

A counterexample to a conjecture by De Giorgi in large dimensions

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Abstract

We consider the Allen–Cahn equation

$$\Delta u + u(1 - u^2) = 0 \quad \text{in } \mathbb{R}^N.$$

A celebrated conjecture by E. De Giorgi (1978) states that if u is a bounded solution to this problem such that $\partial_{x_N} u > 0$, then the level sets $\{u = \lambda\}$, $\lambda \in \mathbb{R}$, must be hyperplanes *at least* if $N \leq 8$. We construct a family of solutions which shows that this statement does not hold true for $N \geq 9$.

Résumé

Un contre-exemple à la conjecture de De Giorgi en grandes dimensions. Nous considérons l'équation d'Allen–Cahn :

$$\Delta u + u(1 - u^2) = 0 \quad \text{dans } \mathbb{R}^N.$$

Une conjecture célèbre de E. De Giorgi (1978) affirme que si u est une solution bornée de ce problème telle que $\partial_{x_N} u > 0$, alors les ensembles de niveau $\{u = \lambda\}$, $\lambda \in \mathbb{R}$, sont des hyperplans *au moins* si $N \leq 8$. Nous construisons une famille de solutions qui montre que cette conjecture n'est pas vraie pour $N \geq 9$. *Pour citer cet article : M. del Pino et al., C. R. Acad. Sci. Paris, Ser. I 346 (2008).*
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Considérons l'équation d'Allen–Cahn :

$$\Delta u + (1 - u^2)u = 0 \quad \text{dans } \mathbb{R}^N. \tag{1}$$

E. De Giorgi [8] formula en 1978 la conjecture célèbre suivante :

(DG) *Soit u une solution bornée de l'équation (1) telle que $\partial_{x_N} u > 0$. Alors les ensembles de niveau $\{u = \lambda\}$ sont des hyperplans, au moins pour des dimensions $N \leq 8$.*

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La conjecture de De Giorgi a été complètement établie en dimension $N = 2$ par Ghoussoub et Gui [14], et en dimension $N = 3$ par Ambrosio et Cabré [2]. Savin [21] a démontré sa validité pour $4 \leq N \leq 8$ sous une condition additionnelle. Un contre-exemple à (DG) en dimension $N \geq 9$ était attendu depuis longtemps, mais la question était restée ouverte. Dans cette Note, nous en construisons un. La conjecture (DG) est analogue au *théorème de Bernstein* pour les graphes minimaux qui, dans sa forme la plus générale due à Simons [23], affirme que dans \mathbb{R}^N , toute surface minimale qui est aussi le graphe d'une fonction de $N - 1$ variables, est un hyperplan si $N \leq 8$. Bombieri, De Giorgi et Giusti [5] ont démontré que ce résultat est faux en dimension $N \geq 9$, en construisant une solution non-triviale du problème,

$$\nabla \cdot \left(\frac{\nabla F}{\sqrt{1 + |\nabla F|^2}} \right) = 0 \quad \text{dans } \mathbb{R}^8, \quad (2)$$

par la méthode des sur- et sous-solutions. Posons

$$x' = (x_1, \dots, x_8) \in \mathbb{R}^8, \quad u = \sqrt{x_1^2 + \dots + x_4^2}, \quad v = \sqrt{x_5^2 + \dots + x_8^2}.$$

La solution BDG est de la forme $F(x') = F(u, v)$ avec la propriété de symétrie $F(u, v) = -F(v, u)$ si $u \geq v$. De plus, nous pouvons montrer que F est asymptotique à une fonction homogène de degré 3 qui s'annule sur le cône $u = v$. Soit $\Gamma = \{x_9 = F(x')\}$ le graphe minimal (BDG) ainsi construit. Considérons pour $\alpha > 0$ sa dilatation $\Gamma_\alpha = \alpha^{-1}\Gamma$, qui est aussi un graphe minimal. Notre principal résultat est le théorème suivant :

