Effective temperature and Gilbert damping of a current-driven localized spin

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Starting from a model that consists of a semiclassical spin coupled to two leads we present a microscopic derivation of the Langevin equation for the direction of the spin. For slowly-changing direction it takes on the form of the stochastic Landau-Lifschitz-Gilbert equation. We give expressions for the Gilbert damping parameter and the strength of the fluctuations, including their bias-voltage dependence. At nonzero bias-voltage the fluctuations and damping are not related by the fluctuation-dissipation theorem. We find, however, that in the low-frequency limit it is possible to introduce a voltage-dependent effective temperature that characterizes the fluctuations in the direction of the spin, and its transport-steady-state probability distribution function.

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I. INTRODUCTION

One of the major challenges in the theoretical description of various spintronics phenomena[1], such as current-induced magnetization reversal[2,3,4,5] and domain-wall motion[6,7,8,9,10,11,12], is their inherent nonequilibrium character. In addition to the dynamics of the collective degree of freedom, the magnetization, the nonequilibrium behavior manifests itself in the quasi-particle degrees of freedom that are driven out of equilibrium by the nonzero bias voltage. Due to this, the fluctuation-dissipation theorem[13,14] cannot be applied to the quasi-particles. This, in part, has led to controversy surrounding the theory of current-induced domain wall motion[15,16].

Effective equations of motion for order-parameter dynamics that do obey the equilibrium fluctuation-dissipation theorem often take the form of Langevin equations, or their corresponding Fokker-Planck equations[13,14,17]. In the context of spintronics the relevant equation is the stochastic Landau-Lifschitz-Gilbert equation for the magnetization direction[18,19,20,21,22,23,24]. In this paper we derive the generalization of this equation to the nonzero-current situation, for a simple microscopic model consisting of a single spin coupled to two leads via an onsite Kondo coupling. This model is intended as a toy-model for a magnetic impurity in a tunnel junction[25,26,27]. Alternatively, one may think of a nanomagnet consisting of a collection of spins that are locked by strong exchange coupling. The use of this simple model is primarily motivated by the fact that it enables us to obtain analytical results. Because the microscopic starting point for discussing more realistic situations has a similar form, however, we believe that our main results apply qualitatively to more complicated situations as well. Similar models have been used previously to explicitly study the violation of the fluctuation-dissipation relation[28], and the voltage-dependence of the Gilbert damping parameter[27]. Starting from this model, we derive an effective stochastic equation for the dynamics of the spin direction using the functional-integral description of the Keldysh-Kadanoff-Baym nonequilibrium theory[29]. (For similar approaches to spin and magnetization dynamics, see also the work by Rebei and Simionato[30], Nussinov *et al.*[31] and Duine *et al.*[32].) This formalism leads in a natural way to the path-integral formulation of stochastic differential equations[33,34]. One of the attractive features of this formalism is that dissipation and fluctuations enter the theory separately. This allows us to calculate the strength of the fluctuations even when the fluctuation-dissipation theorem is not valid.

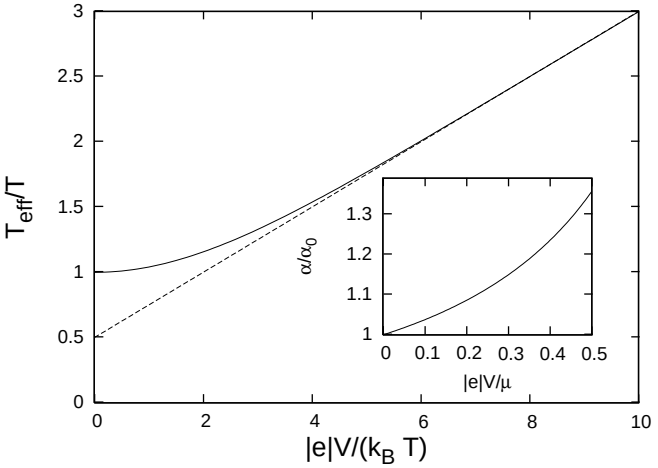

FIG. 1: Effective temperature as a function of bias voltage. The dashed line shows the large bias-voltage asymptotic result $k_{\rm B}T_{\rm eff} \simeq |e|V/4 + k_{\rm B}T/2$. The inset shows the bias-voltage dependence of the Gilbert damping parameter normalized to the zero-bias result.

We find that the dynamics of the direction of the spin is described by a Langevin equation with a damping ker-

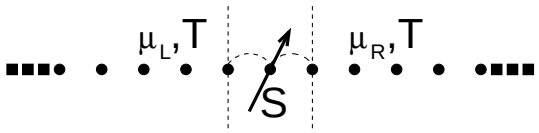

FIG. 2: Model system of a spin S connected to two tight-binding model half-infinite leads. The chemical potential of the left lead is $\mu_{\rm L}$ and different from the chemical potential of the right lead $\mu_{\rm R}$. The temperature T of both leads is for simplicity taken to be equal.

nel and a stochastic magnetic field. We give explicit expressions for the damping kernel and the correlation function of the stochastic magnetic field that are valid in the entire frequency domain. In general, they are not related by the fluctuation-dissipation theorem. In the low-frequency limit the Langevin equation takes on the form of the stochastic Landau-Lifschitz-Gilbert equation. Moreover, in that limit it is always possible to introduce an effective temperature that characterizes the fluctuations and the equilibrium probability distribution for the spin direction. In Fig. 1 we present our main results, namely the bias-voltage dependence of the effective temperature and the Gilbert damping parameter. We find that the Gilbert damping constant initially varies linearly with the bias voltage, in agreement with the result of Katsura *et al.*[27]. The voltage-dependence of the Gilbert damping parameter is determined by the density of states evaluated at an energy equal to the sum of the Fermi energy and the bias voltage. The effective temperature is for small bias voltage equal to the actual temperature, whereas for large bias voltage it is independent of the temperature and proportional to the bias voltage. This bias-dependence of the effective temperature is traced back to shot noise[35].

