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"Bubble-Tower" phenomena in a semilinear elliptic equation with mixed Sobolev growth

Juan F. Campos*

Departamento de Ingeniería Matemática, Universidad de Chile, Casilla 170 Correo 3, Santiago, Chile

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Abstract

In this work we consider the following problem

$$\begin{cases} \Delta u + u^p + u^q = 0 & \text{in } \mathbb{R}^N \\ u > 0 & \text{in } \mathbb{R}^N \\ \lim\limits_{|x|\to\infty} u(x) \to 0 \end{cases}$$

with $N/(N-2) < p < p^* = (N+2)/(N-2) < q$, $N \geq 3$.

We prove that if p is fixed, and q is close enough to the critical exponent p^*, then there exists a radial solution which behaves like a superposition of *bubbles* of different blow-up orders centered at the origin. Similarly when q is fixed and p is sufficiently close to the critical, we prove the existence of a radial solution which resembles a superposition of *flat bubbles* centered at the origin.

Keywords: Bubble-Tower phenomena; Lyapunov–Schmidt reduction; Critical exponent

1. Introduction

Let us consider the problem

$$\begin{cases} \Delta u + u^p + u^q = 0 & \text{in } \mathbb{R}^N \\ u > 0 & \text{in } \mathbb{R}^N \\ \lim\limits_{|x|\to\infty} u(x) \to 0 \end{cases} \tag{1}$$

* Tel.: +56 2 678 0590; fax: +56 2 688 3821.
E-mail address: jcampos@dim.uchile.cl.

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for $N \geq 3$, $1 < p < q$, and Δ denotes the standard Laplacian operator. In the case of a single power, namely $1 < p = q$, (1) is equivalent to the classical Emdem–Fowler–Lane equation

$$\begin{cases} \Delta u + u^p = 0 \quad \text{in } \mathbb{R}^N \\ u > 0 \quad \text{in } \mathbb{R}^N \\ \lim\limits_{|x|\to\infty} u(x) \to 0. \end{cases} \tag{2}$$

This equation was introduced by Lane in the mid-19th century, as a model of the inner structure of stars. A basic question is that of finding *radial ground states* to this problem, namely a solution $u(x) = u(|x|)$ that is finite up to $r = 0$ with $u'(0) = 0$. It is well known that the critical exponent $p = p^* := \frac{N+2}{N-2}$ sets a dramatic shift in the existence of solutions. In [15] the authors showed that in the case $1 < p < p^*$ there is no positive solution of (2). When $p = p^*$, (see [1,17,3]), all the solutions are constituted by the family

$$u_{\lambda,\xi}(x) = \gamma_N \left(\frac{\lambda}{\lambda^2 + |x-\xi|^2}\right)^{\frac{N-2}{2}}, \quad \gamma_N = (N(N-2))^{\frac{N-2}{4}}$$

where $\lambda > 0, \xi \in \mathbb{R}^N$. In the case $\xi = 0$, this solution is radially symmetric and it has *fast decay*, which means that $u_\lambda(r) = O(r^{-(N-2)})$ as $r \to \infty$. If $p > p^*$ then the solutions have the form $u_\lambda(x) = \lambda^{\frac{2}{p-1}} v(\lambda x)$, with $\lambda > 0$, and

$$u_\lambda(x) \sim C_{p,N} |x|^{\frac{-2}{p-1}}$$

where $C_{p,N} = (\frac{2}{p-1}\{\frac{2}{p-1} - (N-2)\})^{\frac{-2}{p-1}}$. This kind of asymptotic behavior is what we call *slow decay*. Let us notice that these solutions still exist when $p = p^*$ but its decay rate is like $r^{-\frac{N-2}{2}}$, which is slower than fast decay.

In the more general case $1 < p < q$, Zou proved in [18] that if $p \leq \frac{N}{N-2}$ then (1) admits no ground states, and if $q < p^*$ then there is no positive solution. He also showed that if $p > p^*$ then (1) admits infinitely many solutions with slow decay, and finally, in the case

$$1 < p < p^* < q \tag{3}$$

he proved that all the ground states of (1) are radial around some point. The first result of existence of radial ground states for (1) under the restriction (3), was given by Lin and Ni in [16]. They found, in the case $q = 2p - 1$, an explicit solution of the form $u(r) = A(B + r^2)^{-1/(p-1)}$ where A, B are positive constants depending on p and N. The question of existence remained open until the work of Bamón, Flores, and del Pino. In [2] the authors proved existence of radial ground states using dynamical systems tools. They proved that for $N/(N-2) < p < p^*$ fixed, given any integer $k \geq 1$, if $q > p^*$ is close enough to p^* then (1) has at least k radial ground states with fast decay. And if $p^* < q$ is fixed, given any integer $k \geq 1$, if $p < p^*$ is sufficiently close to p^*, then (1) has at least k radial ground states with fast decay. They also showed that if $q > p^*$ is fixed there exists $\bar{p} > N/(N-2)$ such that if $1 < p < \bar{p}$ then there are no radial ground states. Let us notice that these results do not cover Lin and Ni's solution since it is of slow decay. It can be shown also that slow decay solutions are unique if they exist and, as discussed in [2, 12], their presence is not expected to be generic. It is worthwhile mentioning that in the case $q = 2p - 1$, if the range of p is further restricted to

$$\frac{N + 2\sqrt{N-1}}{N + 2\sqrt{N-1} - 4} < p \tag{4}$$

then not only does Lin and Ni's solution exists, but also infinitely many solutions with fast decay. Moreover if $\frac{N}{N-2} < p < p^* < q$, p satisfies (4), and there exists a slow decay ground state for (1), then there are infinitely many ground states with fast decay.

