Change of Support for Estimating Local Block Grade Distributions

Xavier Emery

Abstract An important aspect in mineral resource evaluation is the reduction of variance when post-processing the grade distributions defined on the support (volume) of the available data into distributions defined on the support of the proposed selective mining units. Although the volume-variance relationship is well understood for the estimation of global grade distributions, it is still an unsolved issue for local estimation studies based on non-parametric geostatistical methods, such as indicator kriging, for which the support correction is not inherent to the method. To clarify this relationship, the local change of support problem is examined in the scope of two parametric models (multi-Gaussian and discrete Gaussian models). It is shown that the variance reduction factor between point and block-support local distributions depends on the block being considered and is less than the global variance reduction factor. As a consequence, post-processing the local point-support grade distributions on the basis of the latter systematically understates the importance of the change of support at the local scale and makes selective mining appear more economically attractive than it really is. In the light of these results, a methodology is proposed to post-process the local point-support distributions obtained via non-parametric (indicator) methods into block-support distributions. An application to simulated data indicates that this methodology provides an accurate estimation at the block support when dealing with diffusion-type random fields.

Keywords Volume-variance relationship · Support effect · Multi-Gaussian model · Discrete Gaussian model · Non-parametric geostatistics · Mining selectivity

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1 Introduction

A key aspect of selective mining is to decide what material in a deposit is worth extracting and processing, versus what material should be considered as waste, and to estimate the tonnage, metal content, and mean grade of the material selected as ore. To obtain accurate results, the difference in support between the available data (drill hole samples, blast holes, channel samples, etc., regarded as quasi-point supports) and the proposed selective mining units (SMU) must be taken into account in the estimation of recoverable resources.

To address the change-of-support problem, one option is to draw multiple realizations of a random field representing the grade of the element of interest at a point support, conditionally to the available information, and to regularize the realizations to the SMU support (Journel and Kyriakidis 2004; Verly 1984). An alternative to conditional simulation is the use of nonlinear kriging methods. In this respect, one can distinguish between distribution-based methods (e.g., multi-Gaussian, disjunctive and bi-Gaussian kriging; uniform conditioning) and distribution-free or non-parametric methods, mainly multiple indicator kriging and its variants. The former are often associated with change-of-support models and have been applied to the estimation of recoverable resources for a long time (Emery 2005; Guibal and Remacre 1984; Marcotte and David 1985; Maréchal 1984). However, the difficulty of parametric methods is their ability to properly model the spatial distribution of grades and to incorporate secondary information (e.g., trends, rock types codified as a categorical variable, layouts of the mineralization and alteration profiles, soft data corresponding to imprecise measurements, etc.).

In contrast, the indicator approach is more flexible and therefore applicable to a wider range of situations. However, it does not estimate grade distributions on a larger support than that of the data, and therefore requires using a separate change-of-support model, which may hinder the quality of recoverable resource estimates (Rossi and Parker 1994). The goal of this work is to develop a methodology for post-processing a local point-support grade distribution into a SMU grade distribution, which is applicable when the former is estimated via a non-parametric (indicator) method. To fulfill this objective, it will be important to distinguish between global (prior) and local (conditional or posterior) distributions and to design a model for the latter when passing from a point support to a block (SMU) support. The analysis of a few simple parametric models, specifically the multi-Gaussian and discrete Gaussian, will help to point out the distinction between the global and local frameworks and to design the intended local change-of-support model.

In the sequel, we will denote by $\{Z(\mathbf{x}),\ \mathbf{x} \in \mathbb{D}\}$ the random field that represents the point-support grades over a domain $\mathbb{D}$ of interest, and by $\{Z(v), v \in \mathbb{D}\}$ the regularized field defined on the SMU support. When referring to the local distributions, the same random fields conditioned to point-support data are considered; in this occasion, a tilde will be used to identify the conditional fields

$$\begin{cases} \forall \mathbf{x} \in \mathbb{D}, & \tilde{Z}(\mathbf{x}) = \{Z(\mathbf{x}) \mid \text{data}\}, \\ \forall v \in \mathbb{D}, & \tilde{Z}(v) = \{Z(v) \mid \text{data}\}. \end{cases} \tag{1}$$

2 Global Versus Local Change of Support

2.1 Global Framework

Geostatistical change-of-support models for global grade distributions are based on Cartier's relation, which states that the expected grade at a point-support location $\underline{\mathbf{x}}$ uniformly distributed in a block v with known grade is equal to the block grade (Matheron 1984, p. 425). This relation implies that $Z(v)$ has the same mean as $Z(\underline{\mathbf{x}})$ and a smaller variance (support effect). Examples of models that fulfill Cartier's relation include the affine, indirect lognormal, and discrete Gaussian corrections (Emery 2004; Isaaks and Srivastava 1989; Lajaunie 2000; Matheron 1976, 1985).

2.2 Local Framework

Change-of-support is more problematical in the case of distributions conditioned to a set of data. A common practice consists in estimating the point-support grade distribution at the center of the block, then in post-processing this distribution by applying one of the previous corrections. However, the relationship between the local distributions at point and block supports is not as simple as in the global framework. For instance, if the central point corresponds to a data location, the point-support distribution has a zero variance (no uncertainty), but this cannot be the case for the true block-support distribution: in such a situation, the change of support is characterized by an increase of the variance when passing from the point-support to the block-support distributions. A solution to this problem is to post-process the grade distribution at a point uniformly located within the block, instead of the central point. The justification is that Cartier's relation still holds at the local scale, so that the means of $\tilde{Z}(\underline{\mathbf{x}})$ and $\tilde{Z}(v)$ (with $\underline{\mathbf{x}} \in v$) are equal, and the variance of the latter is less than that of the former. A second difficulty arising when post-processing local point-support distributions is that the reduction of variance is not the same as in the case of the prior (global) distributions and depends on the amount of conditioning information surrounding each block. This point is illustrated in the next section for the specific situation of a Gaussian random field and a transform of this field.

