SUPERCRITICAL ELLIPTIC PROBLEMS FROM A PERTURBATION VIEWPOINT

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Abstract. We review some recent existence results for the elliptic problem $\Delta u + u^p = 0$, $u > 0$ in an exterior domain, $\Omega = \mathbb{R}^N \setminus \mathcal{D}$ under zero Dirichlet and vanishing conditions, where $\mathcal{D}$ is smooth and bounded, and $p > \frac{N+2}{N-2}$. We prove that the associated Dirichlet problem has infinitely many positive solutions. We establish analogous results for the standing-wave supercritical nonlinear Schrödinger equation $\Delta u - V(x)u + u^p = 0$ where $V \geq 0$ and $V(x) = o(|x|^{-2})$ at infinity. In addition we present existence results for the Dirichlet problem in bounded domains with a sufficiently small spherical hole if p differs from certain sequence of resonant values which tends to infinity.

1. **Introduction and statement of the main results.** In this paper we will review some recent results concerning semilinear elliptic equations with a power nonlinearity which is above the critical exponent. We will mostly deal with two specific model problems. One is the classical Lane-Emden-Fowler equation in a *exterior domain*,

$$\Delta u + u^p = 0\,, \ u > 0 \quad \text{in } \mathbb{R}^N \setminus \bar{\mathcal{D}}\,, \tag{1}$$

$$u = 0 \ \text{ on } \partial\mathcal{D}\,, \quad \lim_{|x|\to+\infty} u(x) = 0 \tag{2}$$

where $\mathcal{D}$ be a bounded domain in $\mathbb{R}^N$ with smooth boundary. Another problem, largely treated in the literature is

$$\Delta u - V(x)u + u^p = 0, \quad u > 0, \quad \lim_{|x|\to+\infty} u(x) = 0, \tag{3}$$

the *standing-wave problem* for a nonlinear Schrödinger equation. Here V is a non-negative potential and we assume $p > 1$.

In both problems, as in nonlinear elliptic equations in general, *solvability above criticality* namely $p > \frac{N+2}{N-2}$ is an issue widely open. A major technical obstacle in understanding such problems stems from the lack of (local) Sobolev embeddings suitably fit to a weak formulation of this problem. Direct tools of the calculus of variation, very useful in subcritical, and even critical cases, see for instance [1, 2, 3, 4, 14] are not appropriate in the supercritical case.

In this paper we find existence results for problems (1)-(2) and (3) from the optic of singular perturbations. We find that these problems "hide" a parameter which

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indexes a continuum of solutions which asymptotically vanish over compact sets. To describe our results our starting point is the problem in entire space

$$\Delta u + u^p = 0, \quad u > 0 \qquad \text{in } \mathbb{R}^N, \tag{4}$$

which for radially symmetric solutions $u = u(r)$, $r = |x|$ reduces to the equation

$$u'' + \frac{N-1}{r} u' + u^p = 0 \, . \tag{5}$$

This equation can be analyzed through phase plane analysis after a transformation introduced by Fowler [10] in 1931: $v(s) = r^{\frac{2}{p-1}} u(r)$, $r = e^s$, which transforms equation (5) into the autonomous ODE

$$v'' + \alpha v' - \beta v + v^p = 0 \tag{6}$$

where

$$\alpha = N - 2 - \frac{4}{p-1}, \quad \beta = \frac{2}{p-1}\left(N - 2 - \frac{2}{p-1}\right). \tag{7}$$

Since α and β are positive for $p > \frac{N+2}{N-2}$, the Hamiltonian energy

$$E(v) = \frac{1}{2}\dot{v}^2 + \frac{1}{p+1} v^{p+1} - \frac{\beta}{2} v^2$$

strictly decreases along trajectories. Using this it is easy to see the existence of a heteroclinic orbit which connects the equilibria $(0,0)$ and $(0, \beta^{\frac{1}{p-1}})$ in the phase plane (v, v'), which correspond respectively to a saddle point and an attractor. A solution $v(s)$ of (6) corresponding this orbit satisfies $v(-\infty) = 0$, $v(+\infty) = \beta^{\frac{1}{p-1}}$ and $w(r) = r^{-\frac{2}{p-1}} v(\log r)$ solves (5) and is bounded at $r = 0$. Then all radial solutions of (4) defined in all $\mathbb{R}^N$ have the form

$$w_\lambda(x) := \lambda^{\frac{2}{p-1}} w(\lambda |x|), \quad \lambda > 0. \tag{8}$$

We denote in what follows by $w(x)$ the unique positive radial solution

$$\Delta w + w^p = 0 \quad \text{in } \mathbb{R}^N, \quad w(0) = 1. \tag{9}$$

While Problems (1)-(2) and (3) do not carry any parameter explicitly, we can make a parameter appear by means of replacing the variable u in both equations by $\lambda^{\frac{2}{p-1}} u(\lambda x)$, in such a way that Problem (1)-(2) becomes

$$\Delta u + u^p = 0 \, , \; v > 0 \quad \text{in } \mathbb{R}^N \setminus \bar{\mathcal{D}}_\lambda \, , \tag{10}$$

$$u = 0 \;\; \text{on } \partial \mathcal{D}_\lambda \, , \qquad \lim_{|x| \to +\infty} u(x) = 0 \tag{11}$$

where $\mathcal{D}_\lambda := \lambda \mathcal{D}$ is now a small region. Similarly Problem (3) becomes

$$\Delta u - V_\lambda(x) u + u^p = 0, \quad u > 0, \quad \lim_{|x| \to +\infty} u(x) = 0, \tag{12}$$

where

$$V_\lambda(x) = \lambda^{-2} V(\lambda x). \tag{13}$$

If the potential V is assumed to satisfy the asymptotic behavior

$$V(x) = o(|x|^{-2}) \quad \text{as } |x| \to +\infty$$

then we observe that away from the origin $V_\lambda(x) \to 0$ as $\lambda \to 0$. Thus both problems may be regarded, away from the origin, as small perturbations of problem (4) when $\lambda > 0$ is sufficiently small.

Our aim is to establish that solutions to problem (10)-(11) and to (12) which lie close to the radial solution $w(x)$ of (9) indeed exist for any sufficiently small $\lambda > 0$, in spite of the singular behavior of the problems near the origin. We will do this by linearization and a perturbation argument in suitable functional spaces. Thus the key element is to understand the invertibility properties of the operator $\Delta + pw^{p-1}$ in entire space.

Equivalently, we will find solutions to (1)-(2) and (12) which lie close to $w_\lambda(x)$ given by (8) for any small λ. We have the validity of the following results.

Theorem 1. [6], [7] *For any $p > \frac{N+2}{N-2}$ there is a continuum of solutions u_λ, $\lambda > 0$, to Problem (1)-(2), such that*

$$u_\lambda(x) = \beta^{\frac{1}{p-1}} |x|^{-\frac{2}{p-1}} (1 + o(1)) \quad \text{as } |x| \to \infty \tag{14}$$

and $u_\lambda(x) \to 0$ as $\lambda \to 0$, uniformly in $\mathbb{R}^N \setminus \mathcal{D}$.

In reality this result hides two situations that are quite different. The continuum of solutions in this result turns out to be a two-parameter family, dependent not only on all small λ but also on a point $\xi \in \mathbb{R}^N$ used as the reference origin. The bottom line is that an inverse for the linearized operator $\Delta + pw^{p-1}$ indeed exists for $p > \frac{N+1}{N-3}$. However if $\frac{N+2}{N-2} < p \leq \frac{N+1}{N-3}$ the operator is not surjective, having a range orthogonal to the generators of translations. It turns out that a further adjustment of the origin ξ still yields the result.

Concerning the Schrodinger equation (3) we thus assume that $V \geq 0$ satisfies condition (13).

Theorem 2. *Assume that $V \geq 0$, $V \in L^\infty(\mathbb{R}^N)$ and that (13) holds. Let $N \geq 4$, $p > \frac{N+1}{N-3}$. Then problem (3) has a continuum of solutions $u_\lambda(x)$ such that (14) holds and $u_\lambda(x) \to 0$ as $\lambda \to 0$, uniformly in $\mathbb{R}^N$.*

Again, the situation in the remaining supercritical range is more difficult. We do not know if the decay condition (13) of V suffices alone, but we have for instance, the validity of the following result.

