Carleman inequalities and inverse problems for the Schrödinger equation

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Abstract

In this Note, we derive new Carleman inequalities for the evolution Schrödinger equation under a weak pseudoconvexity condition, which allows us to use weights with a linear spatial dependence. As a result, less restrictive boundary or internal observation regions may be used to obtain the stability for the inverse problem consisting in retrieving a stationary potential in the Schrödinger equation from a single boundary or internal measurement, respectively.

Résumé

Inégalités de Carleman et problèmes inverses pour l'équation de Schrödinger. Dans cette Note, nous établissons de nouvelles inégalités de Carleman pour l'équation d'évolution de Schrödinger sous une hypothèse de pseudoconvexité faible, qui permet d'utiliser des poids affines en la variable d'espace. Comme application, nous pouvons définir des régions d'observabilité moins restrictives dans le problème inverse consistant à retrouver un potentiel stationnaire dans l'équation de Schrödinger à partir d'une mesure simple effectuée au bord ou à l'intérieur du domaine.

Version française abrégée

Dans cette Note, on considère le problème inverse consistant à retrouver un potentiel stationnaire dans l'équation d'évolution de Schrödinger à partir d'une mesure simple effectuée sur le bord ou à l'intérieur du domaine. Ce problème a été étudié dans [1] pour un domaine borné, et dans [4] pour un domaine non borné, par le biais des inégalités de Carleman et de la méthode de Bukhgeim–Klibanov [2] pour établir la stabilité du potentiel par rapport à la mesure.

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La localisation de la région d'observation dépend du choix du poids dans l'inégalité de Carleman. Dans [1] et [6], le poids est supposé satisfaire une condition de pseudoconvexité *stricte*, moins sévère que la condition de pseudoconvexité *forte* considérée dans [5] pour l'obtention d'inégalités de Carleman locales. Dans cette Note, nous établissons des inégalités de Carleman globales, frontière ou internes, sous la condition de pseudoconvexité faible (7), qui peut dégénérer en certains points. Cette condition permet d'utiliser des poids affines en la variable d'espace, et autorise des régions d'observation plus petites que celles généralement considérées dans la littérature.

Soit $\Omega \subset \mathbb{R}^N$ un ouvert borné de frontière Lipschitzienne, et soient S^+ une partie ouverte de $\partial\Omega$ et $S^- = \partial\Omega \setminus S^+$. On suppose qu'il existe une fonction $\psi \in C^4(\overline{\Omega})$ vérifiant les conditions (6), (7) et (8) ci-dessous, et on définit les fonctions $\theta(x,t) := \exp(\lambda\psi(x))/(t(T-t))$, $\varphi(x,t) := (\exp(\lambda C_\psi) - \exp(\lambda\psi(x))/(t(T-t))$, où $C_\psi = 2\|\psi\|_{L^\infty(\Omega)}$. On peut alors établir l'inégalité de Carleman suivante :

Proposition 0.1. *Supposons qu'il existe une fonction* $\psi \in C^4(\overline{\Omega})$ *telle que* (6), (7) *et* (8) *aient lieu pour une région* $S^+ \subset \partial\Omega$. *Alors il existe des constantes* $\lambda_0 \geqslant 1$, $s_0 \geqslant 1$ *et* $C_0 > 0$ *telles que pour tout* $\lambda \geqslant \lambda_0$, *tout* $s \geqslant s_0$, *et tout* $q \in C^{2,1}(\overline{\Omega} \times [0,T])$ *vérifiant* $q = 0$ *sur* $\partial\Omega \times [0,T]$, *on ait*