Théorème 1. *Soit $N \geq 9$. Pour tout $\alpha > 0$ assez petit, il existe une solution bornée $u_\alpha(x)$ de l'équation (1) telle que $u_\alpha(0) = 0$, $\partial_{x_N} u_\alpha(x) > 0$ pour tout $x \in \mathbb{R}^N$, et $|u_\alpha(x)| \rightarrow 1$ quand $\text{dist}(x, \Gamma_\alpha) \rightarrow +\infty$, uniformément pour tout $\alpha > 0$ petit.*

La démonstration fournit des informations très précises sur u_α . Si $t = t(y)$ dénote un choix de distance signée au graphe Γ_α , alors pour un nombre $\delta > 0$ fixé, assez petit, notre solution se comporte comme $u_\alpha(x) \sim w(t)$, si $|t| < \frac{\delta}{\alpha}$ avec $w(t) = \tanh(\frac{t}{\sqrt{2}})$, solution uni-dimensionnelle hétérocline de (1). La démonstration utilise une variante en dimension infinie de la réduction de Lyapunov-Schmidt. Considérons des coordonnées de Fermi pour décrire des points proches de Γ_α , $x = y + z\nu_\alpha(y)$, $y \in \Gamma_\alpha$, $|z| < \frac{\delta}{\alpha}$ où ν_α est le vecteur unitaire normal à Γ_α tel que $\nu_{\alpha 9} > 0$. Nous choisissons alors comme première approximation $w(x) := w(z + h(\alpha y))$ où h est une fonction régulière à déterminer, petite, sur $\Gamma = \Gamma_1$. En cherchant une solution de la forme $w + \phi$, le problème se réduit essentiellement à :

$$\Delta_{\Gamma_\alpha} \phi + \partial_{zz} \phi + f'(w(z))\phi + E + N(\phi) = 0 \quad \text{dans } \Gamma_\alpha \times \mathbb{R}$$

où $E = \chi_{|z| < \alpha^{-1}\delta} S(w)$, $S(w) = \Delta w + \mathcal{E}(w)$, $N(\phi) = f(w + \phi) - f(w) - f'(w)\phi + B(\phi)$, $f(w) = w(1 - w^2)$, et $B(\phi)$ est un opérateur linéaire du second ordre avec des coefficients petits, tronqués pour $|z| > \delta\alpha^{-1}$. Plutôt que de résoudre directement le problème précédent, nous en considérons une version projetée :

$$\mathcal{L}(\phi) := \Delta_{\Gamma_\alpha} \phi + \partial_{zz} \phi + f'(w(z))\phi = -E - N(\phi) + c(y)w'(z) \quad \text{dans } \Gamma_\alpha \times \mathbb{R}, \quad (3)$$

$$\int \phi(y, z)w'(z) dz = 0 \quad \text{pour tout } y \in \Gamma_\alpha. \quad (4)$$

Une solution de ce problème peut être obtenue en respectant la taille et le taux de décroissance de l'erreur E , qui est, grossièrement, de l'ordre de $r(\alpha y)^{-3}e^{-|z|}$. Ceci est établi de manière précise à l'aide d'une théorie linéaire pour le problème projeté dans des espaces de Sobolev à poids et en utilisant un principe de contraction. Puis, h est déterminée de sorte que $c(y) \equiv 0$. Nous avons $c(y) \int w'^2 dz = \int (E + N(\phi))w' dz$ et nous réduisons le problème à une EDP non linéaire (non-locale) sur Γ de la forme

$$\mathcal{J}(h) := \Delta_\Gamma h + |A|^2 h = O(\alpha)r(y)^{-3} + M_\alpha(h) \quad \text{dans } \Gamma, \quad h = 0 \quad \text{sur } \Gamma \cap [u = v], \quad (5)$$

où $M(h)$ est un opérateur petit qui contient des termes non-locaux. Une théorie de résolubilité pour l'opérateur de Jacobi dans des espaces de Sobolev à poids développée en utilisant l'ingrédient crucial qui consiste en la présence de barrières explicites pour les inégalités impliquant l'opérateur linéaire défini précédemment. En utilisant cette théorie, le problème (5) est finalement résolu au moyen du principe de l'application contractante. La propriété de monotonie découle facilement du principe du maximum.