3 Local Change of Support in the Multi-Gaussian Model

3.1 First Example: One-Dimensional Gaussian Random Field

Suppose that the random field $\{Z(\mathbf{x}),\ \mathbf{x} \in \mathbb{D}\}$ has multivariate Gaussian distributions, and let us denote by $\rho(\mathbf{x}, \mathbf{x}')$ the covariance between the variables located at $\mathbf{x}$ and $\mathbf{x}'$. For any $\mathbf{x}$ and v in $\mathbb{D}$, the point-support and block-support conditional variables $\tilde{Z}(\mathbf{x})$ and $\tilde{Z}(v)$ are Gaussian, with means and variances equal to their simple kriging estimates and kriging variances, respectively.

To allow explicit calculations, we will consider the case when v is the interval $[0, L]$ ($L > 0$) in $\mathbb{R}$, $\rho(\mathbf{x}, \mathbf{x}') = \exp(-|\mathbf{x} - \mathbf{x}'|)$ is an exponential covariance function, and there is a single conditioning datum located at $\mathbf{x}_1 \leq 0$, with a value equal to

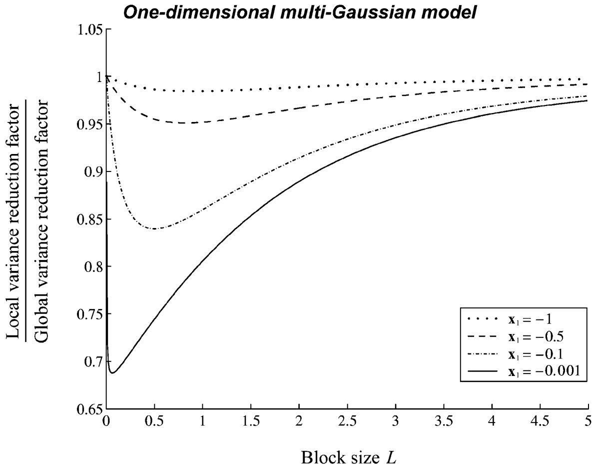


Fig. 1 Ratio between local and global variance reduction factors, as a function of the block size. Case of a one-dimensional Gaussian random field with an exponential covariance function and a single conditioning datum at location $\mathbf{x}_1$

the prior mean (0). In this case, the means of $\tilde{Z}(\mathbf{x})$ and $\tilde{Z}(v)$ are still zero and their variances are

$$\begin{aligned}\sigma^{*2}(\mathbf{x}) &= 1 - \rho(\mathbf{x}, \mathbf{x}_1)^2 = 1 - e^{-2|\mathbf{x}_1 - \mathbf{x}|},\\ \sigma^{*2}(v) &= \frac{1}{L^2}\int_0^L\int_0^L \rho(\mathbf{x}, \mathbf{x}')\,d\mathbf{x}\,d\mathbf{x}' - \left(\frac{1}{L}\int_0^L \rho(\mathbf{x}, \mathbf{x}_1)\,d\mathbf{x}\right)^2 \\ &= \frac{2}{L^2}\left\{L + e^{-L} - 1 - e^{2\mathbf{x}_1 - L}\left[\cosh(L) - 1\right]\right\}.\end{aligned} \tag{2}$$

If $\underline{\mathbf{x}}$ is a random point uniformly distributed in $[0, L]$, the variable $\tilde{Z}(\underline{\mathbf{x}})$ appears as a mixture of Gaussian variables with the same zero mean. Its variance is the average value of the kriging variance $\sigma^{*2}(\mathbf{x})$ when $\mathbf{x}$ belongs to $[0, L]$. Accordingly, the local variance reduction factor between $\tilde{Z}(\underline{\mathbf{x}})$ and $\tilde{Z}(v)$ is

$$f_{\underline{\mathbf{x}}} = \frac{\sigma^{*2}(v)}{\frac{1}{L}\int_0^L \sigma^{*2}(\mathbf{x})\,d\mathbf{x}} = \frac{2}{L}\,\frac{L + e^{-L} - 1 - e^{2\mathbf{x}_1 - L}[\cosh(L) - 1]}{L - e^{2\mathbf{x}_1 - L}\sinh(L)}, \tag{3}$$

while the global variance reduction factor (between $Z(\underline{\mathbf{x}})$ and $Z(v)$) is

$$f = \frac{1}{L^2}\int_0^L\int_0^L \rho(\mathbf{x}, \mathbf{x}')\,d\mathbf{x}\,d\mathbf{x}' = \frac{2}{L^2}\left(L + e^{-L} - 1\right). \tag{4}$$

Figure 1 plots the ratio between both variance reduction factors as a function of the block length L, for several values of $\mathbf{x}_1$. One observes that this ratio depends on the location of the conditioning datum and is always less than one: the local change-of-support correction is stronger than the global one. The ratio between the local and global variance reduction factors is close to one if

- the block is very large ($L \to \infty$): in this case, the variance reduction factors tend to zero

- the block is very small ($L \to 0$): the variance reduction factors are equal to one (no support correction is needed)
- the datum is located "far" from the block (case $\mathbf{x}_1 = -1$): the conditioning information has almost no effect on the local distributions.

On the contrary, the discrepancy between the global and local variance reduction factors increases when the data location is close to the block, and therefore $Z(\mathbf{x}_1)$ is highly correlated with $Z(v)$. This result indicates that post-processing the local point-support distribution on the basis of the global variance reduction factor would understate the true support correction in the vicinity of the conditioning data. In the next section, the analysis is extended to a more general setting, corresponding to the case of a transformed Gaussian random field.

3.2 Second Example: Transformed Gaussian Random Field

Assume the following:

(1) The random field under study $\{Z(\mathbf{x}),\ \mathbf{x} \in \mathbb{D}\}$ can be transformed into a standard Gaussian field $\{Y(\mathbf{x}),\ \mathbf{x} \in \mathbb{D}\}$ (with zero mean and unit variance)

$$\forall \mathbf{x} \in \mathbb{D}, \quad Z(\mathbf{x}) = \phi(Y(\mathbf{x})). \tag{5}$$

(2) The discrete Gaussian correction is suitable at the global scale, with r as its change-of-support coefficient.
(3) The discrete Gaussian correction is also suitable at the local scale, with a change-of-support coefficient $r_{\mathbf{x}}$ to be determined for each $\mathbf{x}$ in $\mathbb{D}$.

