# Asymptotic velocity of one dimensional diffusions with periodic drift

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**Abstract**

We consider the asymptotic behaviour of the solution of one dimensional stochastic differential equations and Lagevin equations in periodic backgrounds with zero average. We prove that in several such models, there is genrically a non vanishing asymptotic velocity, despite of the fact that the average of the background is zero.

## 1 Introduction.

We consider the one dimensional diffusion problem

$$\partial_t u = \partial_x \left( \frac{1}{2} \partial_x u + b(t,x) u \right) \tag{1}$$

where $b$ is a regular function periodic of period $T$ in the time variable $t \in \mathbf{R}^+$ and of period $L$ in the space variable $x \in \mathbf{R}$. This equation and related equations discussed below appear in many natural questions like molecular motors, population dynamics, pulsed dielectrophoresis, etc. See for example [6], [5], [10] [13], [14] and references therein.

We assume that the initial condition $u(0,x)$ is non negative and of integral one with $|x|u(0,x)$ integrable, and denote by $u(x,t)$ the solution at time $t$. Note that the integral of $u$ with respect to $x$ is invariant by time evolution, the integral of $xu(x,t)$ is finite.

One of the striking phenomenon is that even if the drift has zero average, this system may show a non zero average speed. There are many results on homogenization theory which can be applied to equation (1), see for example [3], [11], and references therein.

These results say that the large time asymptotic is given by the solution of a homogenized problem. It remains however to understand if this homogenized problem leads to a non zero asymptotic averaged velocity. For this purpose we will consider the quantity

$$I(b) = \lim_{t\to\infty} \frac{1}{t} \int xu(t,x)dx \tag{2}$$

which describes the asymptotic average displacement of the particle per unit time (provided the limit exists).

Our first theorem states that the average asymptotic velocity is typically non zero.

**Theorem 1.1.** *The quantity $I(b)$ is independent of the initial condition, and the set of $b \in C^1$ with space average and time average equal to zero where $I(b) \neq 0$ is open and dense.*

*Remark* 1.1. By assuming that the space average (which may depend on time) and the time average (which may depend on space) are both zero we restrict the problem to a smaller set of possible drifts. One can prove a similar result with weaker constraints. Note also that it is well known that if $b$ does not depend on time, then $I(b) = 0$ (see for example [13]).

*Remark* 1.2. The theorem can be extended in various directions, for example by using different topologies on $b$, by including a non-constant periodic diffusion coefficient, or by considering almost periodic time dependence.

Another common model for molecular motor is the two states model which describes the time evolution of two non negative function $\rho_1$ and $\rho_2$. In this model, the “molecule” can be in two states: 1 or 2 which have different interaction with the landscape described by the drift. We denote by $\rho_1(t,x)$

the probability to find the molecule at site $x$ ate time $t$ in state 1, and similarly for $\rho_2$. We refer to [13] for more details. The evolution equations are given by

$$\begin{aligned} \partial_t \rho_1 = &\ \partial_x \big( D \partial_x \rho_1 + b_1(x) \rho_1 \big) - \nu_1 \rho_1 + \nu_2 \rho_2 \\ \partial_t \rho_2 = &\ \partial_x \big( D \partial_x \rho_2 + b_2(x) \rho_2 \big) + \nu_1 \rho_1 - \nu_2 \rho_2 \end{aligned} \tag{3}$$

where $D$, $\nu_1$ and $\nu_2$ are positive constants, $b_1$ and $b_2$ are $C^1$ periodic functions of $x$ of period $L$ the last two with average zero.

The asymptotic average displacement per unit time of the particle is now defined by

$$I\big(\nu_1, \nu_2, b_1, b_2\big) = \lim_{t \to \infty} \frac{1}{t} \int x \left( \rho_1(t, x) + \rho_2(t, x) \right) dx \ .$$

We have the equivalent of Theorem 1.1. As before, we assume that $|x|(\rho_1 + \rho_2)$ is integrable.

**Theorem 1.2.** *For any constants $\nu_1 > 0$ and $\nu_2 > 0$, $I(\nu_1, \nu_2, b_1, b_2)$ is independent of the initial condition, and the set of $b_1$ and $b_2 \in C^1$ with space average equal to zero where $I(\nu_1, \nu_2, b_1, b_2) \neq 0$ is open and dense.*

Another model of a particle interacting with a thermal bath is given by the Langevin equation

$$\begin{aligned} dx &= v \, dt \\ dv &= (-\gamma \, v + F(t, x)/m) \, dt + \sigma \, dW_t \end{aligned} \tag{4}$$

where $m$ is the mass of the particle, $\gamma > 0$ the friction coefficient, $F(t, x)$ the force, $W_t$ the Brownian motion and $\sigma = \sqrt{2D}$ where $D$ is the diffusion coefficient. We refer to [6] and [10] for more details.

For the time evolution of the probability density $f(t, v, x)$ of the position and velocity of the particle one gets the so called Kramers equation

$$\partial_t f = -v \, \partial_x f + \partial_v \big[ (\gamma v - F(t, x)/m) f \big] + \frac{D}{2} \partial_v^2 f \ . \tag{5}$$

We refer to [6] and references therein for more details on these equations. By changing scales, we can assume that $m = 1$ and $D = 1$ and we will only consider this situation below. Moreover we will assume as before that $F(t, x)$ is periodic of period $T$ in time, $L$ in space and with zero average in space and time. We can now define the average asymptotic displacement per unit time by

$$I(\gamma, F) = \lim_{t \to \infty} \frac{1}{t} \int_0^t d\tau \iint v \, f(\tau, v, x) \, dv \, dx \ . \tag{6}$$

As for the previous models the average asymptotic velocity is typically non zero. As usual, we denote by $H^1( dv\, dx)$ the Sobolev space of square integrable functions of $x$ and $v$ with square integrable gradient.

**Theorem 1.3.** *For $\gamma > 0$, $I(\gamma, F)$ is independent of the initial condition, the set of $F \in C^1$ with space average and time average equal to zero where $I(\gamma, F) \neq 0$ is open and dense.*

One can also consider a situation where the particle can be in two states which interact differently with the landscape. This leads to the following system of Kramers equation.

$$\begin{aligned} \partial_t f_1 = &\ \tfrac{1}{2}\partial_v^2 f_1 - v\partial_x f_1 + \partial_v\Big[\big(\gamma v - F_1(x)\big)f_1\Big] - \nu_1 f_1 + \nu_2 f_2 \\ \partial_t f_2 = &\ \tfrac{1}{2}\partial_v^2 f_2 - v\partial_x f_2 + \partial_v\Big[\big(\gamma v - F_2(x)\big)f_2\Big] + \nu_1 f_1 - \nu_2 f_2 \ . \end{aligned} \tag{7}$$

In this equation, $F_1$ and $F_2$ are two periodic functions representing the different interaction forces between the two states of the particle and the substrate. The positive constants $\nu_1$ and $\nu_2$ are the transition rates between the two states. The non negative functions $f_1$ and $f_2$ are the probability densities of being in state one and two respectively. The total probability density of the particle is the function $f_1 + f_2$ which is normalised to one. The asymptotic displacement per unit time for this model is given by

$$I(\gamma, F_1, F_2, \nu_1, \nu_2) = \lim_{t\to\infty} \frac{1}{t}\int_0^t ds \iint v\,\big(f_1(s, v, x) + f_2(s, v, x)\big)\, dv\, dx \ , \tag{8}$$

and we will prove the following result

**Theorem 1.4.** *For $\gamma > 0$, $\nu_1 > 0$ and $\nu_2 > 0$, $I(\gamma, F_1, F_2, \nu_1, \nu_2)$ is independent of the initial condition, and the set of $F_1$ and $F_2 \in C^1$ with space average equal to zero where $I(\gamma, F_1, F_2, \nu_1, \nu_2) \neq 0$ is open and dense.*

# 2 Elimination of the spatial average.

Before we start with the proof of the Theorems, we first show that the result does not depend on the spatial average of the drift $b$.

**Proposition 2.1.** *Assume $b$ has space time average zero, namely*

$$\frac{1}{TL}\int_0^T \int_0^L b(t, x)\, dt\, dx = 0\ .$$

*Then the drift* $\tilde{b}$ *given by*

$$\tilde{b}(t,x) = b(t, x + a(t)) - \frac{1}{L}\int_0^L b(t,y)dy$$

*where*

$$a(t) = -\frac{1}{L}\int_0^t ds \int_0^L b(s,y)dy \ .$$

*is periodic of period* $T$ *in time and of period* $L$ *in* $x$*. This drift has zero space average and leads to the same asymptotic displacement per unit time.*

*Proof.* Note first that since the space time average of $b$ is zero, the function $a$ is periodic of period $T$. Let $u$ be a solution of (1), and define the function

$$v(t,x) = u(t, x + a(t)) \ .$$

An easy computation leads to

$$\partial_t v = \partial_x \left(\frac{1}{2}\partial_x v + \tilde{b}(t,x)v\right) \ .$$

Since $a(t)$ is periodic and bounded we have by a simple change of variable

$$\lim_{t\to\infty}\frac{1}{t}\int x\, u(t,x)dx = \lim_{t\to\infty}\frac{1}{t}\int x\, v(t,x)dx \ .$$

