Singular limits for the bi-Laplacian operator with exponential nonlinearity in $\mathbb{R}^4$

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Abstract

Let Ω be a bounded smooth domain in $\mathbb{R}^4$ such that for some integer $d \geqslant 1$ its d-th singular cohomology group with coefficients in some field is not zero, then problem

$$\begin{cases} \Delta^2 u - \rho^4 k(x) e^u = 0 & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases}$$

has a solution blowing-up, as $\rho \to 0$, at m points of Ω, for any given number m.

1. Introduction and statement of main results

Let Ω be a bounded smooth domain in $\mathbb{R}^4$. We are interested in studying existence and qualitative properties of positive solutions to the following boundary value problem

$$\begin{cases} \Delta^2 u - \rho^4 k(x) e^u = 0 & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $k \in C^2(\bar{\Omega})$ is a non-negative, not identically zero function, and $\rho > 0$ is a small, positive parameter which tends to 0.

In a four-dimensional manifold, this type of equations and similar ones arise from the problem of prescribing the so-called Q-curvature, which was introduced in [7]. More precisely, given (M, g) a four-dimensional Riemannian manifold, the problem consists in finding a conformal metric $\tilde{g}$ for which the corresponding Q-curvature $Q_{\tilde{g}}$ is a priori prescribed. The Q-curvature for the metric g is defined as

$$Q_g = -\frac{1}{2}\left(\Delta_g R_g - R_g^2 + 3|\operatorname{Ric}_g|^2\right),$$

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where R_g is the scalar curvature and Ric_g is the Ricci tensor of (M, g). Writing $\tilde{g} = e^{2w} g$, the problem reduces to finding a scalar function w which satisfies

$$P_g w + 2Q_g = 2Q_{\tilde{g}} e^{4w}, \tag{1.2}$$

where P_g is the Paneitz operator [32,10] defined as

$$P_g w = \Delta_g^2 w + \mathrm{div}\left(\frac{2}{3} R_g g - 2\,\mathrm{Ric}_g\right) dw.$$

Problem (1.2) is thus an elliptic fourth-order partial differential equation with exponential non-linearity. Several results are already known for this problem [9,10] and related ones [1,18,30]. When the metric g is not Riemannian, the problem has been recently treated by Djadli and Malchiodi in [19] via variational methods.

In the special case where the manifold is the Euclidean space and g is the Euclidean metric, we recover the equation in (1.1), since (1.2) takes the simplified form

$$\Delta^2 w - 2Qe^{4w} = 0.$$

Problem (1.1) has a variational structure. Indeed, solutions of (1.1) correspond to critical points in $H^2(\Omega) \cap H_0^1(\Omega)$ of the following energy functional

$$J_\rho(u) = \frac{1}{2} \int_\Omega |\Delta u|^2 - \rho^4 \int_\Omega k(x) e^u.$$

For any ρ sufficiently small, the functional above has a local minimum which represents a solution to (1.1) close to 0. Furthermore, the Moser–Trudinger inequality assures the existence of a second solution, which can be obtained as a mountain pass critical point for J_ρ. Thus, as $\rho \to 0$, this second solution turns out not to be bounded. The aim of the present paper is to study multiplicity of solutions to (1.1), for ρ positive and small, under some topological assumption on Ω, and to describe the asymptotic behavior of such solutions as the parameter ρ tends to zero. Indeed, we prove that, if some cohomology group of Ω is not zero, then given any integer m we can construct solutions to (1.1) which concentrate and blow-up, as $\rho \to 0$, around some given m points of the domain. These are the singular limits.

Let us mention that concentration phenomena of this type, in domains with topology, appear also in other problems. As a first example, the two-dimensional version of problem (1.1) is the boundary value problem associated to Liouville's equation [25]

$$\begin{cases} \Delta u + \rho^2 k(x) e^u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where $k(x)$ is a non-negative function and now Ω is a smooth bounded domain in $\mathbb{R}^2$. In [14] it is proved that problem (1.3) admits solutions concentrating, as $\rho \to 0$, around some given set of m points of Ω, for any given integer m, provided that Ω is not simply connected. See also [5,6,21,20,11,8,24,29,31,35,38,36,37] for related results. A similar result holds true for another semilinear elliptic problem, still in dimension 2, namely

$$\begin{cases} \Delta u + u^p = 0, \quad u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where p now is a parameter converging to $+\infty$. Again in this situation, if Ω is not simply connected, then for p large there exists a solution to (1.4) concentrating around some set of m points of Ω, for any positive integer m [22].

In higher dimensions, the analogy is with the classical Bahri–Coron problem. In [2], Bahri and Coron show that, if $N \geqslant 3$ and $\Omega \subset \mathbb{R}^N$ is a bounded domain, then the presence of topology in the domain guarantees existence of solutions to

$$\begin{cases} \Delta u + u^{\frac{N+2}{N-2}} = 0, \quad u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.5}$$

Partial results in this direction are also known in the slightly super critical version of Bahri–Coron's problem, namely

$$\begin{cases} \Delta u + u^{\frac{N+2}{N-2}+\varepsilon} = 0 & \text{in } \Omega, \\ u > 0, \quad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.6}$$

with $\varepsilon > 0$ small. In [12] it is proved that, under the assumption that Ω is a bounded smooth domain in $\mathbb{R}^N$ with a sufficiently small hole, a solution to (1.6) exhibiting concentration in two points is present. See also [3,23,34,13,33].

The main point of this paper is to show that the presence of topology in the domain implies strongly existence of blowing-up solutions for problem (1.1).

We denote by $H^d(\Omega)$ the d-th cohomology group of Ω with coefficients in some field $\mathbb{K}$. We shall prove the following

Theorem 1. *Assume that there exists $d \geqslant 1$ such that $H^d(\Omega) \neq 0$ and that $\inf_\Omega k > 0$. Then, given any integer $m \geqslant 1$, there exists a family of solutions u_ρ to problem (1.1), for ρ small enough, with the property that*

$$\lim_{\rho \to 0} \rho^4 \int_\Omega k(x) e^{u_\rho(x)}\, dx = 64\pi^2 m.$$

Furthermore, there are m points $\xi_1^\rho, \dots, \xi_m^\rho$ in Ω, separated at uniform positive distance from each other and from the boundary as $\rho \to 0$, for which u_ρ remains uniformly bounded on $\Omega \setminus \bigcup_{j=1}^m B_\delta(\xi_j^\rho)$ and

$$\sup_{B_\delta(\xi_j^\rho)} u_\rho \to +\infty,$$

for any $\delta > 0$.

The general behavior of arbitrary families of solutions to (1.1) has been studied by C.S. Lin and J.-C. Wei in [26], where they show that, when blow-up occurs for (1.1) as $\rho \to 0$, then it is located at a finite number of peaks, each peak being isolated and carrying the energy $64\pi^2$ (at a peak, $u \to +\infty$ and outside a peak, u is bounded). See [27] and [28] for related results.

We shall see that the sets of points where the solution found in Theorem 1 blows-up can be characterized in terms of Green's function for the biharmonic operator in Ω with the appropriate boundary conditions. Let $G(x, \xi)$ be the Green function defined by

$$\begin{cases} \Delta_x^2 G(x,\xi) = 64\pi^2 \delta_\xi(x), & x \in \Omega, \\ G(x,\xi) = \Delta_x G(x,\xi) = 0, & x \in \partial\Omega \end{cases} \tag{1.7}$$

and let $H(x, \xi)$ be its *regular part*, namely, the smooth function defined as

$$H(x,\xi) := G(x,\xi) + 8 \log|x - \xi|.$$

The location of the points of concentration is related to the set of critical points of the function

$$\varphi_m(\xi) = -\sum_{j=1}^m \bigl\{2 \log k(\xi_j) + H(\xi_j, \xi_j)\bigr\} - \sum_{i \neq j} G(\xi_i, \xi_j), \tag{1.8}$$

defined for points $\xi = (\xi_1, \dots, \xi_m)$ such that $\xi_i \in \Omega$ and $\xi_i \neq \xi_j$ if $i \neq j$.

In [4] the authors prove that for each *non-degenerate* critical point of φ_m there exists a solution to (1.1), for any small ρ, which concentrates exactly around such critical point as $\rho \to 0$. We shall show the existence of a solution under a weaker assumption, namely, that φ_m has a *minimax value in an appropriate subset.*

More precisely, we consider the following situation. Let Ω^m denote the Cartesian product of m copies of Ω. Note that in any compact subset of Ω^m, we may define, without ambiguity,

$$\varphi_m(\xi_1, \dots, \xi_m) = -\infty \quad \text{if } \xi_i = \xi_j \text{ for some } i \neq j.$$

We shall assume that there exists an open subset U of Ω with smooth boundary, compactly contained in Ω, and such that $\inf_U k > 0$, with the following properties:

(P1) *U^m contains two closed subsets $B_0 \subset B$ such that*

$$\sup_{\xi \in B_0} \varphi_m(\xi) < \inf_{\gamma \in \Gamma} \sup_{\xi \in B} \varphi_m\bigl(\gamma(\xi)\bigr) =: c_0,$$

where $\Gamma := \{\gamma \in \mathcal{C}(B, \bar{U}^m)\colon\ \gamma(\xi) = \xi$ for every $\xi \in B_0\}$.

(P2) *For every* $\xi = (\xi_1, \dots, \xi_m) \in \partial U^m$ *with* $\varphi_m(\xi) = c_0$, *there exists an* $i \in \{1, \dots, m\}$ *such that*

$$\begin{aligned} &\nabla_{\xi_i}\varphi_m(\xi) \neq 0 && \text{if } \xi_i \in U,\\ &\nabla_{\xi_i}\varphi_m(\xi)\cdot\tau \neq 0 \quad \text{for some } \tau \in T_{\xi_i}(\partial U) && \text{if } \xi_i \in \partial U, \end{aligned}$$

where $T_{\xi_i}(\partial U)$ *denotes the tangent space to* ∂U *at the point* ξ_i.

We will show that, under these assumptions, φ_m has a critical point $\xi \in U^m$ with critical value c_0. Moreover, the same is true for any small enough $\mathcal{C}^1$-perturbation of φ_m. Property (P1) is a common way of describing a change of topology of the sublevel sets of φ_m at the level c_0, and c_0 is called a minimax value of φ_m. It is a critical value if U^m is invariant under the negative gradient flow of φ_m. If this is not the case, we use property (P2) to modify the gradient vector field of φ_m near ∂U^m at the level c_0 and thus obtain a new vector field with the same stationary points, and such that $\bar{U}^m$ is invariant and φ_m is a Lyapunov function for the associated negative flow near the level c_0 (see Lemmas 6.3 and 6.4 below). This allows us to prove Theorem 1 and the following.

Theorem 2. *Let* $m \geqslant 1$ *and assume that there exists an open subset* U *of* Ω *with smooth boundary, compactly contained in* Ω, *with* $\inf_U k > 0$, *which satisfies* (P1) *and* (P2). *Then, for* ρ *small enough, there exists a solution* u_ρ *to problem* (1.1) *with*

$$\lim_{\rho\to 0}\rho^4\int_\Omega k(x)e^{u_\rho} = 64\pi^2 m.$$

Moreover, there is an m*-tuple* $(x_1^\rho, \dots, x_m^\rho) \in U^m$, *such that as* $\rho \to 0$

$$\nabla\varphi_m\big(x_1^\rho, \dots, x_m^\rho\big) \to 0, \qquad \varphi_m\big(x_1^\rho, \dots, x_m^\rho\big) \to c_0,$$

for which u_ρ *remains uniformly bounded on* $\Omega \setminus \bigcup_{j=1}^m B_\delta(x_i^\rho)$, *and*

$$\sup_{B_\delta(x_i^\rho)} u_\rho \to +\infty,$$

for any $\delta > 0$.

We will show that, for every $m \geqslant 1$, the set $U := \{\xi \in \Omega\colon \operatorname{dist}(\xi, \partial\Omega) > \delta\}$ has property (P2) at a given c_0, for δ small enough (see Lemma 6.2). Thus, if $\inf_\Omega k > 0$, and if there exist closed subsets $B_0 \subset B$ of Ω^m with

$$\sup_{\xi\in B_0}\varphi_m(\xi) < \inf_{\gamma\in\Gamma}\sup_{\xi\in B}\varphi_m\big(\gamma(\xi)\big),$$

then both conditions (P1) and (P2) hold. Condition (P1) holds, for example, if φ_m has a (possibly degenerate) local minimum or local maximum. So a direct consequence of Theorem 2 is that in any bounded domain Ω with $\inf_\Omega k > 0$, problem (1.1) has at least one solution concentrating exactly at one point, which corresponds to the minimum of the regular Green function H. Moreover if, for example, Ω is a contractible domain obtained by joining together m disjoint bounded domains through thin enough tubes, then the function φ_m has a (possibly degenerate) local minimum, which gives rise to a solution exhibiting m points of concentration.

Finally, recall that problem (1.1) corresponds to a standard case of *uniform singular convergence*, in the sense that the associated non-linear coefficient in problem (1.1) – $\rho^4 k(x)$ – goes to 0 uniformly in $\bar{\Omega}$ as $\rho \to 0$, property that is also present in problem (1.3). Non-trivial topology strongly determines existence of solutions. However, we expect that this strong influence should decay under an inhomogeneous and *non-uniform* singular behavior, where critical points of an *external* function determine existence and multiplicity of solutions. See [16] for a recent two-dimensional case of this phenomenon.

The paper is organized as follows. Section 2 is devoted to describing a first approximation for the solution and to estimating the error. Furthermore, problem (1.1) is written as a fixed point problem, involving a linear operator. In Section 3 we study the invertibility of the linear problem. In Section 4 we solve a projected non-linear problem. In Section 5 we show that solving the entire non-linear problem reduces to finding critical points of a certain functional. Section 6 is devoted to the proofs of Theorems 1 and 2.

2. Preliminaries and ansatz for the solution

This section is devoted to construct a reasonably good approximation U for a solution of (1.1). The shape of this approximation will depend on some points ξ_i, which we leave as parameters yet to be adjusted, where the spikes are meant to take place. As we will see, a convenient set to select $\xi = (\xi_1, \ldots, \xi_m)$ is

$$\mathcal{O} := \Big\{\xi \in \Omega^m \colon \operatorname{dist}(\xi_j, \partial\Omega) \geqslant 2\delta_0,\ \forall j = 1, \ldots, m,\ \text{and}\ \min_{i \neq j} |\xi_i - \xi_j| \geqslant 2\delta_0\Big\} \tag{2.1}$$

where $\delta_0 > 0$ is a small fixed number. We thus fix $\xi \in \mathcal{O}$.