Theorem 3. *Assume that $V \geq 0$, $V \in L^\infty(\mathbb{R}^N)$ and $\frac{N+2}{N-2} < p \leq \frac{N+1}{N-3}$. Then the result of Theorem 2 also holds true if there exist $C > 0$ and $\mu > N$ such that*

$$V(x) \leq C|x|^{-\mu}, \quad |x| \geq 1.$$

Let us go back for a moment to the phase portrait associated to the radial problem $\Delta u + u^p = 0$. We observe that the radial cases supercritical and subcritical with $p > \frac{N}{N-2}$ are completely dual: In equation (6) β remains positive but α becomes negative. The effect of this is basically to make the phase portraits equivalent, just with arrows inverted in the orbits, with obvious dual consequences. For instance, inner-subcritical in a ball has a classical solution, which in the phase diagram is represented by the unstable manifold of $(0,0)$. Correspondingly, in the supercritical case, to the orbit representing the stable manifold of $(0,0)$, it corresponds to the unique solution w_* to the exterior problem with *fast decay*, namely w_* satisfies

$$\Delta w_* + w_*^p = 0\,, \; w_* > 0 \quad \text{in } \mathbb{R}^N \setminus \bar{B}_1(0)\,, \tag{15}$$

$$w_* = 0 \;\text{ on } \partial B_1(0)\,, \quad \limsup_{|x|\to+\infty} |x|^{2-N} w_*(x) < +\infty\,, \tag{16}$$

Given that we are finding solutions in exterior domains which decay at infinity, it is reasonable to ask whether we can also find solutions in bounded domains with small holes. We consider the Lane-Emden-Fowler equation in for exponents p above critical in a domain Ω with a small hole drilled (A Coron type domain [4]). Thus we assume that Ω has the form

$$\Omega = \mathcal{D} \setminus B_\delta(Q) \tag{17}$$

where $\mathcal{D}$ is a bounded domain with smooth boundary, $B_\delta(Q) \subset \mathcal{D}$ and $\delta > 0$ is to be taken small. Thus we consider the problem of finding classical solutions of

$$\Delta u + u^p = 0\,, \;\; u > 0 \quad \text{ in } \mathcal{D} \setminus B_\delta(Q)\,, \tag{18}$$

$$u = 0 \quad \text{on } \partial\mathcal{D} \cup \partial B_\delta(Q)\,. \tag{19}$$

Our main result states that there is a sequence of *resonant exponents*,

$$\frac{N+2}{N-2} < p_1 < p_2 < p_3 < \cdots\,, \quad \text{with} \lim_{k\to+\infty} p_k \;=\; +\infty \tag{20}$$

such that if p is supercritical and differs from all elements of this sequence then Problem (1)-(2) is solvable whenever δ is sufficiently small.

Theorem 4. [9] *There exists a sequence of the form (20) such that if $p > \frac{N+2}{N-2}$ and $p \neq p_j$ for all j, then there is a $\delta_0 > 0$ such that for any $\delta < \delta_0$, Problem (18)-(19) possesses at least one solution.*

In the background of our result is problem (15). The solutions we find have a profile similar to w suitably rescaled. More precisely, Let us observe that

$$w_{*\delta}(x) = \delta^{-\frac{2}{p-1}} w_*(\delta^{-1}|x-Q|) \tag{21}$$

solves uniquely the same problem with $B_1(0)$ replaced with $B_\delta(Q)$. The idea is to consider $w_{*\delta}$ as a first approximation for a solution of Problem (18)-(19), provided that $\delta > 0$ is chosen small enough. What we shall prove is that an actual solution of the problem, which differs little from $w_{*\delta}$ does exist. To this end, it is necessary to understand the linearized operator around $w_{*\delta}$.

The rest of this paper presents the main elements involved in the proofs of the above results. Full details are provided in the articles [6, 7, 8, 9].

2. **The operator $\Delta + pw_\lambda^{p-1}$ in $\mathbb{R}^N$.** The results of Theorems 1-3 are based on a suitable linear theory devised for the linearized operator associated to the equation $\Delta u + u^p = 0$ at $u = w_\lambda$ in entire space $\mathbb{R}^N$ and in the application of perturbation arguments. As we have mentioned, a critical number $p = \frac{N+1}{N-3}$ is present where the invertibility nature of the operator changes drastically.

2.1. **The case $p > \frac{N+1}{N-3}$.** Let us consider the norms

$$\|\phi\|_{*,\lambda} = \lambda^\sigma \sup_{|x|\le\frac{1}{\lambda}} |x|^\sigma|\phi(x)| + \lambda^{\frac{2}{p-1}} \sup_{|x|\ge\frac{1}{\lambda}} |x|^{\frac{2}{p-1}}|\phi(x)|$$

$$\|h\|_{**,\lambda} = \lambda^\sigma \sup_{|x|\le\frac{1}{\lambda}} |x|^{2+\sigma}|h(x)| + \lambda^{\frac{2}{p-1}} \sup_{|x|\ge\frac{1}{\lambda}} |x|^{2+\frac{2}{p-1}}|h(x)|.$$

The proof is based on a similar result valid in entire $\mathbb{R}^N$: Let us consider the problem

$$\Delta\phi + pw_\lambda^{p-1}\phi = h \quad \text{in } \mathbb{R}^N. \tag{22}$$

Lemma 5. *Assume $N \geq 4$ and $p > \frac{N+1}{N-3}$. For $0 < \sigma < N-2$ there exists a constant $C > 0$ such that for any $\lambda > 0$ and h with $\|h\|_{**,\lambda} < +\infty$, equation (22) has a solution $\phi = T_\lambda(h)$ such that T_λ defines a linear map and*

$$\|T_\lambda(h)\|_{*,\lambda} \leq C\|h\|_{**,\lambda}.$$

The invertibility analysis in this range is in strong analogy with one carried out in [13] in the construction of singular solutions with prescribed singularities for $\frac{N}{N-2} < p < \frac{N+2}{N-2}$ in bounded domains.

2.2. **The proof of Lemma 5.** By scaling out λ and using the definitions of the norms, we just need to prove the result for $\lambda = 1$. We denote the norms involved simply by $\|\cdot\|_*$ and $\|\cdot\|_{**}$. Let us consider h with $\|h\|_{**} < +\infty$ and decompose it in the form

$$h(x) = \sum_{k=0}^{\infty} h_k(r)\Theta_k(\theta), \quad r > 0,\, \theta \in S^{N-1} \tag{23}$$

where Θ_k, $k \geq 0$ are eigenfunctions of the Laplace-Beltrami operator in S^{N-1}, normalized so that they constitute an orthonormal system in $L^2(S^{N-1})$. We take Θ_0 to be a positive constant, associated to the eigenvalue 0 and Θ_i, $1 \leq i \leq N$ is an appropriate multiple of $\frac{x_i}{|x|}$ which has eigenvalue $\lambda_i = N-1$, $1 \leq i \leq N$. We recall that the set of eigenvalues is given by $\{j(N-2+j) \,|\, j \geq 0\}$.

We look for a solution ϕ to (22) in the form

$$\phi(x) = \sum_{k=0}^{\infty} \phi_k(r)\Theta_k(\theta)\,. \tag{24}$$

Then ϕ satisfies (22) if and only if

$$\phi_k'' + \frac{N-1}{r}\phi_k' + \left(pw^{p-1} - \frac{\lambda_k}{r^2}\right)\phi_k = h_k, \quad \text{for all } r > 0,\ \text{for all } k \geq 0. \tag{25}$$

To construct solutions of this ODE we need to consider two linearly independent solutions $z_{1,k}$, $z_{2,k}$ of the homogeneous equation

$$\phi_k'' + \frac{N-1}{r}\phi_k' + \left(pw^{p-1} - \frac{\lambda_k}{r^2}\right)\phi_k = 0, \quad r \in (0,\infty). \tag{26}$$

Once these generators are identified, the general solution of the equation can be written through the variation of parameters formula as

$$\phi(r) = z_{1,k}(r)\int z_{2,k}h_k r^{N-1}dr - z_{2,k}(r)\int z_{1,k}h_k r^{N-1}dr$$

where the symbol $\int$ designates arbitrary antiderivatives, which we will specify in the choice of the operators. It is helpful to recall that if one solution $z_{1,k}$ to (26) is known, a second, linearly independent solution can be found in any interval where $z_{1,k}$ does not vanish as

$$z_{2,k}(r) = z_{1,k}(r)\int z_{1,k}(r)^{-2}r^{1-N}dr\,. \tag{27}$$

One can get the asymptotic behaviors of any solution z as $r \to 0$ and as $r \to +\infty$ by examining the indicial roots of the associated Euler equations. It is known that

$r^2 w(r)^{p-1} \to \beta$ as $r \to +\infty$ where

$$\beta = \frac{2}{p-1}\left(N - 2 - \frac{2}{p-1}\right).$$

Thus we get the limiting equation, for $r \to \infty$,

$$r^2\phi'' + (N-1)r\phi' + (p\beta - \lambda_k)\phi = 0, \tag{28}$$

while as $r \to 0$,

$$r^2\phi'' + (N-1)r\phi' - \lambda_k\phi = 0. \tag{29}$$

In this way the respective behaviors will be ruled by $z(r) \sim r^{-\mu}$ as $r \to +\infty$ where μ solves

$$\mu^2 - (N-2)\mu + (p\beta - \lambda_k) = 0$$

while as $r \to 0$ μ satisfies

$$\mu^2 - (N-2)\mu - \lambda_k = 0.$$

Next we shall construct each of the ϕ_k's in the expansion (24), in such a way that they define bounded linear operators of h_k in the norms considered. This method is reminiscent to that in [13], see also [12].

