

# Copula-based measures of dependence structure in assets returns

Viviana Fernandez\*

*Center for Applied Economics (CEA), Department of Industrial Engineering, University of Chile, Chile  
Trinity College, Dublin, Ireland*

## Abstract

Copula modeling has become an increasingly popular tool in finance to model assets returns dependency. In essence, copulas enable us to extract the dependence structure from the joint distribution function of a set of random variables and, at the same time, to isolate such dependence structure from the univariate marginal behavior. In this study, based on US stock data, we illustrate how tail-dependency tests may be misleading as a tool to select a copula that closely mimics the dependency structure of the data. This problem becomes more severe when the data is scaled by conditional volatility and/or filtered out for serial correlation. The discussion is complemented, under more general settings, with Monte Carlo simulations and portfolio management implications.

*Keywords:* Copulas; Tail dependence; Value-at-risk; Expected shortfall

## 1. Introduction

Modeling the dependence structure of assets returns has become an active line of research in the finance field in recent years. In particular, some researchers have resorted to extreme-value theory to model tail dependency and have concluded that the degree of such dependency is generally asymmetric (i.e., bearish (left tail) versus bullish markets (right tail)). In addition, they have found that correlated conditional return volatilities may partially account for extreme-value dependency (e.g., Poon, Rockinger, and Tawn [1] and [2]).

A more general methodology, which enables us to study not only the tail behavior but also the whole structure of dependency of a set of random variables, is copula modeling. Specifically, copulas are joint distribution functions of standard uniforms, which make it possible to extract the dependence structure from the joint probability distribution function of a set of random variables and, simultaneously, to isolate such dependence structure from the univariate marginal behavior. Examples of recent applications of copulas in finance are Cherubini and Luciano [3,4] and [5], Embrechts, Lindskog, and McNeil [6], Giesecke [7], Panchenko [8] Junker, Szimayer, and Wagner [9], Rosenberg and Schuermann [10], Mendes, Leal and Carvalhal-da-Silva [11], Fantazzini [12], Bartram, Taylor, and Wang [13], and Fernandez [14], among others. The textbook by Cherubini, Luciano, and Vecchiato [15] provides a complete discussion on the use of copulas in finance.

\* Corresponding address: Center for Applied Economics (CEA), Department of Industrial Engineering, University of Chile, Avenida Republica 701, Fourth floor, 6521122 Santiago, Region Metropolitana, Chile. Tel.: +56 2 978 4047; fax: +56 2 978 4011.

*E-mail address:* vfernand@dii.uchile.cl.

In this article, we illustrate how tail-dependency tests may be misleading tools to select a suitable copula function. This problem may become more severe when the data is scaled by volatility and/or filtered out for serial correlation. The discussion is complemented under different scenarios by means of Monte Carlo simulations. In addition, Value-at-Risk (VaR) and Expected Shortfall (ES) estimates are provided for alternative copula functions and marginal specifications. Our research is somehow connected with a recent publication by Fantazzini [12], who studies the impact of copula and marginal misspecification on VaR estimation. He concludes that biases in VaR estimates are predominantly due to copula misspecification in large samples.

This article is organized as follows. Section 2 presents background material on tail-dependency tests and copulas. Section 3 concentrates on an application of copula selection for daily data (June 1992–June 2006) of four US stock indices elaborated by Morgan Stanley, namely, US Investable Market Value, US Large Cap 300, US Mid Cap 450, and US Small Cap 1750. In a more general setting, the performance of tail-dependency tests in suggesting a proper copula function is assessed by Monte Carlo simulations. In addition, the sensitivity of a portfolio VaR and ES to alternative copula and marginal functions is illustrated. Section 4 concludes.

## 2. Methodology

### 2.1. Tail-dependency tests

#### 2.1.1. Asymptotic dependence and asymptotic independence

Extreme-value theory (EVT) has arisen as one of the most important statistical disciplines for the applied sciences over the past fifty years, and more recently for the finance field. The distinguishing feature of EVT is to quantify the stochastic behavior of a process at unusually large or small levels. Specifically, EVT usually requires estimation of the probability of events that are more extreme than any other that has been previously observed.

A customary approach in EVT, which is denominated as “Peaks over threshold” or POT, consists of modeling the behavior of extreme values above a high threshold. Let  $X_1, X_2, \dots$  be a sequence of independent and identically distributed random variables with unknown distribution function  $F$ . The excess distribution, above a threshold  $u$ , is given by the conditional probability distribution

$$F_u(y) = \Pr(X - u \leq y | X > u) = \frac{F(y + u) - F(u)}{1 - F(u)}, \quad y > 0. \quad (1)$$

Under some regularity conditions, there exists a positive function  $\beta(u)$ , for a large enough  $u$ , such that (1) is well approximated by the generalized Pareto distribution (GPD):

$$H_{\zeta, \beta(u)}(y) = \begin{cases} 1 - \left(1 + \frac{\zeta y}{\beta(u)}\right)^{-1/\zeta} & \zeta \neq 0 \\ 1 - \exp\left(-\frac{y}{\beta(u)}\right) & \zeta = 0, \end{cases} \quad (2)$$

where  $\beta(u) > 0$ , and  $y \geq 0$  when  $\zeta \geq 0$ , and  $0 \leq y \leq -\beta(u)/\zeta$  when  $\zeta < 0$  (e.g., Coles [16] or Embrechts, Klüpperberg and Mikosch [17]). If  $\zeta > 0$ ,  $F$  is said to be in the Fréchet family and  $H_{\zeta, \beta(u)}$  is a Pareto distribution. In most applications of risk management, the data comes from a heavy-tailed distribution, so that  $\zeta > 0$ .

Poon, Rockinger, and Tawn [1,2] introduce a special case of threshold modeling for the Fréchet family. In this particular case, the tail of a random variable  $Z$  above a (high) threshold  $u$  can be approximated as

$$1 - F(z) = \Pr(Z > z) \sim z^{-1/\eta} L(z), \quad \text{for } z > u \quad (3)$$

where  $L(z)$  is a slowly varying function of  $z$ ,<sup>1</sup> and  $\eta > 0$ . If  $L(z)$  is treated as a constant for all  $z > u$ , such that  $L(z) = c$ , and under the assumption of  $n$  independent observations, the maximum-likelihood estimators of  $\eta$  and  $c$  are

$$\hat{\eta} = \frac{1}{n_u} \sum_{j=1}^{n_u} \log\left(\frac{z_{(j)}}{u}\right) \quad \hat{c} = \frac{n_u}{n} u^{1/\hat{\eta}}, \quad (4)$$

where  $z_{(1)}, \dots, z_{(n_u)}$ , are the  $n_u$  observations above the threshold  $u$ , and  $\hat{\eta}$  is known as the Hill estimator.

<sup>1</sup> A function of  $L$  on  $(0, \infty)$  is slowly varying if  $\lim_{z \rightarrow \infty} L(tz)/L(z) = 1$  for  $t > 0$ .

In order to study dependency of paired returns, Poon et al. suggest transforming the original variables to a common marginal distribution. If  $(X, Y)$  are bivariate returns with corresponding cumulative distribution functions  $F_X$  and  $F_Y$ , the following transformation turns them into unit Fréchet marginals  $(S, T)$ <sup>2</sup>:

$$S = -\frac{1}{\ln F_X(X)} - 1 \quad T = -\frac{1}{\ln F_Y(Y)} - 1 \quad S > 0, T > 0. \quad (5)$$

Poon et al. define the following measure of asymptotic dependence:

$$\chi = \lim_{s \rightarrow \infty} \Pr(T > s | S > s) \quad 0 \leq \chi \leq 1. \quad (6)$$

In particular, two random variables are called asymptotically dependent if  $\chi > 0$ , and asymptotically independent if  $\chi = 0$ .