For numbers $\mu_j > 0$, $j = 1, \ldots, m$, yet to be chosen, $x \in \mathbb{R}^4$ and $\varepsilon > 0$ we define

$$u_j(x) = 4 \log \frac{\mu_j(1+\varepsilon^2)}{\mu_j^2 \varepsilon^2 + |x - \xi_j|^2} - \log k(\xi_j), \tag{2.2}$$

so that u_j solves

$$\Delta^2 u - \rho^4 k(\xi_j) e^u = 0 \quad \text{in } \mathbb{R}^4, \tag{2.3}$$

with

$$\rho^4 = \frac{384 \varepsilon^4}{(1+\varepsilon^2)^4}, \tag{2.4}$$

that is, $\rho \sim \varepsilon$ as $\varepsilon \to 0$.

Since u_j and Δu_j are not zero on the boundary $\partial\Omega$, we will add to it a bi-harmonic correction so that the boundary conditions are satisfied. Let $H_j(x)$ be the smooth solution of

$$\begin{cases} \Delta^2 H_j = 0 & \text{in } \Omega, \\ H_j = -u_j & \text{on } \partial\Omega, \\ \Delta H_j = -\Delta u_j & \text{on } \partial\Omega. \end{cases}$$

We define our first approximation $U(\xi)$ as

$$U(\xi) \equiv \sum_{j=1}^{m} U_j, \quad U_j \equiv u_j + H_j. \tag{2.5}$$

As we will rigorously prove below, $(u_j + H_j)(x) \sim G(x, \xi_j)$ where $G(x, \xi)$ is the Green function defined in (1.7).

While u_j is a good approximation to a solution of (1.1) near ξ_j, it is not so much the case for U, unless the remainder $U - u_j = (H_j + \sum_{k \neq j} u_k)$ vanishes at main order near ξ_j. This is achieved through the following precise choice of the parameters μ_k

$$\log \mu_j^4 = \log k(\xi_j) + H(\xi_j, \xi_j) + \sum_{i \neq j} G(\xi_i, \xi_j). \tag{2.6}$$

We thus fix μ_j *a priori* as a function of ξ. We write

$$\mu_j = \mu_j(\xi)$$

for all $j = 1, \ldots, m$. Since $\xi \in \mathcal{O}$,

$$\frac{1}{C} \leqslant \mu_j \leqslant C, \quad \text{for all } j = 1, \ldots, m, \tag{2.7}$$

for some constant $C > 0$.

The following lemma expands U_j in Ω.

Lemma 2.1. *Assume $\xi \in \mathcal{O}$. Then we have*

$$H_j(x) = H(x, \xi_j) - 4 \log \mu_j (1+\varepsilon^2) + \log k(\xi_j) + O(\mu_j^2 \varepsilon^2), \tag{2.8}$$

uniformly in Ω, and

$$u_j(x) = 4\log\mu_j(1+\varepsilon^2) - \log k(\xi_j) - 8\log|x-\xi_j| + \mathrm{O}(\mu_j^2\varepsilon^2), \tag{2.9}$$

uniformly in the region $|x-\xi_j| \geqslant \delta_0$, so that in this region,

$$U_j(x) = G(x,\xi_j) + \mathrm{O}(\mu_j^2\varepsilon^2). \tag{2.10}$$

Proof. Let us prove (2.8). Define $z(x) = H_j(x) + 4\log\mu_j(1+\varepsilon^2) - \log k(\xi_j) - H(x,\xi_j)$. Then z is a bi-harmonic function which satisfies

$$\begin{cases} \Delta^2 z = 0 & \text{in } \Omega, \\ z = -u_j + 4\log\mu_j(1+\varepsilon^2) - \log k(\xi_j) - 8\log|\cdot-\xi_j| & \text{on } \partial\Omega, \\ \Delta z = -\Delta u_j - \frac{16}{|\cdot-\xi_j|^2} & \text{on } \partial\Omega. \end{cases}$$

Let us define $w \equiv -\Delta z$. Thus w is harmonic in Ω and

$$\sup_{\Omega}|w| \leqslant \sup_{\partial\Omega}|w| \leqslant C\mu_j^2\varepsilon^2.$$

We also have $\sup_{\partial\Omega}|z| \leqslant C\mu_j^2\varepsilon^2$. Standard elliptic regularity implies

$$\sup_{\Omega}|z| \leqslant C\Big(\sup_{\Omega}|w| + \sup_{\partial\Omega}|z|\Big) \leqslant C\mu_j^2\varepsilon^2,$$

as desired. The second estimate is direct from the definition of u_j. □

Now, let us write

$$\Omega_\varepsilon = \varepsilon^{-1}\Omega, \qquad \xi_j' = \varepsilon^{-1}\xi_j. \tag{2.11}$$

Then u solves (1.1) if and only if $v(y) \equiv u(\varepsilon y) + 4\log\rho\varepsilon$ satisfies

$$\begin{cases} \Delta^2 v - k(\varepsilon y)e^v = 0 \quad \text{in } \Omega_\varepsilon, \\ v = 4\log\rho\varepsilon, \quad \Delta v = 0 \quad \text{on } \partial\Omega_\varepsilon. \end{cases} \tag{2.12}$$

Let us define $V(y) = U(\varepsilon y) + 4\log\rho\varepsilon$, with U our approximate solution (2.5). We want to measure the size of the error of approximation

$$R \equiv \Delta^2 V - k(\varepsilon y)e^V. \tag{2.13}$$

It is convenient to do so in terms of the following norm

$$\|v\|_* = \sup_{y\in\Omega_\varepsilon}\left|\left[\sum_{j=1}^m \frac{1}{(1+|y-\xi_j'|^2)^{7/2}} + \varepsilon^4\right]^{-1} v(y)\right|. \tag{2.14}$$

Here and in what follows, C denotes a generic constant independent of ε and of $\xi \in \mathcal{O}$.

Lemma 2.2. *The error R in* (2.13) *satisfies*

$$\|R\|_* \leqslant C\varepsilon \quad \text{as } \varepsilon \to 0.$$

Proof. We assume first $|y-\xi_k'| < \delta_0/\varepsilon$, for some index k. We have

$$\Delta^2 V(y) = \rho^4\sum_{j=1}^m k(\xi_j)e^{u_j(\varepsilon y)} = \frac{384\mu_k^4}{(\mu_k^2+|y-\xi_k'|^2)^4} + \mathrm{O}(\varepsilon^8).$$

Let us estimate $k(\varepsilon y)e^{V(y)}$. By (2.8) and the definition of $\mu_j's$,

$$H_k(x) = H(\xi_k, \xi_k) - 4\log\mu_k + \log k(\xi_j) + \mathrm{O}(\mu_k^2\varepsilon^2) + \mathrm{O}\big(|x-\xi_k|\big)$$
$$= -\sum_{j\neq k} G(\xi_j, \xi_k) + \mathrm{O}(\mu_k^2\varepsilon^2) + \mathrm{O}\big(|x-\xi_k|\big),$$

and if $j \neq k$, by (2.10)

$$U_j(x) = u_j(x) + H_j(x) = G(\xi_j, \xi_k) + \mathrm{O}\big(|x-\xi_k|\big) + \mathrm{O}(\mu_j^2\varepsilon^2).$$

Then

$$H_k(x) + \sum_{j\neq k} U_j(x) = \mathrm{O}(\varepsilon^2) + \mathrm{O}\big(|x-\xi_k|\big). \tag{2.15}$$

Therefore,

$$k(\varepsilon y)e^{V(y)} = k(\varepsilon y)\varepsilon^4\rho^4 \exp\Big\{u_k(\varepsilon y) + H_k(\varepsilon y) + \sum_{j\neq k} U_j(\varepsilon y)\Big\}$$
$$= \frac{384\mu_k^4 k(\varepsilon y)}{(\mu_k^2 + |y-\xi_k'|^2)^4 k(\xi_k)}\big\{1 + \mathrm{O}\big(\varepsilon|y-\xi_k'|\big) + \mathrm{O}(\varepsilon^2)\big\}$$
$$= \frac{384\mu_k^4}{(\mu_k^2 + |y-\xi_k'|^2)^4}\big\{1 + \mathrm{O}\big(\varepsilon|y-\xi_k'|\big)\big\}.$$

We can conclude that in this region

$$\big|R(y)\big| \leqslant C\frac{\varepsilon|y-\xi_k'|}{(1+|y-\xi_k'|^2)^4} + \mathrm{O}(\varepsilon^4).$$

If $|y-\xi_j'| \geqslant \delta_0/\varepsilon$ for all j, using (2.8), (2.9) and (2.10) we obtain

$$\Delta^2 V = \mathrm{O}(\varepsilon^4\rho^4) \quad \text{and} \quad k(\varepsilon y)e^{V(y)} = \mathrm{O}(\varepsilon^4\rho^4).$$

Hence, in this region,

$$R(y) = \mathrm{O}(\varepsilon^8)$$

so that finally

$$\|R\|_* = \mathrm{O}(\varepsilon). \qquad \square$$

Next we consider the energy functional associated with (1.1)

$$J_\rho[u] = \frac{1}{2}\int_\Omega (\Delta u)^2 - \rho^4 \int_\Omega k(x)e^u, \quad u \in H^2(\Omega) \cap H_0^1(\Omega). \tag{2.16}$$

We will give an asymptotic estimate of $J_\rho[U]$, where $U(\xi)$ is the approximation (2.5). Instead of ρ, we use the parameter ε (defined in (2.4)) to obtain the following expansion:

Lemma 2.3. *With the election of μ_j's given by* (2.6)*,*

$$J_\rho[U] = -128\pi^2 m + 256\pi^2 m|\log\varepsilon| + 32\pi^2\varphi_m(\xi) + \varepsilon\Theta_\varepsilon(\xi), \tag{2.17}$$

where $\Theta_\varepsilon(\xi)$ is uniformly bounded together with its derivatives if $\xi \in \mathcal{O}$, and φ_m is the function defined in (1.8).

Proof. We have

$$J_\rho[U]=\frac{1}{2}\sum_{j=1}^{m}\int_\Omega (\Delta U_j)^2+\frac{1}{2}\sum_{j\neq i}\int_\Omega \Delta U_j\Delta U_i-\rho^4\int_\Omega k(x)e^U$$
$$\equiv I_1+I_2+I_3;$$

Note that $\Delta^2 U_j=\Delta^2 u_j=\rho^4 k(\xi_j)e^{u_j}$ in Ω and $U_j=\Delta U_j=0$ in $\partial\Omega$. Then

$$I_1=\frac{1}{2}\rho^4\sum_{j=1}^{m}k(\xi_j)\int_\Omega e^{u_j}U_j \quad \text{and} \quad I_2=\frac{1}{2}\rho^4\sum_{j\neq i}k(\xi_j)\int_\Omega e^{u_j}U_i.$$

Let us define the change of variables $x=\xi_j+\mu_j\varepsilon y$, where $x\in\Omega$ and $y\in\Omega_j\equiv(\mu_j\varepsilon)^{-1}(\Omega-\xi_j)$. Using Lemma 2.1 and the definition of ρ in terms of ε in (2.4) we obtain

$$I_1=192\sum_{j=1}^{m}\int_{\Omega_j}\frac{1}{(1+|y|^2)^4}\left\{4\log\frac{1}{1+|y|^2}-8\log\mu_j\varepsilon+H(\xi_j,\xi_j)+\mathrm{O}\big(\mu_j\varepsilon|y|\big)\right\}$$
$$=32\pi^2\sum_{j=1}^{m}\big\{H(\xi_j,\xi_j)-8\log\mu_j\varepsilon\big\}-64\pi^2 m+\mathrm{O}\left(\varepsilon\mu_j\int_{\Omega_j}\frac{|y|}{(1+|y|^2)^4}\right)$$
$$=32\pi^2\sum_{j=1}^{m}\big\{H(\xi_j,\xi_j)-8\log\mu_j\varepsilon\big\}-64\pi^2 m+\varepsilon\Theta(\xi),$$

where $\Theta_\varepsilon(\xi)$ is bounded together with its derivatives if $\xi\in\mathcal{O}$. Besides we have used the explicit values

$$\int_{\mathbb{R}^4}\frac{1}{(1+|y|^2)^4}=\frac{\pi^2}{6}, \quad \text{and} \quad \int_{\mathbb{R}^4}\frac{\log(1+|y|^2)}{(1+|y|^2)^4}=\frac{\pi^2}{12}.$$

We consider now I_2. As above,

$$\frac{1}{2}\rho^4\int_\Omega e^{u_j}U_i=\int_{\Omega_j}\frac{192}{(1+|y|^2)^4}\big\{u_i(\xi_j+\mu_j\varepsilon y)+H_i(\xi_j+\mu_j\varepsilon y)\big\}$$
$$=\int_{\Omega_j}\frac{192}{(1+|y|^2)^4}\big\{u_i(\xi_j+\mu_j\varepsilon y)-4\log\mu_i(1+\varepsilon^2)+\log k(\xi_i)+8\log|\xi_j-\xi_i|\big\}$$
$$+\int_{\Omega_j}\frac{192}{(1+|y|^2)^4}\big\{H_i(\xi_j+\mu_j\varepsilon y)-H_i(\xi_j)\big\}$$
$$+\int_{\Omega_j}\frac{192}{(1+|y|^2)^4}\big\{H_i(\xi_j)-H(\xi_j,\xi_i)+4\log\mu_i(1+\varepsilon^2)-\log k(\xi_i)\big\}$$
$$+G(\xi_j,\xi_i)\int_{\Omega_j}\frac{192}{(1+|y|^2)^4}$$
$$=32\pi^2 G(\xi_i,\xi_j)+\mathrm{O}\left(\varepsilon\mu_j\int_{\Omega_j}\frac{|y|}{(1+|y|^2)^4}\right)+\mathrm{O}(\mu_j^2\varepsilon^2)$$
$$=32\pi^2 G(\xi_i,\xi_j)+\varepsilon\Theta_\varepsilon(\xi).$$

Thus

$$I_2=32\pi^2\sum_{j\neq i}G(\xi_i,\xi_j)+\varepsilon\Theta_\varepsilon(\xi). \tag{2.18}$$

