Block transitivity and degree matrices

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Abstract

We say that a square matrix M is a degree matrix of a given graph G if there is a so called equitable partition of its vertices into r blocks such that whenever two vertices belong to the same block, they have the same number of neighbors inside any block.

We ask now whether for a given degree matrix M, there exists a graph G such that M is a degree matrix of G, and in addition, for any two edges e, f spanning between the same pair of blocks there exists an automorphism of G that sends e to f. In this work, we fully characterize the matrices for which such a graph exists and show a way to construct one.

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1 Introduction

Definition 1.1 *We call a square matrix M of order r a degree matrix of a graph G if there is a partition (that we will call* equitable partition *or simply* block partition*) of $V(G)$ into blocks $\mathcal{G} = (G_1, \ldots, G_r)$ such that, for every i and every $v \in G_i$, we have:*

$$\forall j : |\mathcal{N}_G(v) \cap G_j| = m_{i,j}.$$

Degree matrices are fully characterized in the following way [1]:

Lemma 1.2 *A non-negative integer square matrix M of order r is a degree matrix if and only if the following conditions are satisfied simultaneously:*

(i) *(Plus-symmetry) For every $1 \le i, j \le r$, $m_{i,j} = 0 \Longrightarrow m_{j,i} = 0$.*

(ii) *(Cycle product identity) For every sequence of indices $i_1, i_2, \ldots, i_k, i_{k+1}$, $k \ge 2$, such that $i_{k+1} = i_1$,*

$$\prod_{j=1}^{k} m_{i_j, i_{j+1}} = \prod_{j=1}^{k} m_{i_{j+1}, i_j}.$$

Definition 1.3 *We say that a graph G with degree matrix M of order r and block partition $\mathcal{G}$ is* block transitive*, if for each pair of edges e and f connecting nodes from the same pair of blocks, there exists an automorphism φ of G that preserves the partition, i.e., if $v \in G_i$, then $\varphi(v) \in G_i$, and that* sends e to f.

The main result in this work is that the answer is always positive. i.e. for every matrix $M \in \mathcal{M}$ we can construct a block transitive graph that has M as its degree matrix.

2 Block product

Definition 2.1 *Let G and H be graphs with block partition $\mathcal{G}$ and $\mathcal{H}$ respectively, both of size r. We construct the* block product graph $J = G \otimes H$ *associated to the partitions $\mathcal{G}$ and $\mathcal{H}$ as follows:*

(i) $V(J) = (G_1 \times H_1) \cup (G_2 \times H_2) \cup \ldots \cup (G_r \times H_r)$.

(ii) $E(J) = \{(u, x)(v, y) \,|\, uv \in E(G), xy \in E(H)\}$.

In other words, $G \otimes H$ is the subgraph of $G \times H$ induced by $\bigcup_{i=1}^{r} G_i \times H_i$.

We denote by $\mathcal{G} \otimes \mathcal{H}$ the natural partition of $V(J)$ induced by this product, i.e. $\mathcal{G} \otimes \mathcal{H} = (G_1 \times H_1, G_2 \times H_2, \ldots, G_r \times H_r)$. Note that the number of neighbors of any node (u, x) in $G_i \times H_i$ is exactly the product of the number of neighbors of u in G_i with the number of neighbors of x in H_i. Consequently, this partition is a degree partition and the following observation holds:

Claim 2.2 *If G is a graph with degree matrix M of order r associated to a partition $\mathcal{G}$, and H is a graph with degree matrix N of order r associated to a partition $\mathcal{H}$ then, $\mathcal{G} \otimes \mathcal{H}$ is a block partition of $G \otimes H$, and the degree matrix associated to this partition is the* coordinate product $M \otimes N$*, defined as:*

$$\forall i, j : \ (M \otimes N)_{i,j} = M_{i,j} N_{i,j}$$

We show that this product behaves well with respect to block transitivity.

Theorem 2.3 *Let G and H be two block transitive graphs with block partitions of size r, and degree matrices M and N, then $G \otimes H$ is also a block transitive graph with degree matrix $M \otimes N$.*

Proof. Let $\mathcal{G}$ and $\mathcal{H}$, respectively, be the block partition associated to the graph G and H, resp., with degree matrix M and N, resp.

Let $e_1 = (u, x)(v, y)$ and $e_2 = (\bar{u}, \bar{x})(\bar{v}, \bar{y})$ be two edges of $G \otimes H$ between the same pair of blocks of the partition, say $(u, x), (\bar{u}, \bar{x}) \in G_i \times H_i$ and $(v, y), (\bar{v}, \bar{y}) \in G_j \times H_j$. Since G is block transitive, there exists an automorphism $\varphi \in \mathrm{Aut}(G)$ that sends uv to $\bar{u}\bar{v}$. Similarly there is $\psi \in \mathrm{Aut}(H)$ that sends xy to $\bar{x}\bar{y}$. Let's consider the function $\varphi \otimes \psi$ from the set $V(G \otimes H)$ to itself defined by:

$$(\varphi \otimes \psi)(w, z) = (\varphi(w), \psi(z))$$

It is straightforward to verify that $\varphi \otimes \psi$ is an automorphism and, by construction, it sends e_1 to e_2. $\square$

3 Construction of block transitive graphs

Due to space restrictions some proofs are sketched or omitted in the sequel.