Even though the question of existence of solutions for (1) under the restriction (3) has been partly answered in the slightly sub-supercritical case with geometrical dynamical systems tools, the result presented in this paper recovers the existence theorems given in [2] and also gives an asymptotic approximation of the solutions with a simpler method.

More precisely we prove the existence of a solution which asymptotically resembles a superposition of *bubbles* of different blow-up orders, centered at the origin. First we consider the case

$$\begin{cases} \Delta u + u^p + u^{p^*+\varepsilon} = 0 \quad \text{in } \mathbb{R}^N \\ u > 0 \quad \text{in } \mathbb{R}^N \\ \lim\limits_{|x|\to\infty} u(x) \to 0 \end{cases} \tag{5}$$

where $\frac{N}{N-2} < p$ is fixed and $\varepsilon > 0$. Then we have

Theorem 1. *Let $N \geq 3$ and $\frac{N}{N-2} < p$. Then for any $k \in \mathbb{N}$ there exists, for all sufficiently small $\varepsilon > 0$, a solution u_ε of (5) of the form*

$$u_\varepsilon(y) = \gamma_N \sum_{i=1}^{k} \left(\frac{1}{1 + \alpha_i^{\frac{4}{N-2}} \varepsilon^{-\left(i-1+\frac{1}{p^*-p}\right)\frac{4}{N-2}} |y|^2} \right)^{\frac{N-2}{2}} \alpha_i \varepsilon^{-\left(i-1+\frac{1}{p^*-p}\right)} (1 + o(1)),$$

with $o(1) \to 0$ uniformly in $\mathbb{R}^N$, as $\varepsilon \to 0$. The constants α_i can be computed explicitly and depend only on N and p.

This kind of concentration phenomena is known as *bubble-tower*, and it has been detected for some semilinear elliptic equations with radial symmetry, see for example [6,4,5]. The existence of bubble-tower solutions in the case of a generic domain has been established for the Brezis–Nirenberg problem in [14], see also [10].

In the case

$$\begin{cases} \Delta u + u^{p^*-\varepsilon} + u^q = 0 \quad \text{in } \mathbb{R}^N \\ u > 0 \quad \text{in } \mathbb{R}^N \\ \lim\limits_{|x|\to\infty} u(x) \to 0 \end{cases} \tag{6}$$

with $p^* < q$ fixed, for any $k \in \mathbb{N}$, we prove the existence of a solution which behaves like the superposition of k *flat bubbles* with a small maximum value which approaches zero uniformly as $\varepsilon \to 0$. In [8] the authors detected these kind of solutions to the problem of finding radially symmetric solutions of the prescribed mean curvature equation.

Theorem 2. *Let $N \geq 3$ and $p^* < q$ be fixed. Then given $k \in \mathbb{N}$ exists, for $\varepsilon > 0$ small enough, a solution u_ε of the problem (6) of the form*

$$u_\varepsilon(y) = \gamma_N \sum_{i=1}^{k} \left(\frac{1}{1 + \beta_i^{\frac{4}{N-2}} \varepsilon^{\left(i-1+\frac{1}{q-p^*}\right)\frac{4}{N-2}} |y|^2} \right)^{\frac{N-2}{2}} \beta_i \varepsilon^{\left(i-1+\frac{1}{q-p^*}\right)} (1 + o(1))$$

with $o(1) \to 0$ uniformly on $\mathbb{R}^N$, where the constants β_i depend only on N y q, can be computed explicitly.

To prove these theorems we use the so-called Emden–Fowler transformation, first introduced in [13], which converts dilations into translations, so the problem of finding a k-bubble solution for (1) becomes equivalent to the problem of finding a k-bump solution of a second-order equation on $\mathbb{R}$. Then a variation of Lyapunov–Schmidt procedure reduces the construction of these solutions to a finite-dimensional variational problem on $\mathbb{R}$. This kind of reduction, first introduced in [11], has been used to detect bubbling concentration phenomena in [6,7], and it also can be adapted to certain situations without radial symmetry, for example when symmetry with respect to N axes at a point of the domain is assumed, see for example [9,7].

The rest of the paper is organized as follows, the next three sections are devoted to the proof of the Theorem 1: we first state an asymptotical estimate of the energy of the ansatz, which is the key to the method, after that we solve a nonlinear linear problem corresponding to the finite-dimensional reduction, and then solve the finite-dimensional variational problem. The last section is the proof of Theorem 2, which is similar to the first one, except for some minor variations.

2. The asymptotic expansion

We are interested in the problem of finding a solution u of (5) with fast decay. We can assume that u is radial around the origin, and then (5) becomes equivalent to

$$\begin{cases} u''(r) + \dfrac{N-1}{r}u'(r) + u^p(r) + u^{p^*+\varepsilon}(r) = 0 \quad r \in (0,\infty) \\ u'(0) = 0 \\ \lim\limits_{r\to\infty} u(r) \to 0. \end{cases} \tag{7}$$

Introducing the change of variable

$$v(x) = r^{\frac{2}{p^*-1}} u(r)\,\big|_{r=\mathrm{e}^{-\frac{p^*-1}{2}x}} \tag{8}$$

for $x \in \mathbb{R}$, which is the so-called Emden–Fowler transformation, the problem (7) becomes

$$\begin{cases} v''(x) + \beta[\mathrm{e}^{\varepsilon x} v^{p^*+\varepsilon}(x) + \mathrm{e}^{-(p^*-p)x} v^p(x)] - v = 0 \quad \text{in } \mathbb{R} \\ 0 < v(x) \to 0 \quad \text{as } x \to \pm\infty \end{cases} \tag{9}$$

with $\beta = (\frac{2}{N-2})^2$. The functional associated to (9) is

$$E_\varepsilon(\psi) = I_\varepsilon(\psi) - \frac{\beta}{p+1}\int_{-\infty}^{\infty} \mathrm{e}^{-(p^*-p)x}|\psi|^{p+1}\mathrm{d}x \tag{10}$$

where

$$I_\varepsilon(\psi) = \frac{1}{2}\int_{-\infty}^{\infty} |\psi'|^2\mathrm{d}x + \frac{1}{2}\int_{-\infty}^{\infty} |\psi|^2\mathrm{d}x - \frac{\beta}{p^*+\varepsilon+1}\int_{-\infty}^{\infty} \mathrm{e}^{\varepsilon x}|\psi|^{p^*+\varepsilon+1}\mathrm{d}x.$$