1. Introduction and statement of main result

This Note deals with entire bounded solutions of the Allen–Cahn equation:

$$\Delta u + (1 - u^2)u = 0 \quad \text{in } \mathbb{R}^N. \quad (6)$$

E. De Giorgi [8] formulated in 1978 the following celebrated conjecture:

(DG) *Let u be a bounded solution of Eq. (6) such that $\partial_{x_N} u > 0$. Then the level sets $\{u = \lambda\}$ are hyperplanes, at least for dimension $N \leq 8$.*

Equivalently, u depends on just one Euclidean variable so that it must have the form $u(x) = \tanh((x - p) \cdot b/\sqrt{2})$, for some $p, b \in \mathbb{R}^N$ with $|b| = 1$ and $b_N > 0$.

Eq. (6) arises in the gradient theory of phase transitions by Cahn–Hilliard and Allen–Cahn, in connection with the energy functional:

$$J_\varepsilon(u) = \frac{\varepsilon}{2} \int_{\Omega} |\nabla u|^2 + \frac{1}{4\varepsilon} \int_{\Omega} (1 - u^2)^2, \quad \varepsilon > 0, \quad \text{where } \Omega \subset \mathbb{R}^N \text{ is a bounded domain.} \quad (7)$$

This problem has been extensively studied by Γ -convergence methods [16–19]. It is known that a family of local minimizers u_ε with uniformly bounded energy approaches as $\varepsilon \rightarrow 0^+$, in L^1 -loc sense, a function of the form $\chi_S - \chi_{S^c}$ where ∂S is a set with minimal perimeter, so that the limiting interface between the *stable phases* $u = 1$ and $u = -1$, is expected to approach a minimal hypersurface. This connection led De Giorgi to formulate his conjecture, as a parallel with *Bernstein’s theorem* for minimal graphs which in its most general form, due to Simons [23], states that any minimal hypersurface in \mathbb{R}^N , which is also a graph of a function of $N - 1$ variables, must be a hyperplane if $N \leq 8$. Bombieri, De Giorgi and Giusti [5] proved that this fact is false in dimension $N \geq 9$. De Giorgi conjecture has been fully established in dimensions $N = 2$ by Ghoussoub and Gui [14], for $N = 3$ by Ambrosio and Cabré [2]. Savin [21] proved its validity for $4 \leq N \leq 8$ under the additional assumption

$$\lim_{x_N \rightarrow \pm\infty} u(x', x_N) = \pm 1 \quad \text{for all } x'. \quad (8)$$

Condition (8) is related to the so-called Gibbons’ Conjecture: *Let u be a bounded solution of Eq. (6) satisfying $\lim_{x_N \rightarrow \pm\infty} u(x', x_N) = \pm 1$, uniformly in x' . Then the level sets $\{u = \lambda\}$ are all hyperplanes.*

Gibbons’ Conjecture has been proved in all dimensions with different methods by Farina [13], Barlow, Bass and Gui [3], and Berestycki, Hamel, and Monneau [4]. If the uniformity in this conjecture is dropped, a counterexample can be built using the method by Pacard and the authors in [11]. In this context we should also mention a result of Caffarelli and Cordoba which says that *De Giorgi conjecture* holds in any dimension for any solution that has one level set which is a globally Lipschitz graph (see also [3]).

A counterexample to (DG) in dimension $N \geq 9$ has long been believed to exist, but the question has remained open. Partial progress in this direction has been achieved by Jerison and Monneau [15] and by Cabré and Terra [6]. In this note we sketch the construction of a bounded solution of Eq. (6) which is monotone in one direction, whose level sets are not hyperplanes when $N \geq 9$. This example “disproves” De Giorgi conjecture in dimension $N \geq 9$. It suffices to do such a construction in dimension $N = 9$ by extending the solution as a constant dependence in the remaining variables. Our starting point is the existence of non-trivial smooth, entire solutions of the minimal surface equation

$$\nabla \cdot \left(\frac{\nabla F}{\sqrt{1 + |\nabla F|^2}} \right) = 0 \quad \text{in } \mathbb{R}^8, \quad (9)$$

as found in [5] by means of the super-subsolution method. Let us write:

$$x' = (x_1, \dots, x_8) \in \mathbb{R}^8, \quad u = \sqrt{x_1^2 + \dots + x_4^2}, \quad v = \sqrt{x_5^2 + \dots + x_8^2}.$$

