Comparison Results for Reflected Jump-diffusions in the Orthant with Variable Reflection Directions and Stability Applications

Francisco J. Piera*
Department of Electrical Engineering
University of Chile
Av. Tupper 2007
Santiago, 8370451, Chile
fpiera@ing.uchile.cl

Ravi R. Mazumdar†
Department of Electrical
and Computer Engineering
University of Waterloo
Waterloo, ON N2L3G1, Canada
mazum@ece.uwaterloo.ca

Abstract

We consider reflected jump-diffusions in the orthant $\mathbb{R}_+^n$ with time- and state-dependent drift, diffusion and jump-amplitude coefficients. Directions of reflection upon hitting boundary faces are also allow to depend on time and state. Pathwise comparison results for this class of processes are provided, as well as absolute continuity properties for their associated regulator processes responsible of keeping the respective diffusions in the orthant. An important role is played by the boundary property in that regulators do not charge times spent by the reflected diffusion at the intersection of two or more boundary faces. The comparison results are then applied to provide an ergodicity condition for the state-dependent reflection directions case.

Key words: Jump-diffusion processes;pathwise comparisons;state-dependent oblique reflection;Skorokhod maps;stability; ergodicity .

*Research supported in part by VID, U. of Chile, Chile, DI Project REIN 06/05, by CONICYT, Chile, FONDECYT Project 1070797, and by the Millennium Science Nucleus on Information and Randomness, Dept. of Mathematical Engineering, U. of Chile, Chile, Program P04-069-F.

†Research supported in part by NSERC, Canada, through the Discovery Grant Program.

1 Introduction

Reflecting stochastic differential equations have found a wide variety of applications over the last decades. They play an increasingly important role in several disciplines such as economics and operations research, where they serve to model portfolio and consumption processes, option pricing and subsidy phenomena in interdependent economies, among others (see for example [10; 15; 21; 16] and references therein). They also play a central role in several fields of applications in the electrical engineering context, where they are used to model, in conjunction with weak convergence methods, from systems such as adaptive antenna arrays to stochastic communication networks (see for example [12; 13] and references therein).

In the context of stochastic networks, such models appear as heavy-traffic limits of complex network models, otherwise difficult to analyze, giving rise to corresponding approximations in terms of reflected diffusions. Reflections are taken into account via the Skorokhod map, and are due to the non-negativeness requirements for the buffer occupation processes in the network (see for example [26; 4] and references therein). Reflected jump-diffusions appear in this queueing application context when for example network stations are subject to service interruptions (see [11; 25] and references therein). In the same way, time- and state-dependence in the corresponding drift and diffusion coefficients, as well as directions of reflection, obey to the corresponding dependence in network traffic parameters, such as arrival and service rates, and station-to-station routing probabilities (see for example [13; 14] and references therein).

Due to the wide variety of applications of reflected diffusion models, as summarized above, the availability of comparison properties for such class of models have become of practical and theoretical importance. For example, the pathwise comparison between buffer occupation processes in queueing networks or cumulative subsidies transferred among entities in interdependent economies, both considered as the respective model parameters change, are of self-explicatory importance. Such pathwise results in general demand not only the comparison between the respective constrained jump-diffusion processes (constructed in terms of the Skorokhod map, as detailed further in the next section), but also the corresponding establishment of absolute continuity properties for the associated regulator processes (constraining the diffusions to the domain of interest, as for example the non-negative orthant). In this context, comparison results for reflected jump-diffusions in the orthant have traditionally been restricted to the case of normal reflection directions upon hitting boundary faces (see [23]). However a comparison result in the oblique reflection directions case is available in the context of the deterministic Skorokhod problem in the orthant (see [21]), the framework in which it is established makes its application to the diffusion setting only possible when the stochastic integral term driving the respective diffusion is state-independent. We will show in the paper that this requirement is essentially a boundary condition at the faces of the orthant, being able to include then a controlled state-dependency in that term over the interior of the orthant covering for example the important case of a product-form setting (see [17; 19]) in queueing network applications. A crucial role is played here by an appropriate boundary behavior characterization, and in particular by the boundary property in that regulators do not charge times spent by the reflected diffusion at the intersection of two or more boundary faces (see [18; 19]).

A direct application of the comparison results established in the paper is to provide a simple ergodicity condition for continuous reflected diffusions in the orthant with state-dependent reflection directions. Stability conditions available in the literature have been established for the constant reflection directions case (see [1]) and critically depend on the Lipschitz continuity of the corresponding Skorokhod map, which is not ensured in the state-dependent case.

The organization of the paper is as follows. In Section 2 we specify the setting to be considered throughout as well as some related notation. In Section 3 we establish the main comparison results of the paper. In Section 4 we apply those results to derive an ergodicity condition for the continuous case. Finally, in Section 5 we provide the corresponding proofs of the main results in Section 3.

2 Setting and Notation

Let $n \geq 2$ be an integer[1] and $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$ be a stochastic basis satisfying the usual hypotheses, i.e., $\mathscr{F}_0$ contains all the $\mathbb{P}$-null sets of $\mathscr{F}$ and the filtration $(\mathscr{F}_t)_{t\geq 0}$ is right continuous. Throughout the paper we consider pair of processes (X, Z) satisfying reflecting stochastic differential equations (RSDEs) in the orthant $\mathbb{R}^n_+ = \{x = (x_i)_{i=1}^n \in \mathbb{R}^n : x_i \geq 0 \ \forall i\} = \times_{i=1}^n \mathbb{R}_+$, with state- and time-dependent reflection directions upon hitting boundary faces, of the form

$$X_t = X_0 + \int_0^t b(s, X_{s-})ds + \int_0^t \gamma(s, X_{s-})dW_s + \int_0^t \int_E \delta(s, X_{s-}, r)N(ds, dr) + \int_0^t R(s, X_{s-})dZ_s, \quad t \geq 0, \quad (2.1)$$

where[2]

- $X = (X_t)_{t\geq 0} = (X_t^1, \ldots, X_t^n)_{t\geq 0}$ is an $(\mathscr{F}_t)_{t\geq 0}$-adapted $\mathbb{R}^n_+$-valued *càdlàg*[3] semi-martingale.
- $b = (b_i)_{i=1}^n : \mathbb{R}_+ \times \mathbb{R}^n_+ \to \mathbb{R}^n$ and[4] $\gamma = (\gamma_{ij})_{i,j=1}^n : \mathbb{R}_+ \times \mathbb{R}^n_+ \to \mathbb{R}^{n\times n}$ are Borel-measurable functions. As usual, we refer to b and $a = (a_{ij})_{i,j=1}^n \doteq \gamma\gamma^T$ as the drift vector and diffusion matrix, respectively.
- $W = (W_t)_{t\geq 0} = (W_t^1, \ldots, W_t^n)_{t\geq 0}$ is an $(\mathscr{F}_t)_{t\geq 0}$-standard Brownian motion on $\mathbb{R}^n$.
- $E \doteq E_1 \times \cdots \times E_n$ with each E_i being an arbitrary Polish space (e.g., $\mathbb{R}$ with the usual Euclidian distance), $\delta = (\delta_{ij})_{i,j=1}^n : \mathbb{R}_+ \times \mathbb{R}^n_+ \times E \to \mathbb{R}^{n\times n}$ is a Borel-measurable function and $N(ds, dr) \doteq (N_1(ds, dr_1), \cdots, N_n(ds, dr_n))$ with each $N_i(ds, dr_i)$ being an independent Poisson random measure over $[0, \infty) \times E_i$ with intensity measure $\lambda_i ds \otimes G_i(dr_i)$, where $\lambda_i \geq 0$ and G_i is a probability distribution on $(E_i, \mathscr{B}(E_i))$. Note since the Poisson measures are independent, any number of them do not "jump" simultaneously at any time (a. s.), and therefore to ensure that a jump does not take X outside the orthant we ask for each δ_{ij} to be such that $\delta_{ij}(t, x, r_j) \geq -x_i$ for all $t \geq 0$, $x = (x_l)_{l=1}^n \in \mathbb{R}^n_+$ and $r_j \in E_j$.
- $Z = (Z_t)_{t\geq 0} = (Z_t^1, \ldots, Z_t^n)_{t\geq 0}$ is a continuous $(\mathscr{F}_t)_{t\geq 0}$-adapted $\mathbb{R}^n_+$-valued process with each Z^i being non-decreasing and such that $Z_0^i = 0$ and $\int_0^\infty X_s^i dZ_s^i = 0$.

[1] All results in the paper apply to the one-dimensional setting as well ($n = 1$). However, in order not to include trivial cases in our proofs, we simply consider $n \geq 2$.

[2] Throughout, (in)equalities involving vectors and matrices are understood to hold componentwise and elementwise respectively, and vectors are envisioned as column vectors.

[3] Acronym in French standing for *continuous from the right with limits from the left.*

[4] We denote as $\mathbb{R}^{n\times n}$ the collection of $n \times n$ real matrices.

- $R = (R_{ij})_{i,j=1}^n : \mathbb{R}_+ \times \mathbb{R}_+^n \to \mathbb{R}^{n\times n}$ is a Borel-measurable function. As usual, and since the j-th column of R gives the reflection direction upon hitting the interior of the j-th face $F_j \doteq \{x = (x_l)_{l=1}^n \in \mathbb{R}_+^n : x_j = 0\}$, we refer to R as the reflection matrix[5] and assume, without loss of generality, the normalization $R_{ii}(\cdot,\cdot) \equiv 1$ for each i.

We now identify different sets of conditions that will be alternatively considered as assumptions in the results given in the paper.