Finally we consider I_3. Let us denote $A_j \equiv B(\xi_j, \delta_0)$ and $x = \xi_j + \mu_j \varepsilon y$. Then using again Lemma 2.1

$$
\begin{aligned}
I_3 &= -\rho^4 \sum_{j=1}^{m} \int_{A_j} k(x) e^{U} + O(\varepsilon^4) \\
&= -\rho^4 \sum_{j=1}^{m} \int_{B(0, \frac{\delta_0}{\mu_j \varepsilon})} \frac{k(\xi_j + \mu_j \varepsilon y)}{k(\xi_j)(1+|y|^2)^4} \frac{(1+\varepsilon^2)^4}{\varepsilon^4} \big(1 + O(\varepsilon \mu_j |y|)\big) + O(\varepsilon^4) \\
&= -384 m \int_{\mathbb{R}^4} \frac{1}{(1+|y|^2)^4} + O\left(\varepsilon \mu_j \int_{\mathbb{R}^4} \frac{|y|}{(1+|y|^2)^4}\right) \\
&= -64\pi^2 m + \varepsilon \Theta_\varepsilon(\xi),
\end{aligned}
$$

uniformly in $\xi \in \mathcal{O}$. Thus, we can conclude the following expansion of $J_\rho[U]$:

$$
J_\rho[U] = -128 m\pi^2 + 256 m\pi^2 |\log \varepsilon| + 32\pi^2 \varphi_m(\xi) + \varepsilon \Theta_\varepsilon(\xi), \tag{2.19}
$$

where $\Theta_\varepsilon(\xi)$ is a bounded function together with is derivatives in the region $\xi \in \mathcal{O}$, φ_m defined as in (1.8) and $\rho^4 = \frac{384\varepsilon^4}{(1+\varepsilon^2)^4}$. □

In the subsequent analysis we will stay in the expanded variable $y \in \Omega_\varepsilon$ so that we will look for solutions of problem (2.12) in the form $v = V + \psi$, where ψ will represent a lower order correction. In terms of ψ, problem (2.12) now reads

$$
\begin{cases}
\mathcal{L}_\varepsilon(\psi) \equiv \Delta^2 \psi - W\psi = -R + N(\psi) & \text{in } \Omega_\varepsilon, \\
\psi = \Delta \psi = 0 & \text{on } \partial\Omega_\varepsilon,
\end{cases} \tag{2.20}
$$

where

$$
N(\psi) = W[e^{\psi} - \psi - 1] \quad \text{and} \quad W = k(\varepsilon y) e^{V}. \tag{2.21}
$$

Note that

$$
W(y) = \sum_{j=1}^{m} \frac{384\mu_j^4}{(\mu_j^2 + |y - \xi_j'|^2)^4} \big(1 + O(\varepsilon |y - \xi_j'|)\big) \quad \text{for } y \in \Omega_\varepsilon. \tag{2.22}
$$

This fact, together with the definition of $N(\psi)$ given in (2.21), give the validity of the following

Lemma 2.4. *For $\xi \in \mathcal{O}$, $\|W\|_* = O(1)$ and $\|N(\psi)\|_* = O(\|\psi\|_\infty^2)$ as $\|\psi\|_\infty \to 0$.*

3. The linearized problem

In this section we develop a solvability theory for the fourth-order linear operator $\mathcal{L}_\varepsilon$ defined in (2.20) under suitable orthogonality conditions. We consider

$$
\mathcal{L}_\varepsilon(\psi) \equiv \Delta^2 \psi - W(y)\psi, \tag{3.1}
$$

where $W(y)$ was introduced in (2.20). By expression (2.22) and setting $z = y - \xi_j'$, one can easily see that formally the operator $\mathcal{L}_\varepsilon$ approaches, as $\varepsilon \to 0$, the operator in $\mathbb{R}^4$

$$
\mathcal{L}_j(\psi) \equiv \Delta^2 \psi - \frac{384\mu_j^4}{(\mu_j^2 + |z|^2)^4} \psi, \tag{3.2}
$$

namely, equation $\Delta^2 v - e^v = 0$ linearized around the radial solution $v_j(z) = \log \frac{384\mu_j^4}{(\mu_j^2 + |z|^2)^4}$. Thus the key point to develop a satisfactory solvability theory for the operator $\mathcal{L}_\varepsilon$ is the non-degeneracy of v_j up to the natural invariances of the equation under translations and dilations. In fact, if we set

$$Y_{0j}(z) = 4\frac{|z|^2 - \mu_j^2}{|z|^2 + \mu_j^2}, \tag{3.3}$$

$$Y_{ij}(z) = \frac{8z_i}{\mu_j^2 + |z|^2}, \quad i = 1, \dots, 4, \tag{3.4}$$

the only bounded solutions of $\mathcal{L}_j(\psi) = 0$ in $\mathbb{R}^4$ are linear combinations of Y_{ij}, $i = 0, \dots, 4$; see Lemma 3.1 in [4] for a proof.

We define for $i = 0, \dots, 4$ and $j = 1, \dots, m$,

$$Z_{ij}(y) \equiv Y_{ij}(y - \xi_j'), \quad i = 0, \dots, 4.$$

Additionally, let us consider R_0 a large but fixed number and χ a radial and smooth cut-off function with $\chi \equiv 1$ in $B(0, R_0)$ and $\chi \equiv 0$ in $\mathbb{R}^4 \setminus B(0, R_0 + 1)$. Let

$$\chi_j(y) = \chi\big(|y - \xi_j'|\big), \quad j = 1, \dots, m.$$

Given $h \in L^\infty(\Omega_\varepsilon)$, we consider the problem of finding a function ψ such that for certain scalars c_{ij} one has

$$\begin{cases} \mathcal{L}_\varepsilon(\psi) = h + \sum_{i=1}^4 \sum_{j=1}^m c_{ij}\chi_j Z_{ij}, & \text{in } \Omega_\varepsilon, \\ \psi = \Delta\psi = 0, & \text{on } \partial\Omega_\varepsilon, \\ \int_{\Omega_\varepsilon} \chi_j Z_{ij}\psi = 0, & \text{for all } i = 1, \dots, 4,\ j = 1, \dots, m. \end{cases} \tag{3.5}$$

We will establish a priori estimates for this problem. To this end we shall introduce an adapted norm in Ω_ε, which has been introduced previously in [15]. Given $\psi : \Omega_\varepsilon \to \mathbb{R}$ and $\alpha \in \mathbb{N}^m$ we define

$$\|\psi\|_{**} \equiv \sum_{j=1}^m \|\psi\|_{C^{4,\alpha}(r_j<2)} + \sum_{j=1}^m \sum_{|\alpha|\leqslant 3} \|r_j^{|\alpha|} D^\alpha \psi\|_{L^\infty(r_j \geqslant 2)}, \tag{3.6}$$

with $r_j = |y - \xi_j'|$.

Proposition 3.1. *There exist positive constants $\varepsilon_0 > 0$ and $C > 0$ such that for any $h \in L^\infty(\Omega_\varepsilon)$, with $\|h\|_* < \infty$, and any $\xi \in \mathcal{O}$, there is a unique solution $\psi = T(h)$ to problem* (3.5) *for all $\varepsilon \leqslant \varepsilon_0$, which defines a linear operator of h. Besides, we have the estimate*

$$\big\|T(h)\big\|_{**} \leqslant C|\log\varepsilon|\|h\|_*. \tag{3.7}$$

The proof will be split into a series of lemmas which we state and prove next. The first step is to obtain a priori estimates for the problem

$$\begin{cases} \mathcal{L}_\varepsilon(\psi) = h & \text{in } \Omega_\varepsilon, \\ \psi = \Delta\psi = 0 & \text{on } \partial\Omega_\varepsilon, \\ \int_{\Omega_\varepsilon} \chi_j Z_{ij}\psi = 0 & \text{for all } i = 0, \dots, 4,\ j = 1, \dots, m, \end{cases} \tag{3.8}$$

which involves more orthogonality conditions than those in (3.5). We have the following estimate.

Lemma 3.1. *There exist positive constants $\varepsilon_0 > 0$ and $C > 0$ such that for any ψ solution of problem* (3.8) *with $h \in L^\infty(\Omega_\varepsilon)$, $\|h\|_* < \infty$, and $\xi \in \mathcal{O}$, then*

$$\|\psi\|_{**} \leqslant C\|h\|_* \tag{3.9}$$

for all $\varepsilon \in (0, \varepsilon_0)$.

Proof. We carry out the proof by a contradiction argument. If the above fact were false, then, there would exist a sequence $\varepsilon_n \to 0$, points $\xi^n = (\xi_1^n, \dots, \xi_m^n) \in \mathcal{O}$, functions h_n with $\|h_n\|_* \to 0$ and associated solutions ψ_n with $\|\psi_n\|_{**} = 1$ such that

$$\begin{cases} \mathcal{L}_{\varepsilon_n}(\psi_n) = h_n & \text{in } \Omega_{\varepsilon_n}, \\ \psi_n = \Delta\psi_n = 0 & \text{on } \partial\Omega_{\varepsilon_n}, \\ \int_{\Omega_{\varepsilon_n}} \chi_j Z_{ij}\psi_n = 0, & \text{for all } i = 0, \dots, 4,\ j = 1, \dots, m. \end{cases} \tag{3.10}$$

Let us set $\tilde{\psi}_n(x) = \psi_n(x/\varepsilon_n)$, $x \in \Omega$. It is directly checked that for any $\delta' > 0$ sufficiently small $\tilde{\psi}_n$ solves the problem

$$\begin{cases} \Delta^2 \tilde{\psi}_n = \mathrm{O}(\varepsilon_n^4) + \varepsilon_n^{-4} h_n = \mathrm{o}(1), & \text{uniformly in } \Omega \setminus \bigcup_{k=1}^m B(\xi_j^n, \delta'), \\ \tilde{\psi}_n = \Delta \tilde{\psi}_n = 0 & \text{on } \partial\Omega, \end{cases}$$

together with $\|\tilde{\psi}_n\|_\infty \leqslant 1$ and $\|\Delta \tilde{\psi}_n\|_\infty \leqslant C_{\delta'}$, in the considered region. Passing to a subsequence, we then get that $\xi^n \to \xi^* \in \mathcal{O}$ and $\tilde{\psi}_n \to 0$ in the $C^{3,\alpha}$ sense over compact subsets of $\Omega \setminus \{\xi_1^*, \ldots, \xi_m^*\}$. In particular

$$\sum_{|\alpha| \leqslant 3} \frac{1}{\varepsilon_n^{|\alpha|}} \left| D^\alpha \psi_n(y) \right| \to 0, \quad \text{uniformly in } \left| y - (\xi_j^n)' \right| \geqslant \frac{\delta'}{2\varepsilon_n},$$

for any $\delta' > 0$ and $j \in \{1, \ldots, m\}$. We obtain thus that

$$\sum_{j=1}^m \sum_{|\alpha| \leqslant 3} \|r_j^{|\alpha|} D^\alpha \psi_n\|_{L^\infty(r_j \geqslant \delta'/\varepsilon_n)}, \to 0, \tag{3.11}$$

for any $\delta' > 0$. In conclusion, the *exterior portion* of $\|\psi_n\|_{**}$ goes to zero, see (3.6).

Let us consider now a smooth radial cut-off function $\hat{\eta}$ with $\hat{\eta}(s) = 1$ if $s < \frac{1}{2}$, $\hat{\eta}(s) = 0$ if $s \geqslant 1$, and define

$$\hat{\psi}_{n,j}(y) = \hat{\eta}_j(y) \psi_n(y) \equiv \hat{\eta}\left(\frac{\varepsilon_n}{\delta_0} \left| y - (\xi_j^n)' \right| \right) \psi_n(y),$$

such that

$$\operatorname{supp} \hat{\psi}_{n,j} \subseteq B\left((\xi_j^n)', \frac{\delta_0}{\varepsilon_n} \right).$$

We observe that

$$\mathcal{L}_{\varepsilon_n}(\hat{\psi}_{n,j}) = \hat{\eta}_j h_n + F(\hat{\eta}_j, \psi_n),$$

where

$$F(f, g) = g \Delta^2 f + 2 \Delta f \Delta g + 4 \nabla(\Delta f) \cdot \nabla g + 4 \nabla f \cdot \nabla(\Delta g) + 4 \sum_{i,j=1}^4 \frac{\partial^2 f}{\partial y_i \partial y_j} \frac{\partial^2 g}{\partial y_i \partial y_j}. \tag{3.12}$$

Thus we get

$$\begin{cases} \Delta^2 \hat{\psi}_{n,j} = W_n(y) \hat{\psi}_{n,j} + \hat{\eta}_j h_n + F(\hat{\eta}_j, \psi_n) & \text{in } B\big((\xi_j^n)', \frac{\delta_0}{\varepsilon_n}\big), \\ \hat{\psi}_{n,j} = \Delta \hat{\psi}_{n,j} = 0 & \text{on } \partial B\big((\xi_j^n)', \frac{\delta_0}{\varepsilon_n}\big). \end{cases} \tag{3.13}$$

The following intermediate result provides an outer estimate. For notational simplicity *we omit* the subscript n in the quantities involved.

Lemma 3.2. *There exist constants C, $R_0 > 0$ such that for large n*

$$\sum_{|\alpha| \leqslant 3} \|r_j^{|\alpha|} D^\alpha \hat{\psi}_j\|_{L^\infty(r_j \geqslant R_0)} \leqslant C \big\{ \|\hat{\psi}_j\|_{L^\infty(r_j < 2R_0)} + \mathrm{o}(1) \big\}. \tag{3.14}$$

Proof. We estimate the right-hand side of (3.13). If $2 < r_j < \delta_0/\varepsilon$ we get

$$\Delta^2 \hat{\psi}_j = \mathrm{O}\left(\frac{1}{r_j^8} \right) \hat{\psi}_j + \frac{1}{r_j^7} \mathrm{o}(1) + \mathrm{O}(\varepsilon^4) + \mathrm{O}\left(\frac{\varepsilon^3}{r_j} \right) + \mathrm{O}\left(\frac{\varepsilon^2}{r_j^2} \right) + \mathrm{O}\left(\frac{\varepsilon}{r_j^3} \right).$$

From (3.13) and standard elliptic estimates we have

$$\sum_{|\alpha| \leqslant 3} |D^\alpha \hat{\psi}_j| \leqslant C \left\{ \frac{1}{r_j^8} \|\hat{\psi}_j\|_{L^\infty(r_j > 1)} + \frac{1}{r_j^7} \mathrm{o}(1) + \mathrm{O}\left(\frac{\varepsilon}{r_j^3} \right) \right\}, \quad \text{in } 2 \leqslant r_j \leqslant \frac{\delta_0}{\varepsilon}.$$

Now, if $r_j \geqslant 2$

$$|r_j^{|\alpha|} D^\alpha \hat{\psi}_j| \leqslant C\left\{\frac{1}{r_j^5}\|\hat{\psi}_j\|_{L^\infty(r_j>1)} + \mathrm{o}(1)\right\}, \quad |\alpha| \leqslant 3.$$

Finally

$$\frac{1}{r_j^5}\|\hat{\psi}_j\|_{L^\infty(r_j>1)} \leqslant \|\hat{\psi}_j\|_{L^\infty(1<r_j<R_0)} + \frac{1}{R_0^5}\|\hat{\psi}_j\|_{L^\infty(r_j>R_0)},$$

thus fixing R_0 large enough we have

$$\sum_{|\alpha|\leqslant 3} \|r_j^{|\alpha|} D^\alpha \hat{\psi}_j\|_{L^\infty(r_j \geqslant R_0)} \leqslant C\left\{\|\hat{\psi}_j\|_{L^\infty(1<r_j<R_0)} + \mathrm{o}(1)\right\}, \quad 2 < r_j < \frac{\delta_0}{\varepsilon},$$

and then (3.14). □

We continue with the proof of Lemma 3.1.