Effective temperatures for magnetization dynamics have been introduced before on phenomenological grounds in the context of thermally-assisted current-driven magnetization reversal in magnetic nanopillars[36,37,38]. A current-dependent effective temperature enters in the theoretical description of these systems because the current effectively lowers the energy barrier thermal fluctuations have to overcome. In addition to this effect, the presence of nonzero current alters the magnetization noise due to spin current shot noise[35]. Covington *et al.*[39] interpret their experiment in terms of current-dependent noise although this interpretation is still under debate[30]. Foros *et al.*[35] also predict, using a different model and different methods, a crossover from thermal to shot-noise dominated magnetization noise for increasing bias voltage. Our main result in Fig. 1 is an explicit example of this crossover for a specific model.

The remainder of the paper is organized as follows. We start in Sec. II by deriving the general Langevin equation for the dynamics of the magnetic impurity coupled to two leads. In Sec. III and IV we discuss the low-frequency limit in the absence and presence of a current, respectively. We end in Sec. V with our conclusions.

II. DERIVATION OF THE LANGEVIN EQUATION

We use a model that consists of a spin S on a site that is coupled via hopping to two semi-infinite leads, as shown in Fig. 2. The full probability distribution for the direction $\hat{\Omega}$ of the spin on the unit sphere is written as a coherent-state path integral over all electron Grassmann field evolutions $\psi^*(t)$ and $\psi(t)$, and unit-sphere paths $\mathbf{S}(t)$, that evolve from $-\infty$ to t and back on the so-called Keldysh contour $\mathcal{C}^t$. It is given by[29]

$$P[\hat{\Omega},t] \;=\; \int_{\mathbf{S}(t)=\hat{\Omega}} d[\mathbf{S}]\;\delta\Big[|\mathbf{S}|^2-1\Big]\,d[\psi^*]d[\psi] \times\exp\left\{\frac{i}{\hbar}S[\psi^*,\psi,\mathbf{S}]\right\}\;, \tag{1}$$

where the delta functional enforces the length constraint of the spin. In the above functional integral an integration over boundary conditions at $t=-\infty$, weighted by an appropriate initial density matrix, is implicitly included in the measure. We have not included boundary conditions on the electron fields, because, as we shall see, the electron correlation functions that enter the theory after integrating out the electrons are in practice conveniently determined assuming that the electrons are either in equilibrium or in the transport steady state.

The action $S[\psi^*,\psi,\mathbf{S}]$ is the sum of four parts,

$$\begin{aligned} S[\psi^*,\psi,\mathbf{S}] \;&=\; S^{\rm L}\left[\left(\psi^{\rm L}\right)^*,\psi^{\rm L}\right]+S^{\rm R}\left[\left(\psi^{\rm R}\right)^*,\psi^{\rm R}\right] \\ &+\; S^{\rm C}\left[\left(\psi^{0}\right)^*,\psi^{0},\left(\psi^{\rm L}\right)^*,\psi^{\rm L},\left(\psi^{\rm R}\right)^*,\psi^{\rm R}\right] \\ &+\; S^{0}\left[\left(\psi^{0}\right)^*,\psi^{0},\mathbf{S}\right]\;. \end{aligned} \tag{2}$$

We describe the leads using one-dimensional non-interacting electron tight-binding models with the action

$$\begin{aligned} &S^{\rm L/R}\left[\left(\psi^{\rm L/R}\right)^*,\psi^{\rm L/R}\right]= \\ &\int_{\mathcal{C}^t}dt'\left\{\sum_{j,\sigma}\left(\psi^{\rm L/R}_{j,\sigma}(t')\right)^* i\hbar\frac{\partial}{\partial t'}\psi^{\rm L/R}_{j,\sigma}(t') \right. \\ &\left. +J\sum_{\langle j,j'\rangle;\sigma}\left(\psi^{\rm L/R}_{j,\sigma}(t')\right)^*\psi^{\rm L/R}_{j',\sigma}(t')\right\}\;, \end{aligned} \tag{3}$$

where the sum in the second term of this action is over nearest neighbors only and proportional to the nearest-neighbor hopping amplitude J in the two leads. (Throughout this paper the electron spin indices are denoted by $\sigma,\sigma'\in\{\uparrow,\downarrow\}$, and the site indices by j,j'.) The coupling between system and leads is determined by the action

$$\begin{aligned} &S^{\rm C}[\left(\psi^{0}\right)^*,\psi^{0},\left(\psi^{\rm L}\right)^*,\psi^{\rm L},\left(\psi^{\rm R}\right)^*,\psi^{\rm R}]= \\ &\int_{\mathcal{C}^t}dt' J_{\rm C}\sum_{\sigma}\left[\left(\psi^{\rm L}_{\partial{\rm L},\sigma}(t')\right)^*\psi^{0}_{\sigma}(t')+\left(\psi^{0}_{\sigma}(t')\right)^*\psi^{\rm L}_{\partial{\rm L},\sigma}(t')\right]+ \end{aligned}$$

$$\int_{\mathcal{C}^t} dt' J_{\rm C} \sum_\sigma \left[\left(\psi^{\rm R}_{\partial{\rm R},\sigma}(t')\right)^* \psi^0_\sigma(t') + \left(\psi^0_\sigma(t')\right)^* \psi^{\rm R}_{\partial{\rm R},\sigma}(t') \right] \ , \tag{4}$$