Condition 2.1. *b and R are continuous. Also, b, γ, δ and R satisfy a linear growth condition and are Lipschitz continuous, both in the state variable $x \in \mathbb{R}_+^n$ and uniformly in all the other corresponding variables, i.e., there exists $K \in (0,\infty)$ such that, for all $x, y \in \mathbb{R}_+^n$, $t \geq 0$ and $r \in E$,*

$$\|b(t,x)\|^2 + \|\gamma(t,x)\|^2 + \|\delta(t,x,r)\|^2 + \|R(t,x)\|^2 \leq K^2(1+\|x\|^2)$$

and

$$\|b(t,x)-b(t,y)\| + \|\gamma(t,x)-\gamma(t,y)\| \\ + \|\delta(t,x,r)-\delta(t,y,r)\| + \|R(t,x)-R(t,y)\| \leq K\|x-y\|,$$

with the usual Euclidian and Frobenius norms in $\mathbb{R}^n$ and $\mathbb{R}^{n\times n}$, respectively. Moreover[6]*,*

$$\sup_{x\in\mathbb{R}_+^n,\, t\geq 0} \sum_j \lambda_j \int_{E_j} \delta_{ij}^2(t,x,r_j) G_j(dr_j) < \infty \tag{2.2}$$

and

$$\sum_j \lambda_j \int_{E_j} \delta_{ij}(\cdot,\cdot,r_j) G_j(dr_j)$$

is continuous, for each i.

Condition 2.2. *For each i, j, $i \neq j$, there exists $m_{ij} \geq 0$ such that*

$$|R_{ij}(t,x)| \leq m_{ij}, \quad x \in \mathbb{R}_+^n, \quad t \geq 0,$$

and, with $m_{ii} \doteq 0$ for each i and $M \doteq (m_{ij})_{i,j=1}^n \in \mathbb{R}^{n\times n}$, we have $\sigma(M) < 1$ with $\sigma(M)$ denoting the spectral radius of M.

Conditions 2.1 and 2.2 in particular guarantee that, given an $\mathscr{F}_0$-measurable initial condition $X_0 \in \mathbb{R}_+^n$ and b, γ, W, δ, N and R as above, the pair (X,Z) is the pathwise unique strong solution of RSDE (2.1). Indeed, this follows from [6] in the continuous case (i.e., in absence of jumps), jumps being taken then into account via standard piecewise construction arguments (see for example [13, Section 3.7, pp. 134]). Whenever that is the case, we write

$$(X,Z) = RSDE(X_0, b, R),$$

omitting γ, W, δ and N in the notation since in all comparison results between pairs

$$(X,Z) = RSDE(X_0,b,R) \text{ and } (\widetilde{X},\widetilde{Z}) = RSDE(\widetilde{X}_0,\widetilde{b},\widetilde{R})$$

they will be the same for both[7].

[5]One in general may assume each R_{ij} as given on F_j and extended to the whole orthant by setting $R_{ij}(\cdot,x) \doteq \pi_{F_j}(R_{ij}(\cdot,x))$, $x \in \mathbb{R}_+^n \setminus F_j$, with π_{F_j} the orthogonal projector onto F_j.

[6]Note from (2.2) in particular follows that $\sum_{0<s\leq t} |\Delta X_s^i| < \infty$ a.s., each i and $t > 0$, with $\Delta X_s^i \doteq X_s^i - X_{s-}^i$ and $X_{s-}^i \doteq \lim_{u\searrow s} X_u^i$, $s > 0$.

[7]In some of the corresponding proofs we will find useful to emphasize the common γ though, including it explicitly in the notation then.

Conditions 2.1 and 2.2 in fact guarantee the well posedness of the Skorokhod problem (SP) in the orthant with state-dependent reflection directions (see [24] for a detailed treatment of the (modified) SP in the orthant with state-dependent reflection directions), and one may then write

$$X = \Phi(U)$$

with

$$U_{\cdot} \doteq X_0 + \int_0^{\cdot} b(s,X_{s-})ds + \int_0^{\cdot} \gamma(s,X_{s-})dW_s + \int_0^{\cdot}\int_E \delta(s,X_{s-},r)N(ds,dr)$$

and $\Phi : D([0,\infty) : \mathbb{R}^n) \to D([0,\infty) : \mathbb{R}^n_+)$ the Skorokhod map[8], i.e., with (X,Z) solving the SP for U and R on an a.s. pathwise basis[9]. $D([0,\infty) : G)$ denotes here, as usual, the space of *càdlàg* functions mapping $[0,\infty)$ into $G \subseteq \mathbb{R}^n$.

Condition 2.2 is standard in the context of RSDEs and SPs in non-smooth domains (see for example [6; 5; 21; 8]) as it guarantees that $R(t,x)$ is completely-S, for each $x \in \mathbb{R}^n_+$ and $t \geq 0$, in that for each principal sub-matrix $\overline{R}(t,x)$ extracted from $R(t,x)$ there always exists a non-negative vector v, of the corresponding proper dimension, such that $\overline{R}(t,x)v > 0$. Each such $\overline{R}(t,x)$ is also non-singular. This structure, along with Condition 2.1 above and Condition 2.5 below, in particular guarantee that (see [18; 19]) for each $\mathscr{A} \subseteq \{1,\ldots,n\}$ with cardinality $|\mathscr{A}| \geq 2$

$$\int_0^{\infty} \mathbf{1}_{\{X_s^j=0,\ \forall j \in \mathscr{A}\}}\, dZ_s^i = 0 \ \text{ a.s.} \tag{2.3}$$

for each $i \in \mathscr{A}$, with $\mathbf{1}_{\{\cdot\}}$ denoting as usual the corresponding indicator function.

As it will be seen in Section 5, the boundary property in equation (2.3) plays an important role in the establishment of the results in the paper.

Finally, we identify the following conditions on the coefficients of δ and γ and on the diffusion matrix a.

Condition 2.3. *Each δ_{ij} is such that, whenever $x = (x_l)_{l=1}^n \in \mathbb{R}^n_+$ and $y = (y_l)_{l=1}^n \in \mathbb{R}^n_+$ with $x_i \leq y_i$,*

$$x_i + \delta_{ij}(t,x,r_j) \leq y_i + \delta_{ij}(t,y,r_j), \ \ r_j \in E_j, \ \ t \geq 0.$$

Condition 2.4. *There exist measurable functions $\{\eta_{ij}\}_{i,j=1}^n$, mapping $\mathbb{R}^2_+$ into $\mathbb{R}$, such that for each i,j*

$$\gamma_{ij}(t,x) = \eta_{ij}(t,x_i), \ \ x = (x_l)_{l=1}^n \in \mathbb{R}^n_+, \ \ t \geq 0.$$

Condition 2.5. *The diffusion matrix a is positive definite for each $x \in \mathbb{R}^n_+$ and $t \geq 0$, i.e.,*

$$\sum_{i,j} a_{ij}(t,x)\xi_i\xi_j > 0, \ \ \xi = (\xi_l)_{l=1}^n \in \mathbb{R}^n, \ \ \xi \neq 0.$$

[8]The map $U \to (X,Z)$ is generally known as the solution mapping of the SP (for a given R) or, in queueing theory jargon, as the reflection map (see [25; 26]). In this same jargon, the Z component is usually referred to as the regulator process.

[9]Note the continuity of Z is ensured from the facts that $X_0 \in \mathbb{R}^n_+$ and that jumps cannot take X outside the orthant.

As mentioned before, since the Poisson jump measures $\{N_i(ds, dr_i)\}_{i=1}^n$ are independent, any number of them do not "jump" simultaneously at any time (a. s.). Therefore, Condition 2.3 is useful when comparing processes X and $\widetilde{X}$, coming from

$$(X,Z) = RSDE(X_0, b, R) \text{ and } (\widetilde{X}, \widetilde{Z}) = RSDE(\widetilde{X}_0, \widetilde{b}, \widetilde{R}),$$

as it ensures that for each i

$$(X - \widetilde{X})_s^i (X - \widetilde{X})_{s-}^i \geq 0$$

at any jump instant (i.e., that jumps cannot alter the order between corresponding components). In this same comparison context, Condition 2.4 will guarantee for the semi-martingale local time at level zero associated with each difference $(X - \widetilde{X})^i$ to be null. It also makes each γ_{ij} independent of the position over the corresponding i-th face F_i, and hence each diagonal diffusion coefficient a_{ii} too. Condition 2.4 is required for comparisons even in the case of normally reflected jump-diffusions in the orthant (see [23]), and it encompasses the important case of a product-form setting (see [17; 19]) in queueing network applications.

In addition to play a role in the establishment of relationship (2.3) above, Condition 2.5 also guarantees for reflections from the boundary to be instantaneous (see [18; 19]).

3 Main Results: Comparison Properties

We establish in this section the main results of the paper, regarding comparison properties between different pairs

$$(X,Z) = RSDE(X_0, b, R) \text{ and } (\widetilde{X}, \widetilde{Z}) = RSDE(\widetilde{X}_0, \widetilde{b}, \widetilde{R}).$$

The following additional notation will be used from now on in the paper, with I denoting the identity matrix in $\mathbb{R}^{n\times n}$.