Coles, Heffernan and Tawn [18] point out that two random variables, which are asymptotically independent (i.e.,  $\chi = 0$ ), may show, however, different degrees of dependence for finite levels of  $s$ . Therefore, they propose the following measure of asymptotic independence:

$$\bar{\chi} = \lim_{s \rightarrow \infty} \frac{2 \log(\Pr(S > s))}{\log(\Pr(S > s, T > s))} - 1 \quad -1 < \bar{\chi} \leq 1. \quad (7)$$

Values of  $\bar{\chi} > 0$ ,  $\bar{\chi} = 0$  and  $\bar{\chi} < 0$  are an approximate measure of positive dependence, exact independence, and negative dependence in the tails, respectively. In particular,  $\bar{\chi}$  resembles a correlation coefficient, and it is identical to the Pearson correlation coefficient under normality.

Poon et al.'s tail-dependence test is based on the  $(\chi, \bar{\chi})$  pair. Specifically, for asymptotically dependent variables,  $\bar{\chi} = 1$  and the degree of dependence is measured by  $\chi > 0$ . For asymptotic independent variables,  $\chi = 0$  and the degree of dependence is measured by  $\bar{\chi}$ .

The above tail-dependence test rests on the fact that

$$\Pr(Z > z) = z^{-1/\zeta} L(z) \quad \text{for } z > u, \quad (8)$$

for some high threshold  $u$ , where  $Z = \min(S, T)$ . Eq. (8) shows that  $\zeta$  is the tail index of the univariate random variable  $Z$ . Therefore, it can be computed by using the Hill estimator, constrained to the interval  $(0, 1]$ . Under the assumption of independent observations on  $Z$ , Poon et al. show that

$$\hat{\bar{\chi}} = 2\hat{\zeta} - 1 = \frac{2}{n_u} \left( \sum_{j=1}^{n_u} \log \left( \frac{z^{(j)}}{u} \right) \right) - 1 \quad \text{Var}(\hat{\bar{\chi}}) = \frac{(\hat{\bar{\chi}} + 1)^2}{n_u}, \quad (9)$$

where  $\hat{\bar{\chi}}$  is asymptotically normal.

The null hypothesis of asymptotic dependence (i.e.,  $\bar{\chi} = 1$ ) is rejected if  $\hat{\bar{\chi}} + 1.96\sqrt{\text{Var}(\hat{\bar{\chi}})} < 1$ . In that case, we conclude that the two random variables are asymptotically independent (i.e.,  $\chi = 0$ ), and the degree of dependency is measured by  $\bar{\chi}$ . Otherwise, if the null hypothesis cannot be rejected,  $\chi$  is estimated under the assumption that  $\bar{\chi} = \xi = 1$ , where  $\hat{\chi} = \frac{un_u}{n}$  and  $\text{Var}(\hat{\chi}) = \frac{un_u(n-n_u)}{n^3}$ .

### 2.1.2. Threshold selection

In order to make Poon et al.'s statistic in (9) operational, one has to select an optimal threshold ( $u$ ). A straightforward and computationally fast algorithm suitable to that end is based on an exponential regression model discussed in Matthys and Beirlant [19], Beirlant, Diercks, Goegebeur, and Matthys [20], and Matthys and Beirlant [21]. This approach directly derives an estimator for  $u$  based on the representation of the asymptotic mean-squared error (AMSE) of the Hill estimator, as outlined below.

<sup>2</sup> Under this transformation,  $\Pr(S > s) = \Pr(T > s) \sim s^{-1}$ . As both  $S$  and  $T$  are on a common scale, the events  $\{S > s\}$  and  $\{T > s\}$ , for large values of  $s$ , correspond to equally extreme events for each one. Given that  $\Pr(S > s) \rightarrow 0$  as  $s \rightarrow \infty$ , the focus of interest is the conditional probability  $\Pr(T > s | S > s)$ , for large  $s$ . If  $(S, T)$  are perfectly dependent,  $\Pr(T > s | S > s) = 1$ . By contrast, if  $(S, T)$  are exactly independent,  $\Pr(T > s | S > s) = \Pr(T > s)$ , which tends to zero as  $s \rightarrow \infty$ .

Specifically, an exponential regression model for the log-spacings of upper statistics of a set of independent data  $X_1, X_2, \dots, X_n$  coming from a heavy-tailed distribution has shown to be (see Beirlant et al. [20])

$$j(\log(X_{n-j+1,n}) - \log(X_{n-j,n})) \sim \left( \gamma + b_{n,k} \left( \frac{j}{k+1} \right)^{-\rho} \right) f_j \quad 1 \leq j \leq k, \quad (10)$$

where  $X_{1,n} \leq X_{2,n} \leq \dots \leq X_{n,n}$ ,  $b_{n,k} \equiv b(\frac{n+1}{k+1})$  is a positive rate function such that  $b(x) \rightarrow 0$  as  $x \rightarrow \infty$ ,  $1 \leq k \leq n-1$ ,  $(f_1, f_2, \dots, f_k)$  is a vector of independent standard exponential random variables,  $\gamma > 0$  and  $\rho \leq 0$  are real constants.

If the threshold  $u$  is fixed at the  $(k+1)$ th largest observation, the Hill estimator can be rewritten as

$$H_{n,k} = \frac{1}{k} \sum_{j=1}^k j(\log(X_{n-j+1,n}) - \log(X_{n-j,n})) \quad (11)$$

so that it boils down to the maximum-likelihood estimator of  $\gamma$  in the reduced model  $j(\log(X_{n-j+1,n}) - \log(X_{n-j,n})) \sim \gamma f_j$ ,  $1 \leq j \leq k$ .

From the above, the AMSE of the Hill estimator is shown to be

$$\text{AMSE}H_{k,n} = \left( \frac{b_{n,k}}{1-\rho} \right)^2 + \frac{\gamma^2}{k}. \quad (12)$$

The optimal threshold  $k_n^{\text{opt}}$  is defined as the one that minimizes (12):

$$k_n^{\text{opt}} \equiv \arg \min_k (\text{AMSE}H_{k,n}) = \arg \min_k \left( \left( \frac{b_{n,k}}{1-\rho} \right)^2 + \frac{\gamma^2}{k} \right). \quad (13)$$

This can be obtained as follows:

- In expression (10), fix  $\rho$  at  $\rho_0 = -1$  and calculate least-squares estimates  $\hat{\gamma}_k$  and  $\hat{b}_{n,k}$  for each  $k \in \{3, \dots, n\}$ .
- Determine  $\overline{\text{AMSE}H_{k,n}} = \left( \frac{\hat{b}_{n,k}}{1-\hat{\rho}_k} \right)^2 + \frac{\hat{\gamma}_k^2}{k}$  for  $k \in \{3, \dots, n\}$ , with  $\hat{\rho}_k \equiv \rho_0$ .<sup>3</sup>
- Determine  $\hat{k}_n^{\text{opt}} = \arg \min_{3 \leq k \leq n} (\overline{\text{AMSE}H_{k,n}})$  and estimate  $\gamma$  by  $H_{\hat{k}_n^{\text{opt}}}$ .

The first step of the algorithm boils down to running a linear regression of  $j(\log(X_{n-j+1,n}) - \log(X_{n-j,n}))$  on a constant term and  $\frac{j(n+1)}{(k+1)^2}$ , for each  $k \in \{3, \dots, n\}$ .