where ∂L and ∂R denote the end sites of the semi-infinite left and right lead, and the fields $\left(\psi^0(t)\right)^*$ and $\psi^0(t)$ describe the electrons in the single-site system. The hopping amplitude between the single-site system and the leads is denoted by $J_{\rm C}$. Finally, the action for the system reads

$$\begin{aligned} S^0\left[\left(\psi^0\right)^*, \psi^*, \mathbf{S}\right] = \int_{\mathcal{C}^t} dt' \Bigg[& \sum_\sigma \left(\psi^0_\sigma(t')\right)^* i\hbar \frac{\partial}{\partial t'} \psi^0_\sigma(t') \\ & - \hbar S \mathbf{A}(\mathbf{S}(t')) \cdot \frac{d\mathbf{S}(t')}{dt'} + \mathbf{h} \cdot \mathbf{S}(t') \\ & + \Delta \sum_{\sigma,\sigma'} \left(\psi^0_\sigma(t')\right)^* \boldsymbol{\tau}_{\sigma,\sigma'} \cdot \mathbf{S}(t') \ \psi^0_{\sigma'}(t') \Bigg] \ . \end{aligned} \tag{5}$$

The second term in this action is the usual Berry phase for spin quantization[40], with $\mathbf{A}(\mathbf{S})$ the vector potential of a magnetic monopole

$$\epsilon_{\alpha\beta\gamma} \frac{\partial A_\gamma}{\partial S_\beta} = S_\alpha \ , \tag{6}$$

where a sum over repeated Greek indices $\alpha, \beta, \gamma \in \{x, y, z\}$ is implied throughout the paper, and $\epsilon_{\alpha\beta\gamma}$ is the anti-symmetric Levi-Civita tensor. The third term in the action in Eq. (5) describes the coupling of the spin to an external magnetic field, up to dimensionful prefactors given by $\mathbf{h}$. (Note that $\mathbf{h}$ has the dimensions of energy.) The last term in the action models the $s-d$ exchange coupling of the spin with the spin of the conduction electrons in the single-site system and is proportional to the exchange coupling constant $\Delta > 0$. The spin of the conduction electrons is represented by the vector of the Pauli matrices that is denoted by $\boldsymbol{\tau}$.

Next, we proceed to integrate out the electrons using second-order perturbation theory in Δ. This results in an effective action for the spin given by

$$\begin{aligned} S^{\rm eff}[\mathbf{S}] = \int_{\mathcal{C}^t} dt' \Bigg[& S\hbar \mathbf{A}(\mathbf{S}(t')) \cdot \frac{d\mathbf{S}(t')}{dt'} + \mathbf{h} \cdot \mathbf{S}(t') \\ & - \Delta^2 \int_{\mathcal{C}^t} dt'' \Pi(t', t'') \mathbf{S}(t') \cdot \mathbf{S}(t'') \Bigg] \ . \end{aligned} \tag{7}$$

This perturbation theory is valid as long as the electron band width is much larger than the exchange interaction with the spin, i.e., $J, J_{\rm C} \gg \Delta$. The Keldysh quasiparticle response function is given in terms of the Keldysh Green's functions by

$$\Pi(t, t') = -\frac{i}{\hbar} G(t, t') G(t', t) \ , \tag{8}$$

where the Keldysh Green's function is defined by

$$iG(t, t') = \left\langle \psi^0_\uparrow(t) \left(\psi^0_\uparrow(t')\right)^* \right\rangle = \left\langle \psi^0_\downarrow(t) \left(\psi^0_\downarrow(t')\right)^* \right\rangle \ . \tag{9}$$

We will give explicit expressions for the response function and the Green's function later on. For now, we will only make use of the fact that a general function $A(t, t')$ with its arguments on the Keldysh contour is decomposed into its analytic pieces by means of

$$A(t, t') = \theta(t, t') A^>(t, t') + \theta(t', t) A^<(t, t') \ , \tag{10}$$

where $\theta(t, t')$ is the Heaviside step function on the Keldysh contour. There can be also a singular piece $A^\delta \delta(t, t')$, but such a general decomposition is not needed here. Also needed are the advanced and retarded components, denoted respectively by the superscript $(-)$ and $(+)$, and defined by

$$A^{(\pm)}(t, t') \equiv \pm\theta(\pm(t - t')) \left[A^>(t, t') - A^<(t, t')\right] \ , \tag{11}$$

and, finally, the Keldysh component

$$A^{\rm K}(t, t') \equiv A^>(t, t') + A^<(t, t') \ , \tag{12}$$

which, as we shall see, determines the strength of the fluctuations.

Next we write the forward and backward paths of the spin on the Keldysh contour, denoted respectively by $\mathbf{S}(t_+)$ and $\mathbf{S}(t_-)$, as a classical path $\boldsymbol{\Omega}(t)$ plus fluctuations $\delta\boldsymbol{\Omega}(t)$, by means of

$$\mathbf{S}(t_\pm) = \boldsymbol{\Omega}(t) \pm \frac{\delta\boldsymbol{\Omega}(t)}{2} \ . \tag{13}$$