Let w be the positive radial solution of

$$\Delta w + w^{p^*} = 0 \quad \text{in } \mathbb{R}^N$$

with $w(0) = \gamma_N$, given by $u_{1,0}$. Now let U be the transformation of w via (8) given by

$$U(x) = \gamma_N \mathrm{e}^{-x}(1 + \mathrm{e}^{-(p^*-1)x})^{-\frac{N-2}{2}}. \tag{11}$$

Then U satisfies

$$\begin{cases} U'' - U + \beta U^{p^*} = 0 \quad \text{in } \mathbb{R} \\ 0 < U(x) \to 0 \quad \text{as } |x| \to \pm\infty \end{cases} \tag{12}$$

and

$$\gamma_N \mathrm{e}^{-|x|} 2^{-\frac{N-2}{2}} \le U(x) \le \gamma_N \mathrm{e}^{-|x|}$$

and therefore $U(x) = O(\mathrm{e}^{-|x|})$.

Let us consider $0 < \xi_1 < \xi_2 < \cdots < \xi_k$. We look for a solution of (9) of the form

$$v(x) = \sum_{i=1}^{k} U(x - \xi_i) + \phi$$

with ϕ small.

We define

$$U_i(x) = U(x - \xi_i), \qquad V(x) = \sum_{i=1}^{k} U_i(x) \tag{13}$$

with the following choices for the points ξ_i:

$$\xi_1 = -\frac{1}{p^* - p}\log\varepsilon - \log\Lambda_1$$
$$\xi_{i+1} - \xi_i = -\log\varepsilon - \log\Lambda_{i+1} \qquad i = 1, \dots, k-1 \tag{14}$$

where the numbers Λ_i are positive parameters. This choice of the ξ_i's turns out to be convenient in the proof of the following asymptotic expansion of $E_\varepsilon(V)$.

Lemma 1. *Let $N \geq 3$, $p > \frac{N}{N-2}$, $k \in \mathbb{N}$ and $\delta > 0$ be fixed. Assume that*

$$\delta < \Lambda_i < \delta^{-1} \quad \forall\, i = 1, \dots, k. \tag{15}$$

Then for $V(x)$ given by (13), and for the choice (14) of the points ξ_i, there are positives numbers $a_1, \dots, a_5$, depending only on N and p, such that

$$E_\varepsilon(V) = ka_1 + \varepsilon\Psi_k(\Lambda) + ka_4\varepsilon + \varepsilon\Theta_\varepsilon(\Lambda) - \frac{a_3 k}{2(p^* - p)}((1-k)(p^* - p) - 2)\varepsilon\log\varepsilon \tag{16}$$

where

$$\Psi_k(\Lambda) = a_3 k\log\Lambda_1 - a_5\Lambda_1^{(p^*-p)} + \sum_{i=2}^{k}[(k-i+1)a_3\log\Lambda_i - a_2\Lambda_i] \tag{17}$$

with $\Theta_\varepsilon(\Lambda) \to 0$ as $\varepsilon \to 0$, uniformly in the C^1-sense on the set of Λ_i's that satisfy (15).

Proof. Let us estimate $I_\varepsilon(V)$. First we may write

$$I_\varepsilon(V) = I_0(V) - \frac{\beta}{p^*+1}\int_{-\infty}^{\infty}(\mathrm{e}^{\varepsilon x} - 1)|V|^{p^*+\varepsilon+1}\mathrm{d}x + A_\varepsilon.$$

Where

$$A_\varepsilon = \beta\left(\frac{1}{p^*+1} - \frac{1}{p^*+\varepsilon+1}\right)\int_{-\infty}^{\infty}\mathrm{e}^{\varepsilon x}|V|^{p^*+\varepsilon+1}\mathrm{d}x + \frac{\beta}{p^*+1}\int_{-\infty}^{\infty}(|V|^{p^*+1} - |V|^{p^*+\varepsilon+1})\mathrm{d}x.$$

We can prove that

$$A_\varepsilon = k\varepsilon\beta\left(\frac{1}{(1+p^*)^2}\int_{-\infty}^{\infty}|U|^{p^*+1}\mathrm{d}x - \frac{1}{(1+p^*)}\int_{-\infty}^{\infty}|U|^{p^*+1}\log U\mathrm{d}x\right) + o(\varepsilon). \tag{18}$$

In a similar way we find

$$\begin{aligned}\int_{-\infty}^{\infty}(\mathrm{e}^{\varepsilon x} - 1)|V|^{p^*+\varepsilon+1}\mathrm{d}x &= \sum_{l=1}^{k}\int_{\mu_{l-1}}^{\mu_l} x|V|^{p^*+1}\mathrm{d}x \\ &= k\varepsilon\left(\sum_{l=1}^{k}\xi_l\right)\int_{-\infty}^{\infty}U^{p^*+1}\mathrm{d}y + o(\varepsilon).\end{aligned} \tag{19}$$