## 2.2. Copula analysis

### 2.2.1. Basic concepts

A copula function is defined as a multivariate distribution function (df)  $F$  of random variables  $X_1, \dots, X_n$  with standard uniform marginal distribution functions  $F_1, \dots, F_n$  (margins), i.e.,  $X_i \sim F_i$ ,  $i = 1, \dots, n$ . Alternatively, it is defined as any function  $C: [0, 1]^n \rightarrow [0, 1]$  that satisfies the following properties (e.g., Cherubini, Luciano, and Vecchiato [3]; Frees and Valdez [22]): (i)  $C(x_1, \dots, x_n)$  is increasing in each component  $x_i$ ; (ii)  $C(1, \dots, 1, x_i, 1, \dots, 1) = x_i \forall i = 1, \dots, n, x_i \in [0, 1]$ ; (iii)  $\forall (a_1, \dots, a_n), (b_1, \dots, b_n) \in [0, 1]^n, a_i \leq b_i, \sum_{i=1}^2 \dots \sum_{i_n=1}^2 (-1)^{i_1 + \dots + i_n} C(x_{1i_1}, \dots, x_{ni_n}) \geq 0, x_{j1} = a_j$  and  $x_{j2} = b_j \forall j \in \{1, \dots, n\}$ .

In general, let us consider an  $n \times 1$  random vector  $\mathbf{X}$  with a joint df  $F$  and continuous margins  $F_i$ , which are not necessarily standard uniform.<sup>4</sup> Then

$$\begin{aligned} F(x_1, \dots, x_n) &= \Pr(X_1 \leq x_1, \dots, X_n \leq x_n) \\ &= \Pr(F_1(X_1) \leq F_1(x_1), \dots, F_n(X_n) \leq F_n(x_n)) \\ &= C(F_1(x_1), \dots, F_n(x_n)). \end{aligned} \quad (14)$$

<sup>3</sup> Matthys and Beirlant [19] point out that for many distributions the exponential-regression method works better, in MSE-sense, if the nuisance parameter  $\rho$  is fixed at some value  $\rho_0$  rather than estimated.

<sup>4</sup> A well-known result in statistics establishes that if  $X_i$  is a random variable with a continuous distribution function  $F_i$ , the random variable  $F_i(X_i)$  is standard-uniformly distributed, i.e.,  $F_i(X_i) \sim U(0, 1)$ .

Eq. (14) shows that the joint df,  $F$ , can be described by the margins  $F_1, \dots, F_n$  and the copula  $C$ , which captures the dependency structure among  $X_1, \dots, X_n$ . The existence of the function  $C$  is established by Sklar's theorem (see Nelsen [23], Section 2.10). The density function of  $X_1, \dots, X_n$  in turn can be expressed in terms of the density copula and the marginal densities:

$$\begin{aligned} f(x_1, x_2, \dots, x_n) &= \left( \frac{\partial^n C(F_1(x_1), F_2(x_2), \dots, F_n(x_n))}{\partial F_1(x_1) \partial F_2(x_2) \cdots \partial F_n(x_n)} \right) \frac{\partial F_1(x_1)}{\partial x_1} \frac{\partial F_2(x_2)}{\partial x_2} \cdots \frac{\partial F_n(x_n)}{\partial x_n} \\ &= c(F_1(x_1), F_2(x_2), \dots, F_n(x_n)) \prod_{i=1}^n f_i(x_i). \end{aligned} \quad (15)$$

In the bivariate case, which is the focus of this study, a copula function is defined on  $I^2 = [0, 1] \times [0, 1]$ , such that  $F(x, y) = C(F_X(x), F_Y(y)) \equiv C(u, v) = \Pr(U \leq u, V \leq v)$ , where  $U = F_X(x)$  and  $V = F_Y(y)$  are standard uniforms. The joint density function of  $X$  and  $Y$  can be expressed in terms of the copula density  $c(u, v) = \frac{\partial^2}{\partial u \partial v} C(u, v)$  and the corresponding marginal densities of  $X$  and  $Y$  as a particular case of Eq. (15).

Upper- and lower-tail dependency measures are given by (see, for instance, Cherubini et al. [3], Section 3.1.5)

$$\lambda_u = \lim_{q \rightarrow 1^-} \Pr(U > q | V > q) = \lim_{q \rightarrow 1^-} \frac{1 - 2q + C(q, q)}{1 - q} \quad (16a)$$

$$\lambda_l = \lim_{q \rightarrow 0^+} \Pr(U < q | V < q) = \lim_{q \rightarrow 0^+} \frac{C(q, q)}{q}. \quad (16b)$$

$C$  is said to have upper-tail dependence if and only if  $\lambda_u \in (0, 1]$ , and no upper-tail dependence if and only if  $\lambda_u = 0$ . Similarly,  $C$  is said to have lower-tail dependence if and only if  $\lambda_l \in (0, 1]$ , and no lower-tail dependence if and only if  $\lambda_l = 0$ .

Empirical counterparts of  $\lambda_u$  and  $\lambda_l$  can be obtained by plugging the empirical copula into Eqs. (16a) and (16b):

$$\hat{C} \left( \frac{i}{n}, \frac{j}{n} \right) = \frac{1}{n} \sum_{k=1}^n \mathbf{1}_{\{u_k \leq u(i), v_k \leq v(j)\}} \quad i, j = 1, 2, \dots, n, \quad (17)$$

where  $u_{(1)} \leq u_{(2)} \leq \dots \leq u_{(n)}$  and  $v_{(1)} \leq v_{(2)} \leq \dots \leq v_{(n)}$  are the order statistics.<sup>5</sup>

It is worth noticing that the coefficient  $\chi$  in expression (6) can be generalized to

$$\chi = \lim_{q \rightarrow 1^-} \Pr(U > q | V > q),$$

where  $U$  and  $V$  are the transformation of  $(X, Y)$  to uniform margins (see Coles, Heffernan, and Twan [18]). Therefore, under such a transformation,  $\chi$  coincides with  $\lambda_u$ .

One of the most frequently used copulas in the finance field is the Gaussian. For the bivariate case, the Gaussian copula boils down to

$$C(u, v) = \int_{-\infty}^{\Phi^{-1}(u)} dx \int_{-\infty}^{\Phi^{-1}(v)} dy \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{x^2 - 2\rho xy + y^2}{2(1-\rho^2)}\right) = \Phi_\rho(\Phi^{-1}(u), \Phi^{-1}(v)), \quad (18)$$

where  $\Phi_\rho$  is the joint df with correlation coefficient  $\rho$ .

One characteristic of the Gaussian copula is that it does not exhibit either lower- or upper-tail dependence unless  $\rho = 1$ . That is to say,

$$\lambda_u = \lambda_l = \begin{cases} 0 & \text{iff } \rho < 1 \\ 1 & \text{iff } \rho = 1. \end{cases}$$

<sup>5</sup> In general, the empirical copula can be obtained as  $\hat{C} \left( \frac{i_1}{n}, \frac{i_2}{n}, \dots, \frac{i_m}{n} \right) = \frac{1}{n} \sum_{k=1}^n \mathbf{1}_{\{u_{1k} \leq u_1(i_1), u_{2k} \leq u_2(i_2), \dots, u_{mk} \leq u_m(i_m)\}}$ ,  $1 \leq j \leq m$ ,  $i_j = 1, 2, \dots, n$ .