Moreover, it turns out to be convenient to write the delta functional, which implements the length constraint of the spin, as a path integral over a Lagrange multiplier $\Lambda(t)$ defined on the Keldysh contour. Hence we have for the probability distribution in first instance that

$$P[\hat{\Omega}, t] = \int_{\mathbf{S}(t)=\hat{\Omega}} d[\mathbf{S}] d[\Lambda] \exp\left\{ \frac{i}{\hbar} S^{\rm eff}[\mathbf{S}] + \frac{i}{\hbar} S^\Lambda[\mathbf{S}, \Lambda] \right\} \ , \tag{14}$$

with

$$S^\Lambda[\mathbf{S}, \Lambda] = \int_{\mathcal{C}^t} dt' \Lambda(t') \left[|\mathbf{S}(t')|^2 - 1 \right] \ . \tag{15}$$

We then also have to split the Lagrange multiplier into classical and fluctuating parts according to

$$\Lambda(t_\pm) = \lambda(t) \pm \frac{\delta\lambda(t)}{2} \ . \tag{16}$$

Note that the coordinate transformations in Eqs. (13) and (16) have a Jacobian of one. Before we proceed, we note that in principle we are required to expand the action up to all orders in $\delta\boldsymbol{\Omega}$. Also note that for some forward and backward paths $\mathbf{S}(t_+)$ and $\mathbf{S}(t_-)$ on the unit sphere the classical path $\boldsymbol{\Omega}$ is not necessarily on the unit sphere. In order to circumvent these problems we note that the Berry phase term in Eq. (5) is proportional to the area on the unit sphere enclosed by the forward and backward paths. Hence, in

the semi-classical limit $S \to \infty$[27,40] paths whose forward and backward components differ substantially will be suppressed in the path integral. Therefore, we take this limit from now on which allows us to expand the action in terms of fluctuations $\delta\mathbf{\Omega}(t)$ up to quadratic order. We will see that the classical path $\mathbf{\Omega}(t)$ is now on the unit sphere. We note that this semi-classical approximation is not related to the second-order perturbation theory used to derive the effective action.

Splitting the paths in classical and fluctuation parts gives for the probability distribution

$$P[\hat{\Omega},t] = \int_{\mathbf{\Omega}(t)=\hat{\Omega}} d[\mathbf{\Omega}]d[\delta\mathbf{\Omega}]d[\lambda]d[\delta\lambda] \exp\left\{\frac{i}{\hbar}S[\mathbf{\Omega},\delta\mathbf{\Omega},\lambda,\delta\lambda]\right\} , \tag{17}$$

with the action, that is now projected on the real-time axis,

$$\begin{aligned} S[\mathbf{\Omega},\delta\mathbf{\Omega},\lambda,\delta\lambda] &= \int dt \Big\{ \hbar S \epsilon_{\alpha\beta\gamma}\delta\Omega_\beta(t)\frac{d\Omega_\alpha(t)}{dt}\Omega_\gamma(t) \\ &+\delta\Omega_\alpha(t)h_\alpha + 2\delta\Omega_\alpha(t)\Omega_\alpha(t)\lambda(t) \\ &+\delta\lambda(t)\left[|\mathbf{\Omega}(t)|^2 - 1 + |\delta\mathbf{\Omega}(t)|^2/4\right]\Big\} \\ &- \Delta^2 \int dt \int dt' \left\{\delta\Omega_\alpha(t)\left[\Pi^{(-)}(t',t)+\Pi^{(+)}(t,t')\right]\Omega_\alpha(t')\right\} \\ &- \frac{\Delta^2}{2}\int dt \int dt' \left[\delta\Omega_\alpha(t)\Pi^{\rm K}(t,t')\delta\Omega_\alpha(t')\right] . \end{aligned} \tag{18}$$

From this action we observe that the integration over $\delta\lambda(t)$ immediately leads to the constraint

$$|\mathbf{\Omega}(t)|^2 = 1 - \frac{|\delta\mathbf{\Omega}(t)|^2}{4} , \tag{19}$$

as expected. Implementing this constraint leads to terms of order $\mathcal{O}(\delta\mathbf{\Omega}^3)$ or higher in the above action which we are allowed to neglect because of the semi-classical limit. From now on we can therefore take the path integration over $\mathbf{\Omega}(t)$ on the unit sphere.

The physical meaning of the terms linear and quadratic in $\delta\mathbf{\Omega}(t)$ becomes clear after a so-called Hubbard-Stratonovich transformation which amounts to rewriting the action that is quadratic in the fluctuations as a path integral over an auxiliary field $\boldsymbol{\eta}(t)$. Performing this transformation leads to

$$\begin{aligned} P[\hat{\Omega},t] &= \int_{\mathbf{\Omega}(t)=\hat{\Omega}} d[\mathbf{\Omega}]d[\delta\mathbf{\Omega}]d[\boldsymbol{\eta}]d[\lambda] \\ &\times \exp\left\{\frac{i}{\hbar}S[\mathbf{\Omega},\delta\mathbf{\Omega},\lambda,\boldsymbol{\eta}]\right\} , \end{aligned} \tag{20}$$

where the path integration over $\mathbf{\Omega}$ is now on the unit sphere. The action that weighs these paths is given by

$$\begin{aligned} S[\mathbf{\Omega},\delta\mathbf{\Omega},\lambda,\boldsymbol{\eta}] &= \int dt \Big[\hbar S\epsilon_{\alpha\beta\gamma}\delta\Omega_\beta(t)\frac{d\Omega_\alpha(t)}{dt}\Omega_\gamma(t) \\ &+\delta\Omega_\alpha(t)h_\alpha + 2\delta\Omega_\alpha(t)\Omega_\alpha(t)\lambda(t) + \delta\Omega_\alpha(t)\eta_\alpha(t)\Big] \\ &- \Delta^2 \int dt \int dt' \left\{\delta\Omega_\alpha(t)\left[\Pi^{(-)}(t',t)+\Pi^{(+)}(t,t')\right]\Omega_\alpha(t')\right\} \\ &+ \frac{1}{2\Delta^2}\int dt \int dt' \left[\eta_\alpha(t)\left(\Pi^{\rm K}\right)^{-1}(t,t')\eta_\alpha(t')\right] . \end{aligned} \tag{21}$$

Note that the inverse in the last term is defined as $\int dt''\Pi^{\rm K}(t,t'')\left(\Pi^{\rm K}\right)^{-1}(t'',t') = \delta(t-t')$.