Here, one is interested in finding a condition that ensures the consistency between the global support correction and the local one (assumptions (2) and (3)). For recovery estimation, this means that the local resource estimates (tonnage, metal content, mean grade above given cutoffs) are expected to match, on average over many blocks of the deposit, the global resource estimates. To simplify the presentation, the demonstration is reported in Appendix A. In this appendix, it is established that the required consistency condition is not met when the local change-of-support coefficient is equal to the global one, but when it is defined by

$$\sigma^{*2}(\mathbf{x})\left(1 - r_{\mathbf{x}}^2\right) = 1 - r^2, \tag{6}$$

in which $\sigma^{*2}(\mathbf{x})$ is the simple kriging variance of $Y(\mathbf{x})$.

As $\sigma^{*2}(\mathbf{x})$ is less than or equal to one, the local change-of-support coefficient $r_{\mathbf{x}}$ is less than the global coefficient r. This corroborates the comment made in the scope of the one-dimensional Gaussian random field, according to which the local change-of-support correction is stronger than the global one. The relationship that defines the local change-of-support coefficient [(6)] has been found by Emery and Ortiz (2004, p. 254) in the scope of the lognormal model, corresponding to the case when the transformation function ϕ is an exponential function. Equation (6) has no solution if $\sigma^{*2}(\mathbf{x}) < 1 - r^2$. This difficulty stems from the fact that one has post-processed the local point-support distribution at a non-random location $\mathbf{x}$ inside block v. As mentioned earlier, to avoid such a situation, the analysis must be performed on the local point-support distribution at a random location within the block, which is considered in the next section.

4 The Discrete Gaussian Model for Local Estimation

4.1 Overview of the Model

Let us consider the original field with randomized locations $\{Z(\underline{\mathbf{x}}),\ \underline{\mathbf{x}} \in \mathbb{D}\}$, the regularized field $\{Z(v), v \in \mathbb{D}\}$, and their Gaussian transforms $\{Y(\underline{\mathbf{x}}),\ \underline{\mathbf{x}} \in \mathbb{D}\}$ and $\{Y_v, v \in \mathbb{D}\}$.

1. At the global scale (discrete Gaussian correction), one assumes that any pair $\{Y(\underline{\mathbf{x}}),\ Y_v\}$ with $\underline{\mathbf{x}} \in v$ has a standard bi-Gaussian distribution with the same correlation coefficient r for every block (change-of-support coefficient). In general, this is a mild hypothesis and the support correction is suitable for many types of deposits (Chilès and Delfiner 1999, p. 447; Demange et al. 1987; Emery and Soto-Torres 2005).
2. At the local scale, one assumes that the transformed random fields $\{Y(\underline{\mathbf{x}}),\ \underline{\mathbf{x}} \in \mathbb{D}\}$ and $\{Y_v, v \in \mathbb{D}\}$ have joint multivariate Gaussian distributions. The application of the local model is therefore restricted to deposits for which the multi-Gaussian assumption is suited to the Gaussian transform of the grade data.

Because of the latter assumption, the conditional distributions of $Y(\underline{\mathbf{x}})$ and Y_v (with $\underline{\mathbf{x}} \in v$) are Gaussian, with their means equal to their simple kriging estimates $y^*(\underline{\mathbf{x}})$ and y_v^* from the Gaussian data and variances equal to the corresponding kriging variances $\sigma^{*2}(\underline{\mathbf{x}})$ and σ_v^{*2}. Therefore, conditionally to these data

$$\begin{cases} \tilde{Y}(\underline{\mathbf{x}}) = y^*(\underline{\mathbf{x}}) + \sigma^*(\underline{\mathbf{x}})U(\underline{\mathbf{x}}), \\ \tilde{Y}_v = y_v^* + \sigma_v^* U_v, \end{cases} \tag{7}$$

where $U(\underline{\mathbf{x}})$ and U_v are standard Gaussian variables that are independent of the data. The kriging estimates and kriging variances at point and block supports are linked by the following relations (Emery and Ortiz 2004, p. 254)

$$\begin{cases} y^*(\underline{\mathbf{x}}) = r y_v^*, \\ 1 - \sigma^{*2}(\underline{\mathbf{x}}) = r^2\left(1 - \sigma_v^{*2}\right). \end{cases} \tag{8}$$

In particular, by accounting for the second identity and the fact that r is less than 1, it is seen that the block-support kriging variance is less than the point-support one

$$\forall \underline{\mathbf{x}} \in v, \quad \sigma_v^{*2} \leq \sigma^{*2}(\underline{\mathbf{x}}). \tag{9}$$

4.2 Determination of the Local Change-of-Support Correction

For $\underline{\mathbf{x}} \in v$, the random variables $Y(\underline{\mathbf{x}})$ and Y_v have a standard bi-Gaussian distribution with correlation coefficient r. Therefore,

$$\forall \underline{\mathbf{x}} \in v, \quad Y(\underline{\mathbf{x}}) = r Y_v + \sqrt{1 - r^2}\, W(\underline{\mathbf{x}}), \tag{10}$$

where $W(\underline{\mathbf{x}})$ is a standard Gaussian variable independent of Y_v. This variable is also uncorrelated with (hence, due to the multi-Gaussian assumption, independent of) the point-support random field $\{Y(\underline{\mathbf{x}}), \underline{\mathbf{x}} \in \mathbb{D}\}$

$$\begin{aligned}\forall \underline{\mathbf{x}} \in v,\ \forall \underline{\mathbf{x}}' \in \mathbb{D}, \quad & \operatorname{cov}\Big(\sqrt{1-r^2}W(\underline{\mathbf{x}}), Y(\underline{\mathbf{x}}')\Big) \\ &= \operatorname{cov}\big(Y(\underline{\mathbf{x}}) - rY_v, Y(\underline{\mathbf{x}}')\big) \\ &= \operatorname{cov}\big(Y(\underline{\mathbf{x}}), Y(\underline{\mathbf{x}}')\big) - r\operatorname{cov}\big(Y_v, Y(\underline{\mathbf{x}}')\big) \\ &= 0. \end{aligned} \tag{11}$$

The last equality stems from the relationship between the simple and cross-covariances of the Gaussian fields at point and block supports (Rivoirard 1994). In particular, $W(\underline{\mathbf{x}})$ is independent of the point-support Gaussian data, of the kriging estimator Y_v^* (a weighted average of these data) and of the kriging error $\sigma_v^* U_v = Y_v - Y_v^*$.