However, assets returns may exhibit extreme-value dependency in both tails. Therefore, recent studies have focused on the Student's  $t$  copula (e.g., Demarta and McNeil [24]; Mashal, Naldi, and Zeevi [25]). The bivariate  $t$  copula is defined as

$$C(u, v) = \int_{-\infty}^{t_v^{-1}(u)} dx \int_{-\infty}^{t_v^{-1}(v)} dy \frac{1}{2\pi\sqrt{1-\rho^2}} \left(1 + \frac{x^2 - 2\rho xy + y^2}{v(1-\rho^2)}\right)^{-\frac{v+2}{2}} = t_{\rho, v}(t_v^{-1}(u), t_v^{-1}(v)), \quad (19)$$

where  $t_v(x) = \int_{-\infty}^x \frac{\Gamma((v+1)/2)}{\sqrt{\pi v} \Gamma(v/2)} \left(1 + \frac{z^2}{v}\right)^{-\frac{v+1}{2}} dz$ .

Provided that  $\rho > -1$ , the Student's  $t$  copula exhibits asymptotic dependence in both tails, such that<sup>6</sup>

$$\lambda_u = \lambda_l = 2t_{v+1} \left( -\sqrt{\frac{(v+1)(1-\rho)}{1+\rho}} \right).$$

The other two copulas considered in this study are the Gumbel and Clayton copulas, which belong to the family of Archimedean copulas.<sup>7</sup> The Gumbel copula exhibits only upper-tail dependency:

$$C(u, v) = \exp\{-[(-\ln(u))^\delta + (-\ln(v))^\delta]^{1/\delta}\} \delta \geq 1 \quad (20)$$

$$\lambda_u = 2 - 2^{1/\delta}$$

whereas the Clayton copula displays exclusively lower-tail dependency:

$$C(u, v) = (u^{-\delta} + v^{-\delta} - 1)^{-1/\delta} \quad 0 < \delta < \infty \quad (21)$$

$$\lambda_l = 2^{-1/\delta}.$$

### 2.2.2. Maximum-likelihood estimation of a bivariate $t$ copula

Fitting a Student's  $t$  copula to the data is relatively cumbersome as it involves the estimation of two parameters,  $\rho$  and  $v$ , from a highly non-linear functional form. The optimization is hence conducted by maximum likelihood by concentrating out the log-likelihood function of the data in terms of the parameter  $\rho$ .

Specifically, the Student's  $t$  copula density is given by

$$c_{\rho, v}(u, v) = \rho^{-\frac{1}{2}} \frac{\Gamma\left(\frac{v+2}{2}\right) \Gamma\left(\frac{v}{2}\right) \left(1 + \frac{\kappa_1^2 + \kappa_2^2 - 2\rho\kappa_1\kappa_2}{v(1-\rho^2)}\right)^{-(v+2)/2}}{\Gamma\left(\frac{v+1}{2}\right)^2 \prod_{j=1}^2 \left(1 + \frac{\kappa_j^2}{v}\right)^{-(v+2)/2}}, \quad (22)$$

where  $\kappa_1 = t_v^{-1}(u)$  and  $\kappa_2 = t_v^{-1}(v)$  (see, for instance, Cherubini et al. [3], Section 3.2.2).

For a sample of  $n$  independent observations, estimates of  $\rho$  and  $v$  can be obtained by maximizing the log-likelihood function of the sample:

$$\log L = \sum_{i=1}^n \log(c_{\rho, v}(u_i, v_i)). \quad (23)$$

Given that both  $\kappa_{1,i} = t_v^{-1}(u_i)$  and  $\kappa_{2,i} = t_v^{-1}(v_i)$ ,  $i = 1, \dots, n$ , depend on the unknown parameter  $v$ , a grid search is conducted over  $v$  and  $\log L$  is maximized with respect to  $\rho$ , for every fixed value of  $v$ . The solution is the paired combination that maximizes  $\log L$ .

<sup>6</sup> When the number of degrees of freedom is large enough, the  $t$  copula will approximate a normal one.

<sup>7</sup> Archimedean copulas are those that can be represented as  $C(u, v) = \phi^{-1}\{\phi(u) + \phi(v)\}$ , where  $\phi : I \rightarrow R^+$  is a continuous, strictly decreasing, convex function, which satisfies  $\phi(1) = 0$ .

Table 1  
Descriptive statistics of MSCI US returns: June 1992–June 2006

Statistic	US Investable Market Value	US Large Cap 300	US Mid Cap 450	US Small Cap 1750
Minimum	−0.060	−0.072	−0.083	−0.063
Maximum	0.009	0.010	0.010	0.010
1st Q	−0.004	−0.005	−0.004	−0.005
3rd Q	0.005	0.005	0.006	0.006
Mean	0.000	0.000	0.000	0.000
Std. dev	0.051	0.057	0.055	0.050
Skewness	−0.183	−0.080	−0.302	−0.272
Kurtosis−3	3.974	4.324	4.458	2.994
Observations	3.674	3.674	3.674	3.674

Log-returns are daily.

### 3. Data and estimation results

#### 3.1. Copula choice for set of US stock indices

Our data set comprises the following Morgan Stanley Capital Investment (MSCI) indices: US Investable Market Value, US Large Cap 300, US Mid Cap 450, and US Small Cap 1750. The data is measured at a daily frequency and covers 15 years: June 1992–June 2006. All the estimation process is carried out in S-Plus 7.0.

Some descriptive statistics are presented in Table 1. As previously found in other studies, returns exhibit excess kurtosis and negative skewness. All returns series are comparably volatile, as measured by their standard deviation and inter-quartile range.

The presence of extreme-value dependence can be informally assessed by transforming the original return series into standard uniforms, so that they are in a common scale. Specifically, let  $(X_t, Y_t)$ ,  $t = 2, \dots, T$ , be independent observations of paired returns. The random variables  $u_t = F_X(X_t)$  and  $v_t = F_Y(Y_t)$  are both distributed as standard uniforms, where  $F_X$  and  $F_Y$  are the marginal distribution functions. Given that  $F_X$  and  $F_Y$  are both unknown, they can be replaced by the corresponding empirical marginals. Fig. 1, Panels (a) and (b), presents scatter plots of the transformed paired large/mid cap and investable market value/small cap return series. As we see, it seems that the former exhibits more tail dependency than the latter judging by the clustering of points in the left-lower and right-upper corners of each plot.

The above conjecture is supported by Fig. 1, Panels (c) and (d), which presents estimates of the left- and right-tail indices for each pair based on their empirical copulas. Specifically,  $\hat{\lambda}_u = \lim_{q \rightarrow 1^-} \frac{1-2q+\hat{C}(q,q)}{1-q}$  and  $\hat{\lambda}_l = \lim_{q \rightarrow 0^+} \frac{\hat{C}(q,q)}{q}$ , where the empirical copula is computed according to Eq. (17). For the large/mid cap pair, the estimates of upper- and lower-tail dependence indices are around 0.64–0.68 and 0.68–0.71, respectively, as shown in Panel (c) of Fig. 1. The estimated tail-dependency index parameters for the investable market value/small cap are by contrast slightly smaller: around 0.43–0.44 in the upper tail and 0.52–0.55 in the lower tail (Panel (d)).

Table 2 reports the computation of Poon et al.'s tail-dependency test for both pairs based on the discussion of Sections 2.1.1 and 2.1.2. As we see in Panel (a), extreme-value dependency is not rejected for the large/mid cap pair in either tail, whereas for the investable market value/small cap, extreme-value dependency is accepted at the 5 percent significance level in the lower tail, but it is rejected in the upper tail at the 1 percent significance level. For the large/mid cap pair, the estimated tail index parameters are 0.69 and 0.67 in the lower and upper tails, respectively. These are fairly close to those reported above.

As shown in Panel (b) of Table 2, filtering the returns data by an AR(1)–GARCH(1, 1) model leads to rejecting upper-tail dependency in both paired returns series. The null hypothesis of lower-tail dependency continues to be accepted for the large/mid cap pair, whereas it is now rejected at a lower significance level for the investable market value/small cap (i.e., 2% significance level).