Since $\|\psi_n\|_{**} = 1$ and using (3.11) and Lemma 3.2 we have that there exists an index $j \in \{1, \dots, m\}$ such that

$$\liminf_{n\to\infty} \|\psi_n\|_{L^\infty(r_j<R_0)} \geqslant \alpha > 0. \tag{3.15}$$

Let us set $\tilde{\psi}_n(z) = \psi_n((\xi_j^n)' + z)$. We notice that $\tilde{\psi}_n$ satisfies

$$\Delta^2 \tilde{\psi}_n - W\big((\xi_j^n)' + z\big)\tilde{\psi}_n = h_n\big((\xi_j^n)' + z\big), \quad \text{in } \Omega_n \equiv \Omega_\varepsilon - (\xi_j^n)'.$$

Since ψ_n, $\Delta\psi_n$ are bounded uniformly, standard elliptic estimates allow us to assume that $\tilde{\psi}_n$ converges uniformly over compact subsets of $\mathbb{R}^4$ to a bounded, non-zero solution $\tilde{\psi}$ of

$$\Delta^2 \psi - \frac{384\mu_j^4}{(\mu_j^2 + |z|^2)^4}\psi = 0.$$

This implies that $\tilde{\psi}$ is a linear combination of the functions Y_{ij}, $i = 0, \dots, 4$. But orthogonality conditions over $\tilde{\psi}_n$ pass to the limit thanks to $\|\tilde{\psi}_n\|_\infty \leqslant 1$ and dominated convergence. Thus $\tilde{\psi} \equiv 0$, a contradiction with (3.15). This conclude the proof. □

Now we will deal with problem (3.8) lifting the orthogonality constraints $\int_{\Omega_\varepsilon} \chi_j Z_{0j}\psi = 0$, $j = 1, \dots, m$, namely

$$\begin{cases} \mathcal{L}_\varepsilon(\psi) = h & \text{in } \Omega_\varepsilon, \\ \psi = \Delta\psi = 0 & \text{on } \partial\Omega_\varepsilon, \\ \int_{\Omega_\varepsilon} \chi_j Z_{ij}\psi = 0, & \text{for all } i = 1, \dots, 4,\ j = 1, \dots, m. \end{cases} \tag{3.16}$$

We have the following a priori estimates for this problem.

Lemma 3.3. *There exist positive constants ε_0 and C such that, if ψ is a solution of* (3.16)*, with $h \in L^\infty(\Omega_\varepsilon)$, $\|h\|_* < \infty$ and with $\xi \in \mathcal{O}$, then*

$$\|\psi\|_{**} \leqslant C|\log\varepsilon|\|h\|_* \tag{3.17}$$

for all $\varepsilon \in (0, \varepsilon_0)$.

Proof. Let $R > R_0 + 1$ be a large and fixed number. Let us consider $\hat{Z}_{0j}$ be the following function

$$\hat{Z}_{0j}(y) = Z_{0j}(y) - 1 + a_{0j} G(\varepsilon y, \xi_j), \tag{3.18}$$

where $a_{0j} = (H(\xi_j, \xi_j) - 8\log(\varepsilon R))^{-1}$. It is clear that if ε is small enough

$$\hat{Z}_{0j}(y) = Z_{0j}(y) + a_{0j}\big(G(\varepsilon y, \xi_j) - H(\xi_j, \xi_j) + 8\log(\varepsilon R)\big)$$
$$= Z_{0j}(y) + \frac{1}{|\log\varepsilon|}\left(\mathrm{O}(\varepsilon r_j) + 8\log\frac{R}{r_j}\right) \tag{3.19}$$

and $Z_{0j}(y) = \mathrm{O}(1)$. Next we consider radial smooth cut-off functions η_1 and η_2 with the following properties:

$$0 \leqslant \eta_1 \leqslant 1, \quad \eta_1 \equiv 1 \text{ in } B(0, R), \quad \eta_1 \equiv 0 \text{ in } \mathbb{R}^4 \setminus B(0, R+1), \quad \text{and}$$
$$0 \leqslant \eta_2 \leqslant 1, \quad \eta_2 \equiv 1 \text{ in } B\left(0, \frac{\delta_0}{3\varepsilon}\right), \quad \eta_2 \equiv 0 \text{ in } R^4 \setminus B\left(0, \frac{\delta_0}{2\varepsilon}\right).$$

Then we set

$$\eta_{1j}(y) = \eta_1(r_j), \qquad \eta_{2j}(y) = \eta_2(r_j), \tag{3.20}$$

and define the test function

$$\tilde{Z}_{0j} = \eta_{1j} Z_{0j} + (1 - \eta_{1j})\eta_{2j}\hat{Z}_{0j}.$$

Note the $\tilde{Z}_{0j}$'s behavior through Ω_ε

$$\tilde{Z}_{0j} = \begin{cases} Z_{0j}, & r_j \leqslant R, \\ \eta_{1j}(Z_{0j} - \hat{Z}_{0j}) + \hat{Z}_{0j}, & R < r_j \leqslant R+1, \\ \hat{Z}_{0j}, & R+1 < r_j \leqslant \frac{\delta_0}{3\varepsilon}, \\ \eta_{2j}\hat{Z}_{0j}, & \frac{\delta_0}{3\varepsilon} < r_j \leqslant \frac{\delta_0}{2\varepsilon}, \\ 0 & \text{otherwise.} \end{cases} \tag{3.21}$$

In the subsequent, we will label these four regions as

$$\Omega_0 \equiv \{r_j \leqslant R\}, \quad \Omega_1 \equiv \{R < r_j \leqslant R+1\}, \quad \Omega_2 \equiv \left\{R+1 < r_j \leqslant \frac{\delta_0}{3\varepsilon}\right\}, \quad \text{and} \quad \Omega_3 \equiv \left\{\frac{\delta_0}{3\varepsilon} < r_j \leqslant \frac{\delta_0}{2\varepsilon}\right\}.$$

Let ψ be a solution to problem (3.16). We will modify ψ so that the extra orthogonality conditions with respect to Z_{0j}'s hold. We set

$$\tilde{\psi} = \psi + \sum_{j=1}^{m} d_j \tilde{Z}_{0j}. \tag{3.22}$$

We adjust the constants d_j so that

$$\int_{\Omega_\varepsilon} \chi_j Z_{ij}\tilde{\psi} = 0, \quad \text{for all } i = 0, \dots, 4;\ j = 1, \dots, m. \tag{3.23}$$

Then,

$$\mathcal{L}_\varepsilon(\tilde{\psi}) = h + \sum_{j=1}^{m} d_j \mathcal{L}_\varepsilon(\tilde{Z}_{0j}). \tag{3.24}$$

If (3.23) holds, the previous lemma allows us to conclude

$$\|\tilde{\psi}\|_{**} \leqslant C\left\{\|h\|_* + \sum_{j=1}^{m} |d_j| \big\|\mathcal{L}_\varepsilon(\tilde{Z}_{0j})\big\|_*\right\}. \tag{3.25}$$

Estimate (3.17) is a direct consequence of the following claim:

Claim 1. *The constants d_j are well defined,*

$$|d_j| \leqslant C|\log\varepsilon|\|h\|_* \quad \textit{and} \quad \big\|\mathcal{L}_\varepsilon(\tilde{Z}_{0j})\big\|_* \leqslant \frac{C}{|\log\varepsilon|}, \quad \textit{for all } j = 1, \dots, m. \tag{3.26}$$

After these facts have been established, using the fact that

$$\|\tilde{Z}_{0j}\|_{**} \leqslant C,$$

we obtain (3.17), as desired.

Let us prove now Claim 1. First we find d_j. From definition (3.22), orthogonality conditions (3.23) and the fact that $\operatorname{supp}\chi_j\eta_{1k} = \emptyset$ and $\operatorname{supp}\chi_j\eta_{2k} = \emptyset$ if $j \neq k$, we can write

$$d_j \int_{\Omega_\varepsilon} \chi_j Z_{0j}^2 = -\int_{\Omega_\varepsilon} \chi_j Z_{0j}\psi, \quad \forall j = 1, \dots, m. \tag{3.27}$$

Thus d_j is well defined. Note that the orthogonality conditions in (3.23) for $i = 1, \dots, 4$ are also satisfied for $\tilde{\psi}$ thanks to the fact that $R > R_0 + 1$.

We prove now the second inequality in (3.26). From (3.21), (3.18) and estimate (2.22) we obtain,

$$\mathcal{L}_\varepsilon(\tilde{Z}_{0j}) = \begin{cases} \mathrm{O}\big(\frac{\mu_j^4\varepsilon}{(\mu_j^2+r_j^2)^{7/2}}\big) & \text{in } \Omega_0, \\ \eta_{1j}\mathcal{L}_\varepsilon(Z_{0j} - \hat{Z}_{0j}) + \mathcal{L}_\varepsilon(\hat{Z}_{0j}) + F(\eta_{1j}, Z_{0j} - \hat{Z}_{0j}) & \text{in } \Omega_1, \\ \mathcal{L}_\varepsilon(\hat{Z}_{0j}) & \text{in } \Omega_2, \\ \eta_{2j}\mathcal{L}_\varepsilon(\hat{Z}_{0j}) + F(\eta_{2j}, \hat{Z}_{0j}) & \text{in } \Omega_3, \end{cases} \tag{3.28}$$

and where F was defined in (3.12). We compute now $\mathcal{L}_\varepsilon(\tilde{Z}_{0j})$ in Ω_i, $i = 1, 2, 3$. In Ω_1, thanks to (3.19) (we consider R here because we will need this dependence below to prove estimate (3.38))

$$|Z_{0j} - \hat{Z}_{0j}|, \ \big|R\nabla(Z_{0j} - \hat{Z}_{0j})\big| \text{ and } \big|R^2\Delta(Z_{0j} - \hat{Z}_{0j})\big| = \mathrm{O}\bigg(\frac{1}{|\log\varepsilon|}\bigg); \tag{3.29}$$

moreover

$$\big|R\nabla\big(\Delta(Z_{0j} - \hat{Z}_{0j})\big)\big| \text{ and } \big|\Delta^2(Z_{0j} - \hat{Z}_{0j})\big| = \mathrm{O}\bigg(\frac{1}{R^2|\log\varepsilon|}\bigg). \tag{3.30}$$

Thus, using (3.12) and the fact that, in Ω_1, $|D^\alpha\eta_{1j}| \leqslant CR^{-|\alpha|}$, for any multi-index $|\alpha| \leqslant 4$,

$$F(\eta_{1j}, Z_{0j} - \hat{Z}_{0j}) = \mathrm{O}\bigg(\frac{1}{R^4|\log\varepsilon|}\bigg).$$

On the other hand,

$$\mathcal{L}_\varepsilon(Z_{0j} - \hat{Z}_{0j}) = \mathrm{O}\bigg(\frac{1}{R^4|\log\varepsilon|}\bigg), \tag{3.31}$$

and

$$\mathcal{L}_\varepsilon(\hat{Z}_{0j}) = \mathrm{O}(\varepsilon R) + \mathrm{O}\bigg(\frac{1}{R^4|\log\varepsilon|}\bigg). \tag{3.32}$$

In conclusion, if $y \in \Omega_1$,

$$\mathcal{L}_\varepsilon(\tilde{Z}_{0j})(y) = \mathrm{O}\bigg(\frac{1}{R^4|\log\varepsilon|}\bigg). \tag{3.33}$$

In Ω_2,

$$\begin{aligned} W\big(1 - a_{0j}G(\varepsilon y, \xi_j)\big) &= \mathrm{O}\bigg(\frac{\mu_j^4 a_{0j}}{(\mu_j^2 + r_j^2)^4}\bigg\{H(\xi_j, \xi_j) - H(\varepsilon y, \xi_j) + 8\log\frac{r_j}{R}\bigg\}\bigg) \\ &= \mathrm{O}\bigg(\frac{\mu_j^4 a_{0j}}{(\mu_j^2 + r_j^2)^{7/2}}\frac{\log r_j}{(\mu_j^2 + r_j^2)^{1/2}}\bigg) \\ &= \mathrm{O}\bigg(\frac{1}{|\log\varepsilon|}\frac{\mu_j^4}{(\mu_j^2 + r_j^2)^{7/2}}\bigg), \end{aligned}$$

and

$$\mathcal{L}_\varepsilon(\hat{Z}_{0j}) = \mathrm{O}\left(\frac{\mu_j^4 \varepsilon}{(\mu_j^2 + r_j^2)^{7/2}}\right).$$

Thus, in this region

$$\mathcal{L}(\tilde{Z}_{0j}) = \mathrm{O}\left(\frac{\mu_j^4 |\log\varepsilon|^{-1}}{(\mu_j^2 + r_j^2)^{7/2}}\right). \tag{3.34}$$

In Ω_3, thanks to (3.18), $|\hat{Z}_{0j}| = \mathrm{O}(\frac{1}{|\log\varepsilon|})$, $|\nabla\hat{Z}_{0j}| = \mathrm{O}(\frac{\varepsilon}{|\log\varepsilon|})$, $|\Delta\hat{Z}_{0j}| = \mathrm{O}(\frac{\varepsilon^2}{|\log\varepsilon|})$, $|\nabla(\Delta\hat{Z}_{0j})| = \mathrm{O}(\frac{\varepsilon^3}{|\log\varepsilon|})$ and $|\Delta^2\hat{Z}_{0j}| = \mathrm{O}(\frac{\varepsilon^4}{|\log\varepsilon|})$. Thus, $F(\eta_{2j}, \hat{Z}_{0j}) = \mathrm{O}(\frac{\varepsilon^4}{|\log\varepsilon|})$.

