Profiles of fully developed (Airy) waves in different water depths

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A B S T R A C T

Airy waves have a sinusoidal profile in deep water that can be modeled by a time series at any point x and time t, given by $\eta(x,t)=(H_o/2)\cos[2\pi x/L_o-2\pi t/T_w]$, where H_o is the deepwater height, L_o is the deepwater wavelength, and T_w is the wave period. However, as these waves approach the shore they change in form and dimension so that this equation becomes invalid. A method is presented to reconstruct the wave profile showing the correct wavelength, wave height, wave shape, and displacement of the water surface with respect to the still water level for any water depth.

1. Introduction

Airy waves can be defined as first-order, small-amplitude gravity waves with a sinusoidal shape, which can be modeled in deep water using the equations of Airy (1845). Le Roux (2008) applied this concept to a fully developed sea state and proposed a series of equations to calculate the characteristics of such waves as they propagate from deep into shallow water (Table 1). However, no attempt was made to determine the wave profile.

In deep water (denoted by subscript $_o$), the water surface elevation η above or below the still water level (SWL), can be modeled for Airy waves by a time series at any point x (distance from the wave crest in the direction of wave propagation) and time t (Airy, 1845). Using radians:

$$\eta(x,t)=(H_o/2)\ \cos[2\pi x/L_o-2\pi t/T_w]=(H_o/2)\ \cos\theta \tag{1}$$

where H_o is the deepwater height, L_o the deepwater wavelength, and T_w the wave period. In the second form of the equation, θ is the wave phase, given by $(k_o x-\omega t)/2\pi$ where k_o is the deepwater wave number ($2\pi/L_o$) and ω is the radian frequency ($2\pi/T_w$). However, Eq. (1) becomes invalid when such waves propagate into shallow water because of a change in their dimensions and shape. Furthermore, the wave profile is displaced upward with respect to the SWL, so that the crest is higher above the latter than the trough is below it.

Here a new formula is proposed that exactly reproduces the wave profile in deep water as modeled by Eq. (1). However, it has the added advantage that it partly models the change in shape of shoaling waves as computed by the equations of Le Roux (2007a,b, 2008), as well as the upward displacement of the profile with respect to the SWL. This equation is used as a basis to reconstruct the wave profile in any water depth.

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2. Basic equation

The equations proposed below are based on higher-order expressions to calculate the wave profile (e.g. Stokes, 1847, 1880; Boussinesq, 1871), but have been modified and simplified to conform to the formulas for Airy waves proposed by Le Roux (2007a,b, 2008). In deep water

$$\eta(x,t)=L_o\{D\ \cos[2\pi x/L_o-2\pi t/T_w]+E\ \cos[4\pi x/L_o-2\pi t/T_w]\}\ \text{(radians)} \tag{2}$$

where D and E are defined below. Eq. (2) is a time series correctly reproducing wave propagation in deep water.

For shoaling waves, the time-dependency in Eq. (2) may be removed by setting $t=0$. This is necessary because the time factor causes the water surface elevation to move up and down with time at any fixed distance x from the crest, so that asymmetric (with respect to the SWL) wave propagation is not modeled correctly. Therefore, in shallow water

$$\eta_p(x)=L_w\{D\ \cos[2\pi x/L_w]+E\ \cos[4\pi x/L_w]\} \tag{3}$$

where L_w is the wavelength in any water depth d as calculated from Eq. (C2) and associated equations in Table 1. The subscript $_p$ is used with η to indicate that the water surface elevation is preliminary, i.e. that it requires a correction with regard to the wave height as discussed below.

In Eq. (2), D is the value where

$$(L_o/d)(2D + 6\pi D^4 f) = H_o/d \tag{4}$$

and is obtained by iteration, H_o being the deepwater wave height (Table 1). For any fully developed deepwater wave, D is equal to 0.017683.

The value of E in Eq. (2) is obtained from

$$E = \pi^2 D^4 f \tag{5}$$

where f is calculated by

$$f = \left\{[2 + \cosh(4\pi d/L_w)][\cosh(2\pi d/L_w)]/2[\sinh(2\pi d/L_w)]^3\right\}^3 \text{(degrees)} \tag{6}$$

The value of f is equal to 1 in deep water, but increases to about 4621 at breaking depth.

3. Correction to the basic equation

Using Eq. (3) to calculate the wave profile from deep into shallow water (keeping D constant at 0.017683), it is found that it correctly reproduces the wavelength given by Eq. (C2) in Table 1, as well as the general wave shape. The trough flattens out and is displaced upward with respect to the SWL, whereas the crest becomes sharper as in typical Stokes and cnoidal waves (see Le Roux, 2008). The elevation of the wave trough below the SWL (η_t) as calculated by Eq. (3) also agrees to within 3.5% with cnoidal theory (See Appendix). For example, for a 7 s wave at breaking, Eq. (3) yields a trough depth $\eta_t = -0.6745$ m, as compared to −0.6987 m given by cnoidal theory, so that this elevation can be accepted as a fair approximation. However, the crest height above the SWL (η_{pc}) plots below the correct elevation. This can be remedied by simply adding H_w as obtained from Eq. (C3) in Table 1 to η_t (maintaining the negative value of the latter). Thus $\eta_c = \eta_t + H_w$.