2.2.1. *The construction of ϕ_0.*

Lemma 6. *Let $k = 0$ and $p > \frac{N+2}{N-2}$. Then equation (25) has a solution ϕ_0 which depends linearly on h_0 and satisfies*

$$\|\phi_0\|_* \le C\|h_0\|_{**}. \tag{30}$$

Proof. For $k = 0$ the possible behaviors at 0 for a solution $z(r)$ to (26) are simply

$$z(r) \sim 1, \quad z(r) \sim r^{2-N}$$

while at $+\infty$ this behavior is more complicated. The indicial roots of (29) are given by

$$\mu_{0\pm} = \frac{N-2}{2} \pm \frac{1}{2}\sqrt{(N-2)^2 - 4p\beta}.$$

The situation depends of course on the sign of $D = (N-2)^2 - 4p\beta$. It is observed in [11] that $D > 0$ if and only if $N > 10$ and $p > p_c$ where we set

$$p_c = \begin{cases} \frac{(N-2)^2 - 4N + 8\sqrt{N-1}}{(N-2)(N-10)} & \text{if } N > 10 \\ \infty & \text{if } N \le 10. \end{cases}$$

Thus when $p < p_c$, $\mu_{0\pm}$ are complex with negative real part, and the behavior of a solution $z(r)$ as $r \to +\infty$ is oscillatory and given by

$$Z(r) = O(r^{-\frac{N-2}{2}}).$$

When $p > p_c$, we have $\mu_{0+} > \mu_{0-} > \frac{2}{p-1}$.

Independently of the value of p, we have that the function

$$z_{1,0} = rw' + \frac{2}{p-1}w$$

satisfies equation (26) for $k = 0$. Using asymptotic formulae derived for w in [11], we find the estimates

$$\text{if } p < p_c : \qquad |z_{1,0}(r)| \le Cr^{\frac{N-2}{2}} \tag{31}$$

$$\text{if } p = p_c : \qquad z_{1,0}(r) = cr^{-\frac{N-2}{2}} \log r \,(1 + o(1)) \tag{32}$$

$$\text{if } p > p_c : \qquad z_{1,0}(r) = cr^{-\mu_{0-}} \,(1 + o(1)), \tag{33}$$

where $c \neq 0$.

Case $p < p_c$**.** We define $z_{2,0}(r)$ for small $r > 0$ by

$$z_{2,0}(r) = z_{1,0}(r) \int_{r_0}^{r} z_{1,0}(s)^{-2} s^{1-N} \, ds \tag{34}$$

where r_0 is small so that $z_{1,0} > 0$ in $(0, r_0)$ (which is possible because $z_{1,r} \sim 1$ near 0). Then $z_{2,0}$ is extended to $(0, +\infty)$ so that it is a solution to the homogeneous equation (26) (with $k = 0$) in this interval. As mentioned earlier $z_{2,0}(r) = O(r^{-\frac{N-2}{2}})$ as $r \to +\infty$.

We define

$$\phi_0(r) = z_{1,0}(r) \int_1^r z_{2,0} h_0 s^{N-1} \, ds - z_{2,0}(r) \int_0^r z_{1,0} h_0 s^{N-1} \, ds,$$

and omit a calculation that shows that this expression satisfies (30).

Case $p \ge p_c$**.** In this case we let

$$\phi_0(r) = -z_{1,0}(r) \int_1^r z_{1,0}(s)^{-2} s^{1-N} \int_0^s z_{1,0}(\tau) h_0(\tau) \tau^{N-1} \, d\tau \, ds,$$

which is justified because when $p \ge p_c$ we have $z_{1,0}(r) > 0$ for all $r > 0$, which follows from the fact that $\lambda \mapsto \lambda^{\frac{2}{p-1}} w(\lambda r)$ is increasing for $\lambda > 0$, see [11]. Again, a calculation using now (32) and (33) shows that ϕ_0 satisfies the estimate (30). $\square$

2.2.2. *The construction of ϕ_k, $1 \le k \le N$.* All these modes are equivalent, so we only consider $k = 1$. We have the following result.

Lemma 7. *Let $k = 1$ and $p > \frac{N+1}{N-3}$. Then equation (25) has a solution ϕ_1 which defines a linear operator of h_1 and satisfies*

$$\|\phi_1\|_* \le C \|h_1\|_{**}. \tag{35}$$

Proof. In this case the indicial roots that govern the behavior of the solutions $z(r)$ as $r \to +\infty$ of the homogeneous equation (26) are given by $\mu_1 = \frac{2}{p-1} + 1$ and $\mu_2 = N - 3 - \frac{2}{p-1}$. Since we are looking for solutions that decay at a rate $r^{-\frac{2}{p-1}}$ as $r \to +\infty$ we will need $N - 3 - \frac{2}{p-1} \ge \frac{2}{p-1}$, which is equivalent to the hypothesis $p \ge \frac{N+1}{N-3}$. On the other hand the behavior near 0 of $z(r)$ can be $z(r) \sim r$ or $z(r) \sim r^{1-N}$.

Similarly as in the case $k = 0$ we have a solution to (26), namely $z_1(r) = -w'(r)$ and luckily enough it is positive in all $(0, +\infty)$. With it we can build

$$\phi_1(r) = -z_1(r) \int_1^r z_1(s)^{-2} s^{1-N} \int_0^s z_1(\tau) h_1(\tau) \tau^{N-1} \, d\tau \, ds. \tag{36}$$

From this formula and using $p \ge \frac{N+1}{N-3}$ we obtain (35). $\square$

2.2.3. *The construction of ϕ_k, $k > N$.*

Lemma 8. *Let $k > N$ and $p > \frac{N+2}{N-2}$. If $\|h_k\|_{**} < \infty$ equation (25) has a unique solution ϕ_k with $\|\phi_k\|_* < \infty$ and there exists $C_k > 0$ such that*

$$\|\phi_k\|_* \le C_k \|h_k\|_{**}. \tag{37}$$

Proof. Let us write L_k for the operator in (25), that is,

$$L_k\phi = \phi'' + \frac{N-1}{r}\phi' + \left(pw^{p-1} - \frac{\lambda_k}{r^2}\right)\phi.$$

This operator satisfies the maximum principle in any interval of the form $(\delta, \frac{1}{\delta})$, $\delta > 0$. Indeed let $z = -w'$, so that $z > 0$ in $(0, +\infty)$ and it is a supersolution, because

$$L_k z = \frac{N-1-\lambda_k}{r^2} z < 0 \quad \text{in } (0, +\infty), \tag{38}$$

since $\lambda_k \ge 2N$ for $k \ge 2$. To prove solvability of (25) in the appropriate space we construct a supersolution ψ of the form

$$\psi = C_1 z + v, \quad v(r) = \frac{1}{r^\sigma + r^{\frac{2}{p-1}}},$$

Choosing C_1 sufficiently large, we can check that

$$L_k\psi \le -c\min(r^{-\sigma-2}, r^{-\frac{2}{p-1}-2}) \quad \text{in } (0, +\infty).$$

for some $c > 0$.

Given h_k with $\|h_k\|_{**} < \infty$, by the method of sub and supersolutions, there exists, for any $\delta > 0$ a unique solution ϕ_δ of the two-point boundary value problem

$$L_k\phi_\delta = h_k \qquad \text{in } (\delta, \tfrac{1}{\delta})$$
$$\phi_\delta(\delta) = \phi_\delta(\tfrac{1}{\delta}) = 0.$$

This solution satisfies the bound

$$|\phi_\delta| \le C\psi\, \|h_k\|_{**} \quad \text{in } (\delta, \tfrac{1}{\delta}).$$

Using standard estimates we have that, up to subsequences, $\phi_\delta \to \phi_k$ as $\delta \to 0$ uniformly on compact subsets of $(0, +\infty)$ where ϕ_k is a solution of (25) which satisfies

$$|\phi_k| \le C\psi\, \|h_k\|_{**} \quad \text{in } (0, \infty).$$

The maximum principle yields that the solution to (25) bounded in this way is actually unique, and thus defines the desired linear operator. □

