Critical points of the regular part of the harmonic Green's function with Robin boundary condition

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Consider the fundamental solution of the Laplacian in Ω with Robin boundary condition:

$$\begin{cases} -\Delta G_\lambda = d_N \delta_y, & \text{in } \Omega, \\ \dfrac{\partial G_\lambda}{\partial \nu} + \lambda b(x) G_\lambda = 0, & \text{on } \partial\Omega, \end{cases} \tag{1}$$

ν unit normal, $\lambda > 0$ a parameter, $b(x) > 0$,

$$d_N = \begin{cases} 2\pi, & N = 2, \\ N(N-2)\omega_N, & N \geq 3 \end{cases}$$

Let Γ be the fundamental solution to Δ in $\mathbb{R}^N$ i.e.

$$\Gamma(x-y) = \begin{cases} -\log|x-y|, & N=2, \\ \frac{1}{|x-y|^{N-2}}, & N>2. \end{cases}$$

The regular part of G_λ is:

$$S_\lambda(x,y) = G_\lambda(x,y) - \Gamma(x-y). \tag{2}$$

The Robin function is

$$R_\lambda(x) = S_\lambda(x,x). \tag{3}$$

We are interested in the asymptotic behavior of R_λ as $\lambda \to +\infty$, and in particular the number and location of its critical points.

Formally, R_∞ corresponds to the Robin function of Δ with Dirichlet boundary conditions.

It turns out that R_∞ plays an important role in applications.

We expect that R_λ, $\lambda < \infty$ plays a similar role when Robin boundary conditions are imposed.

Consider Liouville's equation in a bounded smooth domain $\Omega \subset \mathbb{R}^2$

$$-\Delta u = \varepsilon^2 e^u \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega$$

where $\varepsilon > 0$.
Suppose u_ε is a familiy of solutions that as $\varepsilon \to 0$ concentrates at a point $\xi \in \Omega$ in the sense that

$$\varepsilon^2 e^{u_\varepsilon} \longrightarrow a\delta_\xi.$$

Then $a = 8\pi$ and

$$\nabla R_\infty(\xi) = 0.$$

If u_ε is a family of solutions to $-\Delta u = \varepsilon^2 e^u$ in $\Omega \subset \mathbb{R}^2$, $u = 0$ on $\partial\Omega$, that concentrates at m points $\xi_1, \ldots, \xi_m$ then $\xi_1, \ldots, \xi_m$ is a critical point of

$$\varphi(y_1, \ldots, y_m) = \sum_{j=1}^{m} R_\infty(y_j) + \sum_{i \neq j} G_\infty(y_i, y_j).$$

K. Nagasaki, T. Suzuki, 1990

H. Brezis, F. Merle, 1991

T. Suzuki, 1992

Y.-Y. Li, I. Shafrir, 1994

L. Ma, J, Wei, 2001

C.-C. Chen, C.-S. Lin, 2003

P. Esposito, M. Grossi, A. Pistoia, 2005

M. del Pino, M. Kowalczyk, M. Musso, 2005

J. Wei, D. Ye, F. Zhou, 2005

$$\Delta u + u^{\frac{N+2}{N-2}-\varepsilon} = 0 \quad \text{in } \Omega \subset \mathbb{R}^N, N > 2$$
$$u > 0 \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega$$

Bubbling occurs at the critical points of R_∞.

H. Brezis, B. Peletier, 1989

Z.-C. Han, 1991

O. Rey, 1990

J. Wei, 1996

survey M. del Pino, M. Musso, 2006

A single vortex in the Ginzurg-Landau equation (F. Bethuel, H. Brezis, Helein, 1994 , M. del Pino, M. Kowalczyk, M. Musso, 2006)

The plasma problem, A. Friedman 1988

Theory of diblock copolymer morphology, X. Ren, J. Wei, 2007, Chen, Oshita, 2007

Consider

$$\Delta u + \varepsilon^2 e^u = 0 \quad \text{in } \Omega$$

$$\frac{\partial u}{\partial \nu} + \lambda b(x) u = 0 \quad \text{on } \partial\Omega$$

where $\lambda, \varepsilon > 0$ and $\Omega \subset \mathbb{R}^2$ is open, bounded, smooth set.

Theorem. For all $\varepsilon > 0$ sufficiently small (depending on λ) the equation possesses solutions bubbling a single point as $\varepsilon \to 0$, corresponding to the critical points of R_λ.

What are the possible locations of the critical points of R_λ ?

Consider the Dirichlet Robin function $R_\infty(x) \equiv -h(x)$ in 2 dimensions. If $\zeta = f(z)$ is a conformal map of Ω into the unit disc such that $f(z_0) = 0$ then $G(z, z_0) = -\log |f(z)|$. Using this one can show that

$$\Delta h = 4e^{2h}$$

i.e. h is subharmonic.

The result of Caffarelli and Friedman (1985) shows then that in the case of *strictly convex* Ω, the level sets of h are strictly convex. Consequently function h has a unique critical point (a minimum).

We show that if $\lambda \gg 1$ but finite the structure of the set of the critical points of R_λ is much richer, even in convex domains.

Recall (4):

$$
\begin{cases} -\Delta G_\lambda = d_N \delta_y, & \text{in } \Omega, \\ \dfrac{\partial G_\lambda}{\partial \nu} + \lambda b(x) G_\lambda = 0, & \text{on } \partial\Omega, \end{cases} \tag{4}
$$

and

$$
R_\lambda(x) = [G_\lambda(x,y) - \Gamma(x-y)]\,|_{y=x}
$$

Theorem 1. For λ sufficiently large R_λ has at least 3 different critical points. Two of them at distance $O(\lambda^{-1})$ from $\partial\Omega$.