Now we define

$$B = \frac{p^*+1}{\beta}\left(I_0(V) - \sum_{i=1}^{k}I_0(U_i)\right).$$

It is not hard to check that

$$B = \int_{-\infty}^{\infty}\left(\sum_{i=1}^{k}U_i^{p^*+1} - \left(\sum_{i=1}^{k}U_i\right)^{p^*+1}\right) + (p^*+1)\int_{-\infty}^{\infty}\sum_{i<j}(U_i^{p^*}U_j).$$

Let us consider

$$\mu_0 = -\infty, \qquad \mu_i = \frac{\xi_i + \xi_{i+1}}{2} \quad i = 1, \ldots, k-1, \qquad \mu_k = \infty \tag{20}$$

and decompose B as

$$B = \sum_{l=1}^{k} (C_1^l - C_0^l + C_2^l)$$

where

$$C_0^l = (p^* + 1) \int_{\mu_{l-1}}^{\mu_l} U_l^{p^*} \sum_{j<l}^{k} U_j$$

$$C_1^l = \int_{\mu_{l-1}}^{\mu_l} \left[U_l^{p^*+1} - \left(\sum_{i=1}^{k} U_i \right)^{p^*+1} + (p^* + 1) U_l^{p^*} \sum_{j \neq l}^{k} U_j \right]$$

$$C_2^l = \int_{\mu_{l-1}}^{\mu_l} \left(\sum_{i \neq l}^{k} U_i^{p^*+1} + (p^* + 1) \sum_{i \neq l} \sum_{i<j} U_i^{p^*} U_j \right).$$

Now, let us estimate C_1^l. From the mean value theorem we get

$$|C_1^l| \leq C \int_{\mu_{l-1}}^{\mu_l} \left(\sum_{i \neq l}^{k} U_i \right)^2 \left(\sum_{i=1}^{k} U_i \right)^{p^*-1} \mathrm{d}x.$$

If $l \in \{2, \ldots, k-1\}$, setting $\rho = -\log \varepsilon$ and using the fact that $U(x) = O(\mathrm{e}^{-|x|})$, we get, using (20)

$$\begin{aligned} |C_1^l| &\leq C \int_0^{\frac{\rho}{2}+K} \mathrm{e}^{-2|\rho - y|} \mathrm{e}^{y(p^*-1)} \mathrm{d}y \\ &\leq C \mathrm{e}^{-2\rho} \int_0^{\frac{\rho}{2}+K} \mathrm{e}^{-(p^*-3)y} \mathrm{d}y = O(\mathrm{e}^{-\frac{p^*+1}{2}\rho}) = o(\varepsilon) \end{aligned}$$

where K depends only on δ. If $l \in \{1, k\}$, we easily check that $C_1^l = o(\varepsilon)$. Similar arguments yield $C_2^l = o(\varepsilon)$. To estimate C_0^l, we notice that

$$C_0^l = (p^* + 1) \int_{\mu_{l-1}}^{\mu_l} U_l^{p^*} U_{l-1} \mathrm{d}x + o(\varepsilon).$$

According to (11), we have $U(x) = C_N \cosh\left(\frac{2x}{N-2}\right)^{-\frac{N-2}{2}}$, with $C_N = \gamma_N 2^{-\frac{N-2}{2}}$. Then

$$|U(x + \xi) - C_N \mathrm{e}^{-|x+\xi|}| = O(\mathrm{e}^{-p^*|x+\xi|})$$

when $\xi \to \infty$. Therefore we obtain

$$C_0^l = (p^* + 1) C_N \mathrm{e}^{\xi_l - \xi_{l-1}} \int_{-\infty}^{\infty} U^{p^*}(x) \mathrm{e}^x \mathrm{d}x + o(\varepsilon).$$

From these estimates we conclude

$$I_0(V) = k I_0(U) - \beta C_N \int_{-\infty}^{\infty} U^{p^*}(x) \mathrm{e}^x \mathrm{d}x \left(\sum_{l=2}^{k} \mathrm{e}^{\xi_l - \xi_{l-1}} \right) + o(\varepsilon). \tag{21}$$

Finally, we easily check

$$\int_{-\infty}^{\infty} \mathrm{e}^{-(p^*-p)x} V^{p+1}(x) \mathrm{d}x = \mathrm{e}^{-(p^*-p)\xi_1} \int_{-\infty}^{\infty} \mathrm{e}^{-(p^*-p)x} U^{p+1}(x) + o(\varepsilon). \tag{22}$$

Now we define

$$\begin{cases} a_1 = \frac{1}{2}\int_{-\infty}^{\infty} |U'(x)|^2 \mathrm{d}x + \frac{1}{2}\int_{-\infty}^{\infty} U^2(x)\mathrm{d}x - \frac{\beta}{p^*+1}\int_{-\infty}^{\infty} U^{p^*+1}(x)\mathrm{d}x \\ a_2 = \beta C_N \int_{-\infty}^{\infty} \mathrm{e}^{x} U^{p^*}(x)\mathrm{d}x \\ a_3 = \frac{\beta}{p^*+1}\int_{-\infty}^{\infty} U^{p^*+1}(x)\mathrm{d}x \\ a_4 = \frac{1}{(p^*+1)^2}\int_{-\infty}^{\infty} U^{p^*+1}(x)\mathrm{d}x - \frac{1}{p^*+1}\int_{-\infty}^{\infty} U^{p^*+1}(x)\log U(x)\mathrm{d}x \\ a_5 = \frac{\beta}{p+1}\int_{-\infty}^{\infty} \mathrm{e}^{-(p^*-p)x} U^{p+1}(x)\mathrm{d}x. \end{cases} \tag{23}$$

Collecting the estimates (18)–(22), we get the validity of the following expansion

$$E_\varepsilon(V) = ka_1 - a_2\sum_{l=2}^{k} \mathrm{e}^{-(\xi_l-\xi_{l-1})} - \varepsilon a_3\left(\sum_{i=1}^{k}\xi_i\right) + k\varepsilon a_4 - a_5\mathrm{e}^{-(p^*-p)\xi_1} + o(\varepsilon).$$