2.2.4. *Conclusion of the construction.* Let $m > 0$ be an integer. By Lemmas 6, 7 and 8 we see that if $\|h\|_{**} < \infty$ and its Fourier series (23) has $h_k \equiv 0$ $\forall k \ge m$ there exists a solution ϕ to (22) that depends linearly with respect to h and moreover

$$\|\phi\|_* \le C_m \|h\|_{**}.$$

We can prove that the constant C_m may actually be taken uniform in m. Indeed, an indirect argument, based upon standard elliptic estimates, allows us to end up with the situation that there exists a nonzero, bounded function ϕ which satisfies the equation $\Delta\phi + pw^{p-1}\phi = 0$ and which has no Fourier components in its first few Fourier components. Arguing mode by mode, we see that ϕ must be identically zero. This shows that the solution ϕ defined by (24) defines an operator in h with the desired property. □

2.3. **The case $\frac{N+2}{N-2} < p \le \frac{N+1}{N-3}$.** The proof presented above fails only in one step: in the construction of ϕ_k for $1 \le k \le N$. Formula (36) for ϕ_1 does not define a solution which decays like $r^{-\frac{2}{p-1}}$ *unless* h_1 *satisfies the orthogonality condition*

$$\int_0^\infty w'(\tau) h_1(\tau) \tau^{N-1}\, d\tau \ = 0. \tag{39}$$

This implies the following: Let us write

$$Z_i = \frac{\partial w}{\partial x_i}. \tag{40}$$

Then if $\frac{N+2}{N-3} < p < \frac{N+1}{N-3}$ and $0 < \sigma < N-2$, there is a linear operator $\phi = T(h)$ defined for h with $\|h\|_{**} < \infty$, with the property that for certain unique scalars $c_1, \ldots, c_N$,

$$\Delta\phi + p w^{p-1}\phi = h + \sum_{i=1}^N c_i Z_i \quad \text{ in } \mathbb{R}^N, \tag{41}$$

and $\|\phi\|_* \le C\|h\|_{**,}$. It turns out that this operator is also bounded in a variation of these norms which allows a singularity at a point different from the origin. We have that given $\Lambda > 0$ there is a $C > 0$ such that for all $\xi \in \mathbb{R}^N$ with $|\xi| \le \Lambda$ we have that $\|\phi\|_{*,\xi} \le C\|h\|_{**,\xi}$, where

$$\|\phi\|_{*,\xi} = \sup_{|x-\xi|\le 1} |x-\xi|^\sigma |\phi(x)| + \sup_{|x-\xi|\ge 1} |x-\xi|^{\frac{2}{p-1}} |\phi(x)|$$

$$\|h\|_{**,\xi} = \sup_{|x-\xi|\le 1} |x-\xi|^{2+\sigma} |h(x)| + \sup_{|x-\xi|\ge 1} |x-\xi|^{2+\frac{2}{p-1}} |h(x)|.$$

3. The Proof of Theorem 1 for $p > \frac{N+1}{N-3}$.

3.1. **The fixed point argument.** We look for a solution of Problem (1)-(2) of the form $u = \eta w_\lambda + \phi$, where η is a smooth cut-off function with $\eta(x) = 0$ for $|x| \le R$, $\eta(x) = 1$ for $|x| \ge R+1$ and $\mathcal{D} \subset B(0,R)$. This u solves (1)-(2) if ϕ satisfies

$$\begin{cases} \Delta\phi + p w_\lambda^{p-1}\phi = N(\phi) + E & \text{in } \mathbb{R}^N \setminus \bar{\mathcal{D}} \\ \phi = 0 & \text{on } \partial\mathcal{D} \\ \phi(x) \to 0 & \text{as } |x| \to +\infty \end{cases} \tag{42}$$

where

$$N(\phi) = -(\eta w_\lambda + \phi)^p + (\eta w_\lambda)^p + p(\eta w_\lambda)^{p-1}\phi + p(1-\eta^{p-1}) w_\lambda^{p-1}\phi,$$

and

$$E = -\Delta(\eta w_\lambda) - (\eta w_\lambda)^p.$$

We write the above problem in fixed point form on the basis of the existence of a right inverse for the linear operator $\Delta + p w_\lambda^{p-1}$ in suitable weighted L^∞-spaces. Thus we consider the linear problem

$$\begin{cases} \Delta\phi + p w_\lambda^{p-1}\phi = h & \text{in } \mathbb{R}^N \setminus \bar{\mathcal{D}} \\ \phi = 0 & \text{on } \partial\mathcal{D} \\ \phi(x) \to 0 & |x| \to +\infty, \end{cases} \tag{43}$$

We have the validity of the following result.

Lemma 9. *Assume that $N \geq 4$ and $p \geq \frac{N+1}{N-3}$. Then there exists a constant $C > 0$ such that for all sufficiently small $\lambda > 0$ and all h with $\|h\|_{**,\lambda} < +\infty$, Problem (43) has a solution $\phi = \mathcal{T}_\lambda(h)$ such that $\mathcal{T}_\lambda$ is a linear map and*

$$\|\mathcal{T}_\lambda(h)\|_{*,\lambda} \leq C\|h\|_{**,\lambda}.$$

By this result, we have a solution to (42) if ϕ solves the fixed point problem

$$\phi = \mathcal{T}_\lambda(N(\phi) + E). \tag{44}$$

We can check the estimates

$$\|N(\phi)\|_{**,\lambda} \leq C(\lambda^2\|\phi\|_{*,\lambda} + \lambda^{-\frac{2}{p-1}}\|\phi\|^2_{*,\lambda} + \lambda^{-2}\|\phi\|^p_{*,\lambda}), \tag{45}$$

and

$$\|E\|_{**,\lambda} \leq C\lambda^{\frac{2}{p-1}+\sigma}. \tag{46}$$

Let $\phi_0 = \mathcal{T}_\lambda(E)$. From Lemma 9, and (46), we get $\|\phi_0\|_{*,\lambda} \leq C\lambda^{\frac{2}{p-1}+\sigma}$. Let us write $\phi = \phi_0 + \phi_1$. Then solving equation (44) is equivalent to solving the fixed point problem $\phi = \mathcal{T}_\lambda(N(\phi_0 + \phi))$. We consider the set

$$\mathcal{F} = \{\phi \in L^\infty(\mathbb{R}^N \setminus \mathcal{D}) \;/\; \|\phi\|_{*,\lambda} \leq \rho\lambda^{\frac{2}{p-1}}\}$$

where $\rho > 0$ is going to be fixed independently of λ, and the operator

$$\mathcal{A}(\phi) = \mathcal{T}_\lambda(N(\phi_0 + \phi)).$$

Next we show that $\mathcal{A}$ has a fixed point in $\mathcal{F}$. For $\phi \in \mathcal{F}$ we have

$$\|\mathcal{A}(\phi)\|_{*,\lambda} \leq C\|N(\phi_0 + \phi)\|_{**,\lambda} \tag{47}$$

$$\leq C(\lambda^2\|\phi_0 + \phi\|_{*,\lambda} + \lambda^{-\frac{2}{p-1}}\|\phi_0 + \phi\|^2_{*,\lambda} + \lambda^{-2}\|\phi_0 + \phi\|^p_{*,\lambda}). \tag{48}$$

Thus for a fixed sufficiently small ρ and all small λ we get

$$\|\mathcal{A}(\phi)\|_{*,\lambda} \leq C\lambda^{\frac{2}{p-1}}(\rho\lambda^2 + \lambda^{2\sigma} + \lambda^{p\sigma} + \rho^2 + \rho^p) \leq \rho\lambda^{\frac{2}{p-1}}. \tag{49}$$

Hence $\mathcal{A}(\mathcal{F}) \subset \mathcal{F}$ for all small λ.

On the other hand, we also check that $\mathcal{A}$ is a contraction mapping in $\mathcal{F}$, and hence a fixed point in this region indeed exists. The solutions u_λ built this way satisfy the requirement of Theorem 1. $\square$

3.2. **The proof of Lemma 9.** We shall solve (43) by writing $\phi = \eta\varphi + \psi$ where η is a smooth cut-off function with

$$\eta(x) = 0 \quad \text{for } |x| \leq R_0, \quad \eta(x) = 1 \quad \text{for } |x| \geq R_0 + 1$$

and $R_0 > 0$ is fixed so that $\mathcal{D} \subseteq B_{R_0}$. We also set $\zeta(x) = \eta(x/2)$, so that $\eta\zeta = \zeta$.