The BDG solution has the form $F(x') = F(u, v)$ with the symmetry property $F(u, v) = -F(v, u)$ if $u \geq v$. A refinement of the result in [5] leads to a precise description of the asymptotic behavior of F :

Proposition 1.1. *Using polar coordinates $u = r \cos \theta$, $v = r \sin \theta$, there exists an explicit, smooth positive function $g(\theta)$, $\theta \in (\frac{\pi}{4}, \frac{\pi}{2})$ with $g(\frac{\pi}{4}) = 0$, $g'(\frac{\pi}{2}) = 0$ such that if we set $F_0(x') = r^3 g(\theta)$, then for some $\sigma > 0$ we have that for $x' = (u, v)$, $v > u$, the BDG-solution can be assumed to satisfy*

$$F_0(x') \leq F(x') \leq F_0(x') + |x'|^{-\sigma} \min\{F_0(x'), 1\} \quad \text{for all } |x'| \gg 1.$$

Let $\Gamma = \{x_9 = F(x')\}$ be the minimal BDG graph predicted above and let us consider for $\alpha > 0$ its dilation $\Gamma_\alpha = \{x_9 = \alpha^{-1} F(\alpha x')\}$, which is also a minimal graph. Our main result states as follows:

Theorem 1. *Let $N \geq 9$. For all $\alpha > 0$ sufficiently small there exists a bounded solution $u_\alpha(x)$ of Eq. (6) such that $u_\alpha(0) = 0$,*

$$\partial_{x_N} u_\alpha(x) > 0, \quad \text{for all } x \in \mathbb{R}^N,$$

and

$$|u_\alpha(x)| \rightarrow 1, \quad \text{as } \text{dist}(x, \Gamma_\alpha) \rightarrow +\infty, \tag{10}$$

uniformly in small $\alpha > 0$.

Property (10) implies that the 0 level set of u_α lies inside the region $\text{dist}(y, \Gamma_\alpha) < R$ for some fixed $R > 0$ and all small α , and hence it cannot be a hyperplane. Moreover, it can be shown that u_α inherits the symmetry of Γ_α , and hence u_α has value 0 and the origin. It should also be noticed that as $x_N \rightarrow \pm\infty$ we have $u_\alpha \rightarrow \pm 1$ and hence, by the result of Albritton, Ambrosio and Cabré [1], the solutions we found are local (or global) minimizers. Much more accurate information on the solution will be drawn from the proof. The idea is simple. If $t(x)$ denotes a choice of signed distance to the graph Γ_α then, for a small fixed number $\delta > 0$, our solution looks like:

$$u_\alpha(x) \sim \tanh\left(\frac{t(x)}{\sqrt{2}}\right) \quad \text{if } |t(x)| < \frac{\delta}{\alpha}.$$

We should mention that Allen–Cahn with a small parameter in a compact manifold was established to have concentration phenomena around non-degenerate minimal submanifolds, as found in [20]. Our proof is based on an infinite-dimensional form of Lyapunov–Schmidt reduction, close in spirit to that in [9,10,12,11].