□

## 3 Proof of Theorems 1.1

We start by giving a more convenient expression for the asymptotic velocity $I(b)$. Using (2) and (1), and integrating by parts we get

$$I(b) = \lim_{t\to\infty}\frac{1}{t}\int_0^t ds \int x\,\partial_s u(s,x)dx = -\lim_{t\to\infty}\frac{1}{t}\int_0^t ds\int b(s,x)\, u(s,x)dx \ .$$

Since $b$ is periodic in $x$, of period $L$, we can write

$$\int b(s,x)\, u(s,x)dx = \int_0^L b(s,x)\, u_{\mathrm{per}}(s,x)dx$$

where

$$u_{\mathrm{per}}(s,x) = \sum_n u(s, x + nL)$$

is a periodic function of $x$ of period $L$. Note that since $b$ is periodic of period $L$, $u_{\mathrm{per}}$ satisfies also equation (1). We now have

$$I(b) = -\lim_{t\to\infty} \frac{1}{t}\int_0^t ds \int_0^L b(s,x)\, u_{\mathrm{per}}(s,x)dx\ . \tag{9}$$

Since the system is non autonomous, although periodic in time, we can only expect that when $t$ tends to infinity, the function $u_{\mathrm{per}}(t,x)$ tends to a periodic function $w_b(t,x)$ of $t$ and $x$. Let $w_b$ be the solution of equation (1) periodic in space and time and with an integral (over $[0,L]$) equal to one. It can be expected (see [13], [3], [11]) that the asymptotic average displacement is given by

$$I(b) = -\frac{1}{T}\int_0^T \int_0^L b(t,x)w_b(t,x)dtdx\ . \tag{10}$$

In order to give a rigorous proof of existence of the function $w_b(t,x)$ and of the above relation, we introduce a new time. We consider the operator $\mathcal{L}$ given by

$$\mathcal{L}\, w = -\partial_s w + \partial_x\left(\frac{1}{2}\partial_x w + b(s,x)w\right)\ ,$$

acting in a suitable domain dense in the space $L^1_{\mathrm{per}}(ds\, dx)$ of integrable functions which are periodic in $s$ and $x$ of periods $T$ and $L$ respectively . This operator is the generator of the diffusion on the two dimensional torus ($[0,T]\times[0,L]$ with the suitable identifications) associated to the stochastic differential equation

$$\begin{cases} ds &= dt \\ dx &= -b(s,x)\, dt + dW_t \end{cases} \tag{11}$$

where $W_t$ is the standard Brownian motion (see [8]). We can now establish the following result.

**Proposition 3.1.** *The diffusion (11) has a unique invariant probability measure with density $w_b(s,x)$. This function is strictly positive. It is periodic of period $T$ in $s$ and of period $L$ in $x$ and satisfies equation (1), and it is the only such solution. The semi group with generator $\mathcal{L}$ associated to the diffusion (11) is compact and strongly continuous. The peripheral spectrum of its generator is composed of the simple eigenvalue zero (with eigenvector $w_b$). In particular, for any function $v\in L^1_{\mathrm{per}}(ds\, dx)$, we have in the topology of $L^1_{\mathrm{per}}(ds\, dx)$*

$$\lim_{\tau\to\infty} e^{\tau\mathcal{L}}v = w_b\int_0^T\int_0^L v(s,x)\, ds\, dx\ .$$

This kind of results is well known, we refer to [15] for an exposition and further references. We can now establish the relation between (9), and (10).

**Proposition 3.2.** *Let $v_0 \geq 0$ be a periodic function of period $L$ in $x$ of integral one. Denote by $v(t,x)$ the solution of (1) which is periodic of period $L$ in $x$ with initial condition $v_0$. Then*

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t ds\int_0^L b(s,x)\,v(s,x)dx = -\frac{1}{T}\int_0^T\int_0^L b(t,x)w_b(t,x)dtdx\ .$$

*Proof.* In order to apply Proposition (3.1), we consider the operator $\mathcal{L}_0$ given by

$$\mathcal{L}_0 = \partial_x\left(\frac{1}{2}\partial_x u + b(s,x)u\right)\ ,$$

and observe that if $w \in L^1_{\text{per}}(ds\,dx)$, we have

$$\left(e^{\tau\mathcal{L}}w\right)(s,x) = \left(e^{\tau\mathcal{L}_0}w(s-\tau,\ \cdot\ )\right)(x)\ , \tag{12}$$

and in particular for any integer $n$, we get

$$\left(e^{nT\mathcal{L}}w\right)(s,x) = \left(e^{nT\mathcal{L}_0}w(s,\ \cdot\ )\right)(x)\ ,$$

since $w$ is of period $T$ in $s$.

We now take for $w$ the function $w(s,x) = v(s,x)$ for $0 \leq s < T$. Although this $w$ may have a jump at $s = T$, we can consider it as a function in $L^1_{\text{per}}(ds\,dx)$. We observe that if $W(\tau,s,x)$ is a solution of

$$\partial_\tau W = -\partial_s W + \partial_x\left(\frac{1}{2}\partial_x W + b(s,x)W\right)\ ,$$

then for each fixed $s_0$, the function $h_{s_0}(\tau,x) = W(\tau, s_0+\tau, x)$ is a solution of

$$\partial_\tau h_{s_0} = \partial_x\left(\frac{1}{2}\partial_x h_{s_0} + b(s_0+\tau,x)h_{s_0}\right)\ .$$

Therefore, by the uniqueness of the solution of (1), we have for any $t \geq 0$ (taking $s_0 = t - T[t/T]$)

$$v(t,x) = \left(e^{[t/L]T\mathcal{L}}w\right)(t-[t/L]T, x)\ .$$

The proposition follows by applying Proposition 3.1.

□

The following proposition is the other main step in the proof of Theorem 1.1.

**Proposition 3.3.** *The function $b \mapsto I(b)$ is (real) analytic in the Banach space $C^1$.*

By this we mean (see [12]) that the function is $C^\infty$, and around any point $b \in C^1$ there is a small ball where the Taylor series converges to the function.

*Proof.* We will establish that the map $b \mapsto w_b$ is real analytic in $L^1_{\text{per}}(ds\,dx)$. For this purpose, we first establish that the operator $A$ defined by

$$Av = \partial_x(bv) = b\,\partial_x v + \partial_x b\, v$$

is relatively bounded with respect to

$$\tilde{\mathcal{L}} = -\partial_s + \frac{1}{2}\partial_x^2 \ ,$$

and with relative bound zero (see [9] for the definition). This is obvious for the operator of multiplication by $\partial_x b$ which is bounded, and since $b$ is bounded it is enough to derive the result for the operator $\partial_x$. We will show that there is a constant $C > 0$ such that for any $\lambda > 0$, $\left\|\partial_x R_\lambda\right\|_{L^1_{\text{per}}(ds\,dx)} < C\lambda^{-1/2}$, where $R_\lambda$ is the resolvent of $\tilde{\mathcal{L}}$. In other words, we will show that for any $\lambda > 0$

$$\left\| \partial_x \int_0^\infty e^{-\lambda\tau} e^{\tau\tilde{\mathcal{L}}} d\tau \right\|_{L^1_{\text{per}}(ds\,dx)} < \frac{C}{\sqrt{\lambda}} \ . \tag{13}$$

Analogously to formula (12) we have for any $w \in L^1_{\text{per}}(ds\,dx)$

$$\partial_x \left( e^{\tau\mathcal{L}'} w \right)(s,x) = \int_0^L \partial_x g_\tau(x,y)\, w(s-\tau,y)\, dy \ ,$$

where $g_\tau(x,y)$ is the heat kernel on the circle of length $L$. We now observe that if $n$ is an integer with $|n| \ge 2$, we have (since $x \in [0,L]$)

$$\sup_{y\in[0,L]} \int_0^L \frac{\left|x-y-nL\right|}{\tau^{3/2}}\, e^{-(x-y-nL)^2/(2\tau)} dx \le \mathcal{O}(1) \frac{e^{-n^2/(4\tau)}}{\tau} \ .$$

From the explicit expression

$$g_\tau(x,y) = \sum_n \frac{1}{\sqrt{2\,\pi\,\tau}}\, e^{-(x-y-nL)^2/(2\tau)}$$

it follows easily that

$$\sup_{y\in[0,L]} \int_0^L \left|\partial_x g_\tau(x,y)\right| dx \le \sum_{|n|\le 1} \sup_{y\in[0,L]} \int_0^L \frac{\left|x-y-nL\right|}{\tau^{3/2}\sqrt{2\pi}}\, e^{-(x-y-nL)^2/(2\tau)} dx$$

$$+\mathscr{O}(1)\sum_{|n|\geq 2}\frac{e^{-n^2/(4\tau)}}{\tau}\leq\frac{\mathscr{O}(1)}{\sqrt{\tau}}\ .$$

Therefore, we get

$$\left\|\partial_x\left(e^{\tau\mathcal{L}'}w\right)\right\|_{L^1_{\text{per}}(ds\,dx)}\leq\frac{\mathscr{O}(1)\|w\|_{L^1_{\text{per}}(ds\,dx)}}{\sqrt{\tau}}\ .$$

Multiplying by $e^{-\lambda\tau}$ and integrating over $\tau$ we get the estimate (13) which implies immediately that $A$ is relatively bounded with respect to $\mathcal{L}'$ with relative bound zero. Since the eigenvalue 0 of $\mathcal{L}$ is simple and isolated, the proposition follows from analytic perturbation theory for holomorphic families of type (A) (see [9]). □

In order to prove Theorem 1.1, we now establish that $I$ is non trivial near the origin.