Using (14), this decomposition reads

$$E_\varepsilon(V) = ka_1 + \varepsilon\Psi_k(\Lambda) - \frac{a_3 k}{2(p^*-p)}((1-k)(p^*-p)-2)\varepsilon\log\varepsilon + ka_4\varepsilon + o(\varepsilon)$$

with Ψ_k given by (17). Moreover, the term $o(\varepsilon)$ is uniform in the set of the Λ_i's that satisfy (15). The fact that differentiation with respect to Λ leaves $o(\varepsilon)$ of the same order follows from very similar computations, so we omit them. □

Let us notice that the only critical point of Ψ_k is given by

$$\Lambda^* = \left(\left[\frac{a_3 k}{a_5(p^*-p)}\right]^{\frac{1}{p^*-p}}, \frac{(k-1)a_3}{a_2}, \frac{(k-2)a_3}{a_2}, \ldots, \frac{a_3}{a_2}\right)$$

and is nondegenerate. This result will be useful since, if $V+\phi$ is a solution of (9), with ϕ small, it is natural to expect that this only occurs if Λ corresponds to a critical point of Ψ_k. This is actually true, as we show in the following sections.

3. The finite-dimensional reduction

In this section we consider $p > \frac{N}{N-2}$, points $0 < \xi_1 < \cdots < \xi_k$, which are for now arbitrary, and we keep the notation V and U_i as in the previous section. We define

$$Z_i(x) = U_i'(x), \quad i = 1, \ldots, k.$$

Consider the problem of finding a function ϕ for which there are constants c_i, $i = 1, \ldots, k$ such that

$$\begin{cases} \sum_{i=1}^{k} c_i Z_i = -(V+\phi)'' + (V+\phi) - \beta\left[\mathrm{e}^{\varepsilon x}(V+\phi)_+^{p^*+\varepsilon} + \mathrm{e}^{-(p^*-p)x}(V+\phi)_+^{p}\right] \\ \int_{-\infty}^{\infty} Z_i\phi = 0 \quad \forall i = 1, \ldots, k, \qquad \lim_{x\to\pm\infty}\phi(x) = 0. \end{cases} \tag{24}$$

Let us consider the operator

$$\mathcal{L}_\varepsilon\phi = -\phi'' + \phi - \beta\left[(p^*+\varepsilon)\mathrm{e}^{\varepsilon x}V^{p^*+\varepsilon-1} + p\mathrm{e}^{-(p^*-p)x}V^{p-1}\right]\phi.$$

The problem (24) gets rewritten as

$$\begin{cases} \mathcal{L}_\varepsilon(\phi) = N_\varepsilon(\phi) + R_\varepsilon + \sum_{i=1}^{k} c_i Z_i \quad \text{in } \mathbb{R} \\ \int_{-\infty}^{\infty} Z_i \phi = 0 \quad \forall i = 1, \ldots, k, \qquad \lim_{x\to\pm\infty} \phi(x) = 0 \end{cases} \tag{25}$$

where

$$N_\varepsilon(\phi) = \beta e^{\varepsilon x}((V+\phi)_+^{p^*+\varepsilon} - V^{p^*+\varepsilon} - (p^*+\varepsilon)V^{p^*+\varepsilon-1}\phi) + \beta e^{-(p^*-p)x}((V+\phi)_+^{p} - V^{p} - pV^{p-1}\phi)$$

$$R_\varepsilon = \beta\left(e^{\varepsilon x} V^{p^*+\varepsilon} + e^{-(p^*-p)x} V^p - \sum_{i=1}^{k} U_i^{p^*}(x)\right).$$

Next we introduce a convenient functional setting to analyze the invertibility of the operator $\mathcal{L}_\varepsilon$ under the conditions of orthogonality. For a small $\sigma > 0$, to be fixed, and a function $\psi : \mathbb{R} \to \mathbb{R}$, we define the norm

$$\|\psi\|_* = \sup_{x\in\mathbb{R}} \left(\sum_{i=1}^{k} e^{-\sigma|x-\xi_i|}\right)^{-1} |\psi(x)|.$$

To solve (24) it is important first to understand its linear part, where we consider the problem of, given a function h, finding ϕ such that

$$\begin{cases} \mathcal{L}_\varepsilon(\phi) = h + \sum_{i=1}^{k} c_i Z_i \quad \text{in } \mathbb{R} \\ \int_{-\infty}^{\infty} Z_i \phi = 0 \quad \forall\, i = 1, \ldots, k, \qquad \lim_{x\to\pm\infty} \phi(x) = 0 \end{cases} \tag{26}$$

for certain constants c_i. The following result holds

Proposition 1. *There exists positive numbers $\varepsilon_0, \delta_0, R_0$ such that, if*

$$R_0 < \xi_1, \qquad R_0 < \min_{i=1,\ldots,k} (\xi_{i+1} - \xi_i), \qquad \xi_k < \frac{\delta_0}{\varepsilon} \tag{27}$$

then for all $0 < \varepsilon < \varepsilon_0$ and $\forall h \in C(\mathbb{R})$ with $\|h\|_* < \infty$, the problem (26) admits a unique solution $\phi = T_\varepsilon(h)$. Besides, there exists $C > 0$ such that

$$\|T_\varepsilon(h)\|_* \le C\|h\|_*, \qquad |c_i| \le C\|h\|_*.$$

For the proof we need the following,

Lemma 2. *Assume that there are sequences $\varepsilon_n \to 0$ and $0 < \xi_1^n < \cdots < \xi_k^n$ with*