Finally,

$$\begin{aligned}
\mathcal{L}_\varepsilon(\hat{Z}_{0j}) &= \mathcal{L}_\varepsilon(Z_{0j}) + W a_{0j}\left(H(\xi_j, \xi_j) - H(\varepsilon y, \xi_j) + 8\log\frac{r_j}{R}\right) \\
&= \mathrm{O}\left(\frac{\mu_j^4 \varepsilon}{(\mu_j^2 + r_j^2)^{7/2}}\right) + \mathrm{O}\left(\frac{\mu_j^4}{(\mu_j^2 + r_j^2)^4}\right) \\
&= \mathrm{O}\left(\frac{\mu_j^4 \varepsilon}{(\mu_j^2 + r_j^2)^{7/2}}\right)
\end{aligned}$$

and then, combining (3.33), (3.34) and the previous estimate, we can again write the estimate (3.28):

$$\mathcal{L}_\varepsilon(\tilde{Z}_{0j}) = \begin{cases} \mathrm{O}(\frac{\mu_j^4 \varepsilon}{(\mu_j^2 + r_j^2)^{7/2}}) & \text{in } \Omega_0, \\ \mathrm{O}(\frac{1}{|\log\varepsilon|}) & \text{in } \Omega_1, \\ \mathrm{O}(\frac{\mu_j^4 |\log\varepsilon|^{-1}}{(\mu_j^2 + r_j^2)^{7/2}}) & \text{in } \Omega_2, \\ \mathrm{O}(\frac{\mu_j^4 \varepsilon}{(\mu_j^2 + r_j^2)^{7/2}}) & \text{in } \Omega_3. \end{cases} \tag{3.35}$$

In conclusion,

$$\left\|\mathcal{L}_\varepsilon(\tilde{Z}_{0j})\right\|_* = \mathrm{O}\left(\frac{1}{|\log\varepsilon|}\right). \tag{3.36}$$

Finally, we prove the bounds of d_j. Testing equation (3.24) against $\tilde{Z}_{0j}$ and using relations (3.25) and the above estimate, we get

$$\begin{aligned}
|d_j|\left|\int_{\Omega_\varepsilon} \mathcal{L}_\varepsilon(\tilde{Z}_{0j})\tilde{Z}_{0j}\right| &= \left|\int_{\Omega_\varepsilon} h\tilde{Z}_{0j} + \int_{\Omega_\varepsilon} \tilde{\psi}\mathcal{L}_\varepsilon(\tilde{Z}_{0j})\right| \\
&\leqslant C\|h\|_* + C\|\tilde{\psi}\|_\infty \left\|\mathcal{L}_\varepsilon(\tilde{Z}_{0j})\right\|_* \\
&\leqslant C\|h\|_*\left\{1 + \left\|\mathcal{L}_\varepsilon(\tilde{Z}_{0j})\right\|_*\right\} + C\sum_{k=1}^m |d_k| \left\|\mathcal{L}_\varepsilon(\tilde{Z}_{0k})\right\|_* \left\|\mathcal{L}_\varepsilon(\tilde{Z}_{0j})\right\|_*
\end{aligned}$$

where we have used that

$$\int_{\Omega_\varepsilon} \frac{\mu_j^4}{(\mu_j^2 + r_j^2)^{7/2}} \leqslant C \quad \text{for all } j.$$

But estimate (3.36) imply

$$|d_j|\left|\int_{\Omega_\varepsilon} \mathcal{L}_\varepsilon(\tilde{Z}_{0j})\tilde{Z}_{0j}\right| \leqslant C\|h\|_* + C\sum_{k=1}^m \frac{|d_k|}{|\log\varepsilon|^2}. \tag{3.37}$$

It only remains to estimate the integral term of the left side. For this purpose, we have the following

Claim 2. *If R is sufficiently large,*

$$\left|\int\limits_{\Omega_\varepsilon} \mathcal{L}_\varepsilon(\tilde{Z}_{0j})\tilde{Z}_{0j}\right| = \frac{E}{|\log\varepsilon|}\big(1+\mathrm{o}(1)\big), \tag{3.38}$$

where E is a positive constant independent of ε and R.

Assume for the moment the validity of this claim. We replace (3.38) in (3.37), we get

$$|d_j| \leqslant C|\log\varepsilon|\|h\|_* + C\sum_{k=1}^{m}\frac{|d_k|}{|\log\varepsilon|}, \tag{3.39}$$

and then,

$$|d_j| \leqslant C|\log\varepsilon|\|h\|_*.$$

Claim 1 is thus proven. Let us proof Claim 2. We decompose

$$\begin{aligned}\int\limits_{\Omega_\varepsilon} \mathcal{L}_\varepsilon(\tilde{Z}_{0j})\tilde{Z}_{0j} &= \mathrm{O}(\varepsilon) + \int\limits_{\Omega_1} \mathcal{L}_\varepsilon(\tilde{Z}_{0j})\tilde{Z}_{0j} + \int\limits_{\Omega_2} \mathcal{L}_\varepsilon(\tilde{Z}_{0j})\tilde{Z}_{0j} + \int\limits_{\Omega_3} \mathcal{L}_\varepsilon(\tilde{Z}_{0j})\tilde{Z}_{0j}\\ &\equiv \mathrm{O}(\varepsilon) + I_1 + I_2 + I_3.\end{aligned}$$

First we estimate I_2. From (3.35),

$$\begin{aligned}I_2 &= \mathrm{O}\left(\frac{1}{|\log\varepsilon|}\int\limits_{\Omega_2}\frac{\mu_j^4\hat{Z}_{0j}}{(\mu_j^2+r_j^2)^{7/2}}\right)\\ &= \mathrm{O}\left(\frac{1}{R^3|\log\varepsilon|}\right).\end{aligned}$$

Now we estimate I_3. From the estimates in Ω_3, $|I_3| = \mathrm{O}(\frac{\varepsilon^4}{|\log\varepsilon|})$. On the other hand, since (3.33) holds true and $\hat{Z}_{0j} = Z_{0j}(1+\mathrm{O}(\frac{1}{R|\log\varepsilon|}))$, we conclude

$$\begin{aligned}|I_1| &= \frac{1}{R^4|\log\varepsilon|}\int\limits_{R<r_j\leqslant R+1}\tilde{Z}_{0j}(y)\,dy\\ &= \frac{1}{R^4|\log\varepsilon|}\int\limits_{R<r_j\leqslant R+1}\left\{\mathrm{O}\left(\frac{1}{R|\log\varepsilon|}\right)+\hat{Z}_{0j}(y)\right\}dy\\ &= \frac{1}{R^5|\log\varepsilon|^2} + \frac{|S^3|}{R^4|\log\varepsilon|}\int\limits_{R}^{R+1} r^3\left(\frac{r^2-\mu_j^2}{\mu_j^2+r^2}\right)\big(1+\mathrm{o}(1)\big)\,dr\\ &= \frac{E}{|\log\varepsilon|}\big(1+\mathrm{o}(1)\big),\end{aligned}$$

where E is a positive constant independent of ε and R. Thus, for fixed R large and ε small, we obtain (3.38). □

Now we can try with the original linear problem (3.5).

Proof of Proposition 3.1. We first establish the validity of the a priori estimate (3.7) for solutions ψ of problem (3.5), with $h \in L^\infty(\Omega_\varepsilon)$ and $\|h\|_* < \infty$. Lemma 3.3 implies

$$\|\psi\|_{**} \leqslant C|\log\varepsilon|\left\{\|h\|_* + \sum_{i=1}^{2}\sum_{j=1}^{m}|c_{ij}|\|\chi_j Z_{ij}\|_*\right\}. \tag{3.40}$$

On the other hand,

$$\|\chi_j Z_{ij}\|_* \leqslant C,$$

then, it is sufficient to estimate the values of the constants c_{ij}. To this end, we multiply the first equation in (3.5) by $Z_{ij}\eta_{2j}$, with η_{2j} the cut-off function introduced in (3.20), and integrate by parts to find

$$\int_{\Omega_\varepsilon} \psi \mathcal{L}_\varepsilon(Z_{ij}\eta_{2j}) = \int_{\Omega_\varepsilon} h Z_{ij}\eta_{2j} + c_{ij}\int_{\Omega_\varepsilon} \eta_{2j} Z_{ij}^2. \tag{3.41}$$

It is easy to see that $\int_{\Omega_\varepsilon} \eta_{2j} Z_{ij} h = \mathrm{O}(\|h\|_*)$ and $\int_{\Omega_\varepsilon} \eta_{2j} Z_{ij}^2 = C > 0$. On the other hand we have

$$\begin{aligned}\mathcal{L}_\varepsilon(\eta_{2j} Z_{ij}) &= \eta_{2j}\mathcal{L}_\varepsilon(Z_{ij}) + F(\eta_{2j}, Z_{ij})\\ &= \mathrm{O}\left(\frac{\mu_j^4 \varepsilon}{(\mu_j^2 + r_j^2)^{7/2}}\right)\eta_{2j}|Z_{ij}| + F(\eta_{2j}, Z_{ij}).\end{aligned}$$

Directly from (3.12) we get

$$F(\eta_{2j}, Z_{ij}) = \mathrm{O}\left(\frac{\varepsilon^4}{(\mu_j^2 + r_j^2)^{1/2}}\right) + \mathrm{O}\left(\frac{\varepsilon^3}{\mu_j^2 + r_j^2}\right) + \mathrm{O}\left(\frac{\varepsilon^2}{(\mu_j^2 + r_j^2)^{3/2}}\right) + \mathrm{O}\left(\frac{\varepsilon}{(\mu_j^2 + r_j^2)^{2}}\right),$$

in the region $\frac{\delta_0}{3\varepsilon} \leqslant r_j \leqslant \frac{\delta_0}{2\varepsilon}$. Thus

$$\begin{aligned}&\big\|\mathcal{L}_\varepsilon(\eta_{2j} Z_{ij})\big\|_* = \mathrm{O}(\varepsilon) \quad \text{and}\\ &\left|\int_{\Omega_\varepsilon} \psi \mathcal{L}_\varepsilon(\eta_{2j} Z_{ij})\right| \leqslant C\varepsilon|\log\varepsilon|\|\psi\|_\infty \leqslant C\varepsilon|\log\varepsilon|\|\psi\|_{**}.\end{aligned} \tag{3.42}$$

Using the above estimates in (3.41), we obtain

$$|c_{ij}| \leqslant C\big\{\varepsilon|\log\varepsilon|\|\psi\|_{**} + \|h\|_*\big\}, \tag{3.43}$$

and then

$$|c_{ij}| \leqslant C\left\{\big(1 + \varepsilon|\log\varepsilon|^2\big)\|h\|_* + \varepsilon|\log\varepsilon|^2 \sum_{l,k} |c_{lk}|\right\}.$$

Then $|c_{ij}| \leqslant C\|h\|_*$ and putting this estimate in (3.40), we conclude the validity of (3.17).

We now prove the solvability assertion. To this purpose we consider the space

$$\mathcal{H} = \left\{\psi \in H^3(\Omega_\varepsilon)\colon\ \psi = \Delta\psi = 0 \text{ on } \partial\Omega_\varepsilon,\ \text{and such that}\right.$$
$$\left.\int_{\Omega_\varepsilon} \chi_j Z_{ij}\psi = 0,\ \text{for all } i = 1, \dots, 4;\ j = 1, \dots, m\right\},$$

endowed with the usual inner product $(\psi, \varphi) = \int_{\Omega_\varepsilon} \Delta\psi\Delta\varphi$. Problem (3.16) expressed in a weak form is equivalent to that of finding a $\psi \in \mathcal{H}$, such that

$$(\psi, \varphi) = \int_{\Omega_s} \{h + W\psi\}\varphi, \quad \text{for all } \varphi \in \mathcal{H}.$$

With the aid of Riesz's representation theorem, this equation can be rewritten in $\mathcal{H}$ in the operator form $\psi = K(W\psi + h)$, where K is a compact operator in $\mathcal{H}$. Fredholm's alternative guarantees unique solvability of this problem for any h provided that the homogeneous equation $\psi = K(W\psi)$ has only the zero solution in $\mathcal{H}$. This last equation is equivalent to (3.16) with $h \equiv 0$. Thus existence of a unique solution follows from the a priori estimate (3.17). This concludes the proof. □

The result of Proposition 3.1 implies that the unique solution $\psi = T(h)$ of (3.5) defines a continuous linear map from the Banach space $\mathcal{C}_*$ of all functions $h \in L^\infty(\Omega_\varepsilon)$ with $\|h\|_* < +\infty$, into $W^{3,\infty}(\Omega_\varepsilon)$, with norm bounded uniformly in ε.