$$\int_0^T\int_\Omega \left[\lambda^2 s\theta|\nabla q\cdot\nabla\psi|^2 + \lambda^4(s\theta)^3|q|^2 + |\widetilde{M}_1 q|^2 + |\widetilde{M}_2 q|^2\right]e^{-2s\varphi}\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_{S^-} \lambda s\theta\left|\frac{\partial\psi}{\partial n}\right|\left|\frac{\partial q}{\partial n}\right|^2 e^{-2s\varphi}\,\mathrm{d}x\,\mathrm{d}t$$
$$\leqslant C_0\left(\int_0^T\int_\Omega |\partial_t q + i\Delta q|^2 e^{-2s\varphi}\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_{S^+} \lambda s\theta\left|\frac{\partial\psi}{\partial n}\right|\left|\frac{\partial q}{\partial n}\right|^2 e^{-2s\varphi}\,\mathrm{d}x\,\mathrm{d}t\right), \tag{1}$$

où $\widetilde{M}_1$ *et* $\widetilde{M}_2$ *désignent les opérateurs*

$$\widetilde{M}_1 q := \left[s(\varphi_t + i\Delta\varphi) - 2is^2|\nabla\varphi|^2\right]q + 2is\nabla\varphi\cdot\nabla q, \tag{2}$$
$$\widetilde{M}_2 q := \left[-s(\varphi_t + i\Delta\varphi) + 2is^2|\nabla\varphi|^2\right]q + q_t - 2is\nabla\varphi\cdot\nabla q + i\Delta q. \tag{3}$$

Considérons à présent le problème inverse consistant à déterminer un potentiel $p = p(x)$ à partir de la mesure sur S^+ de la solution, notée $u(p)$, du système

$$\begin{cases} iu_t + \Delta u + p(x)u = 0 & \text{dans } \Omega \times (0,T), \\ u = h & \text{sur } \partial\Omega \times (0,T), \\ u(0) = u_0 & \text{dans } \Omega. \end{cases} \tag{4}$$

Utilisant la Proposition 0.1, on peut établir la stabilité Lipschitz du potentiel par rapport à la mesure :

Théorème 0.2. *Supposons qu'il existe une fonction* $\psi \in C^4(\overline{\Omega})$ *telle que* (6), (7) *et* (8) *aient lieu pour une région* $S^+ \subset \partial\Omega$. *Supposons aussi que* $p \in L^\infty(\Omega;\mathbb{R})$, $u_0 \in L^\infty(\Omega)$ *et* $r > 0$ *soient tels que*

(i) $u_0(x) \in \mathbb{R}$ *ou* $iu_0(x) \in \mathbb{R}$ *p.p. dans* Ω ;
(ii) $|u_0(x)| \geqslant r > 0$ *p.p. dans* Ω ;
(iii) $u(p) \in H^1(0,T;L^\infty(\Omega))$.

Alors, pour tout $m \geqslant 0$, *il existe une constante* $C = C(m, \|u(p)\|_{H^1(0,T;L^\infty(\Omega))}, r) > 0$ *telle que pour tout* $q \in B_m(0) \subset L^\infty(\Omega;\mathbb{R})$ *vérifiant* $\partial u(p)/\partial n - \partial u(q)/\partial n \in H^1(0,T;L^2(S^+))$ *on ait*

$$\|p - q\|_{L^2(\Omega)} \leqslant C\left\|\frac{\partial u(p)}{\partial n} - \frac{\partial u(q)}{\partial n}\right\|_{H^1(0,T;L^2(S^+))}.$$

Des résultats similaires peuvent être donnés pour une observation interne. Lorsque Ω a la forme d'un stade, on peut réduire la région d'observabilité par rapport à celle considérée dans [1,3] ou [7] grâce à l'utilisation d'un poids de Carleman affine en x (cf. Fig. 1).

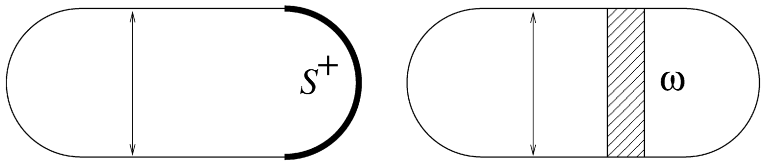

Fig. 1. Observation regions for the Schrödinger equation in a stadium when $\psi(x) = x \cdot e_1 + C$. The left hand figure corresponds to a boundary observation, the right hand figure to an internal observation. The presence of a trapped ray, depicted by an arrow, means that the wave equation is not observable.