$$\xi_1^n \to \infty, \qquad \min_{i=1,\ldots,k} (\xi_{i+1}^n - \xi_i^n) \to \infty, \qquad \xi_k^n = o(\varepsilon_n^{-1})$$

such that for certain functions ϕ_n, h_n with $\|h_n\|_* \to 0$, and scalars c_i^n one has

$$\begin{cases} \mathcal{L}_{\varepsilon_n}(\phi_n) = h_n + \sum_{i=1}^{k} c_i^n Z_i^n \quad \text{in } \mathbb{R} \\ \int_{-\infty}^{\infty} Z_i^n \phi_n = 0 \quad \forall\, i = 1, \ldots, k, \qquad \lim_{x\to\pm\infty} \phi_n(x) = 0 \end{cases} \tag{28}$$

with $Z_i^n(x) = U'(x - \xi_i^n)$. Then

$$\lim_{n\to\infty} \|\phi_n\|_* = 0.$$

Proof. We will establish first the weaker assertion that

$$\lim_{n\to\infty} \|\phi_n\|_\infty = 0.$$

By contradiction, we may assume that $\|\phi_n\|_\infty = 1$. Testing (28) against Z_l^n and integrating by parts we get

$$\sum_{i=1}^k c_i^n \int_{-\infty}^{\infty} Z_i^n Z_l^n \mathrm{d}x = \int_{-\infty}^{\infty} \mathcal{L}_{\varepsilon_n}(Z_l^n)\phi_n \mathrm{d}x - \int_{-\infty}^{\infty} h_n Z_l^n \mathrm{d}x.$$

This defines an "almost diagonal" system in the c_i^n's as $n \to \infty$. Moreover, the fact that $Z_l^n(x) = O(\mathrm{e}^{-|x-\xi_l^n|})$, $p > \frac{N}{N-2}$, and that Z_l^n solves

$$-Z'' + (1 - p^*\beta U_l^{p^*-1})Z = 0$$

yields, after an application of dominated convergence, that $\lim_{n\to\infty} c_i^n = 0$. If we set $x_n \in \mathbb{R}^N$ such that $\phi_n(x_n) = 1$, we can assume that $\exists\, i \in \{1, \ldots, k\}$ such that for n large enough

$$\exists\, R > 0 \quad \text{such that } |x_n - \xi_i^n| < R. \tag{29}$$

Let us fix an index i such that (29) holds. We set

$$\tilde{\phi}_n(x) = \phi_n(x + \xi_i^n).$$

From (28) and (29) and elliptic estimates, we see that passing to a suitable subsequence $\tilde{\phi}_n(x)$ converges uniformly over compacts to a nontrivial bounded solution ϕ of

$$-\phi'' + \phi - \beta p^* U^{p*}\phi = 0 \quad \text{in } \mathbb{R}.$$

Hence $\phi = cU'$, for some $c \neq 0$. However

$$0 = \int_{-\infty}^{\infty} Z_l^n \phi_n \to c \int_{-\infty}^{\infty} [U'(x)]^2$$

which is a contradiction. Then necessarily $\|\phi_n\|_\infty \to 0$.

Let us observe that (28) takes the form

$$-\phi_n'' + \phi_n = g_n(x) \tag{30}$$

with

$$g_n(x) = h_n(x) + \sum_{i=1}^k Z_i^n + \beta[(p^* + \varepsilon_n)\mathrm{e}^{\varepsilon_n x} V^{p^*+\varepsilon_n-1} + p\mathrm{e}^{-(p^*-p)x} V^{p-1}]\phi_n.$$

If $0 < \sigma < \min\{p^* - 1, 1, 2p - 1 - p^*\}$, we have

$$|g_n(x)| \leq \theta_n \sum_{i=1}^k \mathrm{e}^{-\sigma|x-\xi_i^n|}$$

with $\theta_n \to 0$. Choosing $\bar{c} > 0$ large enough we get that

$$\varphi_n(x) = \bar{c}\theta_n \sum_{i=1}^k \mathrm{e}^{-\sigma|x-\xi_i^n|}$$

is a supersolution of (30), and $-\varphi_n(x)$ will be a subsolution of (30). Then

$$|\phi_n| \leq \theta_n \sum_{i=1}^k \mathrm{e}^{-\sigma|x-\xi_i^n|}$$

and the proof of the lemma is concluded. $\square$

Proof of Proposition 1. Consider

$$H = \left\{\phi \in H^1(\mathbb{R}) : \int_{-\infty}^{\infty} Z_i\phi = 0 \ \forall i \in \{1, \ldots, k\}\right\}$$

endowed with the inner product $[\phi, \psi] = \int_{-\infty}^{\infty}(\phi'\psi' + \phi\psi)$. Then the problem (26) expressed in weak form is equivalent to that of finding $\phi \in H$ such that $\forall\psi \in H$

$$[\phi, \psi] = \beta \int_{-\infty}^{\infty} \left[(p^* + \varepsilon)\mathrm{e}^{\varepsilon x} V^{p^*+\varepsilon-1} + p\mathrm{e}^{-(p^*-p)x} V^{p-1}\right]\phi\psi + \int_{-\infty}^{\infty} h\psi.$$

With the aid of Riesz representation theorem, this equation gets rewritten in the operational form

$$[\phi, \psi] = [K_\varepsilon(\phi) + \tilde{h}, \psi]$$

where $\tilde{h}$ depends linearly on h, and $K_\varepsilon(\phi)$ is compact. Fredholm's alternative guarantees unique solvability for any $h \in H$, provided that the equation $\phi = K_\varepsilon(\phi)$ has only the trivial solution in H. This latter statement holds for $R_0, \varepsilon_0, \delta_0$ chosen properly, assuming the opposite would lead us to a contradiction with the previous lemma. Continuity can be deduced in a similar way. $\square$

Now we study some differentiability properties of T_ε on the variables ξ_i, that will be important for later purposes. We shall use the notation $\xi = (\xi_1, \ldots, \xi_k)$, and consider the Banach space

$$C_* = \{f \in C(\mathbb{R}) \mid \|f\|_* < \infty\}$$

endowed with the $\|\cdot\|_*$ norm. We also consider the space $\mathcal{L}(C_*)$ of linear operators of C_*.