At the local scale (conditionally to the data), (10) becomes

$$\forall \underline{\mathbf{x}} \in v, \quad y^*(\underline{\mathbf{x}}) + \sigma^*(\underline{\mathbf{x}})U(\underline{\mathbf{x}}) = r\big(y_v^* + \sigma_v^* U_v\big) + \sqrt{1-r^2}W(\underline{\mathbf{x}}), \tag{12}$$

and the correlation coefficient between $U(\underline{\mathbf{x}})$ and U_v is

$$r_{\underline{\mathbf{x}}} = \frac{r\sigma_v^*}{\sigma^*(\underline{\mathbf{x}})}. \tag{13}$$

In summary, the following statements hold

- The conditioned random fields $\{\tilde{Z}(\underline{\mathbf{x}}), \underline{\mathbf{x}} \in \mathbb{D}\}$ and $\{\tilde{Z}(v), v \in \mathbb{D}\}$ can be transformed into the standard Gaussian random fields $\{U(\underline{\mathbf{x}}), \underline{\mathbf{x}} \in \mathbb{D}\}$ and $\{U_v, v \in \mathbb{D}\}$ [(7)].
- For $\underline{\mathbf{x}} \in v$, the pair $\{U(\underline{\mathbf{x}}), U_v\}$ is bi-Gaussian with correlation coefficient $r_{\underline{\mathbf{x}}}$.

These points are the basis assumptions of the global discrete Gaussian correction (Rivoirard 1994, p. 81–82). This correction remains valid for the local distributions, except that here the change-of-support coefficient is equal to $r_{\underline{\mathbf{x}}}$ instead of r. Since $\sigma^*(\underline{\mathbf{x}}) \geq \sigma_v^*$ [(9)], the local coefficient is smaller than the global one

$$r_{\underline{\mathbf{x}}} \leq r. \tag{14}$$

Such an inequality means that the reduction of variance is more pronounced in the local framework than in the global framework. One consequence is that the approach consisting in post-processing the local point-support distributions using the parameters of the global model is risky, as it makes the block grade distributions appear more selective than they really are. The assumed volume-variance correction is not as strong as it should be and produces biases in the assessment of recoverable resources. Although it has been established in the scope of the discrete Gaussian model, this result is actually very general (Appendix B). The local change-of-support coefficient found in (13) is consistent with that defined in (6) for the multi-Gaussian model, because of the identities given in (8). This coefficient can also be expressed in a simple manner as a function of the block-kriging variance of the Gaussian data, see (31) in Appendix C.

5 Proposed Methodology for Local Change of Support

5.1 General Presentation

The results obtained in the scope of the discrete Gaussian model are used to define a methodology for post-processing local point-support distributions in a more general setting. These distributions can have been estimated via any geostatistical method, for instance indicator kriging or its variants (median indicator kriging, probability kriging, or indicator cokriging). The proposed methodology consists of the following steps:

1. Transform the original Z-data into Gaussian data (Y) and perform a block kriging of the Gaussian data. From the block-kriging variance, derive the local change-of-support coefficient for each block [(31)].
2. For each block, determine the local distribution of the original Z-variable at a point uniformly located within the block. At this step, a non-parametric method (indicator kriging) can be used.
3. Post-process the local point-support distributions by applying a discrete Gaussian correction with the local change-of-support coefficients calculated at step 1. This step requires defining for each block a function that transforms the local point-support distribution into the standard Gaussian distribution. Expansions into Hermite polynomials can be used to determine the local block-support distribution (Rivoirard 1994).

The discrete Gaussian correction is preferred to other change-of-support models that directly work with the original random field (Z), such as the affine correction, as the latter are impractical due to the complicated expression of the local variance reduction factor (Appendix D). Instead, with the recourse to Gaussian transforms, the local change-of-support coefficient is expressed in a straightforward manner from a block-kriging variance [(31)]. One drawback is that the methodology is not applicable to variables with a strong zero effect (i.e., an important proportion of zero values), as the Gaussian transform needed in step 1 is not defined unequivocally (Chilès and Delfiner 1999, p. 406). For the first two steps, there is no need to randomize the data locations in the blocks (at step 2, the random position only concerns the location being estimated), hence one can avoid the loss of information caused by such a randomization. The above methodology aims at avoiding the biases in the estimation of recovery functions that occur when using the global change-of-support coefficient for post-processing the local point-support distributions. Its performance is ensured only when the grade distribution is satisfactorily described by either a multi-Gaussian or a discrete Gaussian model (cases examined in the previous sections).

5.2 Application to Simulated Data

An exercise is now made to assess the robustness of the proposed methodology to departures from the multi-Gaussian assumptions. We will examine two random fields defined in $\mathbb{R}$ with the same univariate gamma distribution and the same exponential covariance function. The first one is a diffusion-type gamma random field (Matheron 1985, p. 156) and the second one is a mosaic-type random field defined by a

Poisson tessellation with independent gamma valuations (Chilès and Delfiner 1999, p. 541). In both cases, we assume that there is a single conditioning datum located at $\mathbf{x} = 0$, with a value equal to the median of the gamma univariate distribution, and that the block v of interest is the union of 50 points regularly spaced in the interval $[-0.5; 0.5]$. The exercise consists of the following steps.

5.2.1 Calculation of the Global Change-of-Support Coefficient

Let $\{Z(\mathbf{x}),\ \mathbf{x} \in \mathbb{R}\}$ denote the gamma random field (diffusion-type or mosaic-type) and $\{Y(\mathbf{x}),\ \mathbf{x} \in \mathbb{R}\}$ its Gaussian transform. The calculation of the global change-of-support coefficient relies on the variance of the block-support variable $Z(v)$ (calculated by regularizing the point-support covariance function) and on the expansion of the point-support transformation function ϕ [(5)] into Hermite polynomials (Rivoirard 1994).

5.2.2 Calculation of the Local Change-of-Support Coefficient

The calculation of the local change-of-support coefficient requires performing a block-kriging of the Gaussian transform from a datum located at $\mathbf{x} = 0$ [(31)]. In the diffusion model, the covariance function of the Gaussian transform is determined from that of the gamma random field (an exponential model) by using the expansion of the inverse transformation function ϕ^{-1} into Laguerre polynomials (Chilès and Delfiner 1999, p. 412). In contrast, in the mosaic model, the covariance function of the Gaussian transform is proportional to that of the gamma random field (Rivoirard 1994).