Our next step consists of fitting a suitable copula to the data based on the dependency tests just reported. We first focus on the raw data and then on the filtered data. Fig. 2 shows the result of fitting normal and Student's  $t$

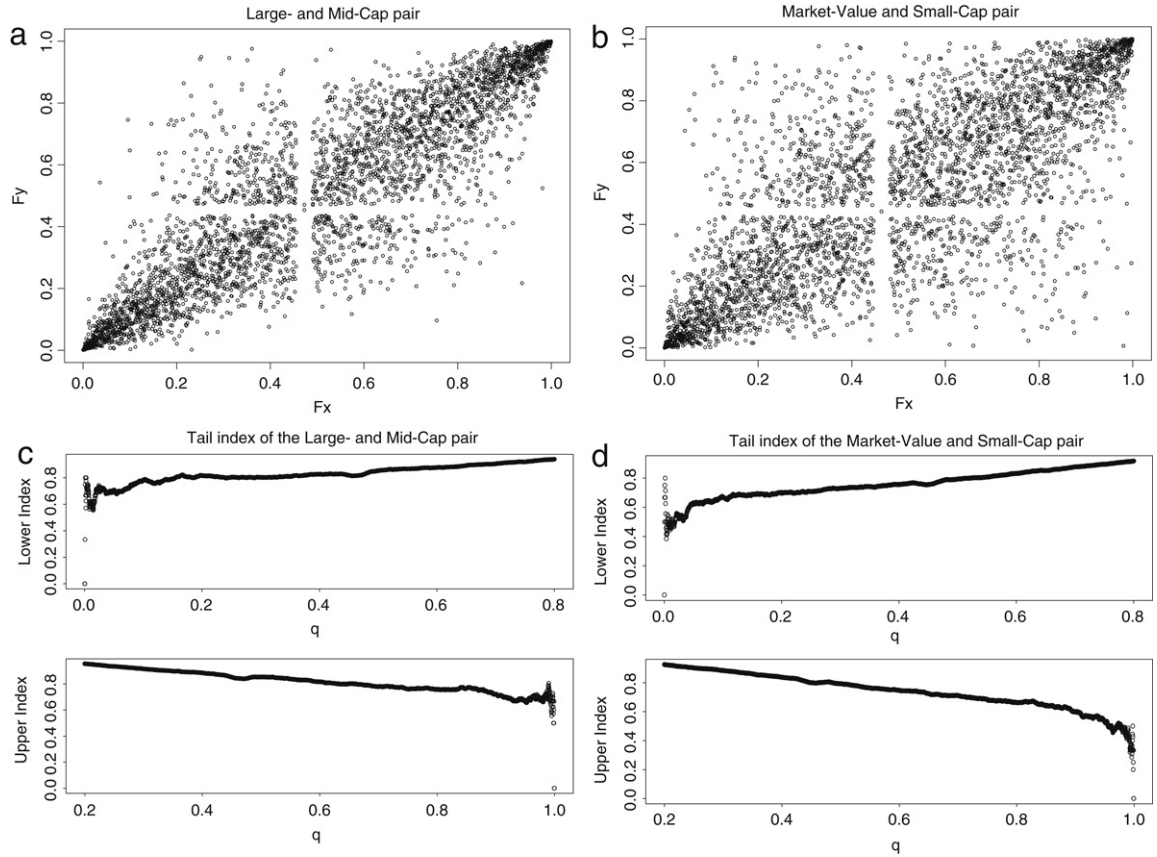


Fig. 1. Tail dependence in the raw returns series. Notes: (1) In panels (a) and (b), the random variables  $u = F_X(X)$  and  $v = F_Y(Y)$  are both distributed as standard uniforms, where  $F_X$  and  $F_Y$  are the marginal distribution functions, which are computed from the empirical distribution functions of  $X$  and  $Y$ . (2) In panels (c) and (d), the upper- and left-tail dependence indices are computed, respectively, as  $\hat{\lambda}_u = \lim_{q \rightarrow 1^-} (1 - 2q + \hat{C}(q, q)) / (1 - q)$  and  $\hat{\lambda}_l = \lim_{q \rightarrow 0^+} \hat{C}(q, q) / q$ , where the empirical copula is computed according to  $\hat{C}\left(\frac{i}{n}, \frac{j}{n}\right) = \frac{1}{n} \sum_{k=1}^n 1_{\{u_k \leq u(i), v_k \leq v(j)\}}$ ,  $i, j = 1, 2, \dots, n$ .

Table 2  
Extreme-value dependency test

Paired return	Lower tail						Upper tail								
	$\rho$	$k^*$	$\bar{\chi}$	s.e	$t$ -test $\bar{\chi} = 1$	$p$ -value	$\chi$	s.e	$k^*$	$\bar{\chi}$	s.e	$t$ -test $\bar{\chi} = 1$	$p$ -value	$\chi$	s.e
(a) Raw data															
Large/mid cap	0.91	191	0.92	0.14	-0.61	0.27	0.69	0.05	250	0.96	0.12	-0.31	0.38	0.67	0.04
Value/small cap	0.81	299	0.83	0.11	-1.63	0.05	0.64	0.03	297	0.75	0.10	-2.48	0.01	0.59	0.03
(b) Filtered data															
Large/mid cap	0.90	221	0.95	0.13	-0.35	0.36	0.68	0.04	299	0.77	0.10	-2.29	0.01	0.68	0.04
Value/small cap	0.80	231	0.76	0.12	-2.1	0.02	0.64	0.04	299	0.66	0.10	-3.5	0.00	0.55	0.03

(1)  $\rho$  is Pearson correlation coefficient for the whole sample period. (2)  $k^*$  represents the optimal threshold obtained by an exponential-regression procedure. (3)  $\bar{\chi}$  is computed based on tail-index estimation of Fréchet transformed margins of co-exceedances of paired returns,  $Z = \min(S, T)$ . Asymptotic dependence cannot be rejected if  $\bar{\chi} = 1$ . In that case, the degree of dependence is measured by  $\chi > 0$ .



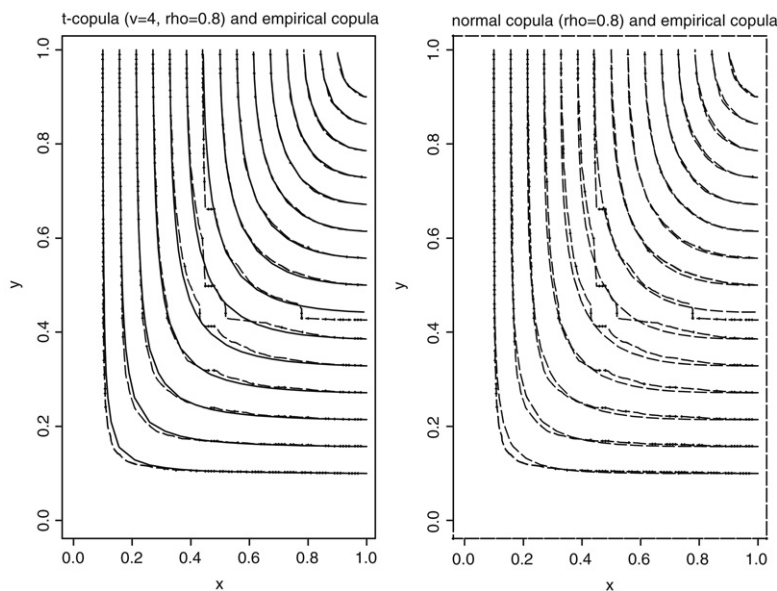


Fig. 2. Student's  $t$  and normal copulas fitted to the investable market value and small cap pair.

copulas to the investable market value/small cap pairs.<sup>8</sup> As a benchmark, the empirical copula is plotted along with each parametric model. As previously discussed, Poon et al.'s test suggests that there is tail independence in this pair. Therefore, a normal copula should be an appropriate choice (right-hand side panel of Fig. 2). However, a  $t$  copula appears to be a better fit as it captures more accurately the dependence structure in the lower tail and in the center of the bivariate distribution. (The degrees of freedom and correlation coefficient are computed by the method of maximum likelihood, which was previously discussed). Indeed, based on the Akaike, Schwarz, and Hannan-Quinn information criteria, the  $t$  copula outperforms the normal copula.<sup>9</sup>

We follow a similar procedure for the large/mid cap pair. Given that in this case Poon et al.'s test does not reject lower- and upper-tail dependency, our choice is a  $t$  copula. As a matter of comparison, we also fit a normal copula. As before, we conclude by computing the above three information criteria that a  $t$  copula provides a better fit.