Lemma 3.1 *Let M be a degree matrix with 0's outside the diagonal. Then we can replace the 0's outside the diagonal with appropriate positive numbers in a way that the resulting matrix M' is a degree matrix.*

Proof. Let $S = (s_1, s_2, \ldots, s_r)$ be the minimal solution for the block sizes problem associated to M. Given that M is a degree matrix we know that there exists a graph G with partition $\mathcal{G}$ of sizes S and degree matrix M. For each pair of different blocks G_i and G_j that are not connected (i.e., that

$m_{i,j} = m_{j,i} = 0$) we insert $h_{i,j} = \gcd(s_i, s_j)$ disjoint copies of the graph $K_{(s_i/h_{i,j}),(s_j/h_{i,j})}$ using the vertices of G_i and G_j. The resulting graph will have the following matrix M':

$$m'_{i,j} = \begin{cases} m_{i,j} & \text{if } m_{i,j} \neq 0 \text{ or } i = j. \\ s_j/h_{i,j} & \text{if } m_{i,j} = 0 \text{ and } i \neq j. \end{cases}$$

□

Lemma 3.2 *Every degree matrix M with zeros on the diagonal can be decomposed into coordinate product of matrices $S_{i,j}(m)$ and $A_I(m)$ of form:*

$$(S_{i,j}(m))_{k,l} = \begin{cases} 0 & \textit{if } k = l. \\ m & \textit{if } \{k, l\} = \{i, j\}. \\ 1 & \textit{in other case.} \end{cases}$$

$$(A_I(m))_{k,l} = \begin{cases} 0 & \textit{if } k = l. \\ m & \textit{if } k \in I, l \notin I. \\ 1 & \textit{in other case.} \end{cases}$$

Proof. According to Lemma 3.1 we construct matrix M' and write

$$M = M' \otimes \bigotimes_{i<j,\, m_{i,j}=0} S_{i,j}(0)$$

In addition, we expand

$$M' = M'' \otimes \bigotimes_{i<j,\, \gcd(m'_{i,j}, m'_{j,i})>1} S_{i,j}(\gcd(m'_{i,j}, m'_{j,i}))$$

so in the resulting matrix M'' the symmetric entries are relative primes.

Let $P = \{p_1, \ldots, p_k\}$ be the set of prime divisors of elements of M''. If P is empty then we are done, otherwise we further decompose

$$M'' = \bigotimes_{p \in P} M_p$$

where $(M_p)_{k,l}$ is the greatest power of p dividing $(M'')_{k,l}$.

It remains to disassemble each matrix M_p into coordinate product of matrices of form $A_I(p)$. For that we iteratively repeat the following procedure:

1. Select I to be the set of indices of rows with an element divisible by p.
2. If I is empty then stop, otherwise divide coordinatewise M_p by $A_I(p)$ and continue by step 1.

□

We now construct a block transitive graph with degree matrix $S_{i,j}(m)$. We take $V = [m+1] \times [r]$ and

$$E = \{(a,k)(a,l) \mid \{k,l\} \neq \{i,j\}\} \cup \{(a,i)(b,j) \mid a \neq b\}.$$

In explanation the first set of edges is a disjoint union of $m+1$ copies of the graph $K_r - e$ joined by the second set of edges that define a bipartite complement of a $(m+1)$-matching between the i-th and the j-th block.

Claim 3.3 *The graph $G_S = (V,E)$ is block transitive and $S_{i,j}(m)$ is its degree matrix.*

Now we construct a block transitive graph with degree matrix $A_I(m)$. Without loss of generality we may assume, that the set I is of size p and that it contains the first p natural numbers, i.e. $I = [p]$.

We take $V = [1] \times [r] \cup [m] \times [p]$ and

$$E = \{(a,i)(a,j)\} \cup \{(a,i)(1,j) \mid i \leq p, j > p\}.$$

In other words, this graph consists of m cliques K_p joined to a single clique K_{r-p} by a complete bipartite graph.

Claim 3.4 *The graph $G_A = (V,E)$ is block transitive and $A_I(m)$ is its degree matrix.*

4 Main Theorem

Theorem 4.1 *For every matrix $M \in \mathcal{M}$ we can construct a block transitive graph G with degree matrix M.*

Proof. We first transform the given matrix M into a matrix M' such that M' contains all non-diagonal entries of M.

As M' has zeros on the diagonal, we now decompose the matrix M' into coordinate product of matrices of form $S_{i,j}(m)$ and $A_I(m)$ according to Lemma 3.2. By Theorem 2.3 we construct a block transitive graph G' with degree matrix M' according to this decomposition.

It remains to further modify G' to incorporate all nonzero diagonal entries of M. Without loss of generality assume that $m_{1,1}, m_{2,2}, \ldots, m_{k,k}$ are all nonzero diagonal entries of M, and put $z = (m_{1,1}+1)(m_{2,2}+1)\dots(m_{k,k}+1)$. We take z copies of the graph G' and distinguish the z copies $u_{(j_1,j_2,\dots,j_k)}$ of every vertex u by indices $(j_1, j_2, \dots, j_k)$ taking range $[m_{1,1}+1] \times [m_{2,2}+1] \times \dots \times [m_{k,k}+1]$.

Now for every index $i = 1, \dots, k$ we join vertices from the block G_i by $\frac{z}{m_{i,i}+1}$ cliques $K_{m_{i,i}+1}$ in the way that two vertices become connected if only if they are copies of the same vertex and their indices differ only in the i-th coordinate.

We claim that by this operation the resulting graph G remains block transitive and that M is its degree matrix. □

References

[1] J. Fiala, D. Paulusma, J. Telle. Locally constrained graph homomorphism and equitable partitions (Preprint).