5.2.3 Conditional Simulation of the Gamma Random Fields

The next step is the simulation of the diffusion-type and mosaic-type gamma random fields at the 50 locations in $[-0.5; 0.5]$ that compose the block of interest, conditionally to the datum located at $\mathbf{x} = 0$. The simulation is performed sequentially by taking advantage of the Markov property of both gamma random fields. By drawing a large number of realizations for each random field, one can determine the true conditional point-support distribution at a location uniformly distributed over the block, as well as the true conditional block-support distribution.

5.2.4 Post-Processing of the True Point-Support Distribution and Comparison with the True Block-Support Distribution

The conditional point-support distributions determined in the previous step are post-processed into block-support distributions, by applying the discrete Gaussian correction with the change-of-support coefficients calculated in the first steps. The post-processed distributions are compared to the true block-support distributions. A simple criterion for comparison consists of examining the metal-tonnage curves, which characterize the distributions and allow comparing their selectivity (Matheron 1984; Rivoirard 1994).

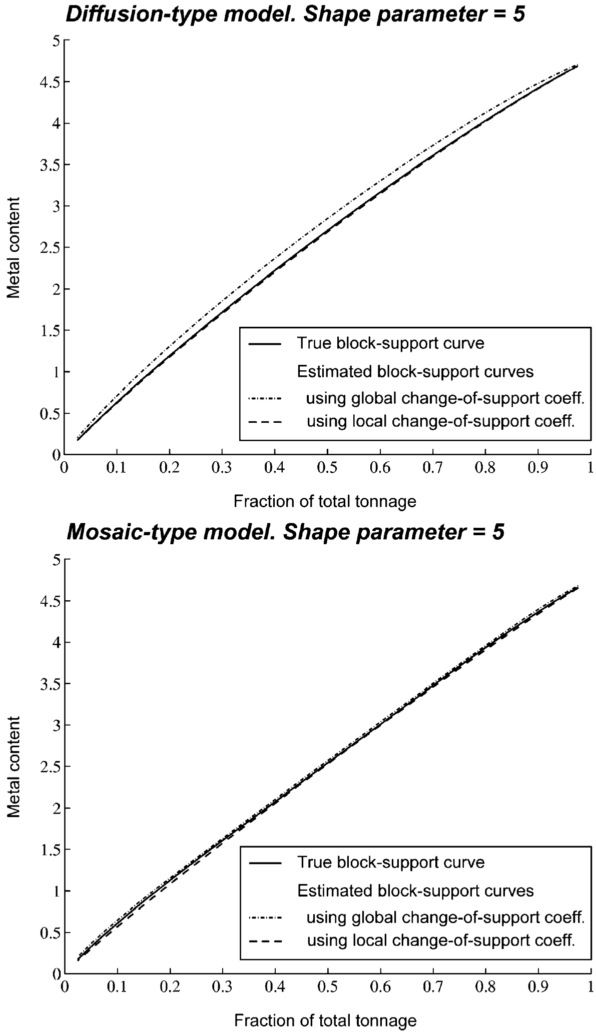
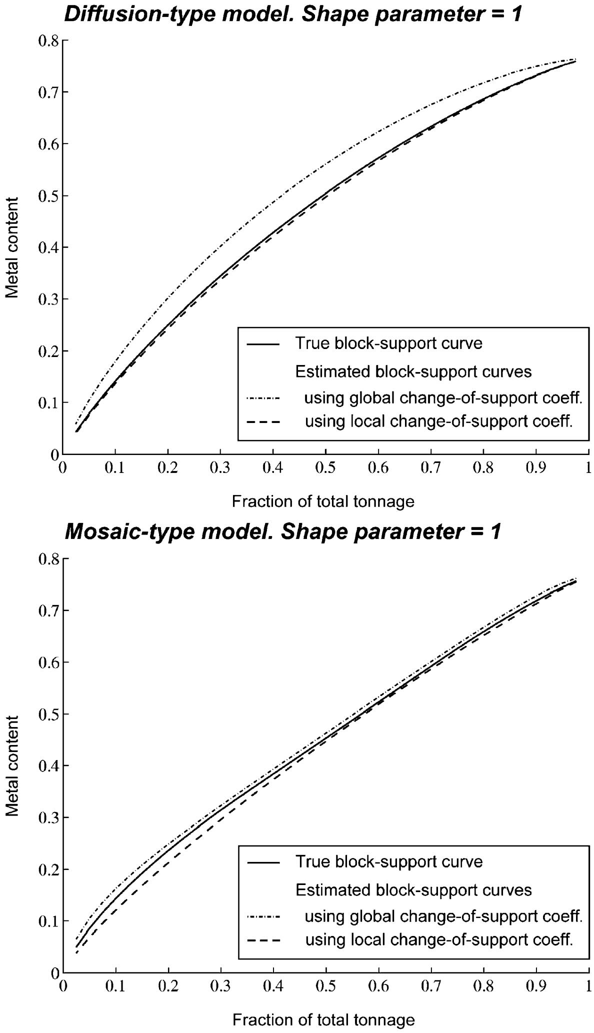


Fig. 2 True and estimated block-support metal-tonnage curves for the one-dimensional diffusion-type and mosaic-type gamma random fields conditioned to a single datum, for three values of the shape parameter of the point-support univariate distribution

The results are shown in Fig. 2 for three values of the shape parameter (5, 1 and 0.5) of the point-support gamma distribution and for a point-support exponential covariance with a practical range of 3. Similar results (not shown here) have been obtained when changing the value of the conditioning datum or the range of the point-support covariance function. The proposed approach provides an accurate estimation of the true block-support distribution for the diffusion-type random field, except maybe when the shape parameter of the point-support distribution is less than 1 (corresponding to a highly skewed distribution). It still performs much better than the approach based on the global change-of-support coefficient. In contrast, the estimation based on the local change-of-support coefficient defined by (31) proves to be biased in the case of the mosaic-type random field, and the proposed approach is therefore not adequate for such random fields. In all the cases, the use of the global