Performing now the path integral over $\delta\mathbf{\Omega}(t)$, we observe that the spin direction $\mathbf{\Omega}(t)$ is constraint to obey the Langevin equation

$$\begin{aligned} \hbar S\epsilon_{\alpha\beta\gamma}\frac{d\Omega_\beta(t)}{dt}\Omega_\gamma(t) &= h_\alpha + 2\lambda(t)\Omega_\alpha(t) \\ &+ \eta_\alpha(t) + \int_{-\infty}^{\infty} dt' K(t,t')\Omega_\alpha(t') , \end{aligned} \tag{22}$$

with the so-called damping or friction kernel given by

$$K(t,t') = -\Delta^2\left[\Pi^{(-)}(t',t)+\Pi^{(+)}(t,t')\right] . \tag{23}$$

Note that the Heaviside step functions in Eq. (11) appear precisely such that the Langevin equation is causal. The stochastic magnetic field is seen from Eq. (21) to have the correlations

$$\begin{aligned} \langle\eta_\alpha(t)\rangle &= 0 ; \\ \langle\eta_\alpha(t)\eta_\beta(t')\rangle &= i\delta_{\alpha\beta}\hbar\Delta^2\Pi^{\rm K}(t,t') . \end{aligned} \tag{24}$$

Using the fact that $\mathbf{\Omega}(t)$ is a unit vector within our semi-classical approximation, the Langevin equation for the direction of the spin $\hat{\Omega}(t)$ is written as

$$\hbar S\frac{d\hat{\Omega}(t)}{dt} = \hat{\Omega}(t)\times\left[\mathbf{h} + \boldsymbol{\eta}(t) + \int_{-\infty}^{\infty} dt' K(t,t')\hat{\Omega}(t')\right] , \tag{25}$$

which has the form of a Landau-Lifschitz equation with a stochastic magnetic field and a damping kernel. In the next sections we will see that for slowly-varying spin direction we get the usual form of the Gilbert damping term.

So far, we have not given explicit expressions for the response functions $\Pi^{(\pm),{\rm K}}(t,t')$. To determine these functions, we assume that the left and right leads are in thermal equilibrium at chemical potentials $\mu_{\rm L}$ and $\mu_{\rm R}$, respectively. Although not necessary for our theoretical approach we assume, for simplicity, that the temperature T of the two leads is the same. The Green's functions for the system are then given by[41,42]

$$\begin{aligned} -iG^<(\epsilon) &= \frac{A(\epsilon)}{2}\sum_{k\in\{\rm L,R\}} N(\epsilon-\mu_k) ; \\ iG^>(\epsilon) &= \frac{A(\epsilon)}{2}\sum_{k\in\{\rm L,R\}} \left[1-N(\epsilon-\mu_k)\right] ; \\ G^{\lessgtr,{\rm K}}(t-t') &= \int\frac{d\epsilon}{(2\pi)}e^{-i\epsilon(t-t')/\hbar}G^{\lessgtr,{\rm K}}(\epsilon) , \end{aligned} \tag{26}$$

with $N(\epsilon) = \{\exp[\epsilon/(k_{\rm B}T)]+1\}^{-1}$ the Fermi-Dirac distribution function with $k_{\rm B}$ Boltzmann's constant, and

$$A(\epsilon) = i\left[G^{(+)}(\epsilon) - G^{(-)}(\epsilon)\right] , \tag{27}$$

the spectral function. Note that Eq. (26) has a particularly simple form because we are dealing with a single-site system. The retarded and advanced Green's functions are determined by

$$\left[\epsilon^{\pm} - 2\hbar\Sigma^{(\pm)}(\epsilon)\right] G^{(\pm)}(\epsilon) = 1 \ , \tag{28}$$

with $\epsilon^{\pm} = \epsilon \pm i0$, and the retarded self-energy due to one lead follows, for a one-dimensional tight-binding model, as

$$\hbar\Sigma^{(+)}(\epsilon) = -\frac{J_{\rm C}^2}{J} e^{ik(\epsilon)a} \ , \tag{29}$$

with $k(\epsilon) = \arccos[-\epsilon/(2J)]/a$ the wave vector in the leads at energy ϵ, and a the lattice constant. The advanced self-energy due to one lead is given by the complex conjugate of the retarded one.