To find a solution of (43) it is sufficient to solve the following system

$$\Delta\varphi + pw_\lambda^{p-1}\varphi = -p\zeta w_\lambda^{p-1}\psi + \zeta h \quad \text{in } \mathbb{R}^N \tag{50}$$

$$\begin{cases} \Delta\psi + p(1-\zeta)w_\lambda^{p-1}\psi = -2\nabla\eta\nabla\varphi - \varphi\Delta\eta + (1-\zeta)h & \text{in } \mathbb{R}^N \setminus \bar{\mathcal{D}} \\ \psi = 0 & \text{on } \partial\mathcal{D} \\ \psi(x) \to 0 & |x| \to +\infty. \end{cases} \tag{51}$$

We assume $\|h\|_{**,\lambda} < \infty$. Let us consider the Banach space X consisting of functions φ such that $\|\varphi\|_{*,\lambda} < \infty$ and that are Lipschitz on $E = B_{2R_0} \setminus B_{R_0}$ equipped with the norm

$$\|\varphi\|_X = \|\varphi\|_{*,\lambda} + \|\nabla\varphi\|_{L^\infty(E)}.$$

Given $\varphi \in X$ we solve first (51) and denote by $\psi(\varphi, h)$ the solution, which is clearly linear in its argument. Then note that $\zeta\psi$ is well defined in $\mathbb{R}^N$ and that $|\psi| \leq \frac{C}{|x|^{N-2}}$ for large $|x|$ so hence the right hand side of (50) has a finite $\| \ \|_{**,\lambda}$ norm. We obtain a solution to the system, which defines a linear operator in h, if we solve the fixed point problem

$$\varphi \,=\, T_\lambda(\, -p\zeta w_\lambda^{p-1}\, \psi(\varphi, h) \,+\, \zeta h\,) \,\equiv\, F(\varphi)\,.$$

where T_λ is the operator in Proposition 5. Then we have the estimate

$$\|F(\varphi)\|_{*,\lambda} \leq C\,\| - p\zeta w_\lambda^{p-1}\psi + \zeta h\|_{**,\lambda} \leq C(\|\zeta w_\lambda^{p-1}\psi\|_{**,\lambda} + \|h\|_{**,\lambda}). \tag{52}$$

But

$$\|\zeta w_\lambda^{p-1}\psi\|_{**,\lambda} = \lambda^\sigma \sup_{R_1\leq|x|\leq\frac{1}{\lambda}} \left(|x|^{2+\sigma} w_\lambda(x)^{p-1}|\psi(x)|\right) + \lambda^{\frac{2}{p-1}} \sup_{|x|\geq\frac{1}{\lambda}} \left(|x|^{2+\sigma+\frac{2}{p-1}} w_\lambda(x)^{p-1}|\psi(x)|\right).$$

Using equation (51) and the fact that $w_\lambda(x) \to 0$ uniformly on compact sets we have

$$|\psi(x)| \leq \frac{C}{|x|^{N-2}}(\|\varphi\|_X + \|h\|_{**,\lambda}). \tag{53}$$

Using this, and the asymptotic behavior of $w(x)$ we then obtain the estimate

$$\|\zeta w_\lambda^{p-1}\psi\|_{**,\lambda} \leq C\lambda^\gamma(\|\varphi\|_X + \|h\|_{**,\lambda}).$$

where $\gamma = \min(2+\sigma, N-2)$. This together with (52) yields

$$\|F(\varphi)\|_{*,\lambda} \leq C(\lambda^\gamma\|\varphi\|_X + \|h\|_{**,\lambda}). \tag{54}$$

While, using elliptic estimates, we get

$$\|\nabla F(\varphi)\|_{L^\infty(E)} \leq C(\|F(\varphi)\|_{*,\lambda} + \|h\|_{**,\lambda} + \lambda^\gamma(\|\varphi\|_X + \|h\|_{**,\lambda})).$$

This and (54) imply that

$$\|F(\varphi)\|_X \leq C(\lambda^\gamma\|\varphi\|_X + \|h\|_{**,\lambda}).$$

It follows that for sufficiently small λ, F defines a contraction mapping of the region

$$\{\varphi \in X \mid \|\varphi\|_X \leq 2C\|h\|_{**,\lambda}\}.$$

A unique fixed point thus exists in this region, which inherits a solution with the required properties. The proof is concluded. □

4. **The proof of Theorem 1 when $\frac{N+2}{N-2} < p \leq \frac{N+1}{N-3}$.** The existence of the operator predicted in §2.3 with similar bounds persists if one drills a small hole in $\mathbb{R}^N$ and imposes Dirichlet boundary conditions on its boundary. Let us consider, for given ξ the set

$$\mathcal{D}_{\lambda,\xi} = \{\xi + \lambda z | z \in \mathcal{D}\}.$$

Then, let us consider the linear problem

$$\begin{cases} \Delta\phi + pw^{p-1}\phi = h + \sum_{i=1}^{N} c_i Z_i & \text{in } \mathbb{R}^N \setminus \mathcal{D}_{\lambda,\xi} \\ \lim_{|x|\to+\infty} \phi(x) = 0, \quad \phi = 0 \text{ on } \partial\mathcal{D}_{\lambda,\xi} \end{cases} \tag{55}$$

We have the following result, whose proof can be carried out with arguments similar to those in Lemma 9.

Lemma 10. *Assume that $\frac{N+2}{N-2} < p \leq \frac{N+1}{N-3}$. Given $\Lambda > 0$ there is a $C > 0$ such that for all $|\xi| \leq \Lambda$, all small $\lambda > 0$, and any h with $\|h\|_{**,\xi} < \infty$, Problem (62) has a solution $\phi = \mathcal{T}(h)$ which depends linearly on h such that*

$$\|\phi\|_{*,\xi} + \max_{1\leq i\leq N} |c_i| \leq C\|h\|_{**,\xi}.$$

In order to apply this result to solve Problem (1)-(2), we observe first that a translation and a dilation makes it equivalent to

$$\begin{cases} \Delta u + u^p = 0 & \text{in } \mathbb{R}^N \setminus \mathcal{D}_{\lambda,\xi} \\ \lim_{|x|\to+\infty} u(x) = 0, \quad u = 0 \text{ on } \partial\mathcal{D}_{\lambda,\xi}. \end{cases} \tag{56}$$

Let $\varphi_\lambda(z)$ be the unique solution of

$$\Delta\varphi_\lambda = 0 \text{ in } \mathbb{R}^N \setminus \mathcal{D}, \quad \varphi_\lambda(z) = w(\xi + \lambda z) \text{ on } \partial\mathcal{D}, \quad \lim_{|x|\to+\infty} \varphi_\lambda(x) = 0. \tag{57}$$

Then $\varphi_\lambda(z) = (w(\xi) + O(\lambda))\varphi_0(z)$ where φ_0 is the unique solution of

$$\Delta\varphi_0 = 0 \text{ in } \mathbb{R}^N \setminus \mathcal{D}, \quad \varphi_0(x) = 1 \text{ on } \partial\mathcal{D}, \lim_{|x|\to+\infty} \varphi_0(x) = 0. \tag{58}$$

We also note that

$$\lim_{|x|\to+\infty} |x|^{N-2}\varphi_0(x) = f_0 := \frac{1}{(N-2)|S^{N-1}|} \int_{\mathbb{R}^N\setminus\mathcal{D}} |\nabla\varphi_0|^2 > 0. \tag{59}$$

The number $\int_{\mathbb{R}^N\setminus\mathcal{D}} |\nabla\varphi_0|^2$ corresponds precisely to the capacity of $\mathcal{D}$.

We look for a solution of the form $u = w - \varphi_\lambda(\frac{x-\xi}{\lambda}) + \phi$, which yields the following equation for ϕ

$$\Delta\phi + pw^{p-1}\phi = N(\phi) + E_\lambda$$

where

$$E_\lambda = pw^{p-1}\varphi_\lambda, \quad N(\phi) = -(w + \phi - \varphi_\lambda)^p + w^p + pw^{p-1}\phi - pw^{p-1}\varphi_\lambda. \tag{60}$$

We consider the intermediate linear problem

$$\begin{cases} \Delta\phi + pw^{p-1}\phi = N(\phi) + E_\lambda + \displaystyle\sum_{i=1}^{N} c_i Z_i \text{ in } \mathbb{R}^N \setminus (\mathcal{D}_{\lambda,\xi}) \\ \phi = 0 \text{ on } \partial\mathcal{D}_{\lambda,\xi}, \displaystyle\lim_{|x|\to+\infty} \phi(x) = 0. \end{cases} \tag{61}$$