Fig. 2 (*Continued*)

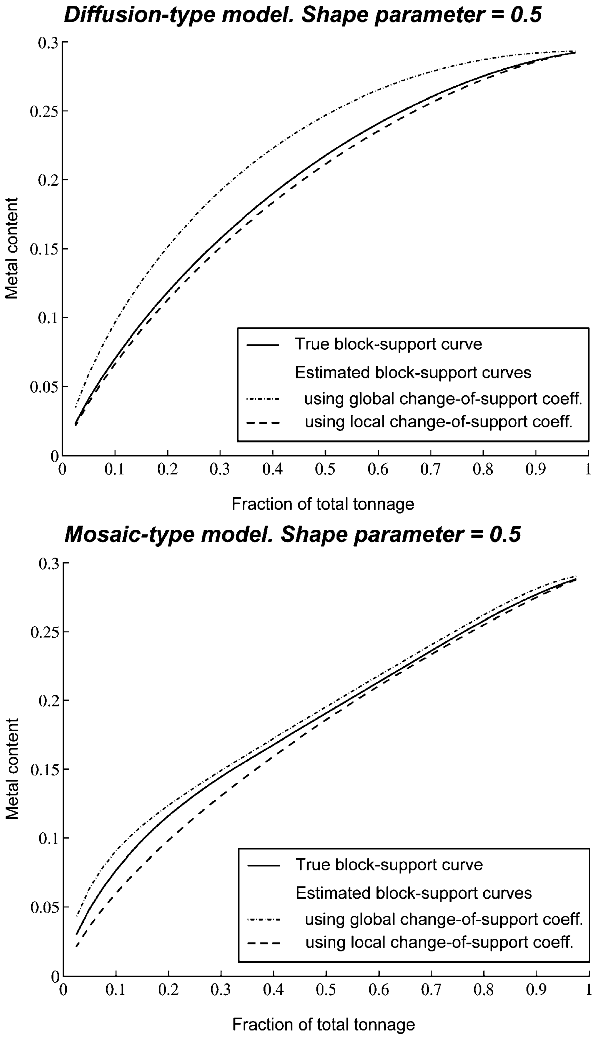


change-of-support coefficient leads to an overestimation of the true block-support metal-tonnage curves or an overestimation of the actual block-support selectivity, especially for the diffusion-type model. This indicates that, as in the multi-Gaussian and discrete Gaussian models, the true local variance reduction factor is less than the global one (Appendix B). In the mosaic-type model, this local factor remains unknown to a great extent, so that non-parametric (distribution-free) methods such as indicator kriging are inadvisable when a change of support has to be considered.

6 Conclusions

Although the volume-variance relationship on global (prior) distributions is well understood and has originated many geostatistical change-of-support models, it turns

Fig. 2 (*Continued*)



out to be more complex when dealing with local distributions, that is distributions conditioned to a set of neighboring data. The multi-Gaussian and discrete Gaussian models offer a simple framework to analyze the change of support at the local scale. The analysis proves that the variance reduction factor is block-dependent and is less than the global one. In the presence of conditioning data, the support correction applied to the point-support distributions is stronger than that used in the global framework. This observation is crucial for mineral resource estimation, as the approach based on the global variance reduction factor systematically understates the importance of the change of support and makes selective mining appear more economically attractive than it really is.

A methodology has been devised to obtain accurate estimates of the recoverable resources, based on the local volume-variance relationship observed in the scope of the discrete Gaussian model. It can be used to post-process the local point-support distributions obtained via any non-parametric (distribution-free) method, but its efficiency is guaranteed only when the discrete Gaussian model is suited to the data under study. The application to simulated random fields with skewed (gamma) univariate distributions indicates that the proposed methodology is robust to departures from the multivariate Gaussian assumptions when the point-support grades can be represented by a diffusion-type random field.

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Appendix A

In this appendix, we assume that the original random field $\{Z(\mathbf{x}), \mathbf{x} \in \mathbb{D}\}$ can be transformed into a standard Gaussian field $\{Y(\mathbf{x}), \mathbf{x} \in \mathbb{D}\}$ [(5)] and that the discrete Gaussian correction is suitable at the global and local scales. Let us examine the implications of these assumptions, in order to define a relationship between the change-of-support coefficients at both scales.

7.1 Global Framework

The distribution of the regularized variable $Z(v)$ is the same as that of $\phi_v(Y_v)$, in which Y_v is a standard Gaussian random variable and ϕ_v is defined by (Chilès and Delfiner 1999, p. 432)

$$\forall y \in \mathbb{R}, \quad \phi_v(y) = \int_{-\infty}^{+\infty} \phi\Big(ry + \sqrt{1-r^2}t\Big) g(t)\, dt, \tag{15}$$

in which $g(.)$ represents the standard Gaussian probability density function and r the global change-of-support coefficient.

7.2 Local Framework

Since the random field $\{Y(\mathbf{x}), \mathbf{x} \in \mathbb{D}\}$ has multivariate Gaussian distributions, the conditional random variable $\tilde{Y}(\mathbf{x})$ is Gaussian, with mean equal to its simple kriging estimate $y^*(\mathbf{x})$ from the Y-data and variance equal to the associated simple kriging variance $\sigma^{*2}(\mathbf{x})$. One can therefore define a local transformation function for the original conditional random field

$$\forall \mathbf{x} \in \mathbb{D}, \quad \tilde{Z}(\mathbf{x}) = \phi\big(\tilde{Y}(\mathbf{x})\big) = \tilde{\phi}_{\mathbf{x}}\big(U(\mathbf{x})\big) \tag{16}$$

with

$$\forall u \in \mathbb{R}, \quad \tilde{\phi}_{\mathbf{x}}(u) = \phi\big(y^*(\mathbf{x}) + \sigma^*(\mathbf{x})u\big). \tag{17}$$