**Lemma 3.4.** *$DI_0=0$ and*

$$D^2I_0(B,B)=\sum_{p,q}\frac{p\,qL^3T^2}{p^2L^4+\pi^2q^4T^2}\left|B_{p,q}\right|^2\ .$$

*Proof.* The first statement is immediate from formula (10). For $b=0$, we have obviously $w_0=1/L$. For $b$ small we use Taylor expansion. We have for $b=\epsilon B$

$$w_b=w^0+\epsilon w^1+\mathscr{O}(\epsilon^2)\ .$$

As we just explained, $w^0=1/L$ and we have for $w^1$ the equation

$$\partial_t w^1=\frac{1}{2}\partial_x^2 w^1+\frac{1}{L}\partial_x B\ .$$

Moreover, $w^1$ must have space time average zero. This equation can be solved using Fourier series in time and space. Namely if

$$w^1(t,x)=\sum_{p,q}e^{2\pi ipt/T}e^{2\pi iqx/L}w^1_{p,q}\ ,$$

we obtain the equation

$$\frac{2\pi ip}{T}w^1_{p,q}=-\frac{2\pi^2q^2}{L^2}w^1_{p,q}+\frac{2\pi iq}{L^2}B_{p,q}\ ,$$

or in other words for any $(p,q)\neq(0,0)$

$$w^1_{p,q}=\frac{\pi iq/L^2}{\pi ip/T+\pi^2q^2/L^2}B_{p,q}\ .$$

Note in particular that the denominator does not vanish except for $p = q = 0$.

Using the Plancherel formula we can now estimate $I(\epsilon B)$. We have

$$I(\epsilon B) = \frac{\epsilon^2}{T} \int B(t,x) w^1(t,x) dt dx + \mathscr{O}(1)\epsilon^3 \ .$$

Therefore

$$I(\epsilon B) = \epsilon^2 \sum_{p,q} \frac{q/L}{p/T - i\pi q^2/L^2} B_{p,q} \overline{B_{p,q}} + \mathbf{O}(\epsilon^3) \ .$$

Since $B$ is real, we have $\overline{B_{p,q}} = B_{-p,-q}$ and this can also be written

$$I(\epsilon B) = \epsilon^2 \sum_{p,q} \frac{p\, q L^3 T^2}{p^2 L^4 + \pi^2 q^4 T^2} \left| B_{p,q} \right|^2 + \mathbf{O}(\epsilon^3) \ .$$

This finishes the proof of the Lemma. □

To prove Theorem 1.1, we observe that since $I$ is continuous, the subset of $C^1$ where it does not vanish is open. If this set is not dense, the zero set of $I$ contains a ball. However since $I$ is real analytic and $C^1$ is pathwise connected we conclude that in that case $I$ should vanish identically contradicting Lemma 3.4. We refer to [12] for more properties of the zero set of analytic functions in Banach spaces.

*Remark* 3.1. We observe that $D^2 I$ is a non definite quadratic form. This leaves the possibility of having non zero drifts $b$ (with space and time average equal to zero) satisfying $I(b) = 0$. Let $b_1$ and $b_2$ be such that $I(b_1) > 0$ and $I(b_2) < 0$. Such $b_1$ and $b_2$ exist, one can for example take them of small enough norm and use Lemma 3.4. Moreover one can assume that $b_2 \notin \mathbf{R} b_1$. Otherwise, by the continuity of $I$, one can perturb slightly $b_2$ such that this relation does not hold anymore, but $I(b_2)$ is still negative. One now considers the function $\varphi(\alpha) = I\big((1-\alpha)b_1 + \alpha b_2\big)$. This function is continuous, it satisfies $\varphi(1) > 0$ and $\varphi(0) < 0$, hence it should vanish at least at one point $\alpha_0 \in ]0,1[$. At this point we have $b_0 = (1-\alpha_0)b_1 + \alpha_0 b_2 \neq 0$ and $I(b_0) = 0$, and $b_0$ is a non trivial periodic function with vanishing space average and time average.

# 4 Proof of Theorem 1.2.

In this section we discuss the model with two components (3). As before, we arrive at the formula

$$I\big(\nu_1, \nu_2, b_1, b_2\big) = - \lim_{t \to \infty} \frac{1}{t} \int_0^t \int_0^L \big(b_1(s,x)\, \rho_1(s,x) + b_2(s,x)\, \rho_2(s,x)\big)\, ds\, dx \ ,$$

where $\rho_1$ and $\rho_2$ are solutions of (3) periodic in $x$ of period $L$. We denote by $L^1_{\rm per}(dx)$ the space of integrable periodic functions of period $L$ in $x$ with value in $\mathbf{R}^2$. The norm is the sum of the $L^1$ norms of the components.

**Proposition 4.1.** *The semi-group defined by (3) is compact in $L^1_{\rm per}(dx)$. It is positivity preserving, and its peripheral spectrum is the simple eigenvalue one. The corresponding eigenvector can be chosen positive with dense support and normalised, it depends analytically on $b_1$, and $b_2$.*

*Proof.* We introduce the three generators

$$\mathcal{L}\begin{pmatrix}\rho_1\\ \rho_2\end{pmatrix} = \begin{pmatrix}\partial_x\big(D\partial_x\rho_1 + b_1(x)\rho_1\big) - \nu_1\rho_1 + \nu_2\rho_2\\ \partial_x\big(D\partial_x\rho_2 + b_2(x)\rho_2\big) + \nu_1\rho_1 - \nu_2\rho_2\end{pmatrix} ,$$

$$\mathcal{L}_0\begin{pmatrix}\rho_1\\ \rho_2\end{pmatrix} = \begin{pmatrix}D\partial_x^2\rho_1\\ D\partial_x^2\rho_2\end{pmatrix} \quad \text{and} \quad \mathcal{L}_1\begin{pmatrix}\rho_1\\ \rho_2\end{pmatrix} = \begin{pmatrix}\partial_x\big(D\partial_x\rho_1 + b_1(x)\rho_1\big)\\ \partial_x\big(D\partial_x\rho_2 + b_2(x)\rho_2\big)\end{pmatrix} .$$

This operator $\mathcal{L}_0$ is the infinitesimal generator of a strongly continuous bounded and compact semi-group in $L^1_{\rm per}(dx)$. It is easy to verify that the operator $A = \mathcal{L} - \mathcal{L}_0$ is $\mathcal{L}_0$ relatively compact and $\mathcal{L}_0$ relatively bounded with relative bound zero (see [9]). Therefore $\mathcal{L}$ is also the infinitesimal generator a strongly continuous and compact semi-group in $L^1_{\rm per}(dx)$, and similarly for $\mathcal{L}_1$.

The semi-group $e^{t\mathcal{L}}$ is positivity improving (see [2]). Indeed, let $M$ be the matrix

$$M = \begin{pmatrix}-\nu_1 & \nu_2\\ \nu_1 & -\nu_2\end{pmatrix} .$$

It is easy to verify for example by direct computation, that the matrix $e^{t\,M}$ has strictly positive entries for any $t > 0$. Moreover, for any $t \geq 0$, we have $e^{t\,M} \geq e^{-t(\nu_1+\nu_2)}\mathrm{Id}$, where the inequality holds for each entry. It immediately follows from the Trotter product formula (see [2] ) that for each $x$ and $y$ in the circle, and any $t > 0$, we have

$$e^{t\mathcal{L}}(x, y) \geq e^{-t(\nu_1+\nu_2)} e^{t\mathcal{L}_1}(x, y) ,$$

again in the sense that the inequality holds between all the entries. Since for each $x$, $y$ and $t > 0$ the diagonal elements of $e^{t\mathcal{L}_1}(x, y)$ are strictly positive (see for example [15]), we conclude that the matrix valued kernel $e^{t\mathcal{L}}(x, y)$ has non negative entries and strictly positive entries on the diagonal.

Since the sum of the integrals of the two components of an element of $L^1_{\rm per}(dx)$ is preserved by the semi-group $e^{t\mathcal{L}}$, it follows that this semi group has norm one in $L^1_{\rm per}(dx)$. It then follows by classical arguments that 0 is a simple isolated eigenvalue of the generator and there is no other eigenvalue with vanishing real part.

The analyticity follows from the uniqueness and simplicity of the eigenvalue 0 as in the proof of proposition (3.3).

□

We denote by $w_1$ and $w_2$ the two (non negative) components of the stationary solution of the system (3) which are periodic of period $L$ and normalised by

$$\int_0^L \big[w_1(x)+w_2(x)\big]\,dx = 1\ .$$

Note that $w_1$ and $w_2$ depend on the constants $\nu_1$, $\nu_2$, and the functions $b_1$, and $b_2$.

It follows immediately from Proposition 4.1 that the average asymptotic velocity is given by

$$I(\nu_1,\nu_2,b_1,b_2) = \int_0^L \big[b_1(x)w_1(x)+b_2(x)w_2(x)\big]\,dx\ . \tag{14}$$

Since the function $I(\nu_1,\nu_2,b_1,b_2)$ is analytic in $\big(b_1,b_2\big)$, to prove that it is non trivial, we look at the successive differentials at the origin.