This nonlinear problem can be solved via contraction mapping principle based on the operator $\mathcal{T}$ above introduced in similar way as in the previous section, to yield existence of a unique solution with

$$\|\phi_\lambda\|_{*,\xi} + \max_{1\le i\le N} |c_i(\lambda,\xi)| \to 0 \quad \text{as } \lambda \to 0,$$

uniformly on $|\xi| \le \Lambda$. Besides, the numbers $c_i(\lambda,\xi)$ define continuous functions of ξ. We also have the estimate

$$\|\phi_\lambda\|_{*,\xi} \le C_\sigma \lambda^\sigma.$$

We recall that in the definition of the norms we are using an arbitrary σ with $0 < \sigma < N-2$. The desired result will be concluded if we manage to choose the point ξ in such a way that

$$c_i(\lambda,\xi) = 0 \qquad \text{for all } i = 1,\dots,N.$$

Testing the equation against Z_i, and using the above stated estimate for ϕ we see that these numbers can be expanded as

$$c_i(\lambda,\xi) = \int_{\mathbb{R}^N\setminus\mathcal{D}_{\lambda,\xi}} E_\lambda Z_i + \lambda^{N-2} o(1),$$

where the quantity $o(1)$ is uniform on $|\xi| \le \Lambda$. Now, we have that

$$\int_{\mathbb{R}^N\setminus\mathcal{D}_{\lambda,\xi}} E_\lambda Z_i = \int_{\mathbb{R}^N\setminus(\mathcal{D}_{\lambda,\xi})} \varphi_\lambda(\tfrac{x}{\lambda}) w^{p-1}(x+\xi) \frac{\partial w}{\partial x_i}(x+\xi) + o(\lambda^{N-2}) =$$
$$\lambda^{N-2}\left(f_0 \int_{\mathbb{R}^N} |x|^{-(N-2)} w^{p-1}(x+\xi) \frac{\partial w}{\partial x_i}(x+\xi) + o(1) \right).$$

Hence we obtain, setting

$$F(\xi) := \frac{f_0}{2} \int_{\mathbb{R}^N} |x|^{2-N} w(x+\xi)^p \, dx.$$

that

$$\mathbf{c}(\xi,\lambda) := (c_1,\dots,c_N) = \lambda^{N-2}\left(\nabla F(\xi) + o(1) \right)$$

where $o(1) \to 0$ uniformly on $|\xi| \le \Lambda$. Observe that F is radial and has a non-degenerate maximum at $\xi = 0$. It follows that the Brouwer degree of $\mathbf{c}(\xi,\lambda)$ in a small ball around the origin is non zero. Hence there exists a point $\xi = \xi_\lambda$, small with λ, that annihilates all c_i's simultaneously. This concludes the proof of the theorem. $\square$

5. **Nonlinear Schrödinger equations.**

5.1. **The operator $\Delta - V_\lambda + pw^{p-1}$ in $\mathbb{R}^N$.** The nonlinear equation, after a change of variables, involves the linearized problem

$$
\begin{cases} \Delta\phi + pw^{p-1}\phi - V_\lambda \phi = h + \displaystyle\sum_{i=1}^N c_i Z_i \quad \text{in } \mathbb{R}^N \\ \displaystyle\lim_{|x|\to+\infty} \phi(x) = 0 \end{cases} \tag{62}
$$

where Z_i is defined in (40) and given $\lambda > 0$ and $\xi \in \mathbb{R}^N$ we denote

$$
V_\lambda(x) = \lambda^{-2} V(\frac{x-\xi}{\lambda}).
$$

Because of the concentration of V_λ at ξ it is desirable to have a linear theory which allows singularities at ξ. Thus, for $\sigma > 0$ and $\xi \in \mathbb{R}^N$ we consider again the norms

$$
\|\phi\|_{*,\xi} = \sup_{|x-\xi|\le 1} |x-\xi|^\sigma |\phi(x)| + \sup_{|x-\xi|\ge 1} |x-\xi|^{\frac{2}{p-1}} |\phi(x)|
$$

$$
\|h\|_{**,\xi} = \sup_{|x-\xi|\le 1} |x-\xi|^{2+\sigma} |h(x)| + \sup_{|x-\xi|\ge 1} |x-\xi|^{2+\frac{2}{p-1}} |h(x)|.
$$

We will also consider ξ with a bound $|\xi| \le \Lambda$ and the estimates we present will depend on Λ.

For the linear theory it suffices to assume

$$
V \in L^\infty(\mathbb{R}^N), \quad V \ge 0, \quad V(x) = o(|x|^{-2}) \quad \text{as } |x| \to +\infty. \tag{63}
$$

Proposition 1. *Let $|\xi| \le \Lambda$. Suppose V satisfies (63) and $\|h\|_{**,\xi} < \infty$.*

(a) If $p > \frac{N+1}{N-3}$ for $\lambda > 0$ sufficiently small equation (55) with $c_i = 0$, $1 \le i \le N$ has a solution $\phi = \mathcal{T}_\lambda(h)$ that depends linearly on h and there is C such that

$$
\|\mathcal{T}_\lambda(h)\|_{*,\xi} \le C\|h\|_{**,\xi}.
$$

(b) If $\frac{N+2}{N-2} < p \le \frac{N+1}{N-3}$ for $\lambda > 0$ sufficiently small equation (55) has a solution $(\phi, c_1, \ldots, c_N) = \mathcal{T}_\lambda(h)$ that depends linearly on h and there is C such that

$$
\|\phi\|_{*,\xi} + \max_{1\le i\le N} |c_i| \le C\|h\|_{**,\xi}.
$$

The constant C is independent of λ.

The proof of this result follows similar lines as those in the previous sections, details can be found in [8]. In what follows we will prove Theorem 3, which the most delicate case. Theorem 2 follows in simpler way with the aid of the above proposition.

5.2. **Sketch of proof of Theorem 3.** Because of the obstruction in the solvability of the linearized operator for p in this range, it will be necessary to do the rescaling about a point ξ suitably chosen. For this reason we make the change of variables $\lambda^{-\frac{2}{p-1}} u(\frac{x-\xi}{\lambda})$ and look for a solution of the form $u = w + \phi$, leading to the following equation for ϕ:

$$
\Delta\phi - V_\lambda\phi + pw^{p-1}\phi = N(\phi) + V_\lambda w
$$

where

$$
V_\lambda(x) = \lambda^{-2} V(\tfrac{x-\xi}{\lambda})
$$

and N is the same as in the previous section, namely

$$
N(\phi) = -(w+\phi)^p + w^p + pw^{p-1}\phi.
$$

We will change slightly the previous notation to make the dependence of the norms in σ explicit. Hence we set

$$\|\phi\|_{*,\xi}^{(\sigma)} = \sup_{|x-\xi|\le 1} |x-\xi|^{\sigma}|\phi(x)| + \sup_{|x-\xi|\ge 1} |x-\xi|^{\frac{2}{p-1}}|\phi(x)|$$

$$\|h\|_{**,\xi}^{(\sigma)} = \sup_{|x-\xi|\le 1} |x-\xi|^{2+\sigma}|h(x)| + \sup_{|x-\xi|\ge 1} |x-\xi|^{2+\frac{2}{p-1}}|h(x)|.$$

In the rest of the section we assume that

$$\frac{N+2}{N-2} < p < \frac{N+1}{N-3}.$$

The case $p = \frac{N+1}{N-3}$ can be handled similarly, with a slight modification of the norms.

Lemma 11. *Let $\frac{N+2}{N-2} < p \le \frac{N+1}{N-3}$ and V satisfy (63) and $\Lambda > 0$. Then there is $\varepsilon_0 >$ such that for $|\xi| < \Lambda$ and $\lambda < \varepsilon_0$ there exist $\phi_\lambda, c_1(\lambda), \dots, c_N(\lambda)$ solution to*

$$\begin{cases} \Delta\phi - V_\lambda\phi + pw^{p-1}\phi = N(\phi) + V_\lambda w + \displaystyle\sum_{i=1}^{N} c_i Z_i \\ \displaystyle\lim_{|x|\to+\infty} \phi(x) = 0. \end{cases} \tag{64}$$

We have in addition

$$\|\phi_\lambda\|_{*,\xi} + \max_{1\le i\le N} |c_i(\lambda)| \to 0 \quad \text{as } \lambda \to 0.$$

If V satisfies also

$$V(x) \le C|x|^{-\mu} \qquad \text{for all } x \tag{65}$$

for some $\mu > 2$, then for $0 < \sigma \le \mu - 2$, $\sigma < N-2$

$$\|\phi_\lambda\|_{*,\xi}^{(\sigma)} \le C_\sigma \lambda^{\sigma}, \qquad \text{for all } 0 < \lambda < \varepsilon_0. \tag{66}$$

Proof. We fix $0 < \sigma < \min(2, \frac{2}{p-1})$ and define for small $\rho > 0$

$$\mathcal{F} = \{\phi : \mathbb{R}^N \to \mathbb{R} \ / \ \|\phi\|_{*,\xi}^{(\sigma)} \le \rho\}$$

and the operator $\phi_1 = \mathcal{A}_\lambda(\phi)$ where $\phi_1, c_1, \dots, c_N$ is the solution in Lemma 1 to

$$\begin{cases} \Delta\phi_1 - V_\lambda\phi_1 + pw^{p-1}\phi_1 = N(\phi) + V_\lambda w + \displaystyle\sum_{i=1}^{N} c_i Z_i \quad \text{in } \mathbb{R}^N \\ \displaystyle\lim_{|x|\to+\infty} |\phi(x)| = 0, \end{cases}$$

where N is given by (60).