Remark 3.1. The operator T is differentiable with respect to the variables ξ'. In fact, computations similar to those used in [14] yield the estimate

$$\left\|\partial_{\xi'} T(h)\right\|_{**} \leqslant C|\log\varepsilon|^2\|h\|_*, \quad \text{for all } l = 1,2;\ k = 1,\dots,m. \tag{3.44}$$

4. The intermediate non-linear problem

In order to solve problem (2.20) we consider first the intermediate non-linear problem.

$$\begin{cases} \mathcal{L}_\varepsilon(\psi) = -R + N(\psi) + \sum_{i=1}^4 \sum_{j=1}^m c_{ij}\chi_j Z_{ij} & \text{in } \Omega_\varepsilon, \\ \psi = \Delta\psi = 0 \quad \text{on } \partial\Omega_\varepsilon, \\ \int_{\Omega_\varepsilon} \chi_j Z_{ij}\psi = 0, \quad \text{for all } i = 1,\dots,4,\ j = 1,\dots,m. \end{cases} \tag{4.1}$$

For this problem we will prove

Proposition 4.1. *Let $\xi \in \mathcal{O}$. Then, there exists $\varepsilon_0 > 0$ and $C > 0$ such that for all $\varepsilon \leqslant \varepsilon_0$ the non-linear problem* (4.1) *has a unique solution $\psi \in$ which satisfies*

$$\|\psi\|_{**} \leqslant C\varepsilon|\log\varepsilon|. \tag{4.2}$$

Moreover, if we consider the map $\xi' \in \mathcal{O} \to \psi \in \mathcal{C}^{4,\alpha}(\bar{\Omega}_\varepsilon)$, the derivative $D_{\xi'}\psi$ exists and defines a continuous map of ξ'. Besides

$$\|D_{\xi'}\psi\|_{**} \leqslant C\varepsilon|\log\varepsilon|^2. \tag{4.3}$$

Proof. In terms of the operator T defined in Proposition 3.1, problem (4.1) becomes

$$\psi = \mathcal{B}(\psi) \equiv T\big(N(\psi) - R\big).$$

Let us consider the region

$$\mathcal{F} \equiv \big\{\psi \in \mathcal{C}^{4,\alpha}(\bar{\Omega}_\varepsilon)\colon \|\psi\|_{**} \leqslant \varepsilon|\log\varepsilon|\big\}.$$

From Proposition 3.1,

$$\left\|\mathcal{B}(\psi)\right\|_{**} \leqslant C|\log\varepsilon|\big\{\left\|N(\psi)\right\|_* + \|R\|_*\big\},$$

and Lemma 2.2 implies

$$\|R\|_* \leqslant C\varepsilon.$$

Also, from Lemma 2.4

$$\left\|N(\psi)\right\|_* \leqslant C\|\psi\|_\infty^2 \leqslant C\|\psi\|_{**}^2.$$

Hence, if $\psi \in \mathcal{F}$, $\|\mathcal{B}(\psi)\|_{**} \leqslant C\varepsilon|\log\varepsilon|$. Along the same way we obtain

$$\left\|N(\psi_1) - N(\psi_2)\right\|_* \leqslant C \max_{i=1,2}\|\psi_i\|_\infty\|\psi_1 - \psi_2\|_\infty \leqslant C \max_{i=1,2}\|\psi_i\|_{**}\|\psi_1 - \psi_2\|_{**},$$

for any $\psi_1, \psi_2 \in \mathcal{F}$. Then, we conclude

$$\left\|\mathcal{B}(\psi_1) - \mathcal{B}(\psi_2)\right\|_{**} \leqslant C|\log\varepsilon|\left\|N(\psi_1) - N(\psi_2)\right\|_* \leqslant C\varepsilon|\log\varepsilon|^2\|\psi_1 - \psi_2\|_{**}.$$

It follows that for all ε small enough $\mathcal{B}$ is a contraction mapping of $\mathcal{F}$, and therefore a unique fixed point of $\mathcal{B}$ exists in this region. The proof of (4.3) is similar to one included in [14] and we thus omit it. $\square$

5. Variational reduction

We have solved the non-linear problem (4.1). In order to find a solution to the original problem (2.20) we need to find ξ such that

$$c_{ij} = c_{ij}(\xi') = 0, \quad \text{for all } i, j, \tag{5.1}$$

where $c_{ij}(\xi')$ are the constants in (4.1). problem (5.1) is indeed variational: it is equivalent to finding critical points of a function of ξ'. In fact, we define the function for $\xi \in \mathcal{O}$

$$\mathcal{F}_\varepsilon(\xi) \equiv J_\rho\big[U(\xi) + \hat{\psi}_\xi\big] \tag{5.2}$$

where J_ρ is defined in (2.16), ρ is given by (2.4), $U = U(\xi)$ is our approximate solution from (2.5) and $\hat{\psi}_\xi = \psi(\frac{x}{\varepsilon}, \frac{\xi}{\varepsilon})$, $x \in \Omega$, with $\psi = \psi_{\xi'}$ the unique solution to problem (4.1) given by Proposition 4.1. Then we obtain that critical points of $\mathcal{F}$ correspond to solutions of (5.1) for small ε. That is,

Lemma 5.1. *$\mathcal{F}_\varepsilon : \mathcal{O} \to \mathbb{R}$ is of class C^1. Moreover, for all ε small enough, if $D_\xi \mathcal{F}_\varepsilon(\xi) = 0$ then ξ satisfies* (5.1).

Proof. We define

$$I_\varepsilon[v] \equiv \frac{1}{2}\int_{\Omega_\varepsilon} (\Delta v)^2 - \int_{\Omega_\varepsilon} k(\varepsilon y)e^v.$$

Let us differentiate the function $\mathcal{F}_\varepsilon$ with respect to ξ. Since $J_\rho[U(\xi) + \hat{\psi}_\xi] = I_\varepsilon[V(\xi') + \psi_{\xi'}]$, we can differentiate directly under the integral sign, so that

$$\begin{aligned}\partial_{(\xi_k)_l}\mathcal{F}_\varepsilon(\xi) &= \varepsilon^{-1} DI_\varepsilon[V + \psi](\partial_{(\xi'_k)_l}V + \partial_{(\xi'_k)_l}\psi)\\ &= \varepsilon^{-1}\sum_{i=1}^{4}\sum_{j=1}^{m}\int_{\Omega_\varepsilon} c_{ij}\chi_j Z_{ij}(\partial_{(\xi'_k)_l}V + \partial_{(\xi'_k)_l}\psi).\end{aligned}$$

From the results of the previous section, this expression defines a continuous function of ξ', and hence of ξ. Let us assume that $D_\xi \mathcal{F}_\varepsilon(\xi) = 0$. Then

$$\sum_{i=1}^{4}\sum_{j=1}^{m}\int_{\Omega_\varepsilon} c_{ij}\chi_j Z_{ij}(\partial_{(\xi'_k)_l}V + \partial_{(\xi'_k)_l}\psi) = 0, \quad \text{for } k = 1, 2, 3, 4;\ l = 1, \ldots, m.$$

Since $\|D_{\xi'}\psi_{\xi'}\| \leqslant C\varepsilon|\log\varepsilon|^2$, we have

$$\partial_{(\xi'_k)_l}V + \partial_{(\xi'_k)_l}\psi = Z_{kl} + o(1),$$

where $o(1)$ is uniformly small as $\varepsilon \to 0$. Thus, we have the following linear system of equation

$$\sum_{i=1}^{4}\sum_{j=1}^{m} c_{ij}\int_{\Omega_\varepsilon} \chi_j Z_{ij}\big(Z_{kl} + o(1)\big) = 0, \quad \text{for } k = 1, 2, 3, 4;\ l = 1, \ldots, m.$$

This system is dominant diagonal, thus $c_{ij} = 0$ for all i, j. This concludes the proof. □

We also have the validity of the following lemma

Lemma 5.2. *Let ρ be given by* (2.4). *For points $\xi \in \mathcal{O}$ the following expansion holds*

$$\mathcal{F}_\varepsilon(\xi) = J_\rho\big[U(\xi)\big] + \theta_\varepsilon(\xi), \tag{5.3}$$

where $|\theta_\varepsilon| + |\nabla\theta_\varepsilon| = o(1)$, uniformly on $\xi \in \mathcal{O}$ as $\varepsilon \to 0$.

Proof. The proof follows directly from an application of Taylor expansion for $\mathcal{F}_\varepsilon$ in the expanded domain Ω_ε and from the estimates for the solution $\psi_{\xi'}$ to problem (4.1) obtained in Proposition 4.1. □

6. Proof of the theorems

In this section we carry out the proofs of our main results.

6.1. Proof of Theorem 1

Taking into account the result of Lemma 5.1, a solution to problem (1.1) exists if we prove the existence of a critical point of $\mathcal{F}_\varepsilon$, which automatically implies that $c_{ij} = 0$ in (2.20) for all i, j. The qualitative properties of the solution found follow from the ansatz.

Finding critical points of $\mathcal{F}_\varepsilon(\xi)$ is equivalent to finding critical points of

$$\tilde{\mathcal{F}}_\varepsilon(\xi) = \mathcal{F}_\varepsilon(\xi) - 256\pi^2 m|\log\varepsilon|. \tag{6.1}$$

On the other hand, if $\xi \in \mathcal{O}$, from Lemmas 2.3 and 5.2 we get the existence of constants $\alpha > 0$ and β such that

$$\alpha\tilde{\mathcal{F}}_\varepsilon(\xi) + \beta = \varphi_m(\xi) + \varepsilon\Theta_\varepsilon(\xi), \tag{6.2}$$

with Θ_ε and $\nabla_\xi\Theta_\varepsilon$ uniformly bounded in the considered region as $\varepsilon \to 0$.

We shall prove that, under the assumptions of Theorems 1 and 2, $\tilde{\mathcal{F}}_\varepsilon$ has a critical point in $\mathcal{O}$ for ε small enough. We start with a topological lemma. We denote by D the diagonal

$$D := \{\xi \in \Omega^m\colon\ \xi_i = \xi_j \text{ for some } i \neq j\},$$

and we write $H^* := H^*(\,\cdot\,;\mathbb{K})$ for singular cohomology with coefficients in a field $\mathbb{K}$.

Lemma 6.1. *If $H^d(\Omega) \neq 0$ for some $d \geqslant 1$, and $H^j(\Omega) = 0$ for $j > d$, then the homomorphism*

$$H^{md}(\Omega^m, D) \longrightarrow H^{md}(\Omega^m),$$

induced by the inclusion of pairs $(\Omega^m, \emptyset) \hookrightarrow (\Omega^m, D)$, is an epimorphism. In particular, $H^{md}(\Omega^m, D) \neq 0$.

Proof. Let us prove first that $H^j(D) = 0$ if $j > (m-1)d$. For this purpose we write

$$D = \bigcup_{1\leqslant i<j\leqslant m} X_{i,j}, \quad \text{where } X_{i,j} := \bigl\{(x_1,\ldots,x_m) \in \Omega^m\colon\ x_i = x_j\bigr\},$$

and consider the sets $\mathcal{F}_0 := \{\Omega^m\}$, $\mathcal{F}_1 := \{X_{i,j}\colon\ 1 \leqslant i < j \leqslant m\}$, and

$$\mathcal{F}_{k+1} := \{Z \cap Z'\colon\ Z, Z' \in \mathcal{F}_k \text{ and } Z \neq Z'\}, \quad k = 1,\ldots,m-2.$$

Note that

$$Z \cong \Omega^{m-k'} \quad \text{for some } k \leqslant k' \leqslant m-1 \text{ if } Z \in \mathcal{F}_k,\ k = 0,\ldots,m-1,$$

where $\cong$ means that the sets are homeomorphic. Künneth's formula

$$H^j(\Omega^{m-k}) = \bigoplus_{p+q=j} \bigl(H^p(\Omega) \otimes H^q(\Omega^{m-k-1})\bigr) \tag{6.3}$$

(see, for example, [17, Proposition 8.18]) yields inductively that, for $0 \leqslant k \leqslant m-1$,

$$H^j(Z) = 0 \quad \text{if } Z \in \mathcal{F}_k \text{ and } j > (m-k)d. \tag{6.4}$$

We claim that, for each $0 \leqslant k \leqslant m-1$, one has that

$$H^j(Z_1 \cup \cdots \cup Z_\ell) = 0 \quad \text{if } Z_1,\ldots,Z_\ell \in \mathcal{F}_k \text{ and } j > (m-k)d. \tag{6.5}$$

Let us prove this claim. Since $\mathcal{F}_{m-1}$ has only one element and (6.4) holds, we have that the claim is true for $k = m-1$. Assume that the claim is true for $k+1$ with $k+1 \leqslant m-1$ and let us then prove it for k. We do this by induction on ℓ. If $\ell = 1$ the assertion reduces to (6.4). Now assume that the assertion is true for every union of at most $\ell - 1$ sets in $\mathcal{F}_k$, and let $Z_1,\ldots,Z_\ell \in \mathcal{F}_k$ be pairwise distinct sets. Consider the Mayer–Vietoris sequence

$$\cdots \to H^{j-1}\Biggl(\bigcup_{i=1}^{\ell-1}(Z_i \cap Z_\ell)\Biggr) \to H^j(Z_1 \cup \cdots \cup Z_\ell) \to H^j(Z_1 \cup \cdots \cup Z_{\ell-1}) \oplus H^j(Z_\ell) \to \cdots. \tag{6.6}$$

Our induction hypothesis on ℓ yields that $H^j(Z_1 \cup \cdots \cup Z_{\ell-1}) = 0$ and $H^j(Z_\ell) = 0$ if $j > (m-k)d$. Since $Z_1, \ldots, Z_\ell$ are pairwise distinct, we have that $Z_i \cap Z_\ell \in \mathcal{F}_{k+1}$ for each $i = 1, \ldots, \ell - 1$ and, since we are assuming that the claim is true for $k+1$ we have that

$$H^{j-1}\left(\bigcup_{i=1}^{\ell-1}(Z_i \cap Z_\ell)\right) = 0 \quad \text{if } j-1 > \big(m-(k+1)\big)d.$$

Note that $j > (m-k)d$ implies $j - 1 > (m-(k+1))d$. This proves that both ends of the exact sequence (6.6) are zero if $j > (m-k)d$, hence the middle term is also zero in this case. This concludes the proof of claim (6.5).

Now, since $D = \bigcup_{Y \in \mathcal{F}_1} Y$, assertion (6.5) with $k = 1$ yields that $H^j(D) = 0$ if $j > (m-1)d$. So the exact cohomology sequence

$$H^{md}\big(\Omega^m, D\big) \longrightarrow H^{md}\big(\Omega^m\big) \longrightarrow H^{md}(D) = 0$$

gives that $H^{md}(\Omega^m, D) \to H^{md}(\Omega^m)$ is an epimorphism. But (6.3) implies that $H^{md}(\Omega^m) \neq 0$. Therefore, $H^{md}(\Omega^m, D) \neq 0$, as claimed. □

For each positive number δ define

$$\Omega_\delta := \big\{\xi \in \Omega\colon\ \operatorname{dist}(\xi, \partial\Omega) > \delta\big\},$$
$$\mathfrak{D}_\delta := \big\{\xi = (\xi_1, \ldots, \xi_m) \in \Omega^m\colon\ \xi_j \in \Omega_\delta\big\}.$$

Lemma 6.2. *Given $K > 0$ there exists $\delta_0 > 0$ such that, for each $\delta \in (0, \delta_0)$, the following holds: For every $\xi = (\xi_1, \ldots, \xi_m) \in \partial\mathfrak{D}_\delta$ with $|\varphi_m(\xi)| \leqslant K$ there exists an $i \in \{1, \ldots, m\}$ such that*

$$\nabla_{\xi_i}\varphi_m(\xi) \neq 0 \qquad \text{if } \xi_i \in \Omega_\delta,$$
$$\nabla_{\xi_i}\varphi_m(\xi)\cdot\tau \neq 0 \quad \text{for some } \tau \in T_{\xi_i}(\partial\Omega_\delta) \quad \text{if } \xi_i \in \partial\Omega_\delta$$

where $T_{\xi_i}(\partial\Omega_\delta)$ denotes the tangent space to $\partial\Omega_\delta$ at the point ξ_i.