**Proposition 4.2.** *$DI_0=0$ and for any $(b_1,b_2)$, $D^2I_0\big((b_1,b_2),(b_1,b_2)\big)=0$.*

*Proof.* The first result is trivial. For the second result one uses perturbation theory as before in the Fourier decomposition. We get with $\sigma = 4\pi^2 D/L^2$

$$\begin{pmatrix} -\sigma n^2-\nu_1 & \nu_2 \\ \nu_1 & -\sigma n^2-\nu_2 \end{pmatrix}\begin{pmatrix} w_1^1(n) \\ w_1^2(n)\end{pmatrix} = -\frac{2\pi\, i\, n}{L(\nu_1+\nu_2)}\begin{pmatrix}\nu_2 b_1(n) \\ \nu_1 b_2(n)\end{pmatrix}$$

Some easy computations using Plancherel identity lead to

$$D^2I_0\big((b_1,b_2),(b_1,b_2)\big) = -\sum_n \frac{2\pi\, i\, n}{L(\nu_1+\nu_2)\big((\sigma n^2+\nu_1)(\sigma n^2+\nu_2)-\nu_1\nu_2\big)}$$
$$\times\Big[(\sigma n^2+\nu_1)\nu_2\bar b_1(n)b_1(n)+(\sigma n^2+\nu_2)\nu_1\bar b_2(n)b_2(n)$$
$$+\nu_1\nu_2\big(\bar b_2(n)b_1(n)+\bar b_1(n)b_2(n)\big)\Big] = 0$$

since $b_1$ and $b_2$ are real ($\bar b_1(n)=b_1(-n)$). □

This result suggests to look at the third differential at the origin which turns out to be a rather involved cubic expression. In order to show that the function $I$ is non trivial, it is enough to find a particular pair $\big(b_1, b_2\big)$ such that $D^3 I_0\big((b_1, b_2), (b_1, b_2), (b_1, b_2)\big) \neq 0$. This was done using a symbolic manipulation program (Maxima). We found that for $L = 2\pi$, $D = 1$, $b_1(x) = \cos(2x)$ and $b_2(x) = \cos(x)$, one gets

$$D^3 I_0\big((b_1, b_2), (b_1, b_2), (b_1, b_2)\big) = -\frac{\nu_1 \nu_2 (\nu_2 - 2\nu_1 + 1)}{4(\nu_2 + \nu_1)(\nu_2 + \nu_1 + 1)^2 (\nu_2 + \nu_1 + 4)} \ .$$

Theorem 1.2 follows as before.

## 4.1 Proof of Theorem 1.3

As in the previous section, we can introduce the periodised function (in $x$)

$$\breve{f}(t, v, x) = \sum_n f(t, v, x + nL) \ .$$

This function is periodic of period $L$ and satisfies also equation (5). We get also $I(\gamma, F)$ by replacing $f$ by $\breve{f}(t, v, x)$ in equation (6) and integrating only on one period. From now on we will work with this periodised function and denote it by $f$ by abuse of notation.

We now introduce a stochastic differential equation on $[0, T] \times [0, L] \times \mathrm{R}$ with periodic boundary conditions in the first two variables $s$ and $x$. This differential equation is given by

$$\begin{cases} ds = dt \\ dx = v\, dt \\ dv = -\gamma v dt + F(s, x) dt + dW_t \ . \end{cases} \tag{15}$$

To this diffusion is associated the infinitesimal generator $\mathcal{L}$ given by

$$\mathcal{L} w = -\partial_s w - v\, \partial_x w + \partial_v \big[(\gamma v - F(t, x)) w\big] + \frac{1}{2} \partial_v^2 w \ .$$

We denote by $\mathscr{B}$ the space $L^2(e^{\gamma v^2} ds\, dv\, dx)$ of functions periodic in $s$ of period $T$ and periodic in $x$ of period $L$. Using an $L^2$ space instead of an $L^1$ space is useful in proving analyticity.

We can now establish the following result.

**Proposition 4.3.** *The diffusion semi-group defined by (15) in $\mathscr{B}$ is compact and the kernel has dense support. It is mixing and has a unique invariant probability measure (in $\mathscr{B}$) with density $\tilde{f}(s, v, x)$. This function is strictly* *positive, satisfies equation (5), and it is the only such solution. In particular, for any function $w \in \mathscr{B}$, we have in the topology of $\mathscr{B}$*

$$\lim_{\tau\to\infty} e^{\tau\mathcal{L}} w = \tilde{f} \int_0^T \int_0^L \int w(s,v,x)\, ds\, dv\, dx\ .$$

*The function $\tilde{f}$ is real analytic in $F \in C^1$.*

*Proof.* Instead of working in the space $\mathscr{B}$ , we can work in the space $L^2(ds\ dv\, dx)$ by using the isomorphism given by the multiplication by function $e^{\gamma v^2/2}$. In that space we obtain the new generator $\mathcal{L}'_F$ given by

$$\mathcal{L}'_F g = -\partial_s g - v\,\partial_x g - F\partial_v g + F\gamma v g - \frac{\gamma^2 v^2}{2} g + \frac{\gamma}{2} g + \frac{1}{2}\partial_v^2 g\ .$$

Using integration by parts, it is easy to verify that

$$\Re \int \bar{g}\, \big(\mathcal{L}'_F\, g\big) ds\ dv\, dx =$$

$$\int \left(\frac{\gamma}{2}|g|^2 + \gamma F\, v\, |g|^2 - \frac{\gamma^2}{2} v^2 |g|^2 - \frac{1}{2}\int \big|\partial_v g\big|^2\right) ds\, dv\, dx$$

$$\leq \left(\frac{\gamma}{2} + \big\|F\big\|_\infty^2\right) \int |g|^2\, ds\, dv\, dx - \int \left(\frac{\gamma^2}{4} v^2 |g|^2 + \frac{1}{2}\big|\partial_v g\big|^2\right) ds\, dv\, dx\ . \quad (16)$$

We see immediately that $-\mathcal{L}'_F$ is quasi accretive (see [9] for the definition and properties of the semi-groups generated by these operators).

Let $g_t = e^{t\mathcal{L}'_F} g_0$, using several integrations by parts one gets easily

$$\partial_t \int \big|g_t\big|^2\, ds\, dv\, dx = 2\gamma \int F\, v\, \big|g_t\big|^2\, ds\, dv\, dx - \gamma^2 \int v^2\, \big|g_t\big|^2\, ds\, dv\, dx$$

$$+\gamma \int \big|g_t\big|^2\, ds\, dv\, dx - \int \big|\partial_v g_t\big|^2\, ds\, dv\, dx$$

$$\leq -\frac{\gamma^2}{2} \int v^2\, \big|g_t\big|^2\, ds\, dv\, dx + \big(\gamma + 4\gamma^2 \big\|F\big\|_\infty^2\big) \int \big|g_t\big|^2\, ds\, dv\, dx\ .$$

We obtain immediately

$$\big\|g_t\big\| \leq e^{\left(\gamma/2 + 2\gamma^2 \|F\|_\infty^2\right) t} \big\|g_0\big\| \quad (17)$$

and

$$\int_0^t d\tau \int v^2\, \big|g_\tau\big|^2\, ds\, dv\, dx \leq \frac{2}{\gamma^2}\, e^{\left(\gamma + 4\gamma^2 \|F\|_\infty^2\right) t}\, \big\|g_0\big\|^2\ . \quad (18)$$

Similarly, we get

$$\partial_t \int v^2 \left|g_t\right|^2 ds\, dv\, dx$$

$$= 4\int F\, v \left|g_t\right|^2 ds\, dv\, dx + 2\gamma \int F\, v^3 \left|g_t\right|^2 ds\, dv\, dx - \gamma^2 \int v^4 \left|g_t\right|^2 ds\, dv\, dx$$

$$+\gamma \int v^2 \left|g_t\right|^2 ds\, dv\, dx + 2\int \left|g_t\right|^2 ds\, dv\, dx - \int v^2 \left|\partial_v g_t\right|^2 ds\, dv\, dx$$

$$\leq C(\gamma, \|F\|_\infty) \int \left|g_t\right|^2 ds\, dv\, dx \tag{19}$$

where $C(\gamma, \|F\|_\infty)$ is a constant independent of $g_t$. For $t > 0$ fixed, we deduce from (18) that there exists $\xi(t) \in [0, t[$ such that

$$\int v^2 \left|g_{\xi(t)}\right|^2 ds\, dv\, dx \leq \frac{2}{\gamma^2\, t}\, e^{\left(\gamma + 4\gamma^2 \|F\|_\infty^2\right) t} \left\|g_0\right\|^2 .$$

Using (4.1) and (17) we get (for any $t > 0$)

$$\int v^2 \left|g_t\right|^2 ds\, dv\, dx = \int v^2 \left|g_{\xi(t)}\right|^2 ds\, dv\, dx + \int_{\xi(t)}^t d\tau\, \partial_\tau \int v^2 \left|g_\tau\right|^2 ds\, dv\, dx$$

$$\leq \left(\frac{2}{\gamma^2\, t} + C\, t\right) e^{\left(\gamma + 4\gamma^2 \|F\|_\infty^2\right) t} \left\|g_0\right\|^2 .$$

In other words, for any $t > 0$ the image of the unit ball by the semi-group is equi-integrable at infinity in $v$. Compactness follows immediately using hypoelliptic estimates (see [15]).