Fig. 1. Régions d'observation pour l'équation de Schrödinger dans un stade lorsque $\psi(x) = x \cdot e_1 + C$. La figure de gauche correspond à une observation au bord, celle de droite à une observation interne. La présence d'un rayon captif, représenté par une flèche, indique que l'équation des ondes n'est pas observable.

1. Introduction

In this Note, we investigate the inverse problem consisting in retrieving a stationary potential in the evolution Schrödinger equation from a single boundary or internal measurement. This problem was considered in [1] and [4] for a bounded and an unbounded domain, respectively. In both papers, the authors established a Carleman inequality, and followed the classical Bukhgeim–Klibanov method (see [2]) to infer the Lipschitz continuity of the map which associates the potential with the measurement.

The observation region is related to the weight involved in the Carleman estimate. In [1] and [6], the authors established their Carleman estimate under the following (strict) pseudoconvexity condition

$$\left|\nabla\psi(x)\cdot\xi\right|^2 + \sum_{i,j=1}^{N}\big(\partial_i\partial_j\psi(x)\big)\xi_i\xi_j \geqslant C|\xi|^2 \quad \forall x\in\overline{\Omega},\ \forall\xi=(\xi_1,\dots,\xi_N)\in\mathbb{R}^N. \tag{5}$$

On the other hand, the local Carleman estimates proved in [5] required a *strong* pseudoconvexity condition, which turned out to be equivalent to the strong convexity of the weight in the space variable. We show here that new Carleman inequalities for the Schrödinger equation may be derived under a *weak* pseudoconvexity condition (see below (7)), which allows us to use weights with a spatial dependence of the type $\psi(x) = x \cdot e$, where e is some fixed direction in $\mathbb{R}^N$. With these new Carleman inequalities the observation region may be dramatically reduced for several examples including the ball, the rectangle, and the stadium (see Fig. 1). Notice that these Carleman estimates have also important consequences in Control Theory. The proofs of all the results in this Note are given in [8].

2. Boundary observation

Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with a Lipschitz boundary $\partial\Omega$, let S^+ be an open subset of $\partial\Omega$, and let $S^- := \partial\Omega \setminus S^+$, $\Sigma := \partial\Omega \times (0,T)$. We assume that there exists a function $\psi \in C^4(\overline{\Omega})$ such that

$$\psi > 0 \quad \text{in } \overline{\Omega}, \qquad \nabla\psi \neq 0 \quad \text{in } \overline{\Omega}, \qquad \partial_n\psi \leqslant 0 \quad \text{on } S^-, \qquad \partial_n\psi > 0 \quad \text{on } S^+, \tag{6}$$

and that

$$\left|\nabla\psi(x)\cdot\xi\right|^2 + \sum_{i,j=1}^{N}\big(\partial_i\partial_j\psi(x)\big)\xi_i\xi_j \geqslant 0 \quad \forall x\in\overline{\Omega},\ \forall\xi=(\xi_1,\dots,\xi_N)\in\mathbb{R}^N. \tag{7}$$

Replacing ψ by $\psi + C$, where $C > 0$ is a large enough number, we may also assume that

$$\psi(x) > \frac{2}{3}\|\psi\|_{L^\infty(\Omega)} \quad \forall x \in \Omega. \tag{8}$$

Set $C_\psi = 2\|\psi\|_{L^\infty(\Omega)}$ and let

$$\theta(x,t) := \frac{e^{\lambda\psi(x)}}{t(T-t)}, \quad \varphi(x,t) := \frac{e^{\lambda C_\psi} - e^{\lambda\psi(x)}}{t(T-t)}, \quad \forall (x,t) \in \Omega \times (0,T) \tag{9}$$

where λ denotes some positive number which will be specified later. We also introduce the set

$$\mathcal{Z} := \big\{q \in C^{2,1}(\overline{\Omega}\times[0,T]);\ q = 0 \text{ on } \Sigma\big\}. \tag{10}$$

