Critical points of the regular part of the harmonic Green’s function with Robin boundary condition

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Consider the fundamental solution of the Laplacian in $\Omega$ with Robin boundary condition:

\[
\begin{aligned}
-\Delta G_\lambda &= d_N \delta_y, \quad \text{in } \Omega, \\
\frac{\partial G_\lambda}{\partial \nu} + \lambda b(x) G_\lambda &= 0, \quad \text{on } \partial \Omega,
\end{aligned}
\]

(1)

$\nu$ unit normal, $\lambda > 0$ a parameter, $b(x) > 0$,

\[
d_N = \begin{cases} 
2\pi, & N = 2, \\
N(N-2)\omega_N, & N \geq 3
\end{cases}
\]
Let $\Gamma$ be the fundamental solution to $\Delta$ in $\mathbb{R}^N$ i.e.

$$\Gamma(x - y) = \begin{cases} 
- \log |x - y|, & N = 2, \\
\frac{1}{|x - y|^{N-2}}, & N > 2.
\end{cases}$$

The regular part of $G_\lambda$ is:

$$S_\lambda(x, y) = G_\lambda(x, y) - \Gamma(x - y). \quad (2)$$

The Robin function is

$$R_\lambda(x) = S_\lambda(x, x). \quad (3)$$
We are interested in the asymptotic behavior of $R_\lambda$ as $\lambda \to +\infty$, and in particular the number and location of its critical points.

Formally, $R_\infty$ corresponds to the Robin function of $\Delta$ with Dirichlet boundary conditions.

It turns out that $R_\infty$ plays an important role in applications.

We expect that $R_\lambda$, $\lambda < \infty$ plays a similar role when Robin boundary conditions are imposed.
Consider Liouville’s equation in a bounded smooth domain $\Omega \subset \mathbb{R}^2$

$$-\Delta u = \varepsilon^2 e^u \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega$$

where $\varepsilon > 0$.

Suppose $u_\varepsilon$ is a family of solutions that as $\varepsilon \to 0$ concentrates at a point $\xi \in \Omega$ in the sense that

$$\varepsilon^2 e^{u_\varepsilon} \to a \delta_\xi.$$

Then $a = 8\pi$ and

$$\nabla R_\infty(\xi) = 0.$$
If $u_\varepsilon$ is a family of solutions to $-\Delta u = \varepsilon^2 e^u$ in $\Omega \subset \mathbb{R}^2$, $u = 0$ on $\partial \Omega$, that concentrates at $m$ points $\xi_1, \ldots, \xi_m$ then $\xi_1, \ldots, \xi_m$ is a critical point of

$$\varphi(y_1, \ldots, y_m) = \sum_{j=1}^{m} R_\infty(y_j) + \sum_{i \neq j} G_\infty(y_i, y_j).$$
H. Brezis, F. Merle, 1991
T. Suzuki, 1992
Y.-Y. Li, I. Shafrir, 1994
L. Ma, J. Wei, 2001
C.-C. Chen, C.-S. Lin, 2003
P. Esposito, M. Grossi, A. Pistoia, 2005
M. del Pino, M. Kowalczyk, M. Musso, 2005
J. Wei, D. Ye, F. Zhou, 2005
\[ \Delta u + u^{\frac{N+2}{N-2}} \epsilon = 0 \quad \text{in } \Omega \subset \mathbb{R}^N, N > 2 \]
\[ u > 0 \quad \text{in } \Omega \]
\[ u = 0 \quad \text{on } \partial \Omega \]

Bubbling occurs at the critical points of \( R_\infty \).

H. Brezis, B. Peletier, 1989
Z.-C. Han, 1991
O. Rey, 1990
J. Wei, 1996
survey M. del Pino, M. Musso, 2006
The plasma problem, A. Friedman 1988
Consider

\[ \Delta u + \varepsilon^2 e^u = 0 \quad \text{in } \Omega \]

\[ \frac{\partial u}{\partial \nu} + \lambda b(x)u = 0 \quad \text{on } \partial \Omega \]

where \( \lambda, \varepsilon > 0 \) and \( \Omega \subset \mathbb{R}^2 \) is open, bounded, smooth set.

**Theorem.** For all \( \varepsilon > 0 \) sufficiently small (depending on \( \lambda \)) the equation possesses solutions bubbling a single point as \( \varepsilon \to 0 \), corresponding to the critical points of \( R_{\lambda} \).
What are the possible locations of the critical points of $R_{\lambda}$?

Consider the Dirichlet Robin function $R_\infty(x) \equiv -h(x)$ in 2 dimensions. If $\zeta = f(z)$ is a conformal map of $\Omega$ into the unit disc such that $f(z_0) = 0$ then $G(z, z_0) = -\log |f(z)|$. Using this one can show that

$$\Delta h = 4e^{2h}$$

i.e. $h$ is subharmonic.
The result of Caffarelli and Friedman (1985) shows then that in the case of \textit{strictly convex} $\Omega$, the level sets of $h$ are strictly convex. Consequently function $h$ has a unique critical point (a minimum).