In plotting the wave profile, the total wavelength is divided by 24, which is then used as the increment distance x_i from the initial wave crest where $x=0$. The preliminary water elevation η_p at each of these 24 points is first calculated from Eq. (3), which is then corrected at 9 locations on both sides of and including the wave crest. The first correction is at the initial wave crest where $x=0$, which is obtained by adding H_w to η_t, as explained above. The 2nd, 3rd, 4th and 5th corrections at increasing increment distances of $L_w/24$ from $x=0$ are calculated by

$$\eta = \eta_p \left(\eta_{pc}/\eta_c\right)^i \tag{7}$$

where the exponent i has values of 0.8, 0.6, 0.457, and 0.2 respectively. The η_{pc}/η_c ratio is 1 in deep water so that $\eta = \eta_p$, which ensures that the crest shape returns to normal for deepwater conditions.

4. A worked example

To demonstrate the use of the equations proposed here for different water depths, a wave period of 7 s is used, but the formulas are valid for any other period. Initial calculations of wave height H_w and length L_w are made to the 4th decimal, as is f. D and E are calculated to the 6th decimal when Eq. (4) is also satisfied to the 6th decimal. The resultant water surface elevation η is shown below to the 2nd decimal (nearest centimeter).

In deep water (taken here as 200 m) according to Eqs. (A4) and (A5) in Table 1, $L_o = 76.5042$ m and $H_o = 2.7058$ m. Inserted into Eq. (1), at the wave crest where $x=0$ and at time $t=0$, this yields $\eta_c = 1.35$ m, whereas at the wave trough where $x = L_o/2 = 38.2521$ m and $t=0$, $\eta_t = -1.35$ m. In Eq. (2), D iterates to 0.017683 when Eq. (4) is satisfied at a value of 0.013529. Eqs. (5) and (6) show that $E = 1 \times 10^{-6}$, and $f=1$.

Table 1
Equations for fully developed (Airy) waves (Airy, 1845; Le Roux, 2007a,b, 2008)

A. Deep water (subscript $_o$)
1) Wind velocity: $U_{a10} = C_o$
2) Wave period: $T_w = 2\pi U_{a10}/g$
3) Wave celerity: $C_o = gT_w/2\pi = L_o/T_w$
4) Wavelength: $L_o = gT_w^2/2\pi$
5) Wave height: $H_o = L_o/9\pi = gT_w^2/18\pi^2$
6) Median crest and trough diameter: $MCD_o = MTD_o = L_o/2 = gT_w^2/4\pi$
7) Horizontal water particle displacement at depth z from SWL: $A_{hoz} = (H_o/2)\exp(2\pi z/L_o)\sin\theta$ (radians)
8) Vertical water particle displacement at depth z from SWL: $A_{voz} = (H_o/2)\exp(2\pi z/L_o)\cos\theta$ (radians)
9) Horizontal particle velocity at wave crest and trough: $U_{hco} = U_{hto} = \pi H_o/T_w = H_o gT_w/2L_o = gT_w/18\pi$
10) Horizontal and vertical water particle velocity under wave crest and trough at depth z from SWL: $U_{hcoz} = U_{vcoz} = U_{htoz} = U_{vtoz} = \pi H_o/T_w \exp(-2\pi z/L_o) = gT_w/18\pi \exp(4\pi^2 z/gT_w^2)$
11) Water surface elevation: $\eta(x,t) = (H_o/2)\cos[2\pi x/L_o - 2\pi t/T_w]$ (radians)