From the standard control arguments (see [15]) applied to the diffusion (15), we obtain that the kernel of the semi-group has dense support. We now observe that integration of a function against $e^{-\gamma v^2}$ is preserved by the semi-group evolution. This implies by standard arguments that the spectral radius of the semi-group is one, the invariant density is unique, and exponential mixing holds (see [15]). Finally, as in equation (4.1), for $g$ in the domain of $\mathcal{L}'$ we have

$$\Re \int \bar{g}\, \left(\mathcal{L}'_F\, g\right) ds\ dv\, dx$$

$$\geq \left(\frac{\gamma}{2} - 2\left\|F\right\|_\infty^2\right) \int |g|^2\, ds\, dv\, dx - \int \left(\frac{\gamma^2}{2} v^2 |g|^2 + \frac{1}{2}\left|\partial_v g\right|^2\right) ds\, dv\, dx\ .$$

This implies for any $\lambda > 0$

$$\int \left|\partial_v g - \gamma v g\right|^2 ds\, dv\, dx \leq \frac{1}{\lambda}\left\|\mathcal{L}' g\right\|_2^2 + \left(4\,\lambda + 3\,\gamma + 4\left\|F\right\|_\infty^2\right)\|g\|_2\ .$$

In other words, the operator $F\left(\partial_v - \gamma v\right)$ is relatively bounded with respect to $\mathcal{L}_0'$ with relative bound zero and this implies the analyticity (see [9]). $\square$

We can now complete the proof of Theorem 1.3.

*Proof.* (of Theorem 1.3).

Repeating the argument in the proof of Proposition 3.2, using Proposition 4.3 we get

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t\int\int v\, f(\tau,v,x)d\tau\, dv\, dx = \frac{1}{\gamma T}\int_0^T\int_0^L\int F(\tau,x)\,\tilde{f}(\tau,v,x)d\tau\, dv\, dx\ ,$$

We can now use perturbation theory to compute the right hand side near $F=0$. For this purpose, it is convenient to fix a $C^1$ function $G$ periodic in space and time and with zero average, and consider $F=\epsilon\, G$ with $\epsilon$ small. Since $\tilde{f}$ is analytic by Proposition 4.3, we can write

$$\tilde{f} = \tilde{f}_0 + \epsilon\tilde{f}_1 + \epsilon^2\tilde{f}_2 + \mathscr{O}(1)\epsilon^3$$

where

$$\tilde{f}_0 = \frac{1}{L}\sqrt{\frac{\gamma}{\pi}}\; e^{-\gamma v^2}$$

and for $n\ge 1$ the $\tilde{f}_n$ are functions of integral zero, periodic in time of period $T$, defined recursively by

$$\partial_t\tilde{f}_n + v\partial_x\tilde{f}_n - \partial_v\big(\gamma v\tilde{f}_n\big) - \frac{1}{2}\partial_v^2\tilde{f}_n = -G(t,x)\partial_v\tilde{f}_{n-1}\ .$$

We get immediately

$$\int_0^T\int_0^L\int G(\tau,x)\,\tilde{f}_0(\tau,v,x)d\tau\, dv\, dx = 0\ ,$$

since $\tilde{f}_0$ is independent of $x$ and $t$ and $G$ has average zero. Therefore we now have to look at the next order perturbation, namely the second order in $\epsilon$ for the average velocity. In other words, we have

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t\int_0^L\int v\, f(\tau,v,x)d\tau\, dv\, dx =$$

$$\frac{\epsilon^2}{\gamma T}\int_0^T\int_0^L\int G(\tau,x)\,\tilde{f}_1(\tau,v,x)d\tau\, dv\, dx\ + \mathscr{O}(1)\epsilon^3\ . \tag{20}$$

We first have to solve

$$\partial_t\tilde{f}_1 + v\partial_x\tilde{f}_1 - \partial_v\big(\gamma v\tilde{f}_1\big) - \frac{1}{2}\partial_v^2\tilde{f}_1 = -G(t,x)\partial_v\tilde{f}_0$$

to get $\tilde{f}_1$. For this purpose, we use Fourier transform in all variables (recall that $t$ and $x$ are periodic variables). We will denote by $\hat{f}_{1,p,q}(k)$ the Fourier transform of $\tilde{f}_1$ (and similarly for other functions), namely

$$\tilde{f}_1(t,v,x) = \sum_{p,q} e^{2\pi\, i\, p\, t/T} e^{2\pi\, i\, q\, x/L} \int e^{i\, k\, v} \hat{f}_{1,p,q}(k)\, dk \ .$$

We get

$$\left(\frac{2\pi ip}{T} + \frac{k^2}{2}\right) \hat{f}_{1,p,q} + \left(\gamma k - \frac{2\pi q}{L}\right) \frac{d}{dk} \hat{f}_{1,p,q} = -\frac{i\, k\, \hat{G}_{p,q}}{2\pi\, L}\, e^{-k^2/(4\gamma)} \ . \tag{21}$$

We now observe that equation (4.1), only involves the integral of $\tilde{f}_1$ with respect to $v$ since $G$ does not depend on $v$. Therefore we need only to compute $\hat{f}_{1,p,q}(0)$. Let $h_{p,q}(k)$ be the function

$$h_{p,q}(k) = e^{k^2/(4\gamma)} e^{\pi\, q\, k/(\gamma^2 L)} \left|1 - \frac{\gamma k L}{2\pi q}\right|^{2\pi^2 q^2/(\gamma^3 L^2) + 2\pi i p/(\gamma T) - 1} \ .$$

For $-2\pi|q|/(\gamma L) < k < 2\pi|q|/(\gamma L)$, this function is a solution of

$$\left(\frac{2\pi ip}{T} + \frac{k^2}{2}\right) h_{p,q}(k) - \frac{d}{dk}\left[\left(\gamma k - \frac{2\pi q}{L}\right) h_{p,q}(k)\right] = 0 \ .$$

For $q > 0$, multiplying (21) by $h_{p,q}(k)$ and integrating over $k$ from 0 to $2\pi q/(\gamma L)$, we get

$$\hat{f}_{1,p,q}(0) = \hat{G}_{p,q} \Gamma_{p,q}$$

where

$$\Gamma_{p,q} = -\frac{i}{4\pi^2 q} \int_0^{2\pi q/(\gamma L)} e^{\pi\, q\, k/(\gamma^2 L)} \left|1 - \frac{\gamma k L}{2\pi q}\right|^{2\pi^2 q^2/(\gamma^3 L^2) + 2\pi i p/(\gamma T) - 1} k\, dk \ .$$

Note that since $q \neq 0$, the integral is convergent. For $q < 0$, one gets a similar result, namely

$$\Gamma_{p,q} = \frac{i}{4\pi^2 q} \int_{2\pi q/(\gamma L)}^0 e^{\pi\, q\, k/(\gamma^2 L)} \left|1 - \frac{\gamma k L}{2\pi q}\right|^{2\pi^2 q^2/(\gamma^3 L^2) + 2\pi i p/(\gamma T) - 1} k\, dk \ ,$$

and it is easy to verify that $\overline{\Gamma}_{p,q} = \Gamma_{-p,-q}$. We now have

$$\int_0^T \int_0^L \int G(\tau,x)\, \tilde{f}_1(\tau,v,x) d\tau\, dv\, dx = \sum_{p,q} \overline{\hat{G}_{p,q}}\, \Gamma_{p,q}\, \hat{G}_{p,q} \ .$$

Since $G(t,x)$ is real, we have $\overline{\hat{G}_{p,q}} = \hat{G}_{-p,-q}$, and therefore

$$\int_0^T \int_0^L \int G(\tau,x)\, \tilde{f}_1(\tau,v,x) d\tau\, dv\, dx = \sum_{p,q} \left|\hat{G}_{p,q}\right|^2 \frac{\Gamma_{p,q}+\Gamma_{-p,-q}}{2} \ .$$

We observe that for $q>0$,

$$\frac{\Gamma_{p,q}+\Gamma_{-p,-q}}{2} = \frac{1}{4\pi^2 q}\int_0^{2\pi q/(\gamma L)} e^{\pi\, q\, k/(\gamma^2 L)} \left|1-\frac{\gamma k L}{2\pi q}\right|^{2\pi^2 q^2/(\gamma^3 L^2)-1}$$

$$\times \sin\big((2\pi p/(\gamma T))\log(1-\gamma k L/(2\pi q))\big)\, k\, dk \ ,$$

$$= \frac{q}{\gamma^2 L}\int_0^1 e^{2\pi^2 q^2 u/(\gamma^3 L^2)}(1-u)^{2\pi^2 q^2/(\gamma^3 L^2)-1} \sin\big((2\pi p/(\gamma T))\log(1-u)\big)\, u\, du \ .$$

Note that this quantity is equal to zero for $p=0$, and it is odd in $p$. It can be expressed in terms of degenerate hypergeometric functions. We now have to prove that for $p \neq 0$, this quantity is not zero, at least for one pair of integers $(p,q)$. For this purpose, we will consider $q$ large. To alleviate the notation, we consider the asymptotic behavior for $\alpha>0$ large and $\beta\in\mathbf{R}$ fixed of the integral

$$J(\alpha) = \int_0^1 e^{\alpha u}(1-u)^{\alpha-1}(1-u)^{i\beta} u\, du \ .$$

Using steepest descent at the critical point $u=0$ (see [7]), one gets

$$\Im J(\alpha) = -\frac{\beta}{(2\alpha)^{3/2}\sqrt{\pi}} + \frac{\mathscr{O}(1)}{\alpha^2} \ .$$

We apply this result with $\alpha = 2\pi^2 q^2/(\gamma^3 L^2)$ and $\beta = 2\pi p/(\gamma T)$, and conclude that for $q$ large enough, $\big(\Gamma_{p,q}+\Gamma_{-p,-q}\big)/2 \neq 0$ as required. □

The proof of Theorem 1.3 is finished as before.