Notation. *Consider*

$$(X,Z) = RSDE(X_0, b, R) \text{ and } (\widetilde{X}, \widetilde{Z}) = RSDE(\widetilde{X}_0, \widetilde{b}, \widetilde{R})$$

with relationship (2.2) in Condition 2.1 holding, respectively, and define the mapping $\psi = (\psi_1, \ldots, \psi_n) : [0,\infty) \times \mathbb{R}_+^n \to \mathbb{R}^n$ *(resp.,* $\widetilde{\psi}$*) by setting each* ψ_i *(resp.,* $\widetilde{\psi}_i$*) as the net-drift including jumps in the i-th coordinate, i.e.,*

$$\psi_i(t,x) \doteq b_i(t,x) + \sum_j \lambda_j \int_{E_j} \delta_{ij}(t,x,r_j) G_j(dr_j), \;\; x \in \mathbb{R}_+^n, \;\; t \geq 0,$$

and similarly for $\widetilde{\psi}$ *with* $\widetilde{b}$ *in place of* b*. We write*

$$(X_0, \psi, R) \preceq (\widetilde{X}_0, \widetilde{\psi}, \widetilde{R})$$

whenever[10]

$$X_0 \leq \widetilde{X}_0 \;\text{ a.s.}, \;\; \psi(t,x) \leq \widetilde{\psi}(t,y) \text{ and } R(t,x) \leq \widetilde{R}(t,y)$$

[10] Recall that inequalities among vectors and matrices are understood to hold componentwise and elementwise, respectively. Since we take any reflection matrix as to have normalized to one diagonal elements, inequalities among them involve then off-diagonal elements only.

as $x, y \in \mathbb{R}^n_+$ with $x \le y$ and $t \ge 0$. If in addition $R(\cdot,\cdot) \le I \le \widetilde{R}(\cdot,\cdot)$, then we write

$$(X_0, \psi, R) \preceq_I (\widetilde{X}_0, \widetilde{\psi}, \widetilde{R}).$$

Finally, we write

$$d\widetilde{Z} << dZ$$

when the random measure each $\widetilde{Z}^i_\cdot$ induces in $[0, \infty)$ is absolutely continuous with respect to the corresponding one associated to $Z^i_\cdot$, denote by

$$\frac{d\widetilde{Z}^i}{dZ^i}$$

the related Radon-Nikodym derivatives, and write

$$\frac{d\widetilde{Z}}{dZ} \le 1 \text{ a.s.}$$

when each Radon-Nikodym derivative above is less than or equal to 1 a.s.

In order not to opaque the continuity in the exposition of the results, we postpone their corresponding proofs to Section 5. We begin with the case when $\widetilde{R}(\cdot,\cdot) \equiv I$.

Theorem 3.1. *Let*

$$(X, Z) = RSDE(X_0, b, R) \text{ and } (\widetilde{X}, \widetilde{Z}) = RSDE(\widetilde{X}_0, \widetilde{b}, I),$$

both under Conditions 2.1 to 2.4, respectively. Assume that

$$(X_0, \psi, R) \preceq (\widetilde{X}_0, \widetilde{\psi}, I).$$

Then we have

$$\mathbb{P}\left\{X_t \le \widetilde{X}_t,\ t \ge 0\right\} = 1,\ \ d\widetilde{Z} << dZ \text{ and } \frac{d\widetilde{Z}}{dZ} \le 1 \text{ a.s.}$$

Remark 3.2. *Note $d\widetilde{Z} << dZ$ with*

$$\frac{d\widetilde{Z}}{dZ} \le 1 \text{ a.s.}$$

is equivalent to

$$\mathbb{P}\left\{Z_t - Z_s \ge \widetilde{Z}_t - \widetilde{Z}_s,\ t \ge s \ge 0\right\} = 1,$$

which in particular implies

$$\mathbb{P}\left\{Z_t \ge \widetilde{Z}_t,\ t \ge 0\right\} = 1.$$

The next result considers the case when $R(\cdot,\cdot) \equiv I$. In that case, a full comparison between the tuples (X, Z) and $(\widetilde{X}, \widetilde{Z})$ is possible when $\widetilde{R}(\cdot,\cdot)$ is constant, a partial comparison being possible otherwise. As it will become clear in Section 5, the main difficulty in getting a full comparison for non-constant $\widetilde{R}(\cdot,\cdot)$ relies on the fact that the usual alternative characterization of $(\widetilde{Z}_t)_{t\ge 0}$ ($\widetilde{Z}_0 = 0$) when $\widetilde{R}(\cdot,\cdot)$ is constant (say $\widetilde{R}(\cdot,\cdot) \equiv \widetilde{R}$), as being the (unique) pathwise-minimum non-decreasing continuous process satisfying (see for example [25; 26])

$$\widetilde{U}_t + \widetilde{R}\widetilde{Z}_t \ge 0,\ \ t \ge 0,$$

with

$$\widetilde{U}_\cdot \doteq \widetilde{X}_0 + \int_0^\cdot \widetilde{b}(s,\widetilde{X}_{s-})ds + \int_0^\cdot \gamma(s,\widetilde{X}_{s-})dW_s + \int_0^\cdot \int_E \delta(s,\widetilde{X}_{s-},r)N(ds,dr),$$

is in general not guaranteed for non-constant $\widetilde{R}(\cdot,\cdot)$ (see for example [21]).

Theorem 3.3. *Let*

$$(X,Z) = RSDE(X_0,b,I) \text{ and } (\widetilde{X},\widetilde{Z}) = RSDE(\widetilde{X}_0,\widetilde{b},\widetilde{R}),$$

both under Conditions 2.1 to 2.4, respectively. Assume that

$$(X_0,\psi,I) \preceq (\widetilde{X}_0,\widetilde{\psi},\widetilde{R}).$$

Then we have

$$\mathbb{P}\left\{X_t \le \widetilde{X}_t,\ t \ge 0\right\} = 1.$$

If moreover $\widetilde{R}(\cdot,\cdot)$ *is constant, then we also have*

$$d\widetilde{Z} << dZ \text{ and } \frac{d\widetilde{Z}}{dZ} \le 1 \text{ a.s.}$$

The following corollary is a direct consequence of Theorems 3.1 and 3.3.

Corollary 3.4. *Let*

$$(X,Z) = RSDE(X_0,b,R) \text{ and } (\widetilde{X},\widetilde{Z}) = RSDE(\widetilde{X}_0,\widetilde{b},\widetilde{R}),$$

both under Conditions 2.1 to 2.4, respectively. Assume that

$$(X_0,\psi,R) \preceq_I (\widetilde{X}_0,\widetilde{\psi},\widetilde{R}).$$

Then we have

$$\mathbb{P}\left\{X_t \le \widetilde{X}_t,\ t \ge 0\right\} = 1.$$

If moreover $\widetilde{R}(\cdot,\cdot)$ *is constant, then we also have*

$$d\widetilde{Z} << dZ \text{ and } \frac{d\widetilde{Z}}{dZ} \le 1 \text{ a.s.}$$

Finally, the next result shows that a full comparison is possible for non-constant oblique reflection directions, provided reflections upon hitting each boundary tend to bring the remaining coordinates closer to the origin. In order to compare X and $\widetilde{X}$ we generally require in this case an ordering at the boundaries on the drift vectors b and $\widetilde{b}$, as it will become clear in Section 5, ensuring in turn a pathwise comparison between the increments of the processes Z and $\widetilde{Z}$. It is in this context where the boundary property in equation (2.3) plays an important role. The result is the following.

Theorem 3.5. *Let*

$$(X,Z) = RSDE(X_0,b,R) \text{ and } (\widetilde{X},\widetilde{Z}) = RSDE(\widetilde{X}_0,\widetilde{b},\widetilde{R}),$$

both under Conditions 2.1 to 2.5, respectively. Assume that

$$(X_0, \psi, R) \preceq (\widetilde{X}_0, \widetilde{\psi}, \widetilde{R})$$

with $\widetilde{R}(\cdot,\cdot) \leq I$, *and that further each pair of drift coefficients* b_i *and* $\widetilde{b}_i$ *satisfies the same ordering as* ψ_i *and* $\widetilde{\psi}_i$ *but only on the corresponding* i*-th face* F_i, *i.e., that*[11]

$$b_i(t,x) \leq \widetilde{b}_i(t,y), \;\; x,y \in F_i, \;\; x \leq y, \;\; t \geq 0,$$

for each i. *Then we have*

$$\mathbb{P}\left\{X_t \leq \widetilde{X}_t, \; t \geq 0\right\} = 1, \;\; d\widetilde{Z} << dZ \;\; \textit{and} \;\; \frac{d\widetilde{Z}}{dZ} \leq 1 \;\; \text{a.s.}$$

4 Stability Applications: An Ergodic Result

In this section we use the comparison results of Section 3 to establish an ergodicity criterium related to solutions of RSDEs as in equation (2.1), the main idea being to exploit those comparison properties, and the stability results in [1] for the constant reflection directions case, to provide a simple but useful ergodicity condition in the context of state-dependent directions of reflection.

For simplicity we consider the continuous case, i.e., in absence of jumps ($\delta(\cdot,\cdot,\cdot) \equiv 0$). Therefore, we consider RSDEs of the form

$$X_t = X_0 + \int_0^t b(X_s)ds + \int_0^t \gamma(X_s)dW_s + \int_0^t R(X_s)dZ_s, \;\; t \geq 0, \tag{4.1}$$

where of course coefficients are assumed to be time-independent. Note when b and γ are constant, X in equation (4.1) reduces to a Semi-martinagle Reflecting Brownian Motion (SRBM) in the orthant with state-dependent reflection directions (see [24]).

We denote by X^x the process X in equation (4.1) when starting from $X_0 = x \in \mathbb{R}^n_+$, whose existence and uniqueness is ensured under Conditions 2.1 and 2.2, and introduce accordingly and as usual the family of distributions $\{\mathbb{P}_x : x \in \mathbb{R}^n_+\}$ on the path space of continuous functions mapping $[0,\infty)$ into $\mathbb{R}^n_+$, denoting as $\mathbb{E}_x$ expectation with respect to $\mathbb{P}_x$. Also, for each $x \in \mathbb{R}^n_+$ and $t \geq 0$, we write $P_x(t,\cdot)$ for the law of X^x_t in $\mathbb{R}^n_+$, i.e.,

$$P_x(t,A) \doteq \mathbb{P}\left\{X^x_t \in A\right\} = \mathbb{P}_x\left\{X_t \in A\right\}, \;\; A \in \mathscr{B}(\mathbb{R}^n_+),$$

abusing notation in the last equality[12]. (Note $P_x(0,\cdot) = \delta_x(\cdot)$, unit mass at $x \in \mathbb{R}^n_+$.)

We will consider in this section a boundedness condition on b and γ, and a uniform non-degeneracy condition on the corresponding diffusion matrix a ($= \gamma\gamma^T$).

[11] Note this condition is in particular implied by the ordering on ψ and $\widetilde{\psi}$ when jump-amplitude coefficients (δ) do not depend on the state.