B. Breaking depth (subscript $_b$)
1) Breaker celerity (horizontal bottom): $C_b = gT_w/3\pi$
2) Breaker celerity (sloping bottom): $C_{b\alpha} = L_{b\alpha}/T_w$
3) Breaker length (horizontal bottom): $L_b = 2L_o/3 = gT_w^2/3\pi$
4) Breaker length (sloping bottom): $L_{b\alpha} = T_w[g(0.5H_{b\alpha} + d_{b\alpha})]^{1/2}$
5) Breaker height (horizontal bottom): $H_b = L_b/16 = gT_w^2/48\pi$
6) Breaker height (sloping bottom): $H_{b\alpha} = d_{b\alpha}(-0.0036\alpha^2 + 0.0843\alpha + 0.835)$
7) Breaking depth (horizontal bottom): $d_b = L_b^2/gT_w^2 - 0.5L_b/16 = gT_w^2/9\pi^2 - gT_w^2/96\pi$
8) Breaking depth (sloping bottom): Iterate d until Eq. (B6) = Eq. (C3) to get $d_{b\alpha}$
9) Median crest diameter (horizontal bottom): $MCD_b = L_o/6 = gT_w^2/12\pi = [(H_o gT_w^2)/8]^{1/2}$
10) Median crest diameter (sloping bottom): $MCD_{b\alpha} = L_{b\alpha} - L_o/2$
11) Median trough diameter (horizontal and sloping bottom): $MTD_b = L_o/2 = gT_w^2/4\pi$
12) Horizontal particle velocity in wave crest (horizontal bottom): $U_{hcb} = C_b = L_b/T_w = 2L_o/3T_w = gT_w/3\pi$
13) Horizontal particle velocity in wave crest (sloping bottom): $U_{hcb}\alpha = H_o gT_w/2MCD_{b\alpha}$
14) Horizontal particle velocity in wave trough (horizontal bottom): $U_{htb} = gT_w/27\pi$
15) Horizontal particle velocity in wave trough (sloping bottom): $U_{htb\alpha} = H_o gT_w/6MTD_{b\alpha}$

C. Any water depth (subscript $_w$)
1) Wave celerity: $C_w = L_w/T_w$
2) Wavelength: $L_w = \{L_{b\alpha}T_w[g(0.5H_{b\alpha} + d)]^{1/2}\}^{1/2}$ (Max. value of $d = L_o/2.965$)
3) Wave height: $H_w = H_o\{A\exp[(H_o/L_o)B]\}$ where $A = 0.5875(d/L_o)^{-0.18}$ when $d/L_o \le 0.0844$; $A = 0.9672(d/L_o)^2 - 0.5013(d/L_o) + 0.9521$ when $0.0844 \le (d/L_o) \le 0.6$; $A = 1$ when $(d/L_o) > 0.6$; $B = 0.0042(d/L_o)^{-2.3211}$
4) Median crest diameter: $MCD_w = L_w - L_o/2$
5) Median trough diameter: $MTD_w = L_o/2 = gT_w^2/4\pi$
6) Horizontal particle velocity in wave crest: $U_{hcw} = H_o gT_w L_w/8MCD_w^2 = g^2T_w^3 L_w/144\pi^2\ MCD_w^2$
7) Horizontal water particle velocity in wave trough: $U_{htw} = H_o gT_w L_w/8MTD_w^2 = L_w/9T_w$
8) Horizontal water particle displacement at depth z from SWL: $A_{hwz} = (H_o/2)\{\cosh[(2\pi(d-z)/MCD_w]/\cosh[2\pi d/MCD_w]\}$
9) Vertical water particle displacement at depth z from SWL: $A_{vwz} = (H_o/2)\{\sinh[(2\pi(d-z)/MCD_w]/\sinh[2\pi d/MCD_w]\}$
10) Horizontal water particle velocity under wave crest at depth z from SWL: $U_{hcwz} = (2A_{hwz}gT_w L_w/8MCD_w^2)$
11) Horizontal water particle velocity under wave trough at depth z from SWL: $U_{htwz} = (2A_{hwz}gT_w L_w/8MTD_w^2)$

Solution of Eq. (2) yields $\eta_c = 1.35$ m and $\eta_t = -1.35$ m at $t=0$. Solving for any combination of x and t, the values of η always correspond exactly to those given by Eq. (1). The typical sinusoidal profile is shown in Fig. 1a.

For a 7 s wave propagating over a horizontal bottom at a water depth of 5 m, Eqs. (C2) and (C3) in Table 1 yield $L_w = 53.5852$ m and $H_w = 2.8238$ m, respectively. In this case $E = 0.000773$ and $f = 801.1379$. Eq. (3) calculates η_{pc} at 0.9890 m and η_t at −0.9061 m at a distance of $x=0$ and 26.7926 m from the crest, respectively. Adding the wave height of 2.8238 to η_t shows that $\eta_c = 1.9177$ m. Fig. 1b portrays the resultant wave shape, the heavy line representing the corrected profile superimposed on the original profile (thin line) obtained directly from Eq. (3). The wave trough is still somewhat rounded as in a typical Stokes wave, with a sharper crest.