The following result holds.

Proposition 2. *Under the assumptions of the Proposition 1, the map $\xi \to T_\varepsilon$ with values in $\mathcal{L}(C_*)$ is of class C^1. Moreover, there is a constant $C > 0$ such that*

$$\|D_\xi T_\varepsilon\|_{\mathcal{L}(C^*)} \leq C$$

uniformly on the vectors ξ that satisfy (27).

Proof. Fix $h \in C_*$, and let $\phi = T_\varepsilon(h)$ for $\varepsilon < \varepsilon_0$. Consider differentiation with respect to ξ_l. Let us recall that ϕ satisfies

$$\mathcal{L}_\varepsilon(\phi) = h + \sum_{i=1}^{k} c_i Z_i \quad \text{in } \mathbb{R}$$

plus orthogonality conditions, for some constants c_i (uniquely determined). For $j \in \{1, \ldots, k\}$ we define the constants α_j as the solution of

$$\sum_{j=1}^{k} \alpha_j \int_{-\infty}^{\infty} Z_j Z_i = 0 \quad \forall i \neq l$$

$$\sum_{j=1}^{k} \alpha_j \int_{-\infty}^{\infty} Z_j Z_l = -\int_{-\infty}^{\infty} \phi\partial_{\xi_l} Z_l.$$

Again this is an almost diagonal system. We define also the function

$$f(x) = \beta\partial_{\xi_l} F_\varepsilon(x)\phi + c_l\partial_{\xi_l} Z_l - \sum_{j=1}^{k} \alpha_j \mathcal{L}_\varepsilon(Z_j)$$

where

$$F_\varepsilon(x) = \beta\left[(p^* + \varepsilon)\mathrm{e}^{\varepsilon x} V^{p^*+\varepsilon-1} + p\mathrm{e}^{-(p^*-p)x} V^{p-1}\right].$$

Hence $\partial_{\xi_l}\phi$ satisfies

$$\partial_{\xi_l}\phi - \sum_{j=1}^{k} \alpha_j Z_j = T_\varepsilon(f).$$

Moreover $|\alpha_i| \le C\|\phi\|_*$, $|c_i| \le C\|h\|_*$, $\|\phi\|_* \le C\|h\|_*$, so that also $\|\partial_{\xi_l}\phi\|_* \le C\|h\|_*$ Besides $\partial_{\xi_l}\phi$ depends continuously on ξ for this norm, and the validity of the result is proved. $\square$

In what follows we assume, for $M > 0$ large and fixed, the validity of the constraints

$$\frac{1}{p^* - p}\log(M\varepsilon)^{-1} < \xi_1, \qquad \log(M\varepsilon)^{-1} < \min_{i=2,\dots,k}(\xi_i - \xi_{i-1}), \qquad \xi_k < k\log(M\varepsilon)^{-1}. \tag{31}$$

For the next purposes it is useful to consider $\|\phi\|_1 \le \frac{1}{\lambda}\|V\|_1$ where

$$\|\psi\|_1 = \sup_{x\in\mathbb{R}} \left(\sum_{i=1}^{k} \mathrm{e}^{-|x-\xi_i|}\right)^{-1} |\psi|$$

and $\lambda > 2^{-\frac{N-2}{2}}$. Under these conditions, provided that σ is fixed and small enough, one can easily check that

$$\|N(\phi)\|_* \le C(\|\phi\|_*^{\min\{p^*,2\}} + \|\phi\|_*^{\min\{2p-p^*,2\}}) \tag{32}$$

$$\left\|\frac{\partial N_\varepsilon}{\partial\phi}\right\|_* \le C(\|\phi\|_*^{\min\{p^*-1,2\}} + \|\phi\|_*^{\min\{2p-p^*-1,2\}}) \tag{33}$$

and

$$\|R_\varepsilon\|_* \le C\varepsilon^\alpha, \qquad \|\partial R_\varepsilon\|_* \le C\varepsilon^\alpha \tag{34}$$

with $\alpha = \frac{1+\lambda}{2}$, for some $\lambda > 0$ small enough.

Proposition 3. *Assume that conditions (31) hold. Then $\exists\, C > 0$ such that, $\forall \varepsilon > 0$ small enough, there exists a unique solution $\phi = \phi(\xi)$ to the problem (24), which besides satisfies*

$$\|\phi\|_* \le C\varepsilon^\alpha.$$

Moreover, the map $\xi \to \phi(\xi)$ is of class C^1 for the $\|\cdot\|_*$-norm, and

$$\|D_\xi\phi\|_* \le C\varepsilon^\alpha.$$

Proof. If we define

$$A_\varepsilon(\phi) := T_\varepsilon(N_\varepsilon(\phi) + R_\varepsilon)$$

then (25) is equivalent to the fixed point problem $\phi = A_\varepsilon(\phi)$. We will show that A_ε is a contraction in a proper region. Let

$$\mathcal{F}_r = \{\phi \in C_* : \|\phi\|_* \le r\varepsilon^\alpha\}$$

where $r > 0$ will be fixed later. We have that

$$\begin{aligned}\|A_\varepsilon(\phi)\|_* &\le \|T_\varepsilon\,(N_\varepsilon(\phi) + R_\varepsilon)\,\|_* \\ &\le C\|N_\varepsilon(\phi) + R_\varepsilon\|_* \\ &\le C_0((r\varepsilon^\alpha)^{\min\{p^*,2\}} + (r\varepsilon^\alpha)^{\min\{2p-p^*,2\}} + \varepsilon^\alpha).\end{aligned}$$