[12] Though $\mathbb{P}_x$ is a probability measure on the path space, the identification $X_0 = x$ under $\mathbb{P}_x$ is standard (through the canonical representation).

Condition 4.1. *b and γ are bounded, i.e.,*

$$\sup_{x\in\mathbb{R}^n_+}\|b(x)\|+\sup_{x\in\mathbb{R}^n_+}\|\gamma(x)\|<\infty.$$

Moreover, the diffusion matrix a is uniformly elliptic, i.e., there exists $\varsigma\in(0,\infty)$ such that

$$\sum_{i,j}a_{ij}(x)\xi_i\xi_j\geq\varsigma\|\xi\|^2$$

for all $x\in\mathbb{R}^n_+$ and $\xi=(\xi_l)_{l=1}^n\in\mathbb{R}^n$.

Conditions 2.1 and 2.2, along with the boundedness requirement in Condition 4.1, guarantee the family $\{X^x:x\in\mathbb{R}^n_+\}$ satisfies the Feller property[13]. Indeed, from [21, Proposition 3.2, pp. 515] and using the Burkholder-Davis-Gundy inequalities [9, Theorem 26.12, pp. 524], it is easy to see that there exists a constant $0<C<\infty$ such that, for all $t\geq 0$ and all $x,y\in\mathbb{R}^n_+$,

$$\mathbb{E}\left[\sup_{0\leq s\leq t}\|X^x_s-X^y_s\|^2\right]\leq C\left\{\|x-y\|^2+t+t^2\right\}. \tag{4.2}$$

Also, from [6, Theorem 5.1, pp. 572] we know for each $0<T<\infty$ there exists a constant $0<C_T<\infty$ such that, for all $0\leq t\leq T$,

$$\mathbb{E}\left[\sup_{0\leq s\leq t}\|X^x_s-X^y_s\|^2\right]\leq C_T\left\{\|x-y\|^2+\int_0^t\mathbb{E}\left[\sup_{0\leq u\leq s}\|X^x_u-X^y_u\|^2\right]ds\right\}.$$

Gronwall's lemma then shows that

$$\mathbb{E}\left[\sup_{0\leq s\leq t}\|X^x_s-X^y_s\|^2\right]\leq C_T\|x-y\|^2\exp\left\{C_T\int_0^t\mathbb{E}\left[\sup_{0\leq u\leq s}\|X^x_u-X^y_u\|^2\right]ds\right\},$$

and therefore, on invoking again (4.2), and the arbitrariness of $0<T<\infty$, we conclude

$$\mathbb{E}[\|X^x_t-X^y_t\|^2]\to 0\ \text{ as } \|x-y\|\searrow 0,$$

for each $t>0$ and all $x,y\in\mathbb{R}^n_+$. The Feller property of the family $\{X^x:x\in\mathbb{R}^n_+\}$ then follows from standard arguments (see for example [15], proof of Lemma 8.1.4, pp. 133-134.).

On the other hand, the uniform ellipticity requirement in Condition 4.1, along with Conditions 2.1 and 2.2, in particular guarantee irreducibility in that the probability measure $P_x(t,\cdot)$ and Lebesgue measure in $\mathbb{R}^n_+$ are, for each $x\in\mathbb{R}^n_+$ and $t>0$, mutually absolutely continuous.

We are now in position to state and prove the advertised ergodic result.

Theorem 4.2. *Consider the family $\{X^x:x\in\mathbb{R}^n_+\}$ as above, under Conditions 2.1, 2.2, 2.4 and 4.1. Set*

$$\widetilde{b}_i\doteq\sup_{x\in\mathbb{R}^n_+}b_i(x)\ \textit{for each } i,$$

[13]Note then, since each X^x is $(\mathscr{F}_t)_{t\geq 0}$-adapted with $(\mathscr{F}_t)_{t\geq 0}$ satisfying the usual hypotheses, the Markov family $\{X^x:x\in\mathbb{R}^n_+\}$ is indeed strong-Markov.

and assume that there exists $\widetilde{R} = (\widetilde{R}_{ij})_{i,j=1}^n \in \mathbb{R}^{n\times n}$ *with* $\widetilde{R} \le I$, $\widetilde{R}_{ii} = 1$ *for each* i *and* $\sigma(\widetilde{R} - I) < 1$, *and such that* $R(\cdot) \le \widetilde{R}$ *and*[14], *with* $\widetilde{b} \doteq (\widetilde{b}_i)_{i=1}^n \in \mathbb{R}^n$,

$$\widetilde{R}^{-1}\widetilde{b} < 0. \tag{4.3}$$

Then there exists a unique invariant distribution for the family $\{X^x : x \in \mathbb{R}^n_+\}$, *in that there exists a unique probability measure* π *on* $(\mathbb{R}^n_+, \mathscr{B}(\mathbb{R}^n_+))$ *such that*

$$\int_{\mathbb{R}^n_+} \mathbb{E}_x \left[f(X_t) \right] \pi(dx) = \int_{\mathbb{R}^n_+} f(x)\pi(dx)$$

for all[15] $f \in \mathscr{C}_b(\mathbb{R}^n_+)$. *Moreover, for each initial distribution* π_0 *on* $(\mathbb{R}^n_+, \mathscr{B}(\mathbb{R}^n_+))$ *we have*

$$\lim_{t\nearrow\infty} \int_{\mathbb{R}^n_+} P_x(t,A)\pi_0(dx) = \pi(A)$$

for each $A \in \mathscr{B}(\mathbb{R}^n_+)$, *and therefore in particular we have that the measure* $\int_{\mathbb{R}^n_+} P_x(t,\cdot)\pi_0(dx)$ *converges weakly to* π *as* t *increases to infinity, i.e.,*

$$\lim_{t\nearrow\infty} \int_{\mathbb{R}^n_+} \mathbb{E}_x \left[f(X_t) \right] \pi_0(dx) = \int_{\mathbb{R}^n_+} f(x)\pi(dx)$$

for each $f \in \mathscr{C}_b(\mathbb{R}^n_+)$.

Before giving the proof of the theorem we make the following remark.

Remark 4.3. *Note the key ergodicity condition in Theorem 4.2, equation (4.3), does not coincide in the case of constant directions of reflection, say* $R(\cdot) \equiv \widetilde{R}$, *with the corresponding one in [1], which reads in this case*

$$\sup_{x\in\mathbb{R}^n_+} \theta_i(x) < 0 \text{ for each } i,$$

with $\theta_i(\cdot)$ *the* i*-th component of* $\widetilde{R}^{-1}b(\cdot)$, *i.e., with the supremun being pulled out in equation (4.3). This is a consequence of supporting our result in an auxiliary comparison, as it will be done in the proof below. However, as mentioned at the beginning of the section, equation (4.3) provides a useful ergodicity condition for the case of applications with state-dependent reflection directions.*

Proof. Consider for each $x \in \mathbb{R}^n_+$ the auxiliary RSDE given by

$$\widetilde{X}^x_t = x + \widetilde{b}t + \int_0^t \gamma(\widetilde{X}^x_s)dW_s + \widetilde{R}\widetilde{Z}^x_t, \ \ t \ge 0,$$

whose well-posedness is straightforwardly ensured, and write $\widetilde{P}_x(t,\cdot)$, $t \ge 0$, for the corresponding laws. From the theorem's assumptions, the Lipschitz continuity of the Skorokhod map in this constant reflection directions case (see [26]) and [1, Theorem 2.16, pp. 8], we conclude the tightness, for each $M \in (0,\infty)$, of the family of probability measures

$$\left\{\widetilde{P}_x(t,\cdot) : \|x\| \le M, t \ge 0\right\},$$

[14]Note the structure of $\widetilde{R}$ not only guarantees for $\widetilde{R}^{-1}$ to exist, but also to be (elementwise) non-negative (see for example [2]).

[15]As usual, $\mathscr{C}_b(\mathbb{R}^n_+)$ denotes the space of bounded continuous functions from $\mathbb{R}^n_+$ into $\mathbb{R}$.

which in turn ensures, since by Theorem 3.5 in Section 3 we have

$$X^x \le \widetilde{X}^x \quad \text{a.s.},$$

the corresponding tightness of the family

$$\left\{P_x(t,\cdot) : \|x\| \le M, t \ge 0\right\}.$$

The above tightness, along with the theorem's assumptions and the Feller structure of the family $\{X^x : x \in \mathbb{R}^n_+\}$, then give the corresponding existence of an invariant distribution π (see [7]). The convergence

$$\lim_{t \nearrow \infty} \int_{\mathbb{R}^n_+} P_x(t,A)\pi_0(dx) = \pi(A)$$

for each initial distribution π_0 and $A \in \mathscr{B}(\mathbb{R}^n_+)$, and therefore the uniqueness of π, follow then in turn from [13, Theorems 1.1. and 1.3, pp. 142 and 144, resp.]. That in particular the measure

$$\int_{\mathbb{R}^n_+} P_x(t,\cdot)\pi_0(dx)$$

converges weakly to π as t increases to infinity, is a direct consequence of Portmanteau's theorem (see for example [3]). ■

5 Proofs of the Main Results

In this section we give the proofs of the main results of the paper in Section 3. To that aim we first establish the following lemma.