At breaking, the breaker depth d_b and height H_b are 3.8176 and 3.1877 m, respectively, as obtained from Eqs. (B7) and (B5) in Table 1, whereas the breaker length $L_b = 51.0022$ m (Eq. (B3) in Table 1). In this case $E = 0.004459$, and $f = 4620.9941$. Eq. (3) in this case gives

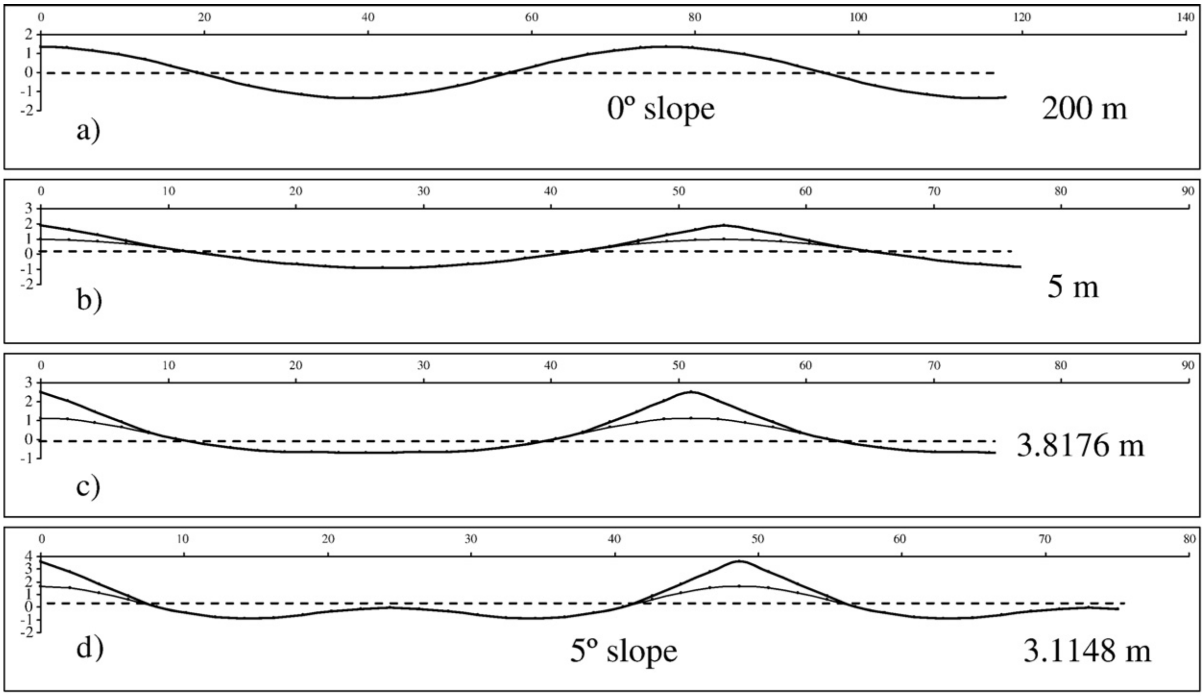

Fig. 1. Wave profiles as modeled in different water depths over a 0° and 5° slope. Thin line shows basic waveform as modeled by Eq. (3). Stippled line represents still water level (SWL).

η_{pc} = 1.1293 m and η_t = −0.6745 m, with η_c = 2.5132 m. Fig. 1c shows a typical cnoidal wave with a flat trough and sharp crest. The water elevation at $MCD_b/2$ is calculated at 0.92 m by Eq. (7), as compared to an independent value of $\eta_c - (H_w/2)$ = 0.92 m.

For a bottom sloping at 5°, d_b resolves to 3.1148 m and H_b to 3.6335 m, with E = 0.016337 and f = 16929.9257. The equations in this case indicate a secondary crest developing in the wave trough, as also modeled by the 5th order Stokes expansion (Fenton, 1985). Fig. 1d shows the resultant wave profile.

5. Conclusion

Eq. (3) models the wave profile exactly for deepwater conditions according to the original Airy (1845) equation, in which case it can also be used as a time series. Plotted in the way explained above, it depicts the wave profile as changing from a sinusoidal, Airy shape in deep water to typical Stokes and cnoidal profiles into the breaker zone. The model can easily be set up on a spreadsheet, whereafter only the water depth needs to be changed to visualize the change in profile.

Appendix A

The crest height above the SWL and trough depth below the SWL can be calculated as follows, based on cnoidal theory and Fig. II-1–14 in Demirbilek and Vincent (2002), as adapted from Wiegel (1960).

The height of the crest above the bottom y_c is given in any water depth by $y_c = H_w(X + d/H_w)$ where $X = 0.5$ when $U_R = 0$; $X = -0.00003U_R^2 + 0.0058U_R + 0.4932$ when $0 \le U_R \le 117$; $X = 0.0821 \ln U_R + 0.37$ when $117 \le U_R \le 830$; and $X = 0.0217 \ln U_R + 0.776$ when $U_R \ge 830$, U_R being the Ursell number given by $U_R = L_w^2 H_w / d^3$. The distance of the wave trough above the bottom y_t can be estimated by $y_t = H_w(X + d/H_w - 1)$. The elevation of the wave crest above the SWL is therefore given by $\eta_c = y_c - d$. For the wave trough, the elevation below the SWL is obtained from $\eta_t = y_t - d$.

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