It is not hard to check that $\mathcal{A}_\lambda$ is a contraction mapping on $\mathcal{F}$ for the above norm for small enough ρ. More precisely, we have

$$\|V_\lambda w\|_{**,\xi}^{(\sigma)} = o(1) \qquad \text{as } \lambda \to 0.$$

And for $\rho = C\|V_\lambda w)\|_{**,\xi}^{(\sigma)}$, suitable C, $\mathcal{A}_\lambda$ possesses a unique fixed point ϕ_λ in $\mathcal{F}$ and it satisfies

$$\|\phi_\lambda\|_{*,\xi}^{(\sigma)} \le C\|V_\lambda w\|_{**,\xi}^{(\sigma)} = o(1). \tag{67}$$

Under assumption (65) and for $0 < \theta \le \mu - 2$ we can also estimate $\|V_\lambda w\|_{**,\xi}^{(\theta)}$ as follows:

$$\|V_\lambda w\|_{**,\xi}^{(\theta)} \le C\lambda^{\theta}. \tag{68}$$

Using this one can argue to find the validity of the desired estimate for ϕ_λ. $\square$

Proof of Theorem 3 We have found a solution $\phi_\lambda, c_1(\lambda), \dots, c_N(\lambda)$ to (64). The solution constructed satisfies for all $1 \le j \le N$:

$$\int_{\mathbb{R}^N} \left(V_\lambda \phi_\lambda + V_\lambda w + N(\phi_\lambda) + \sum_{i=1}^{N} c_i Z_i \right) \frac{\partial w}{\partial x_j}(y) = 0.$$

Thus, for all λ small, we need to find $\xi = \xi_\lambda$ so that $c_i = 0$, $1 \le i \le N$, that is

$$\int_{\mathbb{R}^N} (V_\lambda \phi_\lambda + V_\lambda w + N(\phi_\lambda)) \frac{\partial w}{\partial x_j} = 0 \quad \forall 1 \le j \le N. \tag{69}$$

Condition (69) is actually sufficient under the assumption, which will turn out to be satisfied in our cases, that ξ_λ is bounded as $\lambda \to 0$ because, in this situation, the matrix with coefficients

$$\int_{\mathbb{R}^N} Z_i(y-\xi) \frac{\partial w}{\partial x_j}(y)\, dy$$

is invertible, provided the number R_0 in the definition of Z_i is chosen large enough.

The dominant term in (69) is

$$\lambda^{-2} \int_{\mathbb{R}^N} V(\tfrac{y-\xi}{\lambda}) w \frac{\partial w}{\partial y_j} = \lambda^{-2} \int_{\mathbb{R}^N} V(\tfrac{x}{\lambda}) w(x+\xi) \frac{\partial w}{\partial x_j}(x+\xi) \tag{70}$$

whose asymptotic behavior depends on the decay of $V(x)$ as $|x| \to +\infty$.

We recall that we are assuming $V(x) \le C|x|^{-\mu}$, $\mu > N$. Thus we have

$$\int_{\mathbb{R}^N} V(\tfrac{x}{\lambda}) w(x+\xi) \frac{\partial w}{\partial x_j}(x+\xi) = \lambda^N C_V w(\xi) \frac{\partial w}{\partial x_j}(\xi) + o(\lambda^N) \quad \text{as } \lambda \to 0,$$

where $C_V = \int_{\mathbb{R}^N} V$ and the convergence is uniform with respect to $|\xi| < \varepsilon_0$. We obtain the existence of a solution ξ to (69) thanks to the non-degeneracy of 0 as a critical point of $w^2(\xi)$. Furthermore, the point ξ will be close to 0. After some work we find the other terms in (69) are small compared to (70), in fact it turns out that

$$\left| \int_{\mathbb{R}^N} V_\lambda \phi_\lambda \frac{\partial w}{\partial x_j} \right| + \int_{\mathbb{R}^N} \left| N(\phi_\lambda) \frac{\partial w}{\partial w_j} \right| = o(\lambda^{N-2}) \quad \text{as } \lambda \to 0. \tag{71}$$

Going back to (69) we set

$$F_\lambda^{(j)}(\xi) = \lambda^{-2} \int_{\mathbb{R}^N} V(\tfrac{x}{\lambda}) u_\lambda \frac{\partial w}{\partial x_j}(x+\xi) + \int_{\mathbb{R}^N} N(\phi_\lambda) \frac{\partial w}{\partial x_j}(x+\xi)$$

and $F_\lambda = (F_\lambda^{(1)}, \dots, F_\lambda^{(N)})$. Fix now $\delta > 0$ small and work with $|\xi| = \delta$. Then we have for small λ

$$\langle F_\lambda(\xi), \xi \rangle < 0 \quad \text{for all } |\xi| = \delta.$$

By degree theory we deduce that F_λ has a zero in B_δ.

6. **Sketch of the proof of Theorem 4.** The proof of this result is similar in spirit to that of the previous theorems. Now the basic point is to obtain a suitable invertibility theory for the linearized operator $\Delta + pw^{p-1}$ on $\mathbb{R}^N \setminus B_1(0)$ where, again with abuse of notation, we are calling w the unique solution w_* of Problem (15)-(16). Thus, we consider the problem

$$\Delta\phi + pw^{p-1}\phi = h \qquad \text{in } \mathbb{R}^N \setminus \bar{B}_1(0)\,, \tag{72}$$

$$\phi = 0 \quad \text{on } \partial B_1(0)\,, \qquad \lim_{|x|\to+\infty} \phi(x) = 0. \tag{73}$$

6.1. **Condition for non-resonance.** We want to investigate under what conditions the homogeneous problem with $h = 0$ in (72)-(73) admits only the trivial solution. To this end, we consider the first eigenvalue of the problem

$$\psi'' + \frac{N-1}{r}\psi' + pw^{p-1}\psi + \nu\frac{\psi}{r^2} = 0 \tag{74}$$

$$\psi(1) = 0, \quad \psi(+\infty) = 0\ . \tag{75}$$

This eigenvalue is variationally characterized as

$$\nu(p) = \inf_{\psi\in\mathcal{E}} \frac{\int_1^\infty |\psi'|^2 r^{N-1}dr - p\int_1^\infty w^{p-1}|\psi|^2 r^{N-1}dr}{\int_1^\infty \psi^2 r^{N-3}dr}\,, \tag{76}$$

with

$$\mathcal{E} \;=\; \{\psi \in C^1[1,\infty) \;/\; \psi(1) = 0,\ \int_1^\infty |\psi^{'}(r)|^2 r^{N-1}dr < +\infty\,\}.$$

This quantity is well defined thanks to Hardy's inequality,

$$\frac{(N-2)^2}{4}\int_1^\infty \psi^2 r^{N-3}dr \le \int_1^\infty |\psi'|^2 r^{N-1}dr.$$

The number $\nu(p)$ is negative, since this Rayleigh quotient gets negative when evaluated at $\psi = w$. An extremal is easily found, using the fast decay of $w^{p-1} = O(r^{-4})$. This extremal represents a positive solution to problem (74)-(75) for $\nu = \nu(p)$. Let us consider now Problem (72)-(73) for $h = 0$, and assume that we have a solution ϕ. The symmetry of the domain $\mathbb{R}^N \setminus B_1(0)$ allows us to expand ϕ into spherical harmonics. We write again ϕ as

$$\phi(x) = \sum_{k=0}^{\infty} \phi_k(r)\Theta_k(\theta), \quad r > 0,\, \theta \in S^{N-1}$$

The components ϕ_k then satisfy the differential equations

$$\phi_k'' + \frac{N-1}{r}\phi_k' + \left(pw^{p-1} - \frac{\lambda_k}{r^2}\right)\phi_k = 0, \quad r \in (1,\infty), \tag{77}$$

$$\phi_k(1) = 0, \quad \phi_k(+\infty) = 0.$$

Let us consider first the radial mode $k = 0$, namely $\lambda_k = 0$. We observe that the function

$$Z_1(r) = rw'(r) + \frac{2}{p-1}w$$

satisfies

$$Z_1'' + \frac{N-1}{r}Z_1' + pw^{p-1}Z_1 = 0, \qquad \text{for all } r > 1,$$

but $Z_1(1) \neq 0$. We notice that Z_1 is one-signed for all large r. It follows then that a second generator of the solutions of this ODE is given, for large r, by the reduction of order formula,

$$Z_2 = Z_1(r) \int_R^r \frac{dr}{r^{N-1} Z^2}$$

but since at main order $Z_1(r) \sim cr^{2-N}$ we see that $Z_2(+\infty) \neq 0$. Since ϕ_0 is a linear combination of Z_1 and Z_2 it follows that the only possibility is $\phi_0 = 0$. Let us consider now mode 1, namely $k = 1, \ldots, N-1$, for which $\lambda_k = (N-1)$. In this case we also have an explicit solution which does not vanish at $r = 1$ but it does at $r = +\infty$. Simply $Z_1(r) = w'(r)$. But the same argument as above gives us a second generator $Z_2(r) \sim r$ as $r \to +\infty$, hence again, the only possibility is that $\phi_k \equiv 0$ for all $k = 1, \ldots, N$.