Lemma 5.1. *Let*

$$(X,Z) = RSDE(X_0,b,R) \ \text{and} \ (\widetilde{X},\widetilde{Z}) = RSDE(\widetilde{X}_0,\widetilde{b},\widetilde{R}),$$

both under Conditions 2.1 to 2.4, respectively, and with $X_0 \le \widetilde{X}_0$ a.s. *Then for each constant* $N \ge 0$*, index* i *and* $t \ge 0$ *we have both*[16]

$$\mathbb{E}[(\phi^i_{t\wedge T_N\wedge T})^+] \le \sum_{j\neq i} \mathbb{E}\left[\int_0^{t\wedge T_N\wedge T} \mathbf{1}_{\{\phi^i_s>0\}} R_{ij}(s,X_s)[dZ^j_s - d\widetilde{Z}^j_s]\right] \tag{5.1}$$

and

$$\mathbb{E}[(\phi^i_{t\wedge T_N\wedge T})^+] \le \sum_{j\neq i} \mathbb{E}\left[\int_0^{t\wedge T_N\wedge T} \mathbf{1}_{\{\phi^i_s>0\}} \widetilde{R}_{ij}(s,\widetilde{X}_s)[dZ^j_s - d\widetilde{Z}^j_s]\right], \tag{5.2}$$

where

$$\phi^i_t \doteq X^i_t - \widetilde{X}^i_t, \ \ t \ge 0,$$

and where the $(\mathscr{F}_t)_{t\ge 0}$*-stopping times* T_N *and* T *are defined as*[17]

$$T_N \doteq \inf\left\{t > 0 : \|X_t\|_1 + \|\widetilde{X}_t\|_1 + \|Z_t\|_1 + \|\widetilde{Z}_t\|_1 > N\right\}$$

[16] $(x)^+ \doteq \max\{x,0\}$, $x \wedge y \doteq \min\{x,y\}$ and $x \wedge \infty \doteq x$, $x, y \in \mathbb{R}$.
[17] $\|x\|_1 \doteq \sum_l |x_l|$, $x = (x_l)^n_{l=1} \in \mathbb{R}^n$, and $\inf\emptyset \doteq \infty$.

and

$$T \doteq T^\diamond \wedge T^* \wedge T^\dagger,$$

with $T^\diamond$ *any* $(\mathscr{F}_t)_{t\geq 0}$*-stopping time*[18],

$$T^* \doteq \inf\left\{t > 0 : \psi_l(t,X_t) > \widetilde{\psi}_l(t,\widetilde{X}_t)\ for\ some\ l\right\}$$

and

$$T^\dagger \doteq \inf\left\{t > 0 : R_{lm}(t,X_t) > \widetilde{R}_{lm}(t,\widetilde{X}_t)\ for\ some\ l,m\right\}.$$

Proof. Consider an index i, fixed throughout the proof. Note since ϕ^i is clearly a semi-martingale with

$$\sum_{0<s\leq t} |\Delta\phi^i_s| \leq \sum_{0<s\leq t} [|\Delta X^i_s| + |\Delta\widetilde{X}^i_s|] < \infty \ \text{ a.s.}, \ \ t > 0,$$

the (jointly) right-continuous in y ($\in \mathbb{R}$) and continuous in t ($\in [0,\infty)$) version of the local time associated to ϕ^i, with y indicating the corresponding level, exists (see [20]). We denote it by $L_{\phi^i} = (L_{\phi^i}(t,y))_{t\geq 0, y\in\mathbb{R}}$. In order to prove the lemma, we first verify that

$$L_{\phi^i}(\cdot,0) \equiv 0 \ \text{ a.s.}$$

Indeed, denote by $([\phi^i,\phi^i]^c_t)_{t\geq 0}$ the path-by-path continuous part of the quadratic variation process $([\phi^i,\phi^i]_t)_{t\geq 0}$ with $[\phi^i,\phi^i]^c_0 \doteq 0$, and note that

$$[\phi^i,\phi^i]^c_\cdot = \sum_j \int_0^\cdot [\gamma_{ij}(s,X_s) - \gamma_{ij}(s,\widetilde{X}_s)]^2 ds \ \text{ a.s.}$$

and that, from Conditions 2.1 and 2.4,

$$\sum_j [\gamma_{ij}(s,X_s) - \gamma_{ij}(s,\widetilde{X}_s)]^2 = \sum_j [\eta_{ij}(s,X^i_s) - \eta_{ij}(s,\widetilde{X}^i_s)]^2 \leq K^2[\phi^i_s]^2, \ \ s \geq 0.$$

Define the mapping $\rho : (0,\infty) \to (0,\infty)$ by setting

$$\rho(u) \doteq K^2 u^2, \ \ u \in (0,\infty).$$

Let $\epsilon > 0$ and note that

$$\int_{(0,\epsilon]} [\rho(u)]^{-1} du = \infty. \tag{5.3}$$

Now, with

$$I^i_t \doteq \int_0^t \mathbf{1}_{\{0<\phi^i_s\leq\epsilon\}} [\rho(\phi^i_s)]^{-1} d[\phi^i,\phi^i]^c_s, \ \ t \geq 0,$$

we have a. s.

$$\begin{aligned} I^i_t &= \int_0^t \mathbf{1}_{\{0<\phi^i_s\leq\epsilon\}} [\rho(\phi^i_s)]^{-1} \sum_j [\gamma_{ij}(s,X_s) - \gamma_{ij}(s,\widetilde{X}_s)]^2 ds \\ &\leq K^2 \int_0^t \mathbf{1}_{\{0<\phi^i_s\leq\epsilon\}} [\rho(\phi^i_s)]^{-1} [\phi^i_s]^2 ds \leq t < \infty. \end{aligned} \tag{5.4}$$

[18] For convenience $T^\diamond$ is left here unfixed, being chosen appropriately when applying the lemma.

On the other hand, by the occupation times formula of semi-martingale local times (see [20])

$$I_t^i = \int_{(0,\epsilon]} [\rho(u)]^{-1} L_{\phi^i}(t,u)du \ \text{ a.s.}, \ \ t \geq 0,$$

and therefore, in light of equations (5.3) and (5.4) and since

$$L_{\phi^i}(t,u) \to L_{\phi^i}(t,0) \ \text{ as } \ u \searrow 0 \ \text{ a.s.}, \ \ t \geq 0,$$

we conclude, by the same arguments as in [22, Lemma 3.3, pp. 389], that

$$L_{\phi^i}(t,0) = 0 \ \text{ a.s.}, \ \ t \geq 0.$$

The claim then follows by invoking the sample path continuity of $L_{\phi^i}(\cdot,0)$. We now turn into proving the lemma. From Meyer-Itô's formula (see [20]) we have[19]

$$\begin{aligned}(\phi^i_{\cdot\wedge T_N \wedge T})^+ - (\phi^i_0)^+ = \int_{0+}^{\cdot\wedge T_N\wedge T} \mathbf{1}_{\{\phi^i_{s-}>0\}}\, d\phi^i_s + \frac{1}{2} L_{\phi^i}(\cdot \wedge T_N \wedge T, 0) \\ + \sum_{0<s\leq \cdot\wedge T_N\wedge T} \mathbf{1}_{\{\phi^i_{s-}>0\}} (\phi^i_s)^- + \sum_{0<s\leq \cdot\wedge T_N\wedge T} \mathbf{1}_{\{\phi^i_{s-}\leq 0\}} (\phi^i_s)^+ \ \text{ a.s.}\end{aligned}$$

But, from Condition 2.3 we have

$$\sum_{0<s\leq \cdot\wedge T_N\wedge T} \mathbf{1}_{\{\phi^i_{s-}>0\}} (\phi^i_s)^- = \sum_{0<s\leq \cdot\wedge T_N\wedge T} \mathbf{1}_{\{\phi^i_{s-}\leq 0\}} (\phi^i_s)^+ \equiv 0 \ \text{ a.s.}$$

and, since also $(\phi^i_0)^+ = 0$ and $L_{\phi^i}(\cdot,0) \equiv 0$ a.s., it is therefore easy to see that

$$\mathbb{E}[(\phi^i_{t\wedge T_N\wedge T})^+] \leq \sum_j \mathbb{E}\left[\int_0^{t\wedge T_N\wedge T} \mathbf{1}_{\{\phi^i_s>0\}} [R_{ij}(s,X_s)dZ^j_s - \widetilde{R}_{ij}(s,\widetilde{X}_s)d\widetilde{Z}^j_s]\right]$$

for each $t \geq 0$, where we have replaced X_{s-} by X_s since X is *càdlàg* and Z is continuous, and similarly for $\widetilde{X}$ and $\widetilde{Z}$. Thus, by writing

$$R_{ij}(s,X_s)dZ^j_s - \widetilde{R}_{ij}(s,\widetilde{X}_s)d\widetilde{Z}^j_s = R_{ij}(s,X_s)[dZ^j_s - d\widetilde{Z}^j_s] + [R_{ij}(s,X_s) - \widetilde{R}_{ij}(s,\widetilde{X}_s)]d\widetilde{Z}^j_s$$

and using that

$$\mathbf{1}_{\{\phi^i_s>0\}} R_{ii}(s,X_s)[dZ^i_s - d\widetilde{Z}^i_s] = \mathbf{1}_{\{\phi^i_s>0\}} [dZ^i_s - d\widetilde{Z}^i_s]$$

and that from the definition of Z and $\widetilde{Z}$ we have

$$\begin{aligned}\mathbf{1}_{\{\phi^i_s>0\}}[dZ^i_s - d\widetilde{Z}^i_s] &= \mathbf{1}_{\{X^i_s - \widetilde{X}^i_s>0\}}\, dZ^i_s - \mathbf{1}_{\{X^i_s - \widetilde{X}^i_s>0\}}\, d\widetilde{Z}^i_s \\ &= -\mathbf{1}_{\{X^i_s - \widetilde{X}^i_s>0\}}\, d\widetilde{Z}^i_s \\ &\leq 0\end{aligned}$$

[19] $(x)^- \doteq -\min\{x,0\}$, $x \in \mathbb{R}$, $\int_{0^+}^t \doteq \int_{(0,t]}$, $t > 0$, and $\int_{0^+}^0 = \sum_{0<s\leq 0} \doteq 0$.

and also, on $\{T^\dagger > 0\}$,

$$[R_{ij}(s,X_s) - \widetilde{R}_{ij}(s,\widetilde{X}_s)]d\widetilde{Z}^j_s \le 0, \;\; s \in (0,T^\dagger) \cap (0,\infty),$$

we obtain equation (5.1). In the same way, by writing

$$R_{ij}(s,X_s)dZ^j_s \;-\; \widetilde{R}_{ij}(s,\widetilde{X}_s)d\widetilde{Z}^j_s \quad = \quad [R_{ij}(s,X_s) \;-\; \widetilde{R}_{ij}(s,\widetilde{X}_s)]dZ^j_s \;+\; \widetilde{R}_{ij}(s,\widetilde{X}_s)[dZ^j_s \;-\; d\widetilde{Z}^j_s]$$

we find, by similar arguments than before, equation (5.2). The lemma is then proved. ■

Having established the lemma, we now give the proofs of the results in Section 3. Set $\mathbb{N}^* \doteq \{1,2,\ldots\}$ and, for each $k \in \mathbb{N}^*$, index i and $x = (x_l)_{l=1}^n \in \mathbb{R}^n_+$,

$$x^{(k,i)} \doteq (x_1,\ldots,x_{i-1},x_i - k^{-1},x_{i+1},\ldots,x_n).$$

In addition,

$$F_i^{(k)} \doteq \left\{x = (x_l)_{l=1}^n \in \mathbb{R}^n_+ : x_i \le k^{-1}\right\},$$

for each $k \in \mathbb{N}^*$ and index i as well.