Before proceeding we give a brief physical description of the above results. (More details can be found in Refs. [41] and [42].) They arise by adiabatically eliminating ("integrating out") the leads from the system, assuming that they are in equilibrium at their respective chemical potentials. This procedure reduces the problem to a single-site one, with self-energy corrections for the on-site electron that describe the broadening of the on-site spectral function from a delta function at the (bare) on-site energy to the spectral function in Eq. (27). Moreover, the self-energy corrections also describe the non-equilibrium occupation of the single site via Eq. (26)

For the transport steady-state we have that $\Pi^{(\pm),{\rm K}}(t,t')$ depends only on the difference of the time arguments. Using Eq. (8) and Eqs. (10), (11), and (12) we find that the Fourier transforms are given by

$$\begin{aligned}\Pi^{(\pm)}(\epsilon) &\equiv \int d(t-t') e^{i\epsilon(t-t')/\hbar} \Pi^{(\pm)}(t,t') \\ &= \int \frac{d\epsilon'}{(2\pi)} \int \frac{d\epsilon''}{(2\pi)} \frac{1}{\epsilon^{\pm} + \epsilon' - \epsilon''} \\ &\quad \times \left[G^{<}(\epsilon') G^{>}(\epsilon'') - G^{>}(\epsilon') G^{<}(\epsilon'')\right] \ , \end{aligned} \tag{30}$$

and

$$\begin{aligned}\Pi^{\rm K}(\epsilon) &= -2\pi i \int \frac{d\epsilon'}{(2\pi)} \int \frac{d\epsilon''}{(2\pi)} \delta(\epsilon + \epsilon' - \epsilon'') \\ &\quad \times \left[G^{>}(\epsilon') G^{<}(\epsilon'') + G^{<}(\epsilon') G^{>}(\epsilon'')\right] \ . \end{aligned} \tag{31}$$

In the next two sections we determine the spin dynamics in the low-frequency limit, using these expressions together with the expressions for $G^{\lessgtr}(\epsilon)$. We consider first the equilibrium case.

III. EQUILIBRIUM SITUATION

In equilibrium the chemical potentials of the two leads are equal so that we have $\mu_{\rm L} = \mu_{\rm R} \equiv \mu$. Combining results from the previous section, we find for the retarded and advanced response functions (the subscript "0" denotes equilibrium quantities) that

$$\begin{aligned}\Pi_0^{(\pm)}(\epsilon) &= \int \frac{d\epsilon'}{(2\pi)} \int \frac{d\epsilon''}{(2\pi)} A(\epsilon') A(\epsilon'') \\ &\quad \times \frac{[N(\epsilon'-\mu) - N(\epsilon''-\mu)]}{\epsilon^{\pm} + \epsilon' - \epsilon''} \ . \end{aligned} \tag{32}$$

The Keldysh component of the response function is in equilibrium given by

$$\begin{aligned}\Pi_0^{\rm K}(\epsilon) &= -2\pi i \int \frac{d\epsilon'}{(2\pi)} \int \frac{d\epsilon''}{(2\pi)} A(\epsilon') A(\epsilon'') \delta(\epsilon - \epsilon' + \epsilon'') \\ &\{[1-N(\epsilon'-\mu)]\, N(\epsilon''-\mu) + N(\epsilon'-\mu)\, [1-N(\epsilon''-\mu)]\} \ . \end{aligned} \tag{33}$$

The imaginary part of the retarded and advanced response functions are related to the Keldysh component by means of

$$\Pi_0^{\rm K}(\epsilon) = \pm 2i \left[2N_{\rm B}(\epsilon) + 1\right] {\rm Im}\left[\Pi_0^{(\pm)}(\epsilon)\right] , \tag{34}$$

with $N_{\rm B}(\epsilon) = \{\exp[\epsilon/(k_{\rm B}T)] - 1\}^{-1}$ the Bose distribution function. This is, in fact, the fluctuation-dissipation theorem which relates the dissipation, determined as we shall see by the imaginary part of the retarded and advanced components of the response function, to the strength of the fluctuations, determined by the Keldysh component.

For low energies, corresponding to slow dynamics, we have that

$$\Pi_0^{(\pm)}(\epsilon) \simeq \Pi_0^{(\pm)}(0) \mp \frac{i}{4\pi} A^2(\mu)\epsilon \ . \tag{35}$$

With this result the damping term in the Langevin equation in Eq. (25) becomes

$$\int_{-\infty}^{\infty} dt' K(t,t') \hat{\Omega}(t') = -\frac{\hbar\Delta^2 A^2(\mu)}{2\pi} \frac{d\hat{\Omega}(t)}{dt} \ , \tag{36}$$

where we have not included the energy-independent part of Eq. (35) because it does not contribute to the equation of motion for $\hat{\Omega}(t)$. In the low-energy limit the Keldysh component of the response function is given by

$$\Pi_0^{\rm K}(\epsilon) = \frac{A^2(\mu)}{i\pi} k_{\rm B}T \ . \tag{37}$$

Putting all these results together we find that the dynamics of the spin direction is, as long as the two leads are in equilibrium at the same temperature and chemical potential, determined by the stochastic Landau-Lifschitz-Gilbert equation

$$\hbar S \frac{d\hat{\Omega}(t)}{dt} = \hat{\Omega}(t) \times [\mathbf{h} + \boldsymbol{\eta}(t)] - \hbar\alpha_0 \hat{\Omega} \times \frac{d\hat{\Omega}(t)}{dt} \ , \tag{38}$$

with the equilibrium Gilbert damping parameter

$$\alpha_0 = \frac{\Delta^2 A^2(\mu)}{2\pi} \ . \tag{39}$$

Using Eqs. (24), (37), and (39) we find that the strength of the Gaussian stochastic magnetic field is determined by

$$\langle \eta_\alpha(t)\eta_\beta(t')\rangle = 2\alpha_0 \hbar k_{\rm B} T \delta(t-t')\delta_{\alpha\beta} \ . \qquad (40)$$

Note that these delta-function type noise correlations are derived by approximating the time dependence of $\Pi^{\rm K}(t,t')$ by a delta function in the difference of the time variables. This means that the noisy magnetic field $\boldsymbol{\eta}(t)$ corresponds to a Stratonovich stochastic process[13,14,17].