Then the following Carleman estimate holds:

Proposition 2.1. *Assume that there exists a function $\psi \in C^4(\overline{\Omega})$ such that* (6), (7) *and* (8) *hold for some $S^+, S^- \subset \partial\Omega$. Then, there exist some constants $\lambda_0 \geqslant 1$, $s_0 \geqslant 1$ and $C_0 > 0$ such that for all $\lambda \geqslant \lambda_0$, all $s \geqslant s_0$, and all $q \in \mathcal{Z}$, it holds*

$$\int_0^T \int_\Omega \left[\lambda^2 s\theta |\nabla q \cdot \nabla \psi|^2 + \lambda^4 (s\theta)^3 |q|^2 + |\widetilde{M}_1 q|^2 + |\widetilde{M}_2 q|^2\right] e^{-2s\varphi} \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{S^-} \lambda s\theta \left|\frac{\partial \psi}{\partial n}\right| \left|\frac{\partial q}{\partial n}\right|^2 e^{-2s\varphi} \,\mathrm{d}x\,\mathrm{d}t$$
$$\leqslant C_0 \left(\int_0^T \int_\Omega |\partial_t q + i\Delta q|^2 e^{-2s\varphi} \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{S^+} \lambda s\theta \left|\frac{\partial \psi}{\partial n}\right| \left|\frac{\partial q}{\partial n}\right|^2 e^{-2s\varphi} \,\mathrm{d}x\,\mathrm{d}t \right), \tag{11}$$

where $i = \sqrt{-1}$ and $\widetilde{M}_1$ and $\widetilde{M}_2$ denote the operators

$$\widetilde{M}_1 q := \left[s(\varphi_t + i\Delta\varphi) - 2is^2|\nabla\varphi|^2\right]q + 2is\nabla\varphi \cdot \nabla q, \tag{12}$$
$$\widetilde{M}_2 q := \left[-s(\varphi_t + i\Delta\varphi) + 2is^2|\nabla\varphi|^2\right]q + q_t - 2is\nabla\varphi \cdot \nabla q + i\Delta q. \tag{13}$$

The proof of the above Carleman estimate follows the same pattern as in [9] for the Ginzburg–Landau equation. The first step provides an exact computation of a scalar product in L^2, whereas the second step gives the estimates obtained thanks to the weak pseudoconvexity condition.

Example 1. In all the following examples, we pick $e = e_1 = (1, 0, \dots, 0)$, $S^+ = \{x \in \partial\Omega; n(x) \cdot e_1 > 0\}$, and $\psi(x) = x \cdot e_1 + C$ for $C > 0$ large enough.

(i) Ω IS A BALL: $\Omega := B_R(0)$. S^+ is the half-sphere $\{x \in \mathbb{R}^N; \|x\| = R \text{ and } x_1 = x \cdot e_1 > 0\}$.
(ii) Ω IS A RECTANGLE: $\Omega := (-L_1, L_1) \times \cdots \times (-L_N, L_N)$. S^+ is the side $\{L_1\} \times (-L_2, L_2) \times \cdots \times (-L_N, L_N)$.
(iii) Ω IS A STADIUM: $\Omega = \{x = (x_1, x_2) \in \mathbb{R}^2; x \in (-L_1, L_1) \times (-L_2, L_2) \text{ or } (x_1 \pm L_1)^2 + x_2^2 < L_2^2\}$. S^+ is once again the half-sphere $\{x = (x_1, x_2) \in \mathbb{R}^2; x_1 > L_1 \text{ and } (x_1 - L_1)^2 + x_2^2 = L_2^2\}$.