Proof of Theorem 3.1. We first consider the comparison between[20]

$$(X^{(k)},Z^{(k)}) \doteq RSDE(X_0,b,\gamma^{(k)},R) \tag{5.5}$$

and

$$(\widetilde{X}^{(k)},\widetilde{Z}^{(k)}) \doteq RSDE(\widetilde{X}_0,\widetilde{b}^{(k)},\gamma^{(k)},I), \tag{5.6}$$

where for each $k \in \mathbb{N}^*$, $x \in \mathbb{R}^n_+$ and $t \ge 0$,

$$\widetilde{b}^{(k)}_i(t,x) \doteq \widetilde{b}_i(t,x) + k^{-1}$$

for each i, and

$$\gamma^{(k)}_{ij}(t,x) \doteq \begin{cases} \eta_{ij}(t,0) & \text{if } x \in F_i^{(k)} \\ \gamma_{ij}(t,x^{(k,i)}) = \eta_{ij}(t,x_i - k^{-1}) & \text{elsewhere} \end{cases}$$

for each i,j, with $\{\eta_{ij}\}_{i,j=1}^n$ taken from Condition 2.4. Note (5.5) and (5.6) are clearly well defined since the linear growth and Lipschitz continuity conditions are correspondingly inherited. As in Section 3, we associate $\psi^{(k)}$ and $\widetilde{\psi}^{(k)}$ to (5.5) and (5.6), respectively, and note that $\psi^{(k)} \equiv \psi$ and that, since

$$\psi(t,x) \le \widetilde{\psi}(t,y)$$

as $x,y \in \mathbb{R}^n_+$ with $x \le y$ and $t \ge 0$, we have

$$\psi^{(k)}(t,x) < \widetilde{\psi}(t,y) + k^{-1} = \widetilde{\psi}^{(k)}(t,y)$$

for each $k \in \mathbb{N}^*$, as $x,y \in \mathbb{R}^n_+$ with $x \le y$ and $t \ge 0$ as well. Fix now a $k \in \mathbb{N}^*$. From Lemma 5.1, equation (5.2), applied to

$$(X^{(k)},Z^{(k)}) = ((X^{(k),l})_{l=1}^n,(Z^{(k),l})_{l=1}^n)$$

and

$$(\widetilde{X}^{(k)}, \widetilde{Z}^{(k)}) = ((\widetilde{X}^{(k),l})_{l=1}^{n}, (\widetilde{Z}^{(k),l})_{l=1}^{n}),$$

and with $T^{\diamond} \doteq \infty$, we have[21]

$$\mathbb{E}[(X^{(k),i}_{t\wedge T_N\wedge T^*} - \widetilde{X}^{(k),i}_{t\wedge T_N\wedge T^*})^+] \leq 0, \;\; t \geq 0,$$

for each i. By letting $N \nearrow \infty$, Fatou's lemma then shows that

$$X^{(k)}_{t\wedge T^*} \leq \widetilde{X}^{(k)}_{t\wedge T^*} \;\; \text{a.s.}, \;\; t \geq 0,$$

and therefore right-continuity gives

$$X^{(k)}_{\cdot} \leq \widetilde{X}^{(k)}_{\cdot} \;\; \text{on} \;\; [0, T^*] \cap [0, \infty) \;\; \text{a.s.}$$

In particular,

$$\psi^{(k)}(T^*, X^{(k)}_{T^*}) < \widetilde{\psi}^{(k)}(T^*, \widetilde{X}^{(k)}_{T^*}) \;\; \text{on} \;\; \{T^* < \infty\}. \tag{5.7}$$

Since

$$\psi^{(k)}(0, X^{(k)}_0) = \psi(0, X_0) < \widetilde{\psi}^{(k)}(0, \widetilde{X}_0) = \widetilde{\psi}^{(k)}(0, \widetilde{X}^{(k)}_0) \;\; \text{a.s.},$$

right-continuity of $X^{(k)}$ and $\widetilde{X}^{(k)}$ and continuity of $\psi^{(k)}$ and $\widetilde{\psi}^{(k)}$ then give $T^* > 0$ a.s., and therefore the consideration, on $\{T^* < \infty\}$, of

$$(X^{(k)}_{T^*+t})_{t\geq 0} \;\; \text{and} \;\; (\widetilde{X}^{(k)}_{T^*+t})_{t\geq 0}$$

shows, by a direct argument by contradiction based on equation (5.7), that $T^* = \infty$ a.s. Thus, we conclude that

$$\mathbb{P}\left\{X^{(k)}_t \leq \widetilde{X}^{(k)}_t, \; t \geq 0\right\} = 1. \tag{5.8}$$

Consider now an $s \geq 0$ and write

$$\widetilde{X}^{(k)}_{t\wedge T_N} = \widetilde{X}^{(k)}_{s\wedge T_N} + (\widetilde{U}^{(k)}_{t\wedge T_N} - \widetilde{U}^{(k)}_{s\wedge T_N}) + (\widetilde{Z}^{(k)}_{t\wedge T_N} - \widetilde{Z}^{(k)}_{s\wedge T_N}), \;\; t \geq s, \tag{5.9}$$

i.e., with

$$\widetilde{U}^{(k)}_{\cdot} \;\; \doteq \;\; \widetilde{X}_0 \; + \; \int_0^{\cdot} \widetilde{b}^{(k)}(u, \widetilde{X}^{(k)}_{u-})du \; + \; \int_0^{\cdot} \gamma^{(k)}(u, \widetilde{X}^{(k)}_{u-})dW_u \; + \; \int_0^{\cdot}\int_E \delta(u, \widetilde{X}^{(k)}_{u-}, r)N(du, dr).$$

Set

$$\widetilde{Y}^{(k)}_{t\wedge T_N} \doteq \widetilde{X}^{(k)}_{s\wedge T_N} + (\widetilde{U}^{(k)}_{t\wedge T_N} - \widetilde{U}^{(k)}_{s\wedge T_N}) + (Z^{(k)}_{t\wedge T_N} - Z^{(k)}_{s\wedge T_N}), \;\; t \geq s, \tag{5.10}$$

i.e., with the corresponding replacement of $\widetilde{Z}$ by Z in the right-hand-side of equation (5.9). (Note since $\widetilde{R}(\cdot,\cdot) \equiv I$, it has been therefore correspondingly omitted in equations (5.9) and (5.10).) Then, since $X^{(k)}_{\cdot} \leq \widetilde{X}^{(k)}_{\cdot}$ a.s., by using Meyer-Itô's formula as in the proof of Lemma 5.1, it is easy to see that, for each i,

$$\mathbb{E}[(X^{(k),i}_{t\wedge T_N} - \widetilde{Y}^{(k),i}_{t\wedge T_N})^+] \leq 0, \;\; t \geq s.$$

[21]Note $T^{\dagger}$, T^* and T_N are defined, throughout the proof, with respect to $X^{(k)}$, $\widetilde{X}^{(k)}$, $Z^{(k)}$ and $\widetilde{Z}^{(k)}$. Obviously, $T^{\dagger} = \infty$ a.s.

By letting $N \nearrow \infty$ as before, Fatou's lemma then shows that

$$\widetilde{X}_s^{(k)} + (\widetilde{U}_t^{(k)} - \widetilde{U}_s^{(k)}) + (Z_t^{(k)} - Z_s^{(k)}) \geq 0 \;\text{ a.s.}, \;\; t \geq s,$$

i.e., by right-continuity

$$\widetilde{X}_s^{(k)} + (\widetilde{U}_\cdot^{(k)} - \widetilde{U}_s^{(k)}) + (Z_\cdot^{(k)} - Z_s^{(k)}) \geq 0 \;\text{ on } [s, \infty) \;\text{ a.s.},$$

and therefore, by the alternative characterization of $(\widetilde{Z}_{s+h}^{(k)} - \widetilde{Z}_s^{(k)})_{h \geq 0}$ (see Section 3), we conclude

$$\mathbb{P}\left\{ Z_t^{(k)} - Z_s^{(k)} \geq \widetilde{Z}_t^{(k)} - \widetilde{Z}_s^{(k)}, \; t \geq s \geq 0 \right\} = 1. \tag{5.11}$$

Finally, from the uniform convergence (on $[0, \infty) \times \mathbb{R}_+^n$)

$$\widetilde{b}^{(k)} \to \widetilde{b} \;\text{ and }\; \gamma^{(k)} \to \gamma \;\text{ as } k \nearrow \infty,$$

the regularity requirements from Condition 2.1 and the continuity results for the solution mapping of the SP in the orthant with state-dependent reflection directions from [24], we have that for each $t \geq 0$ there exists a subsequence $\{n_k\}_{k=1}^{\infty}$ along which

$$\lim_{k \nearrow \infty} X_t^{(n_k)} = X_t \;\text{ and }\; \lim_{k \nearrow \infty} Z_t^{(n_k)} = Z_t \;\text{ a.s.}$$

and

$$\lim_{k \nearrow \infty} \widetilde{X}_t^{(n_k)} = \widetilde{X}_t \;\text{ and }\; \lim_{k \nearrow \infty} \widetilde{Z}_t^{(n_k)} = \widetilde{Z}_t \;\text{ a.s.}$$