2. Sketch of the proof of Theorem 1

Let us consider Fermi coordinates to describe points near Γ_α ,

$$x = y + z\nu_\alpha(y), \quad y \in \Gamma_\alpha, \quad |z| < \frac{\delta}{\alpha},$$

where ν_α is the unit normal to Γ_α for which $\nu_{\alpha 9} > 0$. Observe that $\nu_\alpha(y) = \nu_1(\alpha y)$. It is natural to consider the function $w(z)$ as a good first approximation to a solution of (6) whose zero level set equals Γ_α which is of course far from a hyperplane. Rather than taking this as our approximation we choose $w(x) := w(z + h(\alpha y))$, $w(t) = \tanh(\frac{t}{\sqrt{2}})$, where h is a smooth, small function on $\Gamma = \Gamma_1$, to be determined. It is natural to consider Laplace’s operator in \mathbb{R}^9 expressed in these coordinates,

$$\Delta = \partial_z^2 + \Delta_{\Gamma_{\alpha,z}} - H_{\alpha,z} \partial_z.$$

Here

$$\Gamma_{\alpha,z} = \{x \in \mathbb{R}^9 \mid d(x, \Gamma_\alpha) = z\}, \quad |z| < \frac{\delta}{\alpha},$$

$\Delta_{\Gamma_{\alpha,z}}$ is its Laplace–Beltrami operator and $H_{\alpha,z}$ its mean curvature. Let us remark that if $\tilde{p}(y) = p(\alpha y)$ then $\Delta_{\Gamma_{\alpha,z}} \tilde{p}(y) = \Delta_{\Gamma_1, \alpha z} p(\alpha y)$, besides $H_{\alpha,z}(y) = \alpha H_{1, \alpha z}(\alpha y)$. Since Γ_α is a minimal surface, we get that $H_{\alpha,0}(y) = 0$. We can also expand

$$H_{\alpha,z}(y) = \alpha z H_1(\alpha y) + \alpha^2 z^2 H_2(\alpha y) + \alpha^3 z^3 H_3(\alpha y) + \dots,$$

where $H_1 = |A|^2$, the square of the Euclidean norm of the second fundamental form on Γ . We actually have decay in the coefficients H_j of the form $H_j(y) = O(r^{-j-1})$, which follows from curvature estimates for minimal graphs due to L. Simon [22]. Here and henceforth we denote:

$$r(y) = |x'| \quad \text{for } y = (x', F(x')).$$

Besides, we assume (a priori) in h a size $r^2|D_{\Gamma}^2 h| + |h| \sim \alpha r^{-1}$, which gets a posteriori justified as h at last is chosen as the solution of certain fixed problem that yields this size. We expand,

$$S(\mathbf{w}) := \Delta \mathbf{w} + f(\mathbf{w}) = \alpha^2(\Delta_{\Gamma} h + |A|^2 h)(\alpha y) w_z + \alpha^2 |A|^2 z w_z + O(\alpha^3) r^{-3}(\alpha y) e^{-\sigma|z|},$$

where the last term is a smaller order operator which depends on first and second derivatives of h . The term $\alpha^2 |A|^2 z w_z$ can be easily eliminated by an improvement of approximation, adding a suitable (explicit) small term w_1 to the initial approximation. We have now $u = \mathbf{w} + \phi$, $\mathbf{w} = w + w_1$, so that the equation for ϕ reads at the main order,

$$\Delta_{\Gamma_{\alpha}} \phi + \partial_{zz} \phi + f'(w(z)) \phi + E + N(\phi) = 0, \quad \text{in } \Gamma_{\alpha} \times \mathbb{R},$$

where

$$\begin{aligned} E &= \chi_{|z| < \alpha^{-1} \delta} S(\mathbf{w}), & S(\mathbf{w}) &= \Delta(\mathbf{w}) + f(\mathbf{w}), \\ N(\phi) &= f(\mathbf{w} + \phi) - f(\mathbf{w}) - f'(w) \phi + B(\phi), & f(\mathbf{w}) &= \mathbf{w}(1 - \mathbf{w}^2) \end{aligned} \quad (11)$$

and $B(\phi)$ is a second order linear operator with ‘‘small’’ coefficients, also cut-off for $|z| > \delta \alpha^{-1}$. This problem is, of course, not equivalent in entire space to the original one, but a *gluing reduction* procedure, in the same line as those used in [9–11], makes it *essentially so*, to the expense of including small size operators and perturbing N by an exponentially small (in α) non-local operator in ϕ which can be neglected for the remaining of the argument. Rather than solving the above problem directly we consider a projected version of it:

$$\mathcal{L}(\phi) := \Delta_{\Gamma_{\alpha}} \phi + \partial_{zz} \phi + f'(w(z)) \phi = -E - N(\phi) + c(y) w'(z), \quad \text{in } \Gamma_{\alpha} \times \mathbb{R}, \quad (12)$$

$$\int_{\mathbb{R}} \phi(y, z) w'(z) dz = 0, \quad \text{for all } y \in \Gamma_{\alpha}. \quad (13)$$