Further evidence on the goodness of fit of the  $t$  copula is provided by Fig. 3, which depicts QQ-plots of actual and simulated returns for the four indices. In general, we see that the simulated data resembles the actual returns to a great extent.

As mentioned earlier, after filtering the raw data, lower- and upper-tail dependency is rejected for the investable market value/small cap pair at the 5% significance level. Fig. 4, panel (b), shows that the estimates of the tail indices are indeed smaller than in the raw data, particularly so in the upper tail. Therefore, a normal copula seems a suitable choice in this case.<sup>10</sup>

For the large/mid cap paired returns, after filtering Poon et al.'s test suggests choosing a Clayton copula as it only detects lower-tail dependency. As a matter of comparison, we also fit a  $t$  copula to the filtered data because panel (a) of Fig. 4 indicates that the tail index is numerically similar in both tails and approximately equal to 0.6. Our estimation results show that a  $t$  copula, with 6 degrees of freedom and a correlation coefficient of 0.9, mimics the dependency pattern of the data more closely than a Clayton copula.<sup>11</sup>

In sum, for the investable market value/small cap pair filtering does have an impact on our choice of a suitable copula. In particular, filtering washes away tail dependency and a  $t$  copula becomes unsuitable. By contrast, for the

<sup>8</sup> Our S-Plus code draws from that developed by Dean Fantazzini in GAUSS, which is freely available at [http://economia.unipv.it/pagp/pagine\\_personali/dean/programs/t\\_copula\\_simul\\_est\\_new](http://economia.unipv.it/pagp/pagine_personali/dean/programs/t_copula_simul_est_new).

<sup>9</sup> For the normal copula the Akaike, Schwarz, and Hannan-Quinn information criteria are, respectively,  $-1.003$ ,  $-1.001$ , and  $-1.002$ , whereas for the  $t$  copula, they are, respectively,  $-1.092$ ,  $-1.089$ , and  $-1.091$ .

<sup>10</sup> When attempting to fit a  $t$  copula to the paired filtered returns, we faced numerical problems.

<sup>11</sup> The computed Akaike, Schwarz, and Hannan-Quinn information criteria for the  $t$  copula are  $-1.690$ ,  $-1.687$ , whereas for the Clayton copula, they are  $-0.886$ ,  $-0.885$ , and  $-0.886$ , respectively.

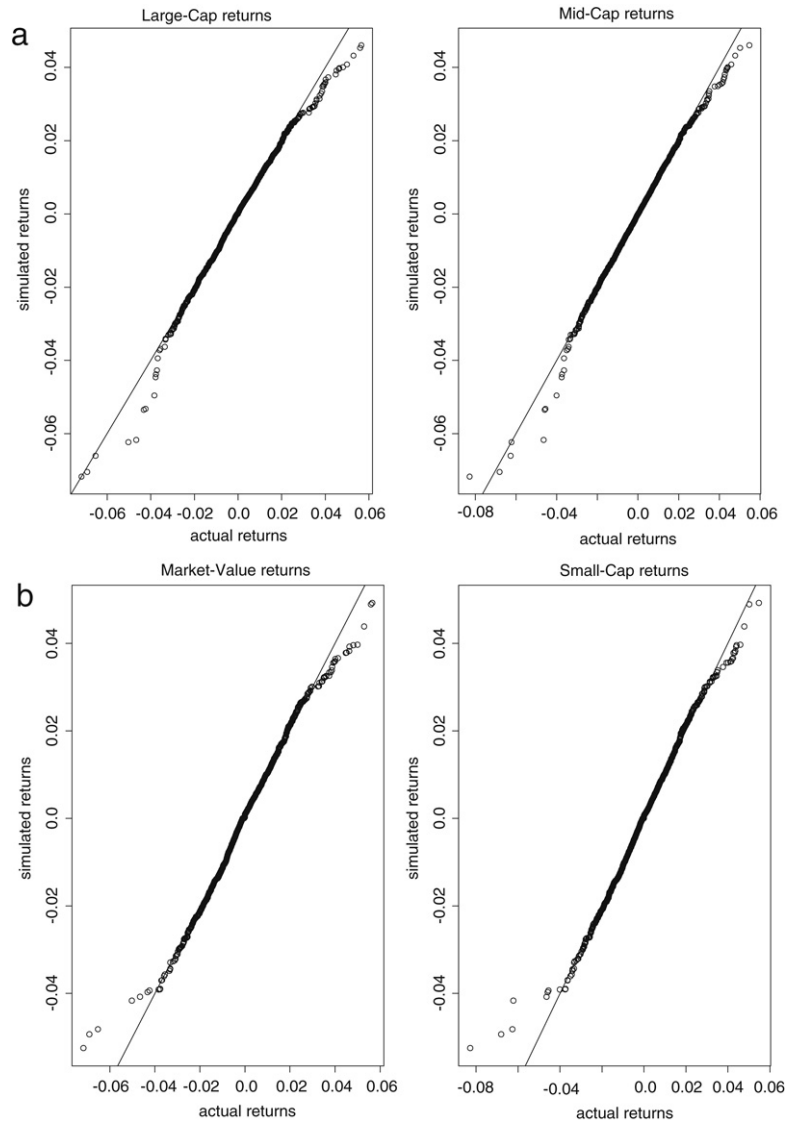


Fig. 3. QQ-plot of actual and simulated returns based on a Student's  $t$  copula.

large/mid cap pair, filtering does not reduce tail dependency considerably and a  $t$  copula continues to provide the best fit. The impact of filtering on tail dependency only translates into a greater estimate of the number of degrees of freedom of the  $t$  copula. As to Poon et al.'s test, in the latter case it proves misleading by suggesting a Clayton copula, whereas in the former case it correctly recommends using a normal copula.

### 3.2. An examination of the size and power of the extreme-value dependence test

In view of the above results, Poon et al.'s test may render unsatisfactory to unveil the true tail dependency in the data. In order to examine this issue in a more general setting, we carry out six Monte Carlo experiments. The first two experiments consist of generating two returns series of 1000 observations each from GARCH(1, 1) processes, whose joint behavior is assumed to be adequately represented by a  $t$  copula with 5 degrees of freedom ( $\nu$ ) and moderate and relatively high correlation coefficients ( $\rho$ ) of 0.5 and 0.8, respectively (Panels (a) and (b)). In both cases, by construction, the returns series exhibit asymptotic dependency in both tails and the null hypothesis holds. Specifically, when  $\nu = 5$  and  $\rho = 0.5$ ,  $\lambda_u = \lambda_l = 0.21$ , whereas when  $\nu = 5$  and  $\rho = 0.8$ ,  $\lambda_u = \lambda_l = 0.45$ .