The theorem then follows from equations (5.8) and (5.11) by the right-continuity of X and $\widetilde{X}$ and the continuity of Z and $\widetilde{Z}$. ■

Theorem 3.3 follows by the same corresponding arguments as in the previous proof (save the corresponding consideration of equation (5.1) in Lemma 5.1), and its proof is therefore omitted. The proof of Corollary 3.4 is also omitted, as it is a direct consequence of Theorems 3.1 and 3.3 by noting that we can always "sandwich" a function $\overline{\psi}$

$$\psi(t, x) \leq \overline{\psi}(t, y) \leq \widetilde{\psi}(t, z), \;\; x, y, z \in \mathbb{R}_+^n, \;\; x \leq y \leq z, \;\; t \geq 0,$$

with $\overline{b}$ defined by

$$\overline{b}_i(\cdot, \cdot) \doteq \overline{\psi}_i(\cdot, \cdot) - \sum_j \lambda_j \int_{E_j} \delta_{ij}(\cdot, \cdot, r_j) G_j(dr_j), \;\text{ each } i,$$

satisfying the regularity requirements in Condition 2.1, the same as b and $\widetilde{b}$. We then proceed to the proof of Theorem 3.5.

Proof of Theorem 3.5. As in the proof of Theorem 3.1, we first consider the comparison between

$$(X^{(k)}, Z^{(k)}) \doteq RSDE(X_0, b, \gamma^{(k)}, R^{(k)}) \tag{5.12}$$

and

$$(\widetilde{X}^{(k)}, \widetilde{Z}^{(k)}) \doteq RSDE(\widetilde{X}_0, \widetilde{b}^{(k)}, \gamma^{(k)}, \widetilde{R}) \tag{5.13}$$

with $\widetilde{b}^{(k)}$ and $\gamma^{(k)}$ defined the same as before and, for each $k \in \mathbb{N}^*$, $x \in \mathbb{R}^n_+$ and $t \geq 0$,

$$R^{(k)}_{ij}(t,x) \doteq \begin{cases} R_{ij}(t,x) - (k+k_0)^{-1} & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

with k_0 sufficiently large so as for $R^{(k)}$ to satisfy Condition 2.2, inherited from[22] R. Also as before, we associate $\psi^{(k)}$ ($\equiv \psi$) to (5.12) and $\widetilde{\psi}^{(k)}$ to (5.13), and fix in what follows a $k \in \mathbb{N}^*$. From Lemma 5.1, equation[23] (5.2), applied to (5.12) and (5.13) above[24], we obtain for each i and $t \geq 0$

$$\mathbb{E}[(X^{(k),i}_{t\wedge T_N \wedge T} - \widetilde{X}^{(k),i}_{t\wedge T_N\wedge T})^+] \leq \sum_{j\neq i} \mathbb{E}\left[\int_0^{t\wedge T_N\wedge T} \mathbf{1}_{\{X^{(k),i}_s - \widetilde{X}^{(k),i}_s > 0\}} \widetilde{R}_{ij}(s,\widetilde{X}^{(k)}_s)[dZ^{(k),j}_s - d\widetilde{Z}^{(k),j}_s]\right], \tag{5.14}$$

where we set $T^\diamond$ in the corresponding definition of T as

$$T^\diamond \doteq \inf\left\{t > 0 : X^{(k),l}_t > \widetilde{X}^{(k),l}_t \; and \; \widetilde{X}^{(k),l}_t = 0 \; for \; some \; l\right\}.$$

We claim that $T^\diamond > 0$ a. s. Indeed, set

$$\mathscr{I}_0 \doteq \left\{l : \widetilde{X}^{(k),l}_0 = 0 \text{ a. s.}\right\}.$$

(Note $X^{(k),l}_0 = 0$ a. s. too for $l \in \mathscr{I}_0$.) If $\mathscr{I}_0 = \emptyset$, then by right-continuity of $\widetilde{X}^{(k)}$ we have $T^\diamond > 0$ a. s. Assume then $\mathscr{I}_0 \neq \emptyset$ and set the $(\mathscr{F}_t)_{t\geq 0}$-stopping times

$$T_1 \doteq \inf\left\{t > 0 : X^{(k),l}_t > k^{-1} \; or \; \widetilde{X}^{(k),l}_t > k^{-1} \; for \; some \; l \in \mathscr{I}_0\right\}$$

and

$$T_2 \doteq \inf\left\{t > 0 : b_l(t, X^{(k)}_t) > \widetilde{b}^{(k)}_l(t, \widetilde{X}^{(k)}_t) \; for \; some \; l \in \mathscr{I}_0\right\}.$$

Right-continuity of $X^{(k)}$ and $\widetilde{X}^{(k)}$, continuity of ψ, $\widetilde{\psi}^{(k)}$, $R^{(k)}$, $\widetilde{R}$, b and $\widetilde{b}^{(k)}$, and the facts that a. s.

$$\psi(0, X_0) < \widetilde{\psi}^{(k)}(0, \widetilde{X}_0),$$

$$R^{(k)}(0, X_0) - I < \widetilde{R}(0, \widetilde{X}_0) - I$$

and

$$b_l(0, X_0) < \widetilde{b}^{(k)}_l(0, \widetilde{X}_0), \;\; l \in \mathscr{I}_0,$$

show that

$$T_1 T_2 T^* T^\dagger > 0 \;\; \text{a. s.}$$

Thus, again by right-continuity of $X^{(k)}$ and $\widetilde{X}^{(k)}$, and the fact that $\widetilde{X}^{(k),l} > 0$ for $l \notin \mathscr{I}_0$, we conclude that for t in some vicinity $[0, \xi)$, with $\xi > 0$ depending on the paths,

$$X^{(k),i}_t = \int_0^t B^{(k)}_i(s,\omega) ds + \Gamma^{(k)}_i(t,\omega) + \sum_{j\in\mathscr{I}_0} \int_0^t \Theta^{(k)}_{ij}(s,\omega) dZ^{(k),j}_s$$

[22]Note analogously as for $\widetilde{b}^{(k)}$ and $\gamma^{(k)}$, $R^{(k)} \to R$ uniformly on $[0,\infty) \times \mathbb{R}^n_+$ as $k \nearrow \infty$.

[23]As the reader will notice, equation (5.1) in Lemma 5.1 serves the same to prove the theorem since off-diagonal elements of $R^{(k)}$ are strictly negative, in particular non-positive.

[24]Note as before, $T^\dagger$, T^* and T_N are defined, throughout the proof, with respect to $X^{(k)}$, $\widetilde{X}^{(k)}$, $Z^{(k)}$ and $\widetilde{Z}^{(k)}$.

and

$$\widetilde{X}^{(k),i}_t = \int_0^t \widetilde{B}^{(k)}_i(s,\omega)ds + \widetilde{\Gamma}_i(t,\omega) + \sum_{j\in\mathscr{I}_0}\int_0^t \widetilde{\Theta}_{ij}(s,\omega)d\widetilde{Z}^{(k),j}_s$$

for each $i\in\mathscr{I}_0$, where for $s\in[0,\xi)$ and $\mathbb{P}$-a.e. $\omega\in\Omega$

$$B^{(k)}_i(s,\omega) \doteq b_i(s,X^{(k)}_s(\omega)) \le \widetilde{b}^{(k)}_i(s,\widetilde{X}^{(k)}_s(\omega)) \doteq \widetilde{B}^{(k)}_i(s,\omega)$$

and

$$\Theta^{(k)}_{ij}(s,\omega) \doteq R^{(k)}_{ij}(s,X^{(k)}_s(\omega)) \le \widetilde{R}_{ij}(s,\widetilde{X}^{(k)}_s(\omega)) \doteq \widetilde{\Theta}^{(k)}_{ij}(s,\omega),$$

and where $\Gamma^{(k)}_i(0,\cdot)=\widetilde{\Gamma}_i(0,\cdot)\doteq 0$ and, for $s,u\in[0,\xi)$, $s\le u$, and $\mathbb{P}$-a.e. $\omega\in\Omega$ as well,

$$\begin{aligned}\Gamma^{(k)}_i(u,\omega)-\Gamma^{(k)}_i(s,\omega)&\\ &\doteq \Big(\sum_j\int_s^u \eta_{ij}(v,0)dW^j_v + \sum_{j\in\mathscr{I}_0^c}\int_s^u R^{(k)}_{ij}(v,X^{(k)}_v)dZ^{(k),j}_v\Big)(\omega)\\ &\le \Big(\sum_j\int_s^u \eta_{ij}(v,0)dW^j_v\Big)(\omega) \doteq \widetilde{\Gamma}_i(u,\omega)-\widetilde{\Gamma}_i(s,\omega)\end{aligned}$$

with $\mathscr{I}_0^c$ denoting the complement of $\mathscr{I}_0$ and $\sum_{j\in\emptyset}\doteq 0$ (recall off-diagonal elements of $R^{(k)}$ are negative, in particular non-positive). Hence, since also $\widetilde{R}(\cdot,\cdot)\le I$, the same arguments leading to the proof of [21, Theorem 4.1, pp. 521] show that

$$X^{(k),l}_\cdot \le \widetilde{X}^{(k),l}_\cdot \;\text{ on }\; [0,\xi)\cap[0,T_N) \;\text{ a.s., each } l\in\mathscr{I}_0, \tag{5.15}$$

