\Omega}^{\mu})$, then $\boldsymbol{\xi}_0$ is a critical point of Φ. We assume the opposite, i.e. that $\boldsymbol{\xi}_0 \in \partial \Sigma_{\Omega}^{\mu}$. Then

$$\Phi_E(\mu^{-1}\xi_{0,1}, \mu^{-1}\xi_{0,2}) = \delta_{\mu}.$$

Firstly, we note that $\nabla_{\boldsymbol{\Lambda}} \Phi(\boldsymbol{\xi}_0, \boldsymbol{\Lambda}_0) = 0$. Then $\boldsymbol{\Lambda}_0$ satisfies

$$\Lambda_{0,i} = \frac{H(\xi_{0,j}, \xi_{0,j}) w(\xi_{0,i}) + G(\xi_{0,i}, \xi_{0,j}) w(\xi_{0,j})}{G^2(\xi_{0,i}, \xi_{0,j}) - H(\xi_{0,i}, \xi_{0,i}) H(\xi_{0,j}, \xi_{0,j})}, \quad i, j = 1, 2, \ i \neq j.$$

Substituting these values in Φ, from (6.3) we obtain

$$c(\Omega) = \Phi(\boldsymbol{\xi}_0, \boldsymbol{\Lambda}_0) = \Phi_{\Omega}(\boldsymbol{\xi}_0)$$

and from (5.8) we deduce that

$$c(\Omega) = \mu^{N+2} \Phi_E(\mu^{-1}\xi_{0,1}, \mu^{-1}\xi_{0,2}) + \theta(\boldsymbol{\xi}_0),$$

where $\theta(\boldsymbol{\xi}_0)$ is small in the C^1 sense, as $\mu > 0$ becomes smaller. Hence, $\nabla_{\boldsymbol{\xi}} \Phi(\boldsymbol{\xi}_0, \boldsymbol{\Lambda}_0) \cdot \tau \sim 0$ for any direction τ tangential to $\partial \Sigma_{\Omega}^{\mu}$. Thus, from the analysis in the previous section, we have that $\xi_{0,1}$ and $\xi_{0,2}$ are in opposite directions; $\Phi(\boldsymbol{\xi}_0, \boldsymbol{\Lambda}_0) \sim \mu^{N+2} \delta_{\mu}$ and $\nabla_{\boldsymbol{\xi}} \Phi(\boldsymbol{\xi}_0, \boldsymbol{\Lambda}_0)$ must be away from 0. Then choosing τ parallel to $\nabla_{\boldsymbol{\xi}} \Phi(\boldsymbol{\xi}_0, \boldsymbol{\Lambda}_0)$ we obtain that $\nabla_{\boldsymbol{\xi}} \Phi(\boldsymbol{\xi}_0, \boldsymbol{\Lambda}_0) \cdot \tau$ must be away from 0, which is a contradiction. Then, the point $\boldsymbol{\xi}_0 \in \operatorname{Int}(\Sigma_{\Omega}^{\mu})$, which implies that the (PS) condition holds and the results follows.

Now we are ready to complete the proof of theorem 1.1.

Proof of theorem 1.1. Let us consider the domain $\boldsymbol{\Sigma}_a^b = \Sigma_{\Omega}^{\mu} \times [\boldsymbol{a}, \boldsymbol{b}]^2$ with $\boldsymbol{a}, \boldsymbol{b}$ to be chosen later. Then the functional $\mathcal{I}$ given by (4.1) is well defined on $\boldsymbol{\Sigma}_a^b$ except on the set

$$\Delta_{\rho} = \{(\boldsymbol{\xi}, \boldsymbol{\Lambda}) \in \boldsymbol{\Sigma}_a^b : |\xi_1 - \xi_2| < \rho\}.$$

From (4.3) we can extend $\mathcal{I}$ to all $\boldsymbol{\Sigma}_a^b$ by extending Φ as in (6.2), and keep relations (4.3) and (4.13) over $\boldsymbol{\Sigma}_a^b$.

From proposition 6.5, Φ satisfies the (PS) condition. There then exist constants $\boldsymbol{b} > 0$, $c > 0$ and $\varrho_0 > 0$, such that if $0 < \varrho < \varrho_0$, and $(\boldsymbol{\xi}, \boldsymbol{\Lambda}) \in \Sigma_{\Omega}^{\mu}$ satisfying $|\boldsymbol{\Lambda}| \geqslant \boldsymbol{b}$ and $c(\Omega) - 2\varrho \leqslant \Phi(\boldsymbol{\xi}, \boldsymbol{\Lambda}) \leqslant c(\Omega) + 2\varrho$, then $|\nabla \Phi(\boldsymbol{\xi}, \boldsymbol{\Lambda})| \geqslant c$.

We now use the min–max characterization of $c(\Omega)$ to choose $\kappa \in \mathcal{K}$ so that

$$c(\Omega) \leqslant \sup_{(\boldsymbol{\xi},\sigma)\in\mathbb{S}\times I_0} \Phi(\kappa(\boldsymbol{\xi},\sigma,1)) \leqslant c(\Omega) + \varrho.$$

By making $\boldsymbol{a}$ small and $\boldsymbol{b}$ large if necessary, we can assume that $\kappa(\boldsymbol{\xi},\sigma,1) \in \boldsymbol{\Sigma}_{2\boldsymbol{a}}^{\boldsymbol{b}/2} \subset \boldsymbol{\Sigma}_a^b$ for all $(\boldsymbol{\xi},\sigma) \in \mathbb{S} \times I_0$.

Consider now $\eta : \boldsymbol{\Sigma}_a^b \times [0,+\infty] \to \boldsymbol{\Sigma}_a^b$, the solution of the equation

$$\dot{\eta} = -h(\eta)\nabla\mathcal{I}(\eta)$$

with initial condition $\eta(\boldsymbol{\xi},\boldsymbol{\Lambda},0) = (\boldsymbol{\xi},\boldsymbol{\Lambda})$. Here the function h is defined in $\boldsymbol{\Sigma}_a^b$ so that $h(\boldsymbol{\xi},\boldsymbol{\Lambda}) = 0$ for all $(\boldsymbol{\xi},\boldsymbol{\Lambda})$ with $\Phi(\boldsymbol{\xi},\boldsymbol{\Lambda}) \leqslant c(\Omega) - 2\varrho$ and $h(\boldsymbol{\xi},\boldsymbol{\Lambda}) = 1$ if $\Phi(\boldsymbol{\xi},\boldsymbol{\Lambda}) \geqslant c(\Omega) - \varrho$, satisfying $0 \leqslant h \leqslant 1$.

Hence, by the choice of $\boldsymbol{a}$ and $\boldsymbol{b}$, and bearing in mind (4.3) and (4.13), we have that $\eta(\boldsymbol{\xi},\boldsymbol{\Lambda},t) \in \boldsymbol{\Sigma}_a^b$ for all $t \geqslant 0$. Then the min–max value

$$C(\Omega) = \inf_{t\geqslant 0} \sup_{(\boldsymbol{\xi},\sigma)\in\mathbb{S}\times I_0} \mathcal{I}(\eta(\kappa(\boldsymbol{\xi},\sigma,1),t))$$

is a critical value for $\mathcal{I}$. We always assume that ε is sufficiently small, in order to make the errors in (4.1) sufficiently small. Theorem 1.1 is thus proven.

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