A solution to this problem can be found in such a way that it respects the size and decay rates of the error E , which is roughly of the order $\sim r(\alpha y)^{-2} e^{-|z|}$. Indeed, let us consider the norm,

$$\|g\|_{p, \sigma, \nu} = \sup_x e^{\sigma|z|} r(\alpha y)^{\mu} \|g\|_{L^p(B(x, 1))}, \quad x = y + z\nu_{\alpha}(y), \quad (14)$$

and the auxiliary linear problem:

$$\mathcal{L}(\phi) = g(y, z) + c(y) w'(z), \quad \text{in } \Gamma_{\alpha} \times \mathbb{R}, \quad \int_{\mathbb{R}} \phi(y, z) w'(z) dz = 0 \quad \text{for all } y \in \Gamma_{\alpha}. \quad (15)$$

We show that there is a unique solution ϕ to the above linear problem with the property that

$$\|\phi\|_{*} := \|D^2 \phi\|_{p, \sigma, \nu} + \|D\phi\|_{\infty, \sigma, \nu} + \|D\phi\|_{\infty, \sigma, \nu} \leq C \|g\|_{p, \sigma, \nu}, \quad (16)$$

where C does not depend on α small, provided that p is fixed large enough. As a conclusion, a direct application of contraction mapping principle yields the desired result: a unique small solution $\phi = \phi(h)$ of problem (12), (13) exists. Its size is controlled by that of E in the sense of the above norms with $\nu = 3$ (after explicit improvement of the approximation is done by adding terms of $O(\alpha^2)$). We will of course have a solution to the original problem (6) if we manage to get $c(y) = 0$. This can indeed be achieved by conveniently adjusting the perturbation h . An equation for this h is obtained by multiplying the above relation by $w'(z)$ and integrating in z -variable: we get

$$c(y) \int_{\mathbb{R}} w'^2 dz = \int_{\mathbb{R}} (E + N(\phi)) w' dz.$$

This relation gets reduced to a (non-local) nonlinear PDE in Γ of the form:

$$\mathcal{J}(h) := \Delta_{\Gamma} h + |A|^2 h = O(\alpha) r(y)^{-3} + M_{\alpha}(h) \quad \text{in } \Gamma, \quad h = 0 \quad \text{on } \Gamma \cap \{u = v\}, \quad (17)$$

where $M(h)$ is a small operator which includes non-local terms. The linear operator $\mathcal{J}(h)$ is nothing but the Jacobi operator of Γ , which in turn is the linearization of its mean curvature operator in the direction of perturbations of the form $h(y)v(y)$. The symmetries of the BDG minimal surface allow us to assume the boundary conditions on h . These boundary conditions are interpreted as eliminating translation effect of the manifold, yielding a form of non-degeneracy: the only decaying Jacobi field of this operator is $h = \frac{1}{\sqrt{1+|\nabla F|^2}}$ which is positive, hence it does not satisfy the boundary conditions. A solvability theory for the Jacobi operator in weighted Sobolev norms, similar to (14), (16), suitably adapted to problem (17) is then built up. In this setting, the presence of explicit barriers for inequalities of the type $-\mathcal{J}(\tilde{h}) \leq r^{-\mu}$ is crucial (roughly with growth $\tilde{h} \sim r^{2-\mu}$). Using this theory, problem (17) is solved by means of contraction mapping principle. Finally, the solution built this way is easily seen to satisfy $\partial_{x_0} u > 0$, using the linear equation satisfied by this function, $\Delta p + f'(u)p = 0$, the fact that it is positive near Γ_α and maximum principle. Full details are provided in the paper [12].

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