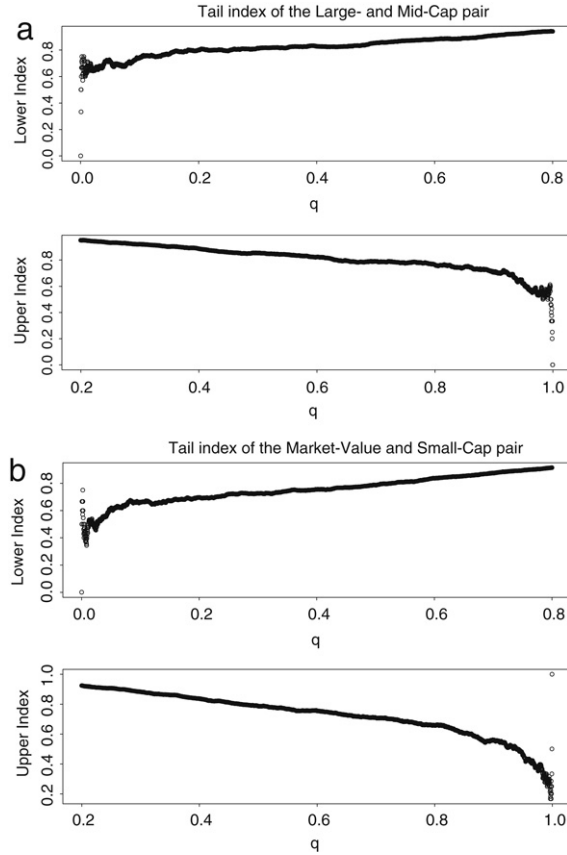


Fig. 4. Tail index parameters computed from empirical copulas of filtered returns. Note: The upper- and left-tail dependence indices are computed, respectively, as  $\hat{\lambda}_u = \lim_{q \rightarrow 1^-} (1 - 2q + \hat{C}(q, q)) / (1 - q)$  and  $\hat{\lambda}_l = \lim_{q \rightarrow 0^+} \hat{C}(q, q) / q$ , where the empirical copula is computed according to  $\hat{C}\left(\frac{i}{n}, \frac{j}{n}\right) = \frac{1}{n} \sum_{k=1}^n 1_{\{u_k \leq u(i), v_k \leq v(j)\}}$ ,  $i, j = 1, 2, \dots, n$ .

The following two other experiments consist of taking normal copulas with the same correlation coefficients and marginals as above (Panels (c) and (d)). In these two cases, by construction, the null hypothesis is false because the normal copula does not exhibit either left- or right-tail dependency unless  $\rho = 1$ .

Finally, two other cases in which the paired returns exhibit either left- or right-tail dependency are considered. To that end, the selected copulas are a Clayton with parameter  $\delta = 0.5$  and a Gumbel with parameter  $\delta = 1.25$ . The tail indices are, respectively,  $\lambda_l = 0.25$ ,  $\lambda_u = 0$  and  $\lambda_l = 0$ ,  $\lambda_u = 0.26$ .

Each of the six Monte Carlo experiments is repeated 100 hundred times, and Poon et al.'s test is computed for the lower and upper tails at each iteration.<sup>12</sup> Our results are reported in Table 3, Panels (a) through (f). As we see, Panel (a) suggests that Poon et al.'s test exhibits a severe size distortion for moderate left- and right-tail dependency. For instance, at the nominal size (i.e., significance level) of 1%, the null hypothesis is rejected 58% and 55% of the time in the upper and lower tail, respectively. Panel (b) in turn shows that the size distortion decreases considerably as the correlation coefficient and, consequently, the left- and right-tail indices increase. That is to say, for given number of degrees of freedom, Poon et al.'s test can discriminate a true hypothesis more effectively when the degree of co-movement of the two series is stronger.

Panels (c) and (d) show power test computations based on a normal copula. For a small correlation coefficient, the power of the test approaches 1. That is, the false null hypothesis is virtually always rejected. However, the power of the test decreases to a great extent for a higher correlation coefficient. For instance, when  $\rho = 0.8$ , the power of the test is only 33 and 28 percent in the lower and upper tail, respectively, for a one percent significance level. The

<sup>12</sup> The necessary time to compute 100 iterations for each experiment is around 50 min on a Pentium 4 with 750 MB of RAM.

Table 3  
Simulation of rejection rates of tail-dependency test

Percentage rejection rate of  $H_0$ : tail dependence

	1% significance level	5% significance level	10% significance level
(a) Data generating process: Student's $t$ copula ( $\nu = 5, \rho = 0.5$ )			
Lower tail	58	82	86
Upper tail	55	80	90
(b) Data generating process: Student's $t$ copula ( $\nu = 5, \rho = 0.8$ )			
Lower tail	4	16	29
Upper tail	4	16	30
(c) Data generating process: Normal copula ( $\rho = 0.5$ )			
Lower tail	98	100	100
Upper tail	98	100	100
(d) Data generating process: Normal copula ( $\rho = 0.8$ )			
Lower tail	33	65	78
Upper tail	28	65	82
(e) Data generating process: Clayton copula ( $\delta = 0.5$ )			
Lower tail	32	56	74
Upper tail	100	100	100
(f) Data generating process: Gumbel copula ( $\delta = 1.25$ )			
Lower tail	100	100	100
Upper tail	45	68	80

(1) Individual return series of 1000 observations each are generated from GARCH(1, 1) processes with variance equations  $\sigma_{1t}^2 = 0.001 + 0.1\varepsilon_{t-1}^2 + 0.8\sigma_{t-1}^2$  and  $\sigma_{2t}^2 = 0.001 + 0.2\varepsilon_{t-1}^2 + 0.7\sigma_{t-1}^2$ . The marginal distribution functions of such GARCH(1, 1) processes are estimated according to Carmona's [26] semi-parametric procedure. (2) Results are obtained from 100 simulations.

reason is that the normal copula will exhibit asymptotic dependence when  $\rho = 1$ . So the test does not discriminate well between high and perfectly positive dependency.

Finally, Panels (e) and (f) of Table 3 again suggest the existence of size distortions when either the left- or right-tail dependency is moderate. However, Poon's test correctly rejects one hundred percent of the time the null hypothesis of extreme-value dependence in the left tail for the Gumbel copula and in the right tail for the Clayton copula.

In sum, we can conclude that when tail dependency is relatively weak, Poon et al.'s test exhibits size distortions or low power, depending upon the case under analysis. Therefore, in empirical applications, it is advisable to complement Poon's et al. test with alternative tools in order to unveil the true data generating process, as illustrated in the previous section.

Finally, Table 4, Panels (a) and (b), presents value-at-risk (VaR) and expected shortfall (ES) estimates for alternative copula and marginal functions. The one-period log-return of an equally weighted two-asset portfolio is computed as  $r = \ln(\omega_1 e^X + \omega_2 e^Y)$ , where  $X = \ln(P_{1,t+1}/P_{1t})$  and  $Y = \ln(P_{2,t+1}/P_{2t})$  are the log-returns on assets 1 and 2, respectively. The VaR and ES are computed as  $\text{VaR}_q = F_{-r}^{-1}(q)$  and  $\text{ES}_q = E(-r | -r > \text{VaR}_q)$  for a given loss probability of  $(1 - q)$  and distribution function  $F_r$ . For instance, in Panel (a), for the Clayton copula, with a 1% probability, the portfolio could lose, on average, 3.4% in one period if the portfolio return was less than -2.9% per period.

As we can see from Table 4, the impact of the copula choice on VaR and ES becomes more noticeable at the 95 and 99% confidence levels. In particular, for the chosen parameter values and given that the Clayton copula displays only lower-tail dependence, the VaR and ES turn out to be highest under this functional form. Panel (b) indicates that VaR and ES are also sensitive to the assumption of GARCH effects in the data. In particular, if one neglected such effects, it would considerably underestimate the portfolio risk.