$\{U(\mathbf{x}), \mathbf{x} \in \mathbb{D}\}$ is a standard Gaussian random field independent of the data, corresponding to the standardized kriging error. Let us now post-process the distribution of $\tilde{Z}(\mathbf{x})$ by applying a discrete Gaussian correction with a change-of-support coefficient $r_\mathbf{x}$, so as to obtain an estimated distribution for $\tilde{Z}(v)$. This distribution is the same as that of $\tilde{\phi}_v(U_v)$, where U_v is a standard Gaussian random variable independent of the data, and $\tilde{\phi}_v$ is defined by [(15) and (17)]

$$\forall u \in \mathbb{R}, \quad \tilde{\phi}_v(u) = \int_{-\infty}^{+\infty} \tilde{\phi}_\mathbf{x}\left(r_\mathbf{x} u + \sqrt{1 - r_\mathbf{x}^2}\, t\right) g(t)\, dt$$
$$= \int_{-\infty}^{+\infty} \phi\left(y^*(\mathbf{x}) + r_\mathbf{x} \sigma^*(\mathbf{x}) u + \sigma^*(\mathbf{x}) \sqrt{1 - r_\mathbf{x}^2}\, t\right) g(t)\, dt. \quad (18)$$

7.3 Consistency between Global and Local Frameworks

The local model is consistent with the global correction (15) if the distribution of $Z(v)$ is the same as that of $\tilde{Z}(v)$ when restoring the randomness to the conditioning data. This is done by replacing the kriging estimate $y^*(\mathbf{x})$ in (18) by the corresponding estimator $Y^*(\mathbf{x})$ (a Gaussian variable with mean zero and variance $1 - \sigma^{*2}(\mathbf{x})$) and by imposing the condition

$$\tilde{\phi}_v(U_v) \equiv \phi_v(Y_v) \text{ (equality of the distributions).} \quad (19)$$

There is no reason why (19) should be fulfilled when $r_\mathbf{x}$ is equal to r. Instead, let us assume that $r_\mathbf{x}$ is defined by (6). By using (18), one obtains

$$\tilde{\phi}_v(U_v) = \int_{-\infty}^{+\infty} \phi\left(Y^*(\mathbf{x}) + r_\mathbf{x} \sigma^*(\mathbf{x}) U_v + \sqrt{1 - r^2}\, t\right) g(t)\, dt$$
$$= \int_{-\infty}^{+\infty} \phi\left(r Y' + \sqrt{1 - r^2}\, t\right) g(t)\, dt, \quad (20)$$

with $Y' = (Y^*(\mathbf{x}) + r_\mathbf{x} \sigma^*(\mathbf{x}) U_v)/r$. Because U_v does not depend on $Y^*(\mathbf{x})$ (as it is independent of the Y-data), it is seen that Y' is a standard Gaussian random variable and has the same distribution as Y_v. By comparing with (15), one concludes that (19) holds and that the global and local viewpoints are consistent.

Appendix B

Let $\underline{\mathbf{x}}$ be a random point uniformly distributed in block v. Denote by

- m and $\tilde{m}$ the means of $Z(\underline{\mathbf{x}})$ and $\tilde{Z}(\underline{\mathbf{x}})$ (these are the same as that of $Z(v)$ and $\tilde{Z}(v)$)
- $\mu_2(\underline{\mathbf{x}})$, $\mu_2(v)$, $\tilde{\mu}_2(\underline{\mathbf{x}})$ and $\tilde{\mu}_2(v)$ the second raw (non-central) moments of $Z(\underline{\mathbf{x}})$, $Z(v)$, $\tilde{Z}(\underline{\mathbf{x}})$, and $\tilde{Z}(v)$, respectively
- f the global variance reduction factor (between $Z(\underline{\mathbf{x}})$ and $Z(v)$), such that

$$\mu_2(v) = f \mu_2(\underline{\mathbf{x}}) + (1 - f) m^2. \quad (21)$$

In general, $\tilde{m}$, $\tilde{\mu}_2(\underline{\mathbf{x}})$, and $\tilde{\mu}_2(v)$ depend on the conditioning data values, and they become random variables when these data values are randomized. For the local and global frameworks to be consistent, the local raw moments are expected to coincide with the corresponding global raw moments

$$E\left(Z^k\right) = E\left\{E\left[Z^k|data\right]\right\}. \tag{22}$$

In particular, for $k = 1$ and $k = 2$

$$\begin{cases} E(\tilde{m}) = m, \\ E\left(\tilde{\mu}_2(\underline{\mathbf{x}})\right) = \mu_2(\underline{\mathbf{x}}), \\ E\left(\tilde{\mu}_2(v)\right) = \mu_2(v). \end{cases} \tag{23}$$

Suppose now that the local variance reduction factor is chosen equal to the global factor f. The corrected block-support second moment is therefore [(21)]

$$\tilde{\mu}_2^*(v) = f\tilde{\mu}_2(\underline{\mathbf{x}}) + (1 - f)\tilde{m}^2. \tag{24}$$

By taking expected values, it becomes [(21) and (23)]

$$E\left(\tilde{\mu}_2^*(v)\right) = f\mu_2(\underline{\mathbf{x}}) + (1 - f)\left(m^2 + \mathrm{var}(\tilde{m})\right) \geq \mu_2(v) = E\left(\tilde{\mu}_2(v)\right). \tag{25}$$

The corrected second moment is over-estimated. To avoid biases, the local variance reduction factor must be less than the global one.

Appendix C

This appendix focuses on the discrete Gaussian model and aims at establishing relationships between the kriging variance $\sigma^{*2}(\underline{\mathbf{x}})$ of the point-support Gaussian variable $Y(\underline{\mathbf{x}})$, in which $\underline{\mathbf{x}}$ is a random location uniformly distributed in block v, the kriging variance σ_v^{*2} of the block-Gaussian variable Y_v, and the kriging variance $\sigma^{*2}(v)$ of the regularized Gaussian variable defined by

$$Y(v) = \frac{1}{|v|}\int_v Y(\mathbf{x})\,d\mathbf{x}, \tag{26}$$

where $|v|$ is the measure of v, and $\{Y(\mathbf{x}), \mathbf{x} \in \mathbb{D}\}$ is the Gaussian transform of the original point-support random field $\{Z(\mathbf{x}), \mathbf{x} \in \mathbb{D}\}$ with non-random locations.