The stationary probability distribution function generated by the Langevin equation in Eqs. (38) and (40) is given by the Boltzmann distribution[18,19,20,21,22,23,24]

$$P[\hat{\Omega}, t\to\infty] \propto \exp\left\{-\frac{E(\hat{\Omega})}{k_{\rm B}T}\right\} \ , \qquad (41)$$

with

$$E[\hat{\Omega}] = -\mathbf{h}\cdot\hat{\Omega} \ , \qquad (42)$$

the energy of the spin in the external field. It turns out that Eq. (41) holds for any effective field $\mathbf{h} = -\partial E[\hat{\Omega}]/\partial\hat{\Omega}$, and in particular for the case that $E[\hat{\Omega}]$ is quadratic in the components of $\hat{\Omega}$ as is often used to model magnetic anisotropy.

It is important to realize that the equilibrium probability distribution has precisely this form because of the fluctuation-dissipation theorem, which ensures that dissipation and fluctuations cooperate to achieve thermal equilibrium[13,14]. Finally, it should be noted that this derivation of the stochastic Landau-Lifschitz-Gilbert equation from a microscopic starting point circumvents concerns regarding the phenomenological form of damping and fluctuation-dissipation theorem, which is subject of considerable debate[22,23].

IV. NONZERO BIAS VOLTAGE

In this section we consider the situation that the chemical potential of the left lead is given by $\mu_{\rm L} = \mu + |e|V$, with $|e|V > 0$ the bias voltage in units of energy, and $\mu = \mu_{\rm R}$ the chemical potential of the right lead. Using the general expressions given for the response functions derived in Sec. II, it is easy to see that the imaginary part of the retarded and advanced components of the response functions are no longer related to the Keldysh component by means of the fluctuation-dissipation theorem in Eq. (34). See also the work by Mitra and Millis[28] for a discussion of this point. As in the previous section, we proceed to determine the low-frequency behavior of the response functions.

Using Eqs. (26), (27), and (30) we find that the retarded and advanced components of the response function are given by

$$\Pi^{(\pm)}(\epsilon) = \mp\frac{i}{8\pi}\left[A^2(\mu+|e|V) + A^2(\mu)\right]\epsilon \ . \qquad (43)$$

In this expression we have omitted the energy-independent part and the contribution following from the principal-value part of the energy integral because, as we have seen previously, these do not contribute to the final equation of motion for the direction of the spin. Following the same steps as in the previous section, we find that the damping kernel in the general Langevin equation in Eq. (25) reduces to a Gilbert damping term with a voltage-dependent damping parameter given by

$$\begin{aligned}\alpha(V) &= \frac{\Delta^2}{4\pi}\left[A^2(\mu+|e|V) + A^2(\mu)\right] \\ &\simeq \alpha_0\left[1+\mathcal{O}\left(\frac{|e|V}{\mu}\right)\right] \ . \end{aligned} \qquad (44)$$

This result is physically understood by noting that the Gilbert damping is determined by the dissipative part of the response function $\Pi^{(+)}(\epsilon)$. In this simple model, this dissipative part gets contributions from processes that correspond to an electron leaving or entering the system, to or from the leads, respectively. The dissipative part is in general proportional to the density of states at the Fermi energy. Since the Fermi energy of left and right lead is equal to $\mu+|e|V$ and μ, respectively, the Gilbert damping has two respective contributions corresponding to the two terms in Eq. (44).

Note that the result that the Gilbert damping parameter initially varies linearly with the voltage is in agreement with the results of Katsura *et al.*[27], although these authors consider a slightly different model. In the inset of Fig. 1 we show the Gilbert damping parameter as a function of voltage. The parameters taken are $\Delta/J = 0.1$, $J_{\rm C} = J$, $\mu/J = 1$ and $\mu/(k_{\rm B}T) = 100$.

Although we can no longer make use of the fluctuation-dissipation theorem, we are nevertheless able to determine the fluctuations by calculating the low-energy behavior of the Keldysh component of the response function in the nonzero-voltage situation. It is given by

$$\begin{aligned}\Pi^{\rm K}(\epsilon) = -\frac{i}{2}\int\frac{d\epsilon'}{(2\pi)}A^2(\epsilon')\{&[N(\mu_{\rm L}-\epsilon')+N(\mu_{\rm R}-\epsilon')] \\ &\times[N(\epsilon'-\mu_{\rm L})+N(\epsilon'-\mu_{\rm R})]\} \ . \end{aligned} \qquad (45)$$

We define an effective temperature by means of

$$k_{\rm B}T_{\rm eff}(T,V) \equiv \frac{i\Pi^{\rm K}(\epsilon)\Delta^2}{2\alpha(V)} \ . \qquad (46)$$

This definition is motivated by the fact that, as we mention below, the spin direction obeys the stochastic Landau-Lifschitz-Gilbert equation with voltage-dependent damping and fluctuations characterized by the above effective temperature[43]. From the expression for $\alpha(V)$ and $\Pi^{\rm K}(\epsilon)$ we see that in the limit of zero bias voltage we recover the equilibrium result $T_{\rm eff} = T$. In the situation that $|e|V$ is substantially larger than $k_{\rm B}T$, which is usually approached in experiments, we have that

$$k_{\rm B}T_{\rm eff}(T,V) \simeq \frac{|e|V}{4} + \frac{k_{\rm B}T}{2} \ , \qquad (47)$$