Table 4  
Value-at-Risk (VaR) and Expected Shortfall (ES) estimates

Copula	90% VaR	90% ES	95% VaR	95% ES	99% VaR	99% ES
(a) Normal marginal returns						
Clayton ( $\delta = 0.25$ )	0.016	0.022	0.021	0.026	0.029	0.034
Gumbel ( $\delta = 1.25$ )	0.015	0.020	0.019	0.023	0.026	0.030
Normal ( $\rho = 0.7$ )	0.015	0.021	0.019	0.024	0.027	0.032
$t$ ( $\nu = 5, \rho = 0.7$ )	0.015	0.021	0.019	0.025	0.028	0.033
(b) GARCH(1, 1) marginal returns						
Clayton ( $\delta = 0.25$ )	0.032	0.048	0.043	0.059	0.068	0.082
Gumbel ( $\delta = 1.25$ )	0.035	0.049	0.046	0.059	0.067	0.079
Normal ( $\rho = 0.7$ )	0.036	0.051	0.047	0.061	0.070	0.083
$t$ ( $\nu = 5, \rho = 0.7$ )	0.036	0.052	0.047	0.064	0.073	0.089

(1) In Panel (a), the marginals are simulated from  $N(0.0003, 0.015)$  and  $N(0.0002, 0.011)$ . (2) In Panel (b), the marginals are simulated from GARCH(1,1) processes with variance equations  $\sigma_{1t}^2 = 0.001 + 0.1\varepsilon_{t-1}^2 + 0.8\sigma_{1t-1}^2$  and  $\sigma_{2t}^2 = 0.001 + 0.2\varepsilon_{t-1}^2 + 0.7\sigma_{2t-1}^2$ . The marginal distribution functions of such GARCH(1,1) processes are estimated according to Carmona's [26] semi-parametric procedure. (3) In both panels, the sample size is 1000 and the simulations are repeated 1000 times.

#### 4. Conclusions

We discussed the choice of an optimal copula function of paired returns aimed at adequately capturing the co-movement between two financial series. Our application focused on daily data of four Morgan Stanley US stock indices: US Investable Market Value, US Large Cap 300, US Mid Cap 450, and US Small Cap 1750 for the sample period June 1992–June 2006. Our estimation results showed that a Student's  $t$  copula, which allows for lower- and upper-tail dependency, works well in general, and that, filtering returns may have an impact on the choice of the most suitable copula.

We also computed Poon et al.'s dependency test to complement our analysis, and found that this can be sometimes misleading as a guidance to select a suitable copula. We further discussed this issue by means of Monte Carlo simulations, which showed that Poon et al.'s test may exhibit size distortions and low power. In addition, in line with recent research, we concluded that the choice of a proper copula is essential to an accurate estimation of value-at-risk and expected shortfall.

#### Acknowledgements

Financial support from SOC 06/01-2 Grant (Research Division of the University of Chile) and from an institutional grant of the Hewlett Foundation to CEA is greatly acknowledged. The helpful comments of participants at the 2007 Southwest Finance Association (SWFA) Meeting are also acknowledged. All remaining errors are the author's.

#### References

- [1] S. Poon, M. Rockinger, J. Tawn, Modeling extreme-value dependence in international stock markets, *Statistica Sinica* 13 (2003) 929–953.
- [2] S. Poon, M. Rockinger, J. Tawn, Extreme value dependence in financial markets: Diagnostics, models, and financial implications, *Review of Financial Studies* 17 (2004) 581–610.
- [3] U. Cherubini, E. Luciano, Bivariate option pricing with copulas, *Applied Mathematical Finance* 9 (2) (2002) 69–85.
- [4] U. Cherubini, E. Luciano, Pricing and hedging credit derivatives with copulas, *Economic Notes* 32 (2) (2003) 219–242.
- [5] U. Cherubini, E. Luciano, Pricing vulnerable options with copulas, *Journal of Risk Finance* 5 (1) (2003) 27–39.
- [6] P. Embrechts, F. Lindskog, A. McNeil, Modeling dependence with copulas and applications to risk management, in: S.T. Rachev (Ed.), *Handbooks in Finance*, in: *Handbook of Heavy Tailed Distribution in Finance*, vol. 1, 2003, pp. 331–384 (Chapter 8).
- [7] K. Giesecke, Correlated default with incomplete information, *Journal of Banking & Finance* 28 (7) (2004) 1521–1545.
- [8] V. Panchenko, Goodness-of-fit test of copulas, *Physica A* 355 (2005) 176–182.
- [9] M. Junker, A. Szimayer, N. Wagner, Nonlinear term structure dependence: Copula functions, empirics, and risk implications, *Journal of Banking & Finance* 30 (4) (2006) 1171–1199.
- [10] J. Rosenberg, T. Schuermann, A general approach to integrated risk management with skewed, fat-tailed risks, *Journal of Financial Economics* 79 (3) (2006) 569–614.
- [11] B. Mendes, R. Leal, A. Carvalhal-da-Silva, Clustering in emerging equity markets, *Emerging Markets Review* 8 (3) (2007) 194–205.

- [12] D. Fantazzini, The effects of misspecified marginals and copulas on computing the value at risk: A Monte Carlo study, *Computational Statistics and Data Analysis*, in press (doi:10.1016/j.csda.2008.02.002). Corrected Proof, Available online 8 February 2008.
- [13] S. Bartram, S. Taylor, Y. Wang, The Euro and European financial market dependence, *Journal of Banking & Finance* 31 (5) (2007) 1461–1481.
- [14] V. Fernandez, Multi-period hedge ratios for a multi-asset portfolio when accounting for returns co-movement, *The Journal of Futures Markets* 28 (2) (2008) 182–207.
- [15] U. Cherubini, E. Luciano, W. Vecchiato, *Copula Methods in Finance*, Wiley, 2004.
- [16] S. Coles, *An Introduction to Statistical Modeling of Extreme Values*, in: Springer Series in Statistics, Springer-Verlag, London Limited, 2001.
- [17] P. Embrechts, C. Klüppelberg, T. Mikosch, *Modelling Extremal Events for Insurance and Finance*, Springer-Verlag, Berlin, Heidelberg, 1997.
- [18] S. Coles, J. Heffernan, J. Tawn, Dependence measures for extreme value analyses, *Extremes* 2 (1999) 339–365.
- [19] G. Matthys, J. Beirlant, Adaptive threshold selection in tail index estimation, mimeo, Center for Statistics at the Catholic University Leuven, 2000.
- [20] J. Beirlant, G. Dierckx, Y. Goegebeur, G. Matthys, Tail index estimation and an exponential regression model, *Extremes* 2 (1999) 177–200.
- [21] G. Matthys, J. Beirlant, Estimating the extreme value index and high quantile with exponential regression models, *Statistica Sinica* 13 (2003) 853–880.
- [22] E. Frees, E. Valdez, Understanding relationships using copulas, *North American Actuarial Journal* 2 (1) (1998) 1–25.
- [23] R. Nelsen, *An introduction to copulas*, in: *Lecture Notes in Statistics*, vol. 139, Springer-Verlag, New York, Inc., 1999.
- [24] S. Demarta, A. McNeil, The t copula and related copulas, *International Statistical Review* 73 (1) (2005) 111–129.
- [25] R. Mashal, M. Naldi, A. Zeevi, Comparing the dependence structure of equity and asset returns, *Risk* 16 (2003) 82–87.
- [26] R. Carmona, *Statistical Analysis of Financial Data in Splines*, Springer-Verlag, New York, 2004.