## 4.2 Proof of Theorem 1.4

The scheme of the proof is similar to the proofs of the previous Theorems. We only sketch the argument except for some particular points. We assume $\gamma>0$, $\nu_1>0$ and $\nu_2>0$. One starts by reducing the problem to a periodic boundary condition in $x$. The key result is the analog of Proposition 4.1.

**Proposition 4.4.** *The semi-group defined by (7) is compact in* $L^2\big(e^{\gamma v^2}dv\,dx\big)$ *(the* $x$ *variable being on the circle of length* $L$*). It is positivity preserving positivity improving on the diagonal. Its peripheral spectrum is the simple eigenvalue one. The corresponding eigenvector* $(\tilde f_1, \tilde f_2)$ *can be chosen positive with dense domain and normalised (*$\tilde f_1 + \tilde f_2$ *of integral one). The functions* $\tilde f_1$ *and* $\tilde f_2$ *depend analytically on* $F_1$*, and* $F_2$*.*

*Proof.* The proof is very similar to the proof of Proposition 4.1 and we only sketch the details for the different points.

∎

It follows as before from the evolution equation that

$$I\big(\gamma, F_1, F_2, \nu_1, \nu_2\big) = \int \big(\tilde f_1 F_1 + \tilde f_2 F_2\big)\,dv\,dx\ .$$

For fixed $\gamma > 0$, $\nu_1 > 0$ and $\nu_2 > 0$, this quantity is real analytic in $(F_1, F_2)$, and to check that it is non trivial we investigate its behaviour near the origin. For this purpose, we set $(F_1, F_2) = \epsilon(G_1, G_2)$ with $G_1$ and $G_2$ two $C^1$ functions, periodic of period $L$ and with zero average. We now develop $I\big(\gamma, \epsilon G_1, \epsilon G_2, \nu_1, \nu_2\big)$ in series of $\epsilon$. As for model (3), the terms of order $\epsilon$ and of order $\epsilon^2$ vanish. One then has to find a pair $(G_1, G_2)$ such that the term of order $\epsilon^3$ does not vanish. The computations are rather tedious and will be detailed in the appendix.

# A Proof of Theorem 1.4.

We define the vectors

$$\vec f = \begin{pmatrix} \tilde f_1 \\ \tilde f_2 \end{pmatrix} \quad \text{and} \quad \vec F = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}\ .$$

We then have

$$I\big(\gamma, F_1, F_2, \nu_1, \nu_2\big) = \iint \vec F \bullet \vec f\ dv\,dx\ .$$

The equation for the stationary solution of (7) can be written

$$\frac{1}{2}\partial_v^2 \vec f - v\partial_x \vec f + \partial_v\big[\gamma v \vec f\big] + M\vec f = B(x)\,\partial_v \vec f \tag{22}$$

where as before

$$M = \begin{pmatrix} -\nu_1 & \nu_2 \\ \nu_1 & -\nu_2 \end{pmatrix}$$

and

$$B(x) = \begin{pmatrix} F_1(x) & 0 \\ 0 & F_2(x) \end{pmatrix} .$$

Without loss of generality, we can rescale $v$ and assume $\gamma = 1$, replacing $\nu_1$ and $\nu_2$ by $\nu_1/\gamma$ and $\nu_2/\gamma$ respectively, $L$ by $L\gamma^{-3/2}$, $F_1$ and $F_2$ by $F_1/\sqrt{\gamma}$ and $F_2/\sqrt{\gamma}$ respectively.

For reasons that will become obvious later on, it is more convenient to consider instead of $\vec{f}$ the vector

$$\vec{\psi} = e^{v^2/2}\vec{f} .$$

Let $\mathscr{L}$ be the operator defined by

$$\mathscr{L}h(v) = \partial_v^2 h(v) - v^2 h(v) + h ,$$

then the equation (22) for the stationary solution can be written

$$\mathscr{L}\vec{\psi} - 2v\partial_x\vec{\psi} + 2M\vec{\psi} = 2B(x) \left(\partial_v\vec{\psi} - v\vec{\psi}\right) , \tag{23}$$

and we have

$$I\big(\gamma, F_1, F_2, \nu_1, \nu_2\big) = \iint \vec{F} \bullet \vec{\psi}\, e^{-v^2/2} \; dv\, dx , \tag{24}$$

with $\vec{\psi}$ normalised by

$$\iint \big(\psi_1 + \psi_2\big)\, e^{-v^2/2} \; dv\, dx = 1 . \tag{25}$$

For $\vec{F} = \epsilon\, \vec{G}$ we have the expansion

$$\vec{\psi} = \vec{\psi}^{\,0} + \epsilon\vec{\psi}^{\,1} + \epsilon^2\vec{\psi}^{\,2} + \mathscr{O}\big(\epsilon^3\big) \tag{26}$$

where

$$\vec{\psi}^{\,0}(v,x) = \frac{1}{L\big(\nu_1 + \nu_2\big)\sqrt{\pi}} \begin{pmatrix} \nu_2 \\ \nu_1 \end{pmatrix} e^{-v^2/2} ,$$

and the vectors $\vec{\psi}^{\,j}$ are obtained recursively by solving

$$\mathscr{L}\vec{\psi}^{\,j+1} - 2v\partial_x\vec{\psi}^{\,j+1} + 2M\vec{\psi}^{\,j+1} = 2C(x) \left(\partial_v\vec{\psi}^{\,j} - v\vec{\psi}^{\,j}\right) ,$$

where

$$C(x) = \begin{pmatrix} G_1(x) & 0 \\ 0 & G_2(x) \end{pmatrix} .$$

Note that $\vec{\psi}^{\,0}$ satisfies

$$\mathscr{L}\vec{\psi}^{\,0} - 2v\partial_x\vec{\psi}^{\,0} + 2M\vec{\psi}^{\,0} = \vec{0}$$

and the normalisation condition (25), while for $j \geq 1$, $\vec{\psi}^{\,j}$ should satisfy the normalisation condition

$$\iint \big(\psi_1^j + \psi_2^j\big)\, e^{-v^2/2}\; dv\, dx = 0\ . \tag{27}$$

Corresponding to the expansion (26), we have the expansion

$$I\big(1, \epsilon G_1, \epsilon G_2, \nu_1, \nu_2\big) = \sum_{n=1}^{3} \epsilon^n I_n\big(G_1, G_2, \nu_1, \nu_2\big)\ + \mathscr{O}\big(\epsilon^4\big)\ ,$$

where

$$I_n\big(G_1, G_2, \nu_1, \nu_2\big) = \iint \vec{G} \bullet \vec{\psi}^{\,n-1}\, e^{-v^2/2}\; dv\, dx\ .$$

We have immediately $I_1\big(G_1, G_2, \nu_1, \nu_2\big) = 0$ since the averages of $G_1$ and $G_2$ over $x$ vanish and $\vec{\psi}^{\,0}$ does not depend on $x$. In order to compute $I_1$, we need to compute $\vec{\psi}^{\,1}$ which solves

$$\mathscr{L}\vec{\psi}^{\,1} - 2v\partial_x\vec{\psi}^{\,1} + 2M\vec{\psi}^{\,1} = v\, e^{-v^2/2}\vec{H}(x)$$

where

$$\vec{H}(x) = -\frac{4}{L\big(\nu_1 + \nu_2\big)\sqrt{\pi}} \begin{pmatrix} \nu_2 G_1(x) \\ \nu_1 G_2(x) \end{pmatrix}\ .$$

Since $\vec{H}$ is periodic of period $L$, we can decompose this equation in Fourier series and get

$$\mathscr{L}\vec{\psi}^{\,1}(p) - \frac{4\pi\, i\, p\, v}{L}\vec{\psi}^{\,1}(p) + 2M\vec{\psi}^{\,1}(p) = v\, e^{-v^2/2}\vec{H}(p)\ .$$

We now observe that

$$\mathscr{L} - \frac{4\pi\, i\, p\, v}{L} + 2M = e^{2\pi\, i\, p\, \partial_v/L}\left(\mathscr{L} - \frac{4\pi^2 p^2}{L^2} + 2M\right) e^{-2\pi\, i\, p\, \partial_v/L}\ .$$

This implies

$$\vec{\psi}^{\,1}(p) = e^{2\pi\, i\, p\, \partial_v/L}\left(\mathscr{L} - \frac{4\pi^2 p^2}{L^2} + 2M\right)^{-1} e^{-2\pi\, i\, p\, \partial_v/L}\left(v\, e^{-v^2/2}\vec{H}(p)\right)\ . \tag{28}$$