We show that if $\lambda \gg 1$ but finite the structure of the set of the critical points of $R_{\lambda}$ is much richer, even in convex domains.
Recall (4):

\[
\begin{cases}
- \Delta G_{\lambda} = d_N \delta_y, & \text{in } \Omega, \\
\frac{\partial G_{\lambda}}{\partial \nu} + \lambda b(x) G_{\lambda} = 0, & \text{on } \partial \Omega,
\end{cases}
\]

and

\[R_{\lambda}(x) = \left[ G_{\lambda}(x, y) - \Gamma(x - y) \right] |_{y=x}\]

**Theorem 1.** For \( \lambda \) sufficiently large \( R_{\lambda} \) has at least 3 different critical points. Two of them at distance \( O(\lambda^{-1}) \) from \( \partial \Omega \).
Theorem 2. Let $x_0 \in \partial \Omega$ be a non-degenerate critical point of $b$. Then there exists a $\lambda_0 > 0$ such that for any $\lambda \geq \lambda_0$ there exists an $x_\lambda \in \Omega$ which is a critical point of $R_\lambda$ such that $|x_\lambda - x_0| = O(\lambda^{-\beta})$ for each $\beta \in (0, 1)$. 
Theorem 3. Assume $b \equiv 1$. Let $\kappa(x)$ denote the mean curvature of $\partial \Omega$ at $x$. If $x_0 \in \partial \Omega$ is a non-degenerate critical point of $\kappa$ then there exists a $\lambda_0 > 0$ such that for any $\lambda \geq \lambda_0$ there exists a critical point $x_\lambda \in \Omega$ of $R_\lambda$ such that $|x_\lambda - x_0| = O(\lambda^{-\beta})$ for each $\beta \in (0, 1)$.

Interior critical point remains inside $\Omega$ when $\lambda \to \infty$

The Four Vertex Theorem implies that generically when $\Omega \subset \mathbb{R}^2$ there are at least 5 critical points of $R_\lambda$. 
Consider

\[-\Delta u = \varepsilon^2 e^u \quad \text{in } \Omega\]

\[\frac{\partial u}{\partial \nu} + \lambda u = 0 \quad \text{on } \partial \Omega\]

where \(\lambda, \varepsilon > 0\) and \(\Omega \subset \mathbb{R}^2\) is open, bounded, smooth set.

**Theorem.** There exists \(\lambda_0 > 0\) such that for all \(\lambda > 0\) the following holds: for any \(m\) and for \(\varepsilon > 0\) sufficiently small (depending on \(\lambda, m\)) there is a solution concentrating at \(m\) different points which are at distance \(O(1/\lambda)\) from \(\partial \Omega\).
Indeed, we can minimize

$$\varphi_\lambda(y_1, \ldots, y_m) = \sum_{j=1}^{m} R_\lambda(y_j) + \sum_{i \neq j} G_\lambda(y_i, y_j).$$

for $y_j$ near $\partial \Omega$. On the other hand

$$\varphi_\infty(y_1, \ldots, y_m) = \sum_{j=1}^{m} R_\infty(y_j) + \sum_{i \neq j} G_\infty(y_i, y_j).$$

has a critical point for any $m$ only if $\Omega$ has non-trivial topology (domain with a hole).
Let $b \equiv \text{const}$. 

Asymptotic expansion of $R_\lambda$ in terms of $\lambda$.

Main order of $R_\lambda$ (Theorem 1 and Theorem 2):

$$R_\lambda(x) = \lambda^{N-2} h_\lambda(\lambda d(x)) + O(\lambda^{N-3})$$

where $h_\lambda(t)$ is an explicit function
The Green’s function in the half space

\[
-\Delta G_a(x, y) = \delta_y \quad \text{in } H = \{(x', x_N) : x_N > 0\}
\]

\[
-\frac{\partial G_a}{\partial x_N} + aG_a = 0 \quad \text{on } \partial H
\]

is given explicitly by

\[
G_a(x, y) = \Gamma(x - y) - \Gamma(x - y^*) - 2 \int_0^\infty e^{-as} \frac{\partial}{\partial x_N} \Gamma(x - y^* + e_N s) \, ds
\]
Then it follows

$$h_\lambda(t) = -\log \lambda - \log(2t) + 2 \int_0^\infty e^{-s} \log(2t + s) \, ds,$$

when \( N = 2 \),

$$h_\lambda(t) = (2t)^{2-N} - 2 \int_0^\infty \frac{e^{-s}}{(2t + s)^{N-2}} \, ds,$$

when \( N > 2 \).

In either case \( h_\lambda \) has a unique critical point and \( h_\lambda(+0) = +\infty \).
For Theorem 3 compute the next term in the expansion

\[ R_\lambda(x) = \lambda^{N-2} h_\lambda(\lambda d(x)) \]
\[ + \lambda^{N-3}(N - 1) \kappa(\hat{x}) v(\lambda d(x)) \]
\[ + O(\lambda^{N-3-\alpha}), \quad 0 < \alpha < 1 \]

where \( \kappa \) is the mean curvature of \( \partial \Omega \) and \( v \) is some function depending only dimension and \( \hat{x} \) is the projection of \( x \) onto \( \partial \Omega \).
When $\Omega = B_R$ is a ball there is an explicit Green’s function, if $N = 2$:

$$G_{\lambda,R}(x, y) = -\log |x - y| + \log \left| (x - y^*) \frac{|y|}{R} \right| + \frac{1}{\lambda R}$$

$$+ 2 \int_0^R \left( 1 - \frac{s}{R} \right)^{\lambda R} \frac{\partial}{\partial s} \log \left| x \left( 1 - \frac{s}{R} \right) - y^* \right| \, ds$$

where $y^*$ is reflection across $\partial B_R$ (similarly for $N \geq 3$). By applying the previous expansion when $\Omega$ is a ball and using the explicit Green’s function we can find $v$. 
Final step is a (tricky) topological degree argument. Basically one needs to show that $h'_\lambda(t)$ and $v(t)$ do not have common zeros, where ($N = 2$):

$$h_\lambda(t) = -\log \lambda - \log(2t) + 2 \int_0^\infty e^{-s} \log(2t + s) \, ds$$

$$v(t) = -\frac{t}{2} - 2t^2 \int_0^\infty e^{-s} \frac{ds}{(s + 2t)^2}$$