Proof. We first need to establish some facts related to the regular part of the Green function on the half hyperplane

$$\mathcal{H} := \big\{x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4\colon\ x_4 \geqslant 0\big\}.$$

It is well known that the regular part of the Green function on $\mathcal{H}$ is given by

$$H(x, y) = 8\log|x - \bar{y}|, \quad \bar{y} = (y_1, y_2, y_3, -y_4),$$

for $x, y \in \mathcal{H}$ and the Green function is

$$G(x, y) = -8\log|x - y| + 8\log|x - \bar{y}|.$$

Consider the function of $k \geqslant 2$ distinct points of $\mathcal{H}$

$$\Psi_k(x_1, \ldots, x_k) := -8\sum_{i \neq j}\log|x_i - x_j|,$$

and denote by I_+ and I_0 the set of indices i for which $(x_i)_4 > 0$ and $(x_i)_4 = 0$, respectively. Define also

$$\varphi_{k,\mathcal{H}}(x_1, \ldots, x_k) = -8\sum_{j=1}^{k}\log|x_j - \bar{x_j}| + 8\sum_{i \neq j}\log\frac{|x_i - x_j|}{|x_i - \bar{x_j}|}.$$

Claim 3. *We have the following alternative: Either*

$$\nabla_{x_i}\Psi_k(x_1, \ldots, x_k) \neq 0 \quad \text{for some } i \in I_+,$$

or

$$\partial_{(x_i)_j}\Psi_k(x_1, \ldots, x_k) \neq 0 \quad \text{for some } i \in I_0 \text{ and } j \in \{1, 2, 3\},$$

where $\partial_{(x_i)_j} \equiv \frac{\partial}{\partial(x_i)_j}$.

Proof. We have that

$$\frac{\partial}{\partial\lambda}\Psi_k(\lambda x_1,\dots,\lambda x_k)|_{\lambda=1}=\sum_{i\in I_+}\nabla_{x_i}\Psi_k(x_1,\dots,x_k)\cdot x_i+\sum_{i\in I_0}\nabla_{x_i}\Psi_k(x_1,\dots,x_k)\cdot x_i.$$

On the other hand

$$\frac{\partial}{\partial\lambda}\Psi_k(\lambda x_1,\dots,\lambda x_k)|_{\lambda=1}=-8k(k-1)\neq 0,$$

and Claim 3 follows. □

Claim 4. *For any k distinct points $x_i\in\operatorname{Int}\mathcal{H}$ we have $\nabla\varphi_{k,\mathcal{H}}(x_1,\dots,x_k)\neq 0$.*

Proof. We have that

$$\frac{\partial}{\partial\lambda}\varphi_{k,\mathcal{H}}(\lambda x_1,\dots,\lambda x_k)|_{\lambda=1}=\sum_{i=1}^{k}\nabla_{x_i}\varphi_{k,\mathcal{H}}(x_1,\dots,x_k)\cdot x_i.$$

On the other hand

$$\frac{\partial}{\partial\lambda}\varphi_{k,\mathcal{H}}(\lambda x_1,\dots,\lambda x_k)|_{\lambda=1}=-8k(k-1)\neq 0,$$

and Claim 4 follows. □

Now we will need an estimate for the regular part $H(x,y)$ of the Green's function for points x,y close to $\partial\Omega$.

Claim 5. *There exists $C_1,C_2>0$ constants such that for any $x,y\in\Omega$*

$$\left|\nabla_x H(x,y)\right|+\left|\nabla_y H(x,y)\right|\leqslant C_1\min\left\{\frac{1}{|x-y|},\frac{1}{\operatorname{dist}(y,\partial\Omega)}\right\}+C_2.$$

Proof. For $y\in\Omega$ a point close to $\partial\Omega$ we denote by $\bar y$ its uniquely determined reflection with respect to $\partial\Omega$. Define $\psi(x,y)=H(x,y)+8\log\frac{1}{|x-\bar y|}$. It is straightforward to see that ψ is bounded in $\bar\Omega\times\bar\Omega$ and that $|\nabla_x\psi(x,y)|+|\nabla_y\psi(x,y)|\leqslant C$ for some positive constant C. Claim 5 follows. □

We have now all elements to prove Lemma 6.2. Assume, by contradiction, that for some sequence $\delta_n\to 0$ there are points $\xi^n\in\partial\mathfrak{D}_{\delta_n}$, such that $|\varphi_m(\xi^n)|\leqslant K$ and, for every $i\in\{1,\dots,m\}$,

$$\nabla_{\xi_i^n}\varphi_m(\xi^n)=0\quad\text{if }\xi_i^n\in\Omega_{\delta_n},\tag{6.7}$$

and

$$\nabla_{\xi_i^n}\varphi_m(\xi^n)\cdot\tau=0\quad\text{if }\xi_i^n\in\partial\Omega_{\delta_n},\tag{6.8}$$

for any vector τ tangent to $\partial\Omega_{\delta_n}$ at ξ_i^n. It follows that there exists a point $\xi_l^n\in\partial\Omega_{\delta_n}$ such that $H(\xi_l^n,\xi_l^n)\to-\infty$ as $n\to\infty$. Since $|\varphi_m(\xi^n)|\leqslant K$, there are necessarily two distinct points ξ_i^n and ξ_j^n coming closer to each other, that is,

$$\rho_n:=\inf_{i\neq j}|\xi_i^n-\xi_j^n|\to 0\quad\text{as }n\to\infty.$$

Without loss of generality we can assume $\rho_n=|\xi_1^n-\xi_2^n|$. We define $x_j^n:=(\xi_j^n-\xi_1^n)/\rho_n$. Thus, up to a subsequence, there exists a k, $2\leqslant k\leqslant m$, such that

$$\lim_{n\to\infty}|x_j^n|<+\infty,\quad j=1,\dots,k,\quad\text{and}\quad\lim_{n\to\infty}|x_j^n|=+\infty,\quad j>k.$$

For $j\leqslant k$ we set

$$\bar x_j=\lim_{n\to\infty}x_j^n.$$

We consider two cases:

(1) Either

$$\frac{\operatorname{dist}(\xi_1^n, \partial\Omega_{\delta_n})}{\rho_n} \to +\infty,$$

(2) or there exists $C_0 < +\infty$ such that for almost all n we have

$$\frac{\operatorname{dist}(\xi_1^n, \partial\Omega_{\delta_n})}{\rho_n} < C_0.$$

Case 1. It is easy to see that in this case we actually have

$$\frac{\operatorname{dist}(\xi_j^n, \partial\Omega_{\delta_n})}{\rho_n} \to +\infty, \quad j = 1, \dots, k.$$

Furthermore, the points $\xi_1^n, \dots, \xi_k^n$ are all in the interior of Ω_{δ_n}, hence (6.7) is satisfied for all partial derivatives ∇_{ξ_j}, $j \leqslant k$. Define $\tilde{\varphi}_m(x_1, \dots, x_m) := \varphi_m(\xi_1^n + \rho_n x_1, \xi_1^n + \rho_n x_2, \dots, \xi_1^n + \rho_n x_k, \xi_{k+1}^n + \rho_n x_{k+1}, \dots, \xi_m^n + \rho_n x_m)$, and $x = (x_1, \dots, x_m)$. We have that, for all $l = 1, 2$, $j = 1, \dots, k$, $\partial_{(x_j)_l}\tilde{\varphi}_m(x) = \rho_n \partial_{(\xi_j)_l}\varphi_m(\xi_1^n + \rho_n x_1, \dots, \xi_1^n + \rho_n x_k, \xi_{k+1}^n + \rho_n x_{k+1}, \dots, \xi_m^n + \rho_n x_m)$. Then at $\bar{x} = (\bar{x}_1, \dots, \bar{x}_k, 0, \dots, 0)$ we have

$$\partial_{(x_j)_l}\tilde{\varphi}_m(\bar{x}) = 0.$$

On the other hand, using Claim 5 and letting $n \to \infty$, we obtain

$$\lim_{n\to\infty} \rho_n \partial_{(\xi_j)_l}\varphi_m(\xi_1^n + \rho_n\tilde{x}_1, \dots, \xi_m^n + \rho_n\tilde{x}_m) = 8 \sum_{i \neq j,\, i \leqslant k} \partial_{(x_j)_i} \log|\bar{x}_i - \bar{x}_j| = 0,$$

a contradiction with Claim 3.

Case 2. In this case we actually have

$$\frac{\operatorname{dist}(\xi_j^n, \partial\Omega_{\delta_n})}{\rho_n} < C_1, \quad j = 1, \dots, m,$$

for some constant $C_1 > 0$ and for almost all n. If the points ξ_j^n are all interior to Ω_{δ_n}, we argue as in Case 1 above to reach a contradiction to Claim 4.

Therefore, we assume that for some j^* we have $\xi_{j^*}^n \in \partial\Omega_{\delta_n}$. Assume first that there exists a constant C such that $\delta_n \leqslant C\rho_n$. Consider the following sum

$$s_n := \sum_{i \neq j} G(\xi_j^n, \xi_i^n).$$

In this case it is not difficult to see that $s_n = \mathrm{O}(1)$ as $n \to +\infty$. On the other hand

$$\sum_j H(\xi_j^n, \xi_j^n) \leqslant H(\xi_{j^*}^n, \xi_{j^*}^n) + C \leqslant 8 \log|\xi_{j^*}^n - \bar{\xi}_{j^*}^n| + C,$$

where $\bar{\xi}_{j^*}^n$ is the reflection of the point $\xi_{j^*}^n$ with respect to $\partial\Omega$. Since $|\xi_{j^*}^n - \bar{\xi}_{j^*}^n| \leqslant 2\delta_n$ we have that

$$\sum_j H\big(\xi_j^n, \xi_j^n\big) \to -\infty, \quad \text{as } n \to \infty.$$

But $|\varphi_m(\xi^n)| \leqslant K$, a contradiction.

Finally assume that $\rho_n = \mathrm{o}(\delta_n)$. In this case after scaling with ρ_n around $\xi_{j^*}^n$, and arguing similarly as in Case 1 we get a contradiction with Claim 3 since those points ξ_j^n which lie on $\partial\Omega_{\delta_n}$, after passing to the limit, give rise to points that lie on the same straight line. Thus this case cannot occur. □

We shall now show that we can perturb the gradient vector field of φ_m near $\partial\mathfrak{D}_\delta$ to obtain a new vector field with the same stationary points, such that φ_m is a Lyapunov function for the associated flow and $\mathfrak{D}_\delta \cap \varphi_m^{-1}[-K, K]$ is positively invariant.

We consider the following more general situation. Let U be a bounded open subset of $\mathbb{R}^N$ with smooth boundary, and let $m \in \mathbb{N}$. We consider a decomposition of $\bar{U}^m$ as follows. Let S be the set of all functions $\sigma : \{1, \dots, m\} \to \{U, \partial U\}$, and define

$$\mathcal{Y}_\sigma := \sigma(1) \times \cdots \times \sigma(m) \subset \mathbb{R}^{mN}.$$

Then

$$\bar{U}^m = \bigcup_{\sigma \in S} \mathcal{Y}_\sigma, \quad \partial(U^m) = \bigcup_{\sigma \in S \smallsetminus \sigma_U} \mathcal{Y}_\sigma, \quad \text{and} \quad \mathcal{Y}_\sigma \cap \mathcal{Y}_\zeta = \emptyset \quad \text{if } \sigma \neq \zeta,$$

where σ_U stands for the constant function $\sigma_U(i) = U$. Note that $\mathcal{Y}_\sigma$ is a manifold of dimension $\leqslant mN$. We denote by $T_\xi(\mathcal{Y}_\sigma)$ the tangent space to $\mathcal{Y}_\sigma$ at the point $\xi \in \mathcal{Y}_\sigma$. The following holds.

Lemma 6.3. *Let $\mathcal{F}$ be a function of class $\mathcal{C}^1$ in a neighborhood of $\bar{U}^m \cap \mathcal{F}^{-1}[b, c]$. Assume that*

$$\nabla_\sigma \mathcal{F}(\xi) \neq 0 \quad \text{for every } \xi \in \mathcal{Y}_\sigma \cap \mathcal{F}^{-1}[b, c] \text{ with } \sigma \neq \sigma_U, \tag{6.9}$$

where $\nabla_\sigma \mathcal{F}(\xi)$ is the projection of $\nabla \mathcal{F}(\xi)$ onto the tangent space $T_\xi(\mathcal{Y}_\sigma)$. Then there exists a locally Lipschitz continuous vector field $\chi : \mathcal{U} \to \mathbb{R}^N$, defined in an open neighborhood $\mathcal{U}$ of $\bar{U}^m \cap \mathcal{F}^{-1}[b, c]$, with the following properties: For $\xi \in \mathcal{U}$,

(i) $\chi(\xi) = 0$ *if and only if* $\nabla \mathcal{F}(\xi) = 0$,
(ii) $\chi(\xi) \cdot \nabla \mathcal{F}(\xi) > 0$ *if* $\nabla \mathcal{F}(\xi) \neq 0$,
(iii) $\chi(\xi) \in T_\xi(\mathcal{Y}_\sigma)$ *if* $\xi \in \mathcal{Y}_\sigma \cap \mathcal{F}^{-1}[b, c]$.