We now turn to the inverse problem with boundary measurement. Let us consider the following boundary initial-value problem:

$$\begin{cases} iu_t + \Delta u + p(x)u = 0 & \text{in } \Omega \times (0, T), \\ u = h & \text{on } \partial\Omega \times (0, T), \\ u(0) = u_0 & \text{in } \Omega. \end{cases} \tag{14}$$

In what follows, we shall denote by $u(p)$ the solution of the system (14) associated with the potential p. Using Proposition 2.1 and the classical Bukhgeim–Klibanov method, we obtain the following result:

Theorem 2.2. *Assume that there exists a function $\psi \in C^4(\overline{\Omega})$ such that* (6), (7) *and* (8) *hold for some $S^+ \subset \partial\Omega$. Suppose also that $p \in L^\infty(\Omega; \mathbb{R})$, $u_0 \in L^\infty(\Omega)$ and $r > 0$ are such that*

(i) *$u_0(x) \in \mathbb{R}$ or $iu_0(x) \in \mathbb{R}$ a.e. in Ω;*
(ii) *$|u_0(x)| \geqslant r > 0$ a.e. in Ω;*
(iii) *$u(p) \in H^1(0, T; L^\infty(\Omega))$.*

Then, for any $m \geqslant 0$, there exists a constant $C = C(m, \|u(p)\|_{H^1(0,T;L^\infty(\Omega))}, r) > 0$ such that for any $q \in B_m(0) \subset L^\infty(\Omega; \mathbb{R})$ satisfying $\partial u(p)/\partial n - \partial u(q)/\partial n \in H^1(0, T; L^2(S^+))$ we have that

$$\|p - q\|_{L^2(\Omega)} \leqslant C \left\| \frac{\partial u(p)}{\partial n} - \frac{\partial u(q)}{\partial n} \right\|_{H^1(0,T;L^2(S^+))}.$$

3. Internal observation

Once again, Ω denotes a bounded open set in $\mathbb{R}^N$ with a Lipschitz boundary. Let $\omega \subset \Omega$ be any given open subset. We shall assume that there exists a function $\psi \in C^4(\overline{\Omega})$ such that

$$\nabla\psi \neq 0 \quad \text{in } \overline{\Omega \setminus \omega}, \tag{15}$$

$$\frac{\partial\psi}{\partial n} \leqslant 0 \quad \text{on } \partial\Omega, \tag{16}$$

$$\left|\nabla\psi(x)\cdot\xi\right|^2 + \sum_{i,j=1}^{N}\left(\partial_i\partial_j\psi(x)\right)\xi_i\xi_j \geqslant 0 \quad \forall x \in \overline{\Omega\setminus\omega},\ \forall\xi = (\xi_1,\ldots,\xi_N) \in \mathbb{R}^N, \tag{17}$$

$$\psi(x) > \frac{2}{3}\|\psi\|_{L^\infty(\Omega)} \quad \forall x \in \Omega. \tag{18}$$

Let C_ψ, θ, φ, and $\mathcal{Z}$ be defined as in (9), (10). Then the following Carleman estimate holds:

Proposition 3.1. *Assume that there exists a function $\psi \in C^4(\overline{\Omega})$ such that* (15)–(17) *and* (18) *hold for some $\omega \subset \Omega$. Then there exist some constants $\lambda_0 \geqslant 1$, $s_0 \geqslant 1$ and $C_0 > 0$ such that for all $\lambda \geqslant \lambda_0$, all $s \geqslant s_0$, and all $q \in \mathcal{Z}$, it holds*

$$\int_0^T\int_\Omega \left[\lambda^2 s\theta|\nabla q\cdot\nabla\psi|^2 + \lambda^4(s\theta)^3|q|^2 + |\widetilde{M}_1 q|^2 + |\widetilde{M}_2 q|^2\right]e^{-2s\varphi}\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_{\partial\Omega}\lambda s\theta\left|\frac{\partial\psi}{\partial n}\right|\left|\frac{\partial q}{\partial n}\right|^2 e^{-2s\varphi}\,\mathrm{d}x\,\mathrm{d}t$$
$$\leqslant C_0\left(\int_0^T\int_\Omega |\partial_t q + i\Delta q|^2 e^{-2s\varphi}\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_\omega \left[\lambda s\theta|\nabla q|^2 + \lambda^4(s\theta)^3|q|^2\right]e^{-2s\varphi}\,\mathrm{d}x\,\mathrm{d}t\right), \tag{19}$$

where $\widetilde{M}_1$ and $\widetilde{M}_2$ denote the operators defined in (12)–(13).