The spectrum of the self adjoint operator $\mathscr{L}$ in $L^2(dv)$ is well known (see for example [1] or [4]). The eigenvalues are the numbers $-2n$ with $n \in \mathbf{N}^*$, and the corresponding normalised eigenvectors are the functions

$$e_n(v) = \frac{2^{-n/2}}{\pi^{1/4}\,(n!)^{1/2}}\; H_n(v)\; e^{-v^2/2} = \frac{(-1)^n\; 2^{-n/2}}{\pi^{1/4}\,(n!)^{1/2}}\;\; e^{v^2/2}\; \frac{d^n}{dv^n} e^{-v^2}\;,$$

where the $H_n$ are the Hermite polynomials. We have

$$e^{-2\pi\, i\, p\, \partial_v/L}\left(v\, e^{-v^2/2}\right) = \left(v - \frac{2\pi\, i\, p}{L}\right) e^{-v^2/2} e^{2\pi\, i\, p\, v/L} e^{2\,\pi^2\, p^2/L^2}\;,$$

and we decompose this function on the Hermite basis $(e_n)$. It follows easily using integration by parts that

$$\int e_n(v)\left(v - \frac{2\pi\, i\, p}{L}\right) e^{-v^2/2} e^{2\pi\, i\, p\, v/L} e^{2\,\pi^2\, p^2/L^2}\, dv$$

$$= (-1)^n\; e^{2\,\pi^2\, p^2/L^2} \frac{2^{-n/2}}{\pi^{1/4}\,(n!)^{1/2}} \int \left(v - \frac{2\pi\, i\, p}{L}\right) e^{2\pi\, i\, p\, v/L}\; \frac{d^n}{dv^n} e^{-v^2}\, dv$$

$$= e^{2\,\pi^2\, p^2/L^2} \frac{2^{-n/2}}{\pi^{1/4}\,(n!)^{1/2}} \times$$

$$\int \left[ v\left(\frac{2\pi\, i\, p}{L}\right)^n - \left(\frac{2\pi\, i\, p}{L}\right)^{n+1} + n\left(\frac{2\pi\, i\, p}{L}\right)^{n-1}\right] e^{-v^2} e^{2\pi\, i\, p\, v/L}\; dv$$

$$= e^{\pi^2\, p^2/L^2} \frac{2^{-n/2}\pi^{1/4}}{(n!)^{1/2}}\; \left(n + 2\pi^2 p^2/L^2\right)\; \left(\frac{2\pi\, i\, p}{L}\right)^{n-1}\;.$$

This formula holds for any $p \neq 0$ and any $n \geq 0$. For $p = 0$, the formula holds for any $n \geq 1$ (with the convention $0^0 = 1$). Finally, for $n = 0$ and $p = 0$ the result is zero. We now have after some simple linear algebra

$$\left(\mathscr{L} - \frac{4\pi^2 p^2}{L^2} + 2M\right)^{-1} \left(e_n\, \vec{H}(p)\right)(v) = e_n(v)\left(-2n - \frac{4\pi^2 p^2}{L^2} + 2M\right)^{-1} \vec{H}(p)$$

$$= \frac{e_n(v)}{2\big(n + 2\pi^2 p^2/L^2\big)\big(n + \nu_1 + \nu_2 + 2\pi^2 p^2/L^2\big)}$$

$$\times \begin{pmatrix} -\nu_2 - n - 2\pi^2 p^2/L^2 & -\nu_2 \\ -\nu_1 & -\nu_1 - n - 2\pi^2 p^2/L^2 \end{pmatrix} \vec{H}(p)\;.$$

We can write using Fourier series

$$I_2\big(G_1, G_2, \nu_1, \nu_2\big) = \sum_p \overline{\vec{G}(p)} \bullet \int \vec{\psi}^{\,1}(v, p)\; e^{-v^2/2}\; dv$$

$$
= \sum_p \sum_n \overline{\vec{G}(p)} \bullet \begin{pmatrix} -\nu_2 - n - 2\pi^2 p^2/L^2 & -\nu_2 \\ -\nu_1 & -\nu_1 - n - 2\pi^2 p^2/L^2 \end{pmatrix} \vec{H}(p)
$$

$$
\times \frac{1}{2\big(n + \nu_1 + \nu_2 + 2\pi^2 p^2/L^2\big)} \, e^{\pi^2 p^2/L^2} \frac{2^{-n/2}\pi^{1/4}}{(n!)^{1/2}} \left(\frac{2\pi\, i\, p}{L}\right)^{n-1}
$$

$$
\times \int \left(e^{2\pi\, i\, p\, \partial_v/L} e_n\right)(v) \, e^{-v^2/2} \, dv \, .
$$

Since the operator $i\partial_v$ is self adjoint, we get

$$
\int e^{-v^2/2} \left(e^{2\pi\, i\, p\, \partial_v/L} e_n\right)(v) \, dv = \int \overline{e^{2\pi\, i\, p\, \partial_v/L} e^{-v^2/2}} \, e_n(v) \, dv
$$

$$
= \int e_n(v) \, e^{-\left(v - 2\pi\, i\, p/L\right)^2/2} \, dv = e^{2\,\pi^2\, p^2/L^2} \int e_n(v) \, e^{-v^2/2} \, e^{2\, i\, v\, \pi\, p\, v/L} \, dv \, .
$$

By a computation similar to the above one, we get

$$
\int e^{2\pi\, i\, p\, v/L} e_n(v) \, e^{-v^2/2} \, dv = e^{\pi^2\, p^2/L^2} \frac{2^{-n/2}\pi^{1/4}}{(n!)^{1/2}} \left(\frac{2\pi\, i\, p}{L}\right)^n \, .
$$

After some simple algebra, we obtain

$$
I_2\big(G_1, G_2, 2\nu_1, 2\nu_2\big)
$$

$$
= \frac{2}{L(\nu_1 + \nu_2)} \sum_{n,\, p,\, n+p^2>0} e^{2\,\pi^2\, p^2/L^2} \frac{2^{-n}}{n!} \left(\frac{2\pi\, i\, p}{L}\right)^{2n-1} \frac{R(p)}{\big(n + \nu_1 + \nu_2 + 2\pi^2 p^2/L^2\big)}
$$

where

$$
R(p) = \overline{G_1(p)} G_1(p) \nu_2 \big(\nu_2 + n + 2\pi^2 p^2/L^2\big) + \overline{G_2(p)} G_2(p) \nu_1 \big(\nu_1 + n + 2\pi^2 p^2/L^2\big)
$$

$$
+ \nu_1 \nu_2 \Big(\overline{G_1(p)} G_2(p) + \overline{G_2(p)} G_1(p)\Big) \, .
$$

Since $G_1$ and $G_2$ are real, we have $\overline{G_1(p)} = G_1(-p)$ and $\overline{G_2(p)} = G_2(-p)$. This implies immediately that $R(p)$ is even in $p$ and therefore $I_2\big(G_1, G_2, \nu_1, \nu_2\big) = 0$. A similar computation (involving only $p = 0$) shows that the normalisation condition (27) holds. We now need to compute $I_3\big(G_1, G_2, \nu_1, \nu_2\big)$, and we recall that it is given by

$$
I_3\big(G_1, G_2, \nu_1, \nu_2\big) = \sum_p \overline{\vec{G}(p)} \bullet \int \vec{\psi}^{\,2}(v, p) \, e^{-v^2/2} \, dv \tag{29}
$$

where

$$
\vec{\psi}^{\,2}(v, p) = e^{2\pi\, i\, p\, \partial_v/L} \left(\mathscr{L} - \frac{4\pi^2 p^2}{L^2} + 2M\right)^{-1} e^{-2\pi\, i\, p\, \partial_v/L} \left(\vec{J}(v, p)\right) \, .
$$

and where $\vec{J}(v,p)$ is the Fourier series of the vector

$$\vec{J}(v,x) = 2C(x)\left(\partial_v \vec{\psi}^{\,1}(v,x) - v\vec{\psi}^{\,1}(v,x)\right) \ .$$

As above, using that the operator $i\partial_v$ is self adjoint, we get

$$I_3\big(G_1, G_2, 2\nu_1, 2\nu_2\big) = \sum_p \int \overline{\left(\mathscr{L} - \frac{4\pi^2 p^2}{L^2} + 2M^t\right)^{-1} e^{2\pi\, i\, p\, \partial_v / L}\left(e^{-v^2/2}\vec{G}(p)\right)}$$

$$\bullet\; e^{-2\pi\, i\, p\, \partial_v/L}\left(\vec{J}(v,p)\right)\ dv\ .$$

We can now compute everything in terms of series of Hermite coefficients. Denoting by $C(p)$ the Fourier coefficients of the matrix $C(x)$ we get

$$e^{-2\pi\, i\, p\, \partial_v/L}\vec{J}(v,p) = 2\sum_q C(p-q) e^{-2\pi\, i\, p\, \partial_v/L}\left(\partial_v \vec{\psi}^{\,1}(v,q) - v\vec{\psi}^{\,1}(v,q)\right)$$

$$= 2\sum_q C(p-q)\ e^{-2\pi\, i\, p\, \partial_v/L}$$

$$\big(\partial_v - v\big) e^{2\pi\, i\, q\, \partial_v/L}\left(\mathscr{L} - \frac{4\pi^2 q^2}{L^2} + 2M\right)^{-1} e^{-2\pi\, i\, q\, \partial_v/L}\left(v\, e^{-v^2/2}\vec{H}(q)\right)$$

$$= 2\sum_q C(p-q) e^{2\pi\, i\,(p-q)\,\partial_v/L}$$

$$\big(\partial_v - v + 2\pi\, i\, q/L\big)\left(\mathscr{L} - \frac{4\pi^2 q^2}{L^2} + 2M\right)^{-1} e^{-2\pi\, i\, q\, \partial_v/L}\left(v\, e^{-v^2/2}\vec{H}(q)\right).$$