Proof. Let $\mathcal{N}_\alpha := \{x \in \mathbb{R}^N\colon \operatorname{dist}(x, \partial U) < \alpha\}$. Fix $\alpha > 0$ small enough so that there exists a smooth retraction $r : \mathcal{N}_\alpha \to \partial U$. For every $\sigma \in S$, let $\hat{\sigma} : \{1, \dots, m\} \to \{U, \partial \mathcal{N}_\alpha\}$ be the function $\hat{\sigma}(i) = \sigma(i)$ if $\sigma(i) = U$ and $\hat{\sigma}(i) = \mathcal{N}_\alpha$ if $\sigma(i) = \partial U$. Set

$$\mathcal{U}_\sigma := \hat{\sigma}(1) \times \cdots \times \hat{\sigma}(m).$$

Then $\mathcal{U}_\sigma$ is an open neighborhood of $\mathcal{Y}_\sigma$. Let $r_\sigma : \mathcal{U}_\sigma \to \mathcal{Y}_\sigma$ be the obvious retraction. Assumption (6.9) implies that $\mathcal{F}$ has no critical points on $\partial(U^m) \cap \mathcal{F}^{-1}[b, c]$ and, moreover, that

$$\nabla_\sigma \mathcal{F}(\xi) \cdot \nabla \mathcal{F}(\xi) > 0 \quad \text{if } \xi \in \mathcal{Y}_\sigma \cap \mathcal{F}^{-1}[b, c] \text{ and } \nabla \mathcal{F}(\xi) \neq 0.$$

So taking α even smaller if necessary, we may assume that $\mathcal{F}$ has no critical points in $\mathcal{U}_\sigma \cap \mathcal{F}^{-1}[b, c]$ if $\sigma \neq \sigma_U$, and that

$$\nabla_\sigma \mathcal{F}\big(r_\sigma(\xi)\big) \cdot \nabla \mathcal{F}(\xi) > 0 \quad \text{if } \xi \in \mathcal{U}_\sigma \cap \mathcal{F}^{-1}(b - \alpha, c + \alpha) \text{ and } \nabla \mathcal{F}(\xi) \neq 0.$$

Let $\{\pi_\sigma\colon \sigma \in S\}$ be a locally Lipschitz partition of unity subordinated to the open cover $\{\mathcal{U}_\sigma\colon \sigma \in S\}$. Define

$$\chi(\xi) := \sum_{\sigma \in S} \pi_\sigma(\xi) \nabla_\sigma \mathcal{F}\big(r_\sigma(\xi)\big), \quad \xi \in \mathcal{U} := \bigcup_{\sigma \in S} \mathcal{U}_\sigma \cap \mathcal{F}^{-1}(b - \alpha, c + \alpha).$$

One can easily verify that χ has the desired properties. $\square$

As usual, set $\mathcal{F}^c := \{\xi \in \operatorname{dom} \mathcal{F}\colon \mathcal{F}(\xi) \leqslant c\}$.

Lemma 6.4 *(Deformation lemma). Let $\mathcal{F}$ be a function of class $\mathcal{C}^1$ in a neighborhood of $\bar{U}^m \cap \mathcal{F}^{-1}[b, c]$. Assume that*

$$\nabla_\sigma \mathcal{F}(\xi) \neq 0 \quad \text{for every } \xi \in \mathcal{Y}_\sigma \cap \mathcal{F}^{-1}[b, c] \text{ with } \sigma \neq \sigma_U.$$

If $\mathcal{F}$ has no critical points in $U^m \cap \mathcal{F}^{-1}[b, c]$, then there exists a continuous deformation $\tilde{\eta} : [0, 1] \times (\bar{U}^m \cap \mathcal{F}^c) \to \bar{U}^m \cap \mathcal{F}^c$ such that

$$\tilde{\eta}(0, \xi) = \xi \quad \text{for all } \xi \in \bar{U}^m \cap \mathcal{F}^c,$$
$$\tilde{\eta}(s, \xi) = \xi \quad \text{for all } (s, \xi) \in [0, 1] \times (\bar{U}^m \cap \mathcal{F}^b),$$
$$\tilde{\eta}(1, \xi) \in \bar{U}^m \cap \mathcal{F}^b \quad \text{for all } \xi \in \bar{U}^m \cap \mathcal{F}^c.$$

Proof. Let $\chi : \mathcal{U} \to \mathbb{R}^N$ be as in Lemma 6.3 and consider the flow η defined by

$$\begin{cases} \frac{\partial}{\partial t}\eta(t,\xi) = -\chi(\eta(t,\xi)), \\ \eta(0,\xi) = \xi, \end{cases} \tag{6.10}$$

for $\xi \in \mathcal{U}$ and $t \in [0, t^+(\xi))$, where $t^+(\xi)$ is the maximal existence time of the trajectory $t \mapsto \eta(t,\xi)$ in $\mathcal{U}$. For each $\xi \in \mathcal{U}$, let

$$t_b(\xi) := \inf\{t \geqslant 0\colon\ \mathcal{F}(\eta(t,\xi)) \leqslant b\} \in [0,\infty]$$

be the entrance time into the sublevel set $\mathcal{F}^b$. Property (ii) in Lemma 6.3 implies that

$$\frac{d}{dt}\mathcal{F}(\eta(t,\xi)) = -\nabla\mathcal{F}(\eta(t,\xi)) \cdot \chi(\eta(t,\xi)) \leqslant 0,$$

therefore $\mathcal{F}(\eta(t,\xi))$ is non-increasing in t. This, together with (iii) in Lemma 6.3 yields

$$\eta(t,\xi) \in \bar{U}^m \cap \mathcal{F}^{-1}[b,c] \quad \text{if } \xi \in \bar{U}^m \cap \mathcal{F}^{-1}[b,c] \text{ and } t \in [0, t_b(\xi)].$$

Since $\mathcal{F}$ has no critical points in $U^m \cap \mathcal{F}^{-1}[b,c]$, we have that $t_b(\xi) < \infty$ for every $\xi \in \bar{U}^m \cap \mathcal{F}^{-1}[b,c]$, and the entrance time map $t_b : \bar{U}^m \cap \mathcal{F}^c \cap \mathcal{U} \to [0,\infty)$ is continuous. It follows that the map

$$\tilde{\eta} : [0,1] \times (\bar{U}^m \cap \mathcal{F}^c) \to \bar{U}^m \cap \mathcal{F}^c$$

given by

$$\tilde{\eta}(s,\xi) := \begin{cases} \eta(st_b(\xi),\xi) & \text{if } \xi \in (\bar{U}^m \cap \mathcal{F}^c) \cap \mathcal{U}, \\ \xi & \text{if } \xi \in \bar{U}^m \cap \mathcal{F}^b \end{cases}$$

is a continuous deformation of $\bar{U}^m \cap \mathcal{F}^c$ into $\bar{U}^m \cap \mathcal{F}^b$ which leaves $\bar{U}^m \cap \mathcal{F}^b$ fixed, as claimed. □

Proof of Theorem 1. Fix δ_1 small enough so that the inclusions

$$\mathfrak{D}_{\delta_1} \hookrightarrow \Omega^m \quad \text{and} \quad \mathfrak{D}_{\delta_1} \cap D \hookrightarrow B_{\delta_1}(D) := \{x \in \Omega^m\colon\ \mathrm{dist}(x,D) \leqslant \delta_1\} \tag{6.11}$$

are homotopy equivalences, where $D := \{\xi \in \Omega^m\colon\ \xi_i = \xi_j \text{ for some } i \neq j\}$. Since φ_m is bounded above on $\mathfrak{D}_{\delta_1}$ and bounded below on $\Omega^m \smallsetminus B_{\delta_1}(D)$, we may choose $b_0, c_0 > 0$ such that

$$\mathfrak{D}_{\delta_1} \subset \varphi_m^{c_0} \quad \text{and} \quad \varphi_m^{b_0} \subset B_{\delta_1}(D).$$

Fix $K > \max\{-b_0, c_0\}$ and, for this K, fix $\delta \in (0,\delta_1)$ as in Lemma 6.2. By property (6.2), for each ε small enough, there exist $b < c$ such that

$$\varphi_m^{c_0} \subset \tilde{\mathcal{F}}_\varepsilon^c \subset \varphi_m^K, \qquad \varphi_m^{-K} \subset \tilde{\mathcal{F}}_\varepsilon^b \subset \varphi_m^{b_0},$$

and such that, for every $\xi = (\xi_1,\dots,\xi_m) \in \partial\mathfrak{D}_\delta$ with $\tilde{\mathcal{F}}_\varepsilon(\xi) \in [b,c]$ there is an $i \in \{1,\dots,m\}$ with

$$\begin{aligned} &\nabla_{\xi_i}\tilde{\mathcal{F}}_\varepsilon(\xi) \neq 0 && \text{if } \xi_i \in \Omega_\delta, \\ &\nabla_{\xi_i}\tilde{\mathcal{F}}_\varepsilon(\xi) \cdot \tau \neq 0 \quad \text{for some } \tau \in T_{\xi_i}(\partial\Omega_\delta) && \text{if } \xi_i \in \partial\Omega_\delta. \end{aligned}$$

We wish to prove that $\tilde{\mathcal{F}}_\varepsilon$ has a critical point in $\mathfrak{D}_\delta \cap \tilde{\mathcal{F}}_\varepsilon^{-1}[b,c]$. We argue by contradiction: Assume that $\tilde{\mathcal{F}}_\varepsilon$ has no critical points in $\mathfrak{D}_\delta \cap \tilde{\mathcal{F}}_\varepsilon^{-1}[b,c]$. Then Lemma 6.4 gives a continuous deformation

$$\tilde{\eta} : [0,1] \times (\bar{\mathfrak{D}}_\delta \cap \tilde{\mathcal{F}}_\varepsilon^c) \to \bar{\mathfrak{D}}_\delta \cap \tilde{\mathcal{F}}_\varepsilon^c$$

of $\bar{\mathfrak{D}}_\delta \cap \tilde{\mathcal{F}}_\varepsilon^c$ into $\bar{\mathfrak{D}}_\delta \cap \tilde{\mathcal{F}}_\varepsilon^b$ which keeps $\bar{\mathfrak{D}}_\delta \cap \tilde{\mathcal{F}}_\varepsilon^b$ fixed. Our choices of b and c imply that $\mathfrak{D}_{\delta_1} \subset \bar{\mathfrak{D}}_\delta \cap \tilde{\mathcal{F}}_\varepsilon^c$ and $\tilde{\eta}$ induces a deformation of $\mathfrak{D}_{\delta_1}$ into $\bar{\mathfrak{D}}_\delta \cap \tilde{\mathcal{F}}_\varepsilon^b \subset B_{\delta_1}(D)$, which keeps the diagonal D fixed. Consequently, the homomorphism

$$\iota^* : H^*(\Omega^m, B_{\delta_1}(D)) \to H^*(\mathfrak{D}_{\delta_1}, \mathfrak{D}_{\delta_1} \cap D),$$

induced by th inclusion map $\iota : \mathfrak{D}_{\delta_1} \hookrightarrow \Omega^m$, factors through $H^*(B_{\delta_1}(D), B_{\delta_1}(D)) = 0$. Hence, ι^* is the zero homomorphism. On the other hand, our choice (6.11) of δ_1 implies that ι^* is an isomorphism. Therefore, $H^*(\Omega^m, B_{\delta_1}(D)) = H^*(\Omega^m, D) = 0$. But, by assumption, $H^d(\Omega) \neq 0$ for some $d \geqslant 1$. If we choose d so that $H^j(\Omega) = 0$ for $j > d$, then Lemma 6.1 asserts that $H^{md}(\Omega^m, D) \neq 0$. This is a contradiction. Consequently, $\tilde{\mathcal{F}}_\varepsilon$ must have critical point in $\mathfrak{D}_\delta \cap \tilde{\mathcal{F}}_\varepsilon^{-1}[b,c]$, as claimed. □

6.2. *Proof of Theorem 2*

Assume that there exist an open subset U of Ω with smooth boundary, compactly contained in Ω, and two closed subsets $B_0 \subset B$ of U^m, which satisfy conditions (P1) and (P2) stated in Section 1. By property (6.2), for ε small enough, $\tilde{\mathcal{F}}_\varepsilon$ satisfies those conditions too, that is,

$$b_\varepsilon := \sup_{\xi \in B_0} \tilde{\mathcal{F}}_\varepsilon(\xi) < \inf_{\gamma \in \Gamma} \sup_{\xi \in B} \tilde{\mathcal{F}}_\varepsilon\big(\gamma(\xi)\big) =: c_\varepsilon,$$

where $\Gamma := \{\gamma \in \mathcal{C}(B, \bar{U}^m)\colon\ \gamma(\xi) = \xi$ for every $\xi \in B_0\}$ and, for every $\xi = (\xi_1, \ldots, \xi_m) \in \partial U^m$ with $\tilde{\mathcal{F}}_\varepsilon(\xi) \in [c_\varepsilon - \alpha, c_\varepsilon + \alpha]$, $\alpha \in (0, c_\varepsilon - b_\varepsilon)$ small enough, one has that

$$\begin{aligned} &\nabla_{\xi_i} \tilde{\mathcal{F}}_\varepsilon(\xi) \neq 0 && && \text{if } \xi_i \in U, \\ &\nabla_{\xi_i} \tilde{\mathcal{F}}_\varepsilon(\xi) \cdot \tau \neq 0 && \text{for some } \tau \in T_{\xi_i}(\partial U) && \text{if } \xi_i \in \partial U, \end{aligned}$$

for some $i \in \{1, \ldots, m\}$. If $\tilde{\mathcal{F}}_\varepsilon$ has no critical points in $U^m \cap \tilde{\mathcal{F}}_\varepsilon^{-1}[c_\varepsilon - \alpha, c_\varepsilon + \alpha]$, then Lemma 6.4 gives a continuous deformation

$$\tilde{\eta} : [0,1] \times (\bar{U}^m \cap \tilde{\mathcal{F}}_\varepsilon^{c_\varepsilon + \alpha}) \to \bar{U}^m \cap \tilde{\mathcal{F}}_\varepsilon^{c_\varepsilon + \alpha}$$

of $\bar{U}^m \cap \tilde{\mathcal{F}}_\varepsilon^{c_\varepsilon + \alpha}$ into $\bar{U}^m \cap \tilde{\mathcal{F}}_\varepsilon^{c_\varepsilon - \alpha}$ which keeps $\bar{U}^m \cap \tilde{\mathcal{F}}_\varepsilon^{c_\varepsilon - \alpha}$ fixed. Let $\gamma \in \Gamma$ be such that $\tilde{\mathcal{F}}_\varepsilon(\gamma(\xi)) \leqslant c_\varepsilon + \alpha$ for every $\xi \in B$. Since $b_\varepsilon < c_\varepsilon - \alpha$, the map $\tilde{\gamma}(\xi) := \tilde{\eta}(1, \gamma(\xi))$ belongs to Γ. But $\tilde{\mathcal{F}}_\varepsilon(\tilde{\gamma}(\xi)) \leqslant c_\varepsilon - \alpha$ for every $\xi \in B$, contradicting the definition of c_ε. Therefore, c_ε is a critical value of $\tilde{\mathcal{F}}_\varepsilon$. □

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