Let $\mathbf{R}$ be the covariance matrix of the Gaussian data with randomized locations, and $\mathbf{C}_{\underline{\mathbf{x}}}$ the covariance vector between these data and $Y(\underline{\mathbf{x}})$. The simple kriging variance of $Y(\underline{\mathbf{x}})$ is

$$\sigma^{*2}(\underline{\mathbf{x}}) = 1 - \mathbf{C}_{\underline{\mathbf{x}}}^T\mathbf{R}^{-1}\mathbf{C}_{\underline{\mathbf{x}}}. \tag{27}$$

Because $\underline{\mathbf{x}}$ is uniformly distributed in v, $\mathbf{C}_{\underline{\mathbf{x}}}$ also coincides with the covariance vector between the Gaussian data with randomized locations and the regularized Gaussian variable $Y(v)$. Therefore, the simple kriging variance of this regularized variable is

$$\sigma^{*2}(v) = \mathrm{var}\left(Y(v)\right) - \mathbf{C}_{\underline{\mathbf{x}}}^T\mathbf{R}^{-1}\mathbf{C}_{\underline{\mathbf{x}}}. \tag{28}$$

Now, the variance of $Y(v)$ is the same as the covariance between $Y(\underline{\mathbf{x}})$ and $Y(\underline{\mathbf{x}}')$, where $\underline{\mathbf{x}}$ and $\underline{\mathbf{x}}'$ are random locations uniformly distributed in v, which in turn is equal to the square of the change-of-support coefficient (Emery and Ortiz 2005; Emery 2007)

$$\forall \underline{\mathbf{x}},\ \underline{\mathbf{x}}' \in v, \underline{\mathbf{x}} \neq \underline{\mathbf{x}}', \quad r^2 = \operatorname{cov}\big(Y(\underline{\mathbf{x}}), Y(\underline{\mathbf{x}}')\big) = \operatorname{var}\big(Y(v)\big). \tag{29}$$

From (8) and (27) to (29), it follows

$$\sigma^{*2}(v) = \sigma^{*2}(\underline{\mathbf{x}}) - 1 + r^2 = r^2 \sigma_v^{*2}, \tag{30}$$

so that the local change-of-support coefficient [(13)] can be written as

$$\forall \underline{\mathbf{x}} \in v, \quad r_{\underline{\mathbf{x}}} = \sqrt{\frac{\sigma^{*2}(v)}{\sigma^{*2}(v) + 1 - \operatorname{var}(Y(v))}}. \tag{31}$$

Appendix D

Still in the scope of the discrete Gaussian model, it is of interest to express the local variance reduction factor for the original random field (Z), when passing from the point support to the block support. Let us expand the point- and block-support transformation functions (5) and (15) into the normalized Hermite polynomials $\{H_p, p \in \mathbb{N}\}$ (Rivoirard 1994)

$$\begin{cases} Z(\underline{\mathbf{x}}) = \phi(Y(\underline{\mathbf{x}})) = \sum_{p=0}^{+\infty} \phi_p H_p(Y(\underline{\mathbf{x}})), \\ Z(v) = \phi_v(Y_v) = \sum_{p=0}^{+\infty} \phi_p r^p H_p(Y_v), \end{cases} \tag{32}$$

where r is the global change-of-support coefficient. Accordingly, the original random fields conditioned to the data can be written as follows [(7)]

$$\begin{cases} \tilde{Z}(\underline{\mathbf{x}}) = \sum_{p=0}^{+\infty} \phi_p H_p\big(y^*(\underline{\mathbf{x}}) + \sigma^*(\underline{\mathbf{x}}) U(\underline{\mathbf{x}})\big), \\ \tilde{Z}(v) = \sum_{p=0}^{+\infty} \phi_p r^p H_p\big(y_v^* + \sigma_v^* U_v\big). \end{cases} \tag{33}$$

To express the variances of these conditional random fields, the following translation formula is used (Emery 2005, p. 298)

$$H_p(a + by) = \sum_{n=0}^{p} \sqrt{C_p^n} \big(1 - b^2\big)^{(p-n)/2} b^n H_{p-n}\left(\frac{a}{\sqrt{1 - b^2}}\right) H_n(y). \tag{34}$$

Using (8) and (33)

$$\begin{cases} \tilde{Z}(\underline{\mathbf{x}}) = \sum_{p=0}^{+\infty} \tilde{\phi}_p(1) H_p(U(\underline{\mathbf{x}})), \\ \tilde{Z}(v) = \sum_{p=0}^{+\infty} \tilde{\phi}_p(r) H_p(U_v), \end{cases} \tag{35}$$

with $\forall p \in \mathbb{N}$ and $\forall t \in [0, 1]$

$$\tilde{\phi}_p(t) = \sum_{n=p}^{+\infty} \phi_n \sqrt{C_n^p} \big(1 - \sigma^{*2}(\underline{\mathbf{x}})\big)^{(n-p)/2} \big(\sigma^{*2}(\underline{\mathbf{x}}) + t^2 - 1\big)^{p/2} H_{n-p}\left(\frac{y^*(\underline{\mathbf{x}})}{\sqrt{1-\sigma^{*2}(\underline{\mathbf{x}})}}\right). \tag{36}$$

Because the Hermite polynomials are orthonormal for the standard Gaussian distribution, the local variances of the original random fields can be expressed as

$$\operatorname{var}\big(\tilde{Z}(\underline{\mathbf{x}})\big) = \sum_{p=1}^{+\infty} \tilde{\phi}_p^2(1) \quad \text{and} \quad \operatorname{var}\big(\tilde{Z}(v)\big) = \sum_{p=1}^{+\infty} \tilde{\phi}_p^2(r). \tag{37}$$

There is no reason why the ratio between both local variances should be equal to the global variance reduction factor, as it is often assumed in practical applications (Journel and Kyriakidis 2004, p. 40)

$$\frac{\operatorname{var}(\tilde{Z}(v))}{\operatorname{var}(\tilde{Z}(\underline{\mathbf{x}}))} \neq \frac{\operatorname{var}(Z(v))}{\operatorname{var}(Z(\underline{\mathbf{x}}))}. \tag{38}$$

In general, the local variance reduction factor depends on the configuration of data locations neighboring the location under consideration (through the kriging variance term $\sigma^{*2}(\underline{\mathbf{x}})$) and also on the grade values (through the kriging estimate $y^*(\underline{\mathbf{x}})$). Its calculation is quite tedious, so that a local change-of-support model based on this factor, such as the affine correction, turns out to be impractical.

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