which in the limit that $|e|V \gg k_{\rm B}T$ becomes independent of the actual temperature of the leads. In Fig. 1 the effective temperature as a function of bias voltage is shown, using the expression for $\Pi^{\rm K}(\epsilon)$ given in Eq. (45). The parameters are the same as before, i.e., $\Delta/J = 0.1$, $J_{\rm C} = J$, $\mu/J = 1$ and $\mu/(k_{\rm B}T) = 100$. Clearly the effective temperature changes from $T_{\rm eff} = T$ at zero bias voltage to the asymptotic expression in Eq. (47) shown by the dashed line in Fig. 1. The crossover between actual temperature and voltage as a measure for the fluctuations is reminiscent of the theory of shot noise in mesoscopic conductors[44]. This is not surprising, since in the single-site model we use the noise in the equation of motion ultimately arises because of fluctuations in the number of electrons in the single-site system, and is therefore closely related to shot noise in the current through the system. Foros *et al.*[35] calculate the magnetization noise arising from spin current shot noise in the limit that $|e|V \gg k_{\rm B}T$ and $|e|V \ll k_{\rm B}T$. In these limits our results are similar to theirs.

With the above definition of the effective temperature we find that in the nonzero bias voltage situation the spin direction obeys the stochastic Landau-Lifschitz-Gilbert equation, identical in form to the equilibrium case in Eqs. (38) and (40), with the Gilbert damping parameter and temperature replaced according to

$$\begin{aligned} \alpha_0 \;&\rightarrow\; \alpha(V)\;; \\ T \;&\rightarrow\; T_{\rm eff}(T,V)\;. \end{aligned} \tag{48}$$

Moreover, the transport-steady-state probability distribution for the direction of the spinis a Boltzmann distribution with the effective temperature characterizing the fluctuations.

V. DISCUSSION AND CONCLUSIONS

We have presented a microscopic derivation of the stochastic Landau-Lifschitz-Gilbert equation for a semiclassical single spin under bias. We found that the Gilbert damping parameter is voltage dependent and to lowest order acquires a correction linear in the bias voltage, in agreement with a previous study for a slightly different model[27]. In addition, we have calculated the strength of the fluctuations directly without using the fluctuation-dissipation theorem and found that, in the low-frequency regime, the fluctuations are characterized by a voltage and temperature dependent effective temperature.

To arrive at these results we have performed a low frequency expansion of the various correlation functions that enter the theory. Such an approximation is valid as long as the dynamics is much slower than the times set by the other energy scales in the system such as temperature and the Fermi energy. Moreover, in order for the leads to remain in equilibrium as the spin changes direction, the processes in the leads that lead to equilibration have to be much faster than the precession period of the magnetizationspin. Both these criteria are satisfied in experiments with magnetic materials. In principle however, the full Langevin equation derived in Sec. II also describes dynamics beyond this low-frequency approximation. The introduction of the effective temperature relies on the low-frequency approximation though, and for arbitrary frequencies such a temperature can no longer be uniquely defined[28].

An effective temperature for magnetization dynamics has been introduced before on phenomenological grounds[36,37,38]. Interestingly, the phenomenological expression of Urazhdin *et al.*[36], found by experimentally studying thermal activation of current-driven magnetization reversal in magnetic trilayers, has the same form as our expression for the effective temperature in the large bias-voltage limit [Eq. (47)] that we derived microscopically. Zhang and Li[37], and Apalkov and Visscher[38], have, on phenomenological grounds, also introduced an effective temperature to study thermally-assisted spin-transfer-torque-induced magnetization switching. In their formulation, however, the effective temperature is proportional to the real temperature because the current effectively modifies the energy barrier for magnetization reversal.

Foros *et al.*[35] consider spin current shot noise in the large bias-voltage limit and find for sufficiently large voltage that the magnetization noise is dominated by shot noise. Moreover, they also consider the low bias-voltage limit and predict a crossover for thermal to shot-noise dominated magnetization fluctuations. Our main result in Fig. 1 provides an explicit example of this crossover for a simple model system obtained by methods that are easily generalized to more complicated models. In the experiments of Krivorotov *et al.*[45] the temperature dependence of the dwell time of parallel and anti-parallel states of a current-driven spin valve was measured. At low temperatures $k_{\rm B}T \lesssim |e|V$ the dwell times are no longer well-described by a constant temperature, which could be a signature of the crossover from thermal noise to spin current shot noise. However, Krivorotov *et al.* interpret this effect as due to ohmic heating, which is not taken into account in the model presented in this paper, nor in the work by Foros *et al.*[35]. Moreover, in realistic materials phonons provide an additional heat bath for the magnetization, with an effective temperature that may depend in a completely different manner on the bias voltage than the electron heat-bath effective temperature. Nonetheless, we believe that spin current shot noise may be observable in future experiments and that it may become important for applications as technological progress enables further miniaturization of magnetic materials. Moreover, the formalism presented here is an important step in understanding magnetization noise from a microscopic viewpoint as its generalization to more complicated models is in principle straightforward. Possible interesting generalizations include making one of the leads ferromagnetic (see also Ref. [46]).

Since spin transfer torques will occur on the single spin as a spin-polarized current from the lead interacts with the single-spin system, the resulting model would be a toy model for microscopically studying the attenuation of spin transfer torques and current-driven magnetization reversal by shot noise. Another simple and useful generalization would be enlarging the system to include more than one spin. The formalism presented here would allow for a straightforward microscopic calculation of Gilbert damping and adiabatic and nonadiabatic spin transfer torques which are currently attracting a lot of interest in the context of current-driven domain wall motion[6,7,8,9,10,11,12]. The application of our theory in the present paper is, in addition to its intrinsic physical interest, chosen mainly because of the feasibility of analytical results. The applications mentioned above are more complicated and analytical results may be no longer obtainable. In conclusion, we reserve extensions of the theory presented here for future work.

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