Theorem 2. Let $x_0 \in \partial\Omega$ be a non-degenerate critical point of b. Then there exists a $\lambda_0 > 0$ such that for any $\lambda \geq \lambda_0$ there exists an $x_\lambda \in \Omega$ which is a critical point of R_λ such that
$|x_\lambda - x_0| = O(\lambda^{-\beta})$ for each $\beta \in (0, 1)$.

Theorem 3. Assume $b \equiv 1$. Let $\kappa(x)$ denote the mean curvature of $\partial\Omega$ at x. If $x_0 \in \partial\Omega$ is a non-degenerate critical point of κ then there exists a $\lambda_0 > 0$ such that for any $\lambda \geq \lambda_0$ there exists a critical point $x_\lambda \in \Omega$ of R_λ such that $|x_\lambda - x_0| = O(\lambda^{-\beta})$ for each $\beta \in (0, 1)$.

- Interior critical point remains inside Ω when $\lambda \to \infty$
- The Four Vertex Theorem implies that generically when $\Omega \subset \mathbb{R}^2$ there are at least 5 critical points of R_λ.

Consider

$$-\Delta u = \varepsilon^2 e^u \quad \text{in } \Omega$$

$$\frac{\partial u}{\partial \nu} + \lambda u = 0 \qquad \text{on } \partial\Omega$$

where $\lambda, \varepsilon > 0$ and $\Omega \subset \mathbb{R}^2$ is open, bounded, smooth set.

Theorem. There exists $\lambda_0 > 0$ such that for all $\lambda > 0$ the following holds: for any m and for $\varepsilon > 0$ sufficiently small (depending on λ, m) there is a solution concentrating at m different points which are at distance $O(1/\lambda)$ from $\partial\Omega$.

Indeed, we can minimize

$$\varphi_\lambda(y_1, \ldots, y_m) = \sum_{j=1}^{m} R_\lambda(y_j) + \sum_{i \neq j} G_\lambda(y_i, y_j).$$

for y_j near $\partial\Omega$. On the other hand

$$\varphi_\infty(y_1, \ldots, y_m) = \sum_{j=1}^{m} R_\infty(y_j) + \sum_{i \neq j} G_\infty(y_i, y_j).$$

has a critical point for any m only if Ω has non-trivial topology (domain with a hole).

Let $b \equiv const$.

Asymptotic expansion of R_λ in terms of λ.

Main order of R_λ (Theorem 1 and Theorem 2):

$$R_\lambda(x) = \lambda^{N-2}\mathtt{h}_\lambda(\lambda d(x)) + O(\lambda^{N-3})$$

where $\mathtt{h}_\lambda(t)$ is an explicit function

The Green's function in the half space

$$\begin{cases} -\Delta G_a(x,y) = \delta_y & \text{in } H = \{(x', x_N) : x_N > 0\} \\ -\dfrac{\partial G_a}{\partial x_N} + aG_a = 0 & \text{on } \partial H \end{cases}$$

is given explicitly by

$$G_a(x,y) = \Gamma(x-y) - \Gamma(x-y^*) - 2\int_0^\infty e^{-as} \frac{\partial}{\partial x_N} \Gamma(x - y^* + e_N s)\, ds$$

Then it follows

$$\mathtt{h}_\lambda(t) = -\log\lambda - \log(2t) + 2\int_0^\infty e^{-s}\log(2t+s)\,ds,$$

when $N = 2$,

$$\mathtt{h}_\lambda(t) = (2t)^{2-N} - 2\int_0^\infty \frac{e^{-s}}{(2t+s)^{N-2}}\,ds,$$

when $N > 2$.

In either case h_λ has a unique critical point and $h_\lambda(+0) = +\infty$.

For Theorem 3 compute the next term in the expansion

$$R_\lambda(x) = \lambda^{N-2}\mathbf{h}_\lambda\big(\lambda d(x)\big) + \lambda^{N-3}(N-1)\kappa(\hat{x})\mathbf{v}\big(\lambda d(x)\big) + O(\lambda^{N-3-\alpha}), \quad 0 < \alpha < 1$$

where κ is the mean curvature of $\partial\Omega$ and $\mathbf{v}$ is some function depending only dimension and $\hat{x}$ is the projection of x onto $\partial\Omega$.

When $\Omega = B_R$ is a ball there is an explicit Green's function, if $N = 2$:

$$G_{\lambda,R}(x,y) = -\log|x-y| + \log\left|(x-y^*)\frac{|y|}{R}\right| + \frac{1}{\lambda R}$$
$$+2\int_0^R \left(1-\frac{s}{R}\right)^{\lambda R} \frac{\partial}{\partial s}\log\left|x\left(1-\frac{s}{R}\right)-y^*\right|\,ds$$

where y^* is reflection across ∂B_R (similarly for $N \geq 3$). By applying the previous expansion when Ω is a ball and using the explicit Green's function we can find v.

Final step is a (tricky) topological degree argument. Basically one needs to show that $\mathtt{h}'_\lambda(t)$ and $\mathtt{v}(t)$ do not have common zeros, where ($N = 2$):

$$\mathtt{h}_\lambda(t) = -\log\lambda - \log(2t) + 2\int_0^\infty e^{-s}\log(2t+s)\,ds$$

$$\mathtt{v}(t) = -\frac{t}{2} - 2t^2\int_0^\infty e^{-s}\frac{ds}{(s+2t)^2}$$