and therefore, equation (5.15) holding for each $N\ge 0$,

$$T^\diamond > 0 \;\text{ a.s.},$$

as claimed. Note then $T(=T^\diamond\wedge T^*\wedge T^\dagger)>0$ a.s. as well. Assume now $S(\omega)\in[0,T(\omega))$ is a point of increase of $\widetilde{Z}^{(k),i}$ (some index i). Then $\widetilde{X}^{(k),i}_S=0$ and, since $S<T^\diamond$, $X^{(k),i}_S=0$ too. Thus, by right-continuity of $X^{(k)}$ and $\widetilde{X}^{(k)}$ we have, for t in some vicinity $[S,S+\varsigma)\cap[0,T)$ with $\varsigma>0$ depending on the paths,

$$X^{(k),i}_t = \int_S^t b_i(u,X^{(k)}_u)du + \sum_j\int_S^t \eta_{ij}(u,0)dW^j_u + \sum_{j\ne i}\int_S^t R^{(k)}_{ij}(u,X^{(k)}_u)dZ^{(k),j}_u + (Z^{(k),i}_t - Z^{(k),i}_S)$$

and

$$\widetilde{X}^{(k),i}_t = \int_S^t \widetilde{b}^{(k)}_i(u,\widetilde{X}^{(k)}_u)du + \sum_j\int_S^t \eta_{ij}(u,0)dW^j_u + (\widetilde{Z}^{(k),i}_t - \widetilde{Z}^{(k),i}_S),$$

where for this last expression we have assumed without lost of generality that $\widetilde{X}^{(k),j}_S>0$ for all $j\ne i$, in light of the boundary property (see equation (2.3))

$$\int_0^\infty \mathbf{1}_{\{\widetilde{X}^{(k),l}_u=0,\ \forall l\in\mathscr{A}\}}\,d\widetilde{Z}^{(k),i}_u = 0 \;\text{ a.s.}$$

for each $\mathscr{A} \subseteq \{1,\dots,n\}$ with $|\mathscr{A}| \geq 2$ and $\mathscr{A} \supset \{i\}$. Right-continuity of $X^{(k)}$ and $\widetilde{X}^{(k)}$, continuity of b and $\widetilde{b}^{(k)}$, the ordering

$$b_i(t,x) < \widetilde{b}_i^{(k)}(t,y), \;\; x,y \in F_i, \;\; x \leq y, \;\; t \geq 0,$$

and the standard fixed-point-equation characterization of

$$(Z_{S+h}^{(k)} - Z_S^{(k)})_{h\geq 0} \;\text{ and }\; (\widetilde{Z}_{S+h}^{(k)} - \widetilde{Z}_S^{(k)})_{h\geq 0}$$

(see [24]), then show that for t in some vicinity $[S, S+\epsilon) \cap [0,T)$, with $\epsilon > 0$ depending on the paths as well,

$$Z_t^{(k),i} - Z_S^{(k),i} = \sup_{S\leq u\leq t}(-V_u^{(k),i})^+ \geq \sup_{S\leq u\leq t}(-\widetilde{V}_u^{(k),i})^+ = \widetilde{Z}_t^{(k),i} - \widetilde{Z}_S^{(k),i}, \tag{5.16}$$

where[25]

$$V_u^{(k),i} \doteq \int_S^u b_i(u, X_u^{(k)})du + \sum_j \int_S^u \eta_{ij}(u,0)dW_u^j + \sum_{j\neq i}\int_S^u R_{ij}^{(k)}(u, X_u^{(k)})dZ_u^{(k),j} \tag{5.17}$$

and

$$\widetilde{V}_u^{(k),i} \doteq \int_S^u \widetilde{b}_i^{(k)}(u, \widetilde{X}_u^{(k)})du + \sum_j \int_S^u \eta_{ij}(u,0)dW_u^j.$$

Thus, since also

$$Z_t^{(k)} - Z_s^{(k)} \geq 0 \;\text{ for all } t \geq s \geq 0,$$

we conclude

$$\mathbb{P}\left\{Z_t^{(k)} - Z_s^{(k)} \geq \widetilde{Z}_t^{(k)} - \widetilde{Z}_s^{(k)},\; T > t \geq s \geq 0\right\} = 1.$$

Equation (5.14) then shows, by the same corresponding arguments as in the proof of Theorem 3.1, that indeed $T = \infty$ a.s. and that

$$\mathbb{P}\left\{X_t^{(k)} \leq \widetilde{X}_t^{(k)} \; t \geq 0\right\} = 1.$$

Therefore, we conclude that for each $k \in \mathbb{N}^*$

$$\mathbb{P}\left\{X_t^{(k)} \leq \widetilde{X}_t^{(k)} \; t \geq 0\right\} = \mathbb{P}\left\{Z_t^{(k)} - Z_s^{(k)} \geq \widetilde{Z}_t^{(k)} - \widetilde{Z}_s^{(k)},\; t \geq s \geq 0\right\} = 1,$$

and a limiting argument as in the proof of Theorem 3.1 gives then the desired result. ■

References

[1] Atar, R. and Budhiraja, A. (2002). Stability properties of constrained jump-diffusion processes. *Electronic J. Prob.* **7**, paper no. 22, 1–31. MR1943895

[25] Note from equation (5.16), and by the same corresponding arguments as above, the last sum appearing in (5.17) can then indeed be dropped.

[2] BERMAN, A. AND PLEMMONS, R. J. (1994). *Nonnegative Matrices in the Mathematical Sciences*. Classics in Applied Mathematics, Vol. **9**. Society for Industrial and Applied Mathematics (SIAM), Philadelphia. MR1298430

[3] BILLINGSLEY, P. (1999). *Convergence of Probability Measures*, Second ed. Wiley Series in Probability and Statistics. John Wiley & Sons, Inc., New York. MR1700749

[4] CHEN, H. AND YAO, D. D. (2001). *Fundamentals of Queueing Networks*. Applications of Mathematics, Vol. **46**. Springer-Verlag, New York. MR1835969

[5] DUPUIS, P. AND ISHII, H. (1991). On oblique derivative problems for fully nonlinear second order elliptic PDEs on domains with corners. *Hokkaido Math. J. 20*, 135–164. MR1096165

[6] DUPUIS, P. AND ISHII, H. (1993). SDEs with oblique reflection on nonsmooth domains. *Ann. Prob.* **21**, 1, 554–580. MR1207237

[7] ETHIER, S. N. AND KURTZ, T. G. (1986). *Markov Processes: Characterization and Convergence*. Wiley Series in Probability and Statistics. John Wiley & Sons, Inc., New Jersey. MR0838085

[8] HARRISON, J. M. AND REIMAN, M. I. (1981). Reflected Brownian motion in an orthant. *Ann. Prob. 9*, 302–308. MR0606992

[9] KALLENBERG, O. (2002). *Foundations of Modern Probability*, Second ed. Probability and its Applications. Springer-Verlag, New York. MR1876169

[10] KARATZAS, I. AND SHREVE, S. E. (1991). *Brownian Motion and Stochastic Calculus*. Graduate Texts in Mathematics, Vol. **113**. Springer-Verlag, New York. MR1121940

[11] KELLA, O. AND WHITT, W. (1990). Diffusion approximations for queues with server vacations. *Adv. Appl. Prob. 22*, 706–729. MR1066971

[12] KUSHNER, H. J. (1984). *Approximation and Weak Convergence Methods for Random Processes*. The MIT Press, Massachusetts. MR0741469

[13] KUSHNER, H. J. (2001). *Heavy Traffic Analysis of Controlled Queueing and Communication Networks*. Applications of Mathematics, Vol. **47**. Springer-Verlag, New York. MR1834938

[14] MANDELBAUM, A. AND PATS, G. (1998). State-dependent stochastic networks. Part I: Approximations and applications with continuous diffusion limits. *Ann. Appl. Prob.* **8**, 2, 569–646. MR1624965

[15] OKSENDAL, B. (2000). *Stochastic Differential Equations*, Fifth ed. Springer-Verlag, Berlin.

[16] OKSENDAL, B. AND SULEM, A. (2005). *Applied Stochastic Control of Jump Diffusions*. Springer-Verlag, New York. MR2109687

[17] PIERA, F., MAZUMDAR, R., AND GUILLEMIN, F. (2005). On product-form stationary distributions for reflected diffusions with jumps in the positive orthant. *Adv. Appl. Prob.* **37**, 1, 212–228. MR2135160

[18] PIERA, F., MAZUMDAR, R., AND GUILLEMIN, F. (2006). On the local times and boundary properties of reflected diffusions with jumps in the positive orthant. *Markov Processes Rel. Fields* **12**, 3, 561–582. MR2246264

[19] PIERA, F., MAZUMDAR, R., AND GUILLEMIN, F. (2008). Boundary behavior and product-form stationary distributions of jump-diffusions in the orthant with state-dependent reflections. *Adv. Appl. Prob.* **40**, 2, 529–547. MR2433708

[20] PROTTER, P. (2004). *Stochastic Integration and Differential Equations*, Second ed. Applications of Mathematics, Vol. **21**. Springer-Verlag, Berlin. MR2020294

[21] RAMASUBRAMANIAN, S. (2000). A subsidy-surplus model and the Skorokhod problem in an orthant. *Math. Oper. Res.* **25**, 3, 509–538. MR1855180

[22] REVUZ, D. AND YOR, M. (2005). *Continuous Martingales and Brownian Motion*, Third (corrected third printing) ed. A Series of Comprehensive Studies in Mathematics, Vol. **293**. Springer-Verlag, Berlin.

[23] RONG, S. (2000). *Reflecting Stochastic Differential Equations with Jumps and Applications*. Research Notes in Mathematics, Vol. **408**. Chapman and Hall/CRC, London. MR1714981

[24] SHASHIASHVILI, M. (1994). A lemma of variational distance between maximal functions with application to the Skorokhod problem in a nonnegative orthant with state-dependent reflection directions. *Stochastics Stochastics Rep. 48*, 161–194. MR1782747

[25] WHITT, W. (2001). The reflection map with discontinuities. *Math. Oper. Res. 26*, 447–484. MR1849880

[26] WHITT, W. (2002). *Stochastic-Process Limits*. Springer Series in Operations Research. Springer-Verlag, New York. MR1876437