We have already computed (for $n \ge 0$ if $p \neq 0$ and for $p = 0$ if $n > 0$)

$$\left(\mathscr{L} - \frac{4\pi^2 q^2}{L^2} + 2M\right)^{-1} e^{-2\pi\, i\, q\, \partial_v/L}\left(v\, e^{-v^2/2}\vec{H}(q)\right)$$

$$= \sum_n e^{\pi^2 q^2/L^2}\frac{2^{-n/2}\pi^{1/4}}{(n!)^{1/2}}\left(\frac{2\pi\, i\, q}{L}\right)^{n-1}\frac{e_n(v)}{2\big(n+\nu_1+\nu_2+2\pi^2 q^2/L^2\big)}$$
$$\begin{pmatrix} -\nu_2 - n - 2\pi^2 q^2/L^2 & -\nu_2 \\ -\nu_1 & -\nu_1 - n - 2\pi^2 q^2/L^2 \end{pmatrix}\vec{H}(q) \qquad (30)$$

By a similar computation left to the reader, we get

$$\left(\mathscr{L} - \frac{4\pi^2 p^2}{L^2} + 2M^t\right)^{-1} e^{2\pi\, i\, p\, \partial_v/L}\left(e^{-v^2/2}\vec{G}(p)\right)$$

$$= \sum_m e^{\pi^2 p^2/L^2} \frac{2^{-m/2}\pi^{1/4}}{(m!)^{1/2}} \left(-\frac{2\pi\, i\, p}{L}\right)^m$$

$$\times \frac{e_m(v)}{2\big(m + 2\pi^2 p^2/L^2\big)\big(m + \nu_1 + \nu_2 + 2\pi^2 p^2/L^2\big)}$$

$$\begin{pmatrix} -\nu_2 - m - 2\pi^2 p^2/L^2 & -\nu_1 \\ -\nu_2 & -\nu_1 - m - 2\pi^2 p^2/L^2 \end{pmatrix} \vec{G}(p) \qquad (31)$$

It finally remains to compute

$$\int e_m(v) e^{-2\,\pi\, i\,(p-q)\,\partial_v/L} \big(\partial_v - v + 2\pi\, i\, q/L\big) e_n(v) dv$$

$$= -\sqrt{2}\,\sqrt{n+1} \int e_m(v) e_{n+1}\big(v - 2\,\pi\, i\,(p-q)/L\big)\, dv$$

$$+ \frac{2\pi\, i\, q}{L} \int e_m(v) e_n\big(v - 2\,\pi\, i\,(p-q)/L\big)\, dv\ ,$$

since

$$\left(\frac{d}{dv} - v\right) e_n(v) = -\sqrt{2}\,\sqrt{n+1}\ e_{n+1}(v)\ .$$

We now compute the quantity

$$\gamma_{n,m}(r) = \int e_m(v) e_n\big(v - 2\,\pi\, i\, r/L\big)\, dv$$

$$= \frac{2^{-(n+m)/2} e^{2\,\pi^2\, r^2/L^2}}{\sqrt{\pi}\ (n!\, m!)^{1/2}} \int e^{-v^2}\ e^{2\,\pi\, i\, r\, v/L} H_m(v)\ H_n\big(v - 2\,\pi\, i\, r/L\big)\ dv$$

$$= \frac{2^{-(n+m)/2} e^{\pi^2\, r^2/L^2}}{\sqrt{\pi}\ (n!\, m!)^{1/2}} \int e^{-\left(v - \pi\, i\, r/L\right)^2}\ H_m(v)\ H_n\big(v - 2\,\pi\, i\, r/L\big)\ dv$$

$$= \frac{2^{-(n+m)/2} e^{\pi^2\, r^2/L^2}}{\sqrt{\pi}\ (n!\, m!)^{1/2}} \int e^{-v^2} H_m\big(v +\ \pi\, i\, r/L\big)\ H_n\big(v -\ \pi\, i\, r/L\big)\ dv\ .$$

When $m \le n$, we obtain

$$\gamma_{n,m}(r) = \frac{(m!)^{1/2} 2^{(n-m)/2} e^{\pi^2\, r^2/L^2}}{(n!)^{1/2}} \left(-\frac{i\,\pi\, r}{L}\right)^{n-m} L_m^{n-m} \left(-\frac{2\pi^2\, r^2}{L^2}\right)\ ,$$

where $L_r^s$ denotes a Laguerre polynomial (see [4], formula **7.377**). Similarly, when $m \ge n$ we get

$$\gamma_{n,m}(r) = \frac{(n!)^{1/2} 2^{(m-n)/2} e^{\pi^2\, r^2/L^2}}{(m!)^{1/2}} \left(\frac{i\,\pi\, r}{L}\right)^{m-n} L_n^{m-n} \left(-\frac{2\pi^2\, r^2}{L^2}\right)\ .$$

We can now use this result together with equations (A) and (A) into the expression (29). We get

$$I_3\big(G_1,G_2,\nu_1,\nu_2\big) =$$

$$-\frac{1}{L\big(\nu_1+\nu_2\big)}\sum_{n,m,p,q}\left(\frac{2\pi\, i\, q}{L}\gamma_{n,m}(p-q)-\sqrt{2}\,\sqrt{n+1}\gamma_{n+1,m}(p-q)\right)$$

$$e^{\,\pi^2\, q^2/L^2}\frac{2^{-n/2}\pi^{1/4}}{(n!)^{1/2}}\left(\frac{2\pi\, i\, q}{L}\right)^{n-1} e^{\,\pi^2\, p^2/L^2}\frac{2^{-m/2}\pi^{1/4}}{(m!)^{1/2}}\left(\frac{2\pi\, i\, p}{L}\right)^{m}$$

$$\times\frac{S(p,q)}{\big(m+2\pi^2p^2/L^2\big)\big(m+\nu_1+\nu_2+2\pi^2p^2/L^2\big)\big(n+\nu_1+\nu_2+2\pi^2q^2/L^2\big)}$$

where

$$S(p,q)=\left(\begin{pmatrix}\nu_2+m+2\pi^2p^2/L^2 & \nu_1\\ \nu_2 & \nu_1+m+2\pi^2p^2/L^2\end{pmatrix}\overline{\vec{G}(p)}\right)\bullet$$

$$\left(\begin{pmatrix}G_1(p-q) & 0\\ 0 & G_2(p-q)\end{pmatrix}\begin{pmatrix}\nu_2+n+2\pi^2q^2/L^2 & \nu_2\\ \nu_1 & \nu_1+n+2\pi^2q^2/L^2\end{pmatrix}\right.$$

$$\left.\times\begin{pmatrix}\nu_2G_1(q)\\ \nu_1G_2(q)\end{pmatrix}\right)\ .$$

A numerical simulation using for $G1$ and $G2$ real Fourier polynomials of order two and random coefficients give a nonzero result for $I_3\big(G_1,G_2,1,2\big)$ and $L=10$.

# References


[1] M.Abramowitz, I.Stegun. *Handbook of mathematical functions with formulas, graphs, and mathematical tables*. Dover, New York 1970.

[2] E.B.Davies. *Heat kernels and spectral theory*. Cambridge Tracts in Mathematics, **92**. Cambridge University Press, Cambridge, 1990.

[3] J.Garnier. Homogenization of periodic and time-dependent potential. SIAM J. Appl. Math. **57**, 95-111 (1997).

[4] I.S.Gradshteyn, I.M.Ryzhik. *Table of Integrals Series and Products*. Fourth edition, Academic Press, New York, 1965.

[5] P.Hänggi, F.Marchesoni, F.Nori. Brownian Motors. Ann. Phys. (8), Vol. **14**, 51-70 (2005).

[6] K.Hadeler, T.Hillen and F.Lutscher. The Langevin or Kramers approach to biological modeling. Math. Models Methods Appl. Sci. **14**, 1561-1583 (2004).

[7] L.Hörmander The analysis of linear partial differential operators. I. Distribution theory and Fourier analysis. Second edition. Grundlehren der Mathematischen Wissenschaften **256**. Springer-Verlag, Berlin, 1990.

[8] Ikeda, Watanabe. *Stochastic differential equations and diffusion processes*. Second edition. North-Holland Mathematical Library, 24. North-Holland Publishing Co., Amsterdam; Kodansha, Ltd., Tokyo, 1989.

[9] T.Kato. *Perturbation Theory for Linear Operators*. Springer, New York 1966.

[10] The mechanochemical stochastic processes of molecular motor. B. Gaveau, M. Moreau and B. Schuman. J. de Maths. Pures et Appliquées, **84**, 1758-1793 (2005).

[11] S.Molchanov. *Lectures on Random Media*. Lectures on probability theory (Saint-Flour, 1992), 242-411, Lecture Notes in Math., **1581**, Springer, Berlin, 1994.

[12] J.Mujica. *Complex analysis in Banach spaces. Holomorphic functions and domains of holomorphy in finite and infinite dimensions*. North-Holland Mathematics Studies, 120. Notas de Matemática [Mathematical Notes], 107. North-Holland Publishing Co., Amsterdam, 1986.

[13] P.Reimann. Brownian motors: noisy transport far from equilibrium. Physics Report **361**, 57-265 (2002).

[14] H.Qian. The Mathematical Theory of Molecular Motors. Movement and Chemomechanical Energy Transduction. Journal of Mathematical Chemistry, **27**, 219-234 (2000).

[15] L.Rey-Bellet. Ergodic properties of Markov processes. In *Open Quantum Systems II. The Markovian approach*. Lecture Notes in Mathematics **1881**, Springer, Berlin, 2006.