Example 2.

- (i) Ω IS A BALL: $\Omega = B_R(0)$.
  - (a) $\omega = B_R(0) \cap \{x; 0 < x_1 < \varepsilon\}$, for an arbitrarily small $\varepsilon > 0$ (take $\psi(x) = \pm x\cdot e_1 + C$ on each connected component of $\Omega\setminus\omega$); or
  - (b) ω is a neighborhood of a half-sphere.
- (ii) Ω IS A RECTANGLE and ω is any strip $(a, a+\varepsilon)\times(-L_2, L_2)\times\cdots\times(-L_N, L_N)$.
- (iii) Ω IS A STADIUM:
  - (a) ω is any strip $(a, a+\varepsilon)\times(-L_2, L_2)$; or
  - (b) ω is a neighborhood of a half-sphere.

Let us now turn to the inverse problem with internal measurement. Once again, we denote by $u(p)$ the solution of (14) associated with the potential p. The following result may be deduced from Proposition 3.1.

Theorem 3.2. *Assume that there exists a function $\psi \in C^4(\overline{\Omega})$ such that* (15)–(17) *and* (18) *hold for some $\omega \subset \Omega$. Suppose also that $p \in L^\infty(\Omega;\mathbb{R})$, $u_0 \in L^\infty(\Omega)$ and $r > 0$ satisfy*

- (i) *$u_0(x) \in \mathbb{R}$ or $iu_0(x) \in \mathbb{R}$ a.e. in Ω;*
- (ii) *$|u_0(x)| \geqslant r > 0$ a.e. in Ω;*
- (iii) *$u(p) \in H^1(0,T;L^\infty(\Omega))$.*

Then, for any $m \geqslant 0$, there exists a constant $C = C(m, \|u(p)\|_{H^1(0,T;L^\infty(\Omega))}, r) > 0$ such that for any $q \in B_m(0) \subset L^\infty(\Omega;\mathbb{R})$ satisfying $u(p) - u(q) \in H^1(0,T;H^1(\omega))$ we have that

$$\|p - q\|_{L^2(\Omega)} \leqslant C\left\|u(p) - u(q)\right\|_{H^1(0,T;H^1(\omega))}. \tag{20}$$

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References

[1] L. Baudouin, J.-P. Puel, Uniqueness and stability in an inverse problem for the Schrödinger equation, Inverse Problems 18 (2002) 1537–1554.
[2] A. Bukhgeim, M. Klibanov, Global uniqueness of a class of inverse problems, Sov. Math. Dokl. 24 (1982) 244–247.
[3] N. Burq, M. Zworski, Geometric control in the presence of a black box, J. Amer. Math. Soc. 17 (2) (2004) 443–471.
[4] L. Cardoulis, M. Cristofol, P. Gaitan, Inverse problem for the Schrödinger operator in an unbounded strip, http://arxiv.org/abs/math/0612539v1.
[5] V. Isakov, Carleman type estimates in an anisotropic case and applications, J. Differential Equations 105 (1993) 217–238.
[6] I. Lasiecka, R. Triggiani, X. Zhang, Global uniqueness, observability and stabilization of nonconservative Schrödinger equations via pointwise Carleman estimates. Part I: H1-estimates, J. Inv. Ill-Posed Problems 11 (1) (2004) 43–123.
[7] K. Liu, Locally distributed control and damping for the conservative systems, SIAM J. Control Optim. 35 (5) (1997) 1574–1590.
[8] A. Mercado, A. Osses, L. Rosier, Inverse problems for the Schrödinger equation via Carleman inequalities with degenerate weights, Inverse Problems, in press.
[9] L. Rosier, B.-Y. Zhang, Null controllability of the complex Ginzburg–Landau equation, submitted for publication.