Let us consider now modes $N+1$ or higher. This case is harder. Not only we do not have an explicit solution to the ODE to rely on, but it could be the case that a non-trivial solution exists. Let us assume this is the case for an arbitrary mode $k \geq N$. We claim that ϕ_k cannot change sign in $(1, \infty)$. In fact if it did, we begin by observing that it can only do it a finite number of times, since its behavior at infinity must be eventually like that of a decaying solution of the Euler's ODE

$$Z'' + \frac{N-1}{r} Z' - \frac{\lambda_k}{r^2} Z = 0$$

namely, at main order we must have

$$Z(r) = cr^{-\mu}(1 + o(1)), \quad \mu = -\frac{N-2}{2} - \frac{1}{2}\sqrt{(N-2)^2 + 4\lambda_k}\,.$$

Let $r_0 > 1$ be the last zero of ϕ_k, and let us assume that $\phi > 0$ on (r_0, ∞). We observe now that since $\Delta w < 0$, $w'(r)$ has exactly one zero in $(1, \infty)$. Thanks to Sturm's theorem this zero must be less than r_0. Hence $w' < 0$ in (r_0, ∞). Let us observe now that

$$W(r) = r^{N-1}(w'\phi_k' - w''\phi_k)$$

satisfies in (r, ∞)

$$W'(r) = r^{N-3}(\lambda_k - \lambda_1) w' \phi_k < 0 \quad \text{in } (r_0, \infty),$$

while $W(r_0) < 0$ and $W(+\infty) = 0$, which is impossible. This shows that ϕ_k must be one-signed. Thus the only possibility for equation (77) to have a nontrivial solution for a given $k \geq N$ is that $\lambda_k = -\nu(p)$. Thus we have proven the following result.

Lemma 12. *Assume that p is such that*

$$\nu(p) \neq -j(N-2+j) \quad \textit{for all } j = 2, 3, \ldots \tag{78}$$

where $\nu(p)$ is the principal eigenvalue defined by (76). Then Problem (74)-(75) with $h = 0$ admits only the solution $\phi = 0$.

This non-resonance condition produces a good solvability theory for equation (72)-(73). We can describe qualitatively the set of exponents p for which condition (78) fails. We have:

Lemma 13. *For each $j \geq 2$ the set of numbers p for which $\nu(p) = -j(N-2+j)$ is non-empty and finite. In particular, there exists a sequence of the form*

$$\frac{N+2}{N-2} < p_1 < p_2 < p_3 < \cdots\,; \quad p_j \to +\infty \quad \textit{as } j \to +\infty\,, \tag{79}$$

such that condition (78) holds if and only if $p \neq p_j$ for all $j = 1, 2, \ldots$.

The proof of this result is contained in [9]. It consists of showing that the eigenvalue $\nu(p)$ is a real analytic function of the parameter p. A basic ingredient is the proof of analytic dependence of w as a function of p, in appropriate spaces, which follows basically form an analysis due to Dancer [5].

6.2. **Solvability of (72)-(73).** We consider now the full problem (72)-(73), namely

$$\Delta\phi + pw^{p-1}\phi = h \qquad \text{in } \mathbb{R}^N \setminus \bar{B}_1(0)\,,$$

$$\phi = 0 \ \text{ on } \partial B_1(0)\,, \qquad \lim_{|x|\to+\infty} \phi(x) = 0\,.$$

Let us fix a small number $\sigma > 0$ and consider the norms

$$\|\phi\|_* = \sup_{|x|>1} |x|^{N-2-\sigma}|\phi(x)| + \sup_{|x|>1} |x|^{N-1-\sigma}|\nabla\phi(x)| \tag{80}$$

and

$$\|h\|_{**} = \sup_{|x|>1} |x|^{N-\sigma}|h(x)|. \tag{81}$$

Lemma 14. *Assume that p satisfies condition (78). Then for any h with $\|h\|_{**} < +\infty$, Problem (72)-(73) has a unique solution $\phi = T(h)$ with $\|\phi\|_* < +\infty$. Besides, there exists a constant $C(p) > 0$ such that*

$$\|T(h)\|_* \le C\|h\|_{**}.$$

6.3. **The operator $\Delta + pw^{p-1}$ in $\delta^{-1}\mathcal{D} \setminus B_1(0)$.** We assume that $Q = 0$, and consider the large expanded domain $\mathcal{D}_\delta = \delta^{-1}\mathcal{D}$. We shall carry out a gluing procedure that will permit us to establish the same conclusion of Proposition 14 in this domain, provided that δ is taken sufficiently small. Thus we consider now the linear problem

$$\Delta\phi + pw^{p-1}\phi = h \qquad \text{in } \mathcal{D}_\delta \setminus \bar{B}_1(0)\,, \tag{82}$$

$$\phi = 0 \ \text{ on } \partial B_1(0) \cup \partial\mathcal{D}_\delta. \tag{83}$$

We consider the same norms as in (80), (81) restricted to this domain.

Lemma 15. *Assume that p satisfies condition (78). Then there is a number δ_0 such that for all $\delta < \delta_0$ and any h with $\|h\|_{**} < +\infty$, Problem (82)-(83) has a unique solution $\phi = T_\delta(h)$ with $\|\phi\|_* < +\infty$. Besides, there exists a constant $C(p, \mathcal{D}) > 0$ such that*

$$\|T_\delta(h)\|_* \le C\|h\|_{**}.$$

The proof of this result follows a similar scheme to that of Lemma 9. The point now is that the fact that the linear theory involves faster decays makes the contribution of the far-away part of $\mathcal{D}_\delta$ to enter at a substantially small order. An analysis of this type *is not possible* if the basic cell w was taken as a slow-decaying solution.

6.4. **Conclusion of the proof of Theorem 4.** Let us assume the validity of condition (78) or, equivalently, that $p \neq p_j$ for all j, with p_j the sequence in (79). Problem (1)-(2) is, after setting $v(x) = \delta^{\frac{2}{p-1}} u(\delta x)$, equivalent to

$$\Delta v + v^p = 0 \quad \text{ in } \mathcal{D}_\delta \setminus \bar{B}_1(0)\,, \tag{84}$$

$$v = 0 \;\text{ on } \partial B_1(0) \cup \partial \mathcal{D}_\delta. \tag{85}$$

Let us consider the smooth cut-off function η_δ, introduced in the previous section, which equals 1 in $B(0, 2\delta^{-1})$ and 0 outside $B(0, 3\delta^{-1})$. We search for a solution v to problem (84)-(85) of the form

$$v = \eta_\delta w + \phi,$$

which is equivalent to the following problem for ϕ:

$$\Delta\phi + p w^{p-1}\phi = N(\phi) + E \quad \text{ in } \mathcal{D}_\delta \setminus \bar{B}_1(0)\,, \tag{86}$$

$$\phi = 0 \;\text{ on } \partial B_1(0) \cup \partial \mathcal{D}_\delta. \tag{87}$$

where

$$N(\phi) = N_1(\phi) + N_2(\phi)\,,$$

$$N_1(\phi) = -(\eta_\delta w + \phi)^p + (\eta_\delta w)^p + p(\eta_\delta w)^{p-1}\phi,$$

$$N_2(\phi) = p(1 - \eta_\delta^{p-1}) w^{p-1}\phi,$$

and

$$E = -\Delta(\eta_\delta w) - (\eta_\delta w)^p.$$

According to Proposition 15 we thus have a solution to (84)-(85) if ϕ solves the fixed point problem

$$\phi = T_\delta(N(\phi) + E)\,. \tag{88}$$

We get

$$\|E\|_{**} \leq C\delta^\sigma\,. \tag{89}$$

On the other hand, we also find

$$\|N_2(\phi)\|_{**} \leq C\delta^2 \|\phi\|_*$$

and so that

$$\|N_1(\phi)\|_{**} \leq C\,(\,\|\phi\|_*^p + \|\phi\|_*^2\,)\,. \tag{90}$$

Let us consider now the operator

$$\mathcal{T}(\phi) = T_\delta(N(\phi) + E)$$

defined in the region

$$\mathcal{B} = \{\phi \in C^1(\bar{\mathcal{D}}_\delta \setminus B_1(0)) \;/\; \|\phi\|_* \leq \delta^{\frac{\sigma}{2}}\,\}\,.$$

We immediately get that $\mathcal{T}(\mathcal{B}) \subset \mathcal{B}$, provided that δ is sufficiently small. The existence of a fixed point thus follows from Schauder's theorem. $\square$

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