Besides

$$|N_\varepsilon(\phi_1) - N_\varepsilon(\phi_2)| \le C((r\varepsilon^\alpha)^{\min\{p^*-1,2\}} + (r\varepsilon^\alpha)^{\min\{2p-p^*-1,2\}})|\phi_1 - \phi_2|$$

consequently

$$\|A(\phi_1) - A(\phi_2)\|_* \le C_1((r\varepsilon^\alpha)^{\min\{p^*-1,2\}} + (r\varepsilon^\alpha)^{\min\{2p-p^*-1,2\}})\|\phi_1 - \phi_2\|_*.$$

If we choose $r \ge 3C_0$, then for ε small enough

$$C_0((r\varepsilon^\alpha)^{\min\{p^*,2\}} + (r\varepsilon^\alpha)^{\min\{2p-p^*,2\}} + \varepsilon^\alpha) \le r\varepsilon^\alpha$$
$$C_1((r\varepsilon^\alpha)^{\min\{p^*-1,2\}} + (r\varepsilon^\alpha)^{\min\{2p-p^*-1,2\}}) < 1$$

and so there is a unique fixed point of A in $\mathcal{F}_r$.

Concerning now the differentiability of $\xi \to \phi(\xi)$, let

$$B(\xi, \phi) = \phi - T_\varepsilon(N_\varepsilon(\phi) + R_\varepsilon).$$

Of course we have $B(\xi, \phi(\xi)) = 0$. Now let us write

$$D_\phi B(\xi, \phi)[\theta] = \theta - T_\varepsilon(\theta D_\phi N_\varepsilon(\phi)) = \theta + M(\theta)$$

where

$$M(\theta) = -T_\varepsilon(\theta D_\phi N_\varepsilon(\phi)).$$

From (33) and using the fact that $\phi \in F_r$, we obtain

$$\|M(\theta)\|_* \le C(\varepsilon^{\alpha\min\{p^*-1,2\}} + \varepsilon^{\alpha\min\{2p-p^*-1,2\}})\|\theta\|_*.$$

It follows that for a small ε, the operator $D_\phi B(\varepsilon, \phi)$ is invertible, with uniformly bounded inverse. It also depends continuously on its parameters. Let us differentiate with respect to ξ, we have

$$D_\xi B(\xi, \phi) = -D_\xi T_\varepsilon[N_\varepsilon(\phi) + R_\varepsilon] - T_\varepsilon[D_\xi N_\varepsilon(\xi, \phi) + D_\xi R_\varepsilon]$$

where all these expressions depend continuously on their parameters. Now, the implicit function theorem yields that $\phi(\xi)$ is of class C^1 and

$$D_\xi \phi = -\left(D_\phi B(\xi, \phi)\right)^{-1}[D_\xi B(\xi, \phi)]$$

so that

$$\|D_\xi \phi\|_* \le C\left(\|N_\varepsilon(\phi) + R_\varepsilon\|_* + \|D_\xi N_\varepsilon(\xi, \phi)\|_* + \|D_\xi R_\varepsilon\|_*\right) \le C\varepsilon^\alpha.$$

This concludes the proof. □

4. The finite-dimensional variational problem

In this section we fix $M > 0$ large and assume that conditions (31) hold for $\xi = (\xi_1, \ldots, \xi_k)$. According to the previous sections, our original problem has been reduced to that of finding ξ such that the c_i that appears in (24), given by Proposition 3, are all zero. Thus we need to solve

$$c_i(\xi) = 0 \quad \forall i \in \{1, \ldots, k\}. \tag{35}$$

This problem is equivalent to a variational problem. We define

$$J_\varepsilon(\xi) = E_\varepsilon(V + \phi(\xi)).$$

Lemma 3. *The function $V + \phi$ is a solution of (9) $\Leftrightarrow$ ξ is a critic point of J_ε, where $\phi = \phi(\xi)$ is given by Proposition 3.*

Proof. Assume that $V + \phi$ solves (9), integrating (9) against $\partial_{\xi_l}(V + \phi)$ we get

$$DE_\varepsilon(V + \phi)\partial_{\xi_l}(V + \phi) = 0.$$

Now if ξ is a critic point of J_ε, we have

$$D_{\xi_l} J_\varepsilon(\xi) = 0 \Leftrightarrow DE_\varepsilon(V+\phi)\partial_{\xi_l}(V+\phi) = 0$$
$$\Leftrightarrow \sum_{i=1}^{k} c_i \int_{-\infty}^{\infty} Z_i \partial_{\xi_l}(V+\phi) = 0.$$

But $\partial_{\xi_l}(V+\phi) = Z_l + o(1)$ where $o(1) \to 0$ as $\varepsilon \to 0$, uniformly for the $\|\cdot\|_*$-norm. Therefore

$$D_{\xi_l} J_\varepsilon(\xi) = 0 \quad \forall l = 1,\dots,k \Leftrightarrow \sum_{i=1}^{k} c_i \int_{-\infty}^{\infty} Z_i [Z_l + o(1)] = 0 \quad \forall l = 1,\dots,k$$

which defines an almost diagonal linear system on c_i, and the conclusion follows. $\square$

The next lemma is crucial to find the critical points of J_ε.

Lemma 4. *The following expansion holds*

$$J_\varepsilon(\xi) = E_\varepsilon(V) + o(\varepsilon)$$

where $o(\varepsilon)$ is uniform in the C^1-sense on the vectors ξ which satisfy (31).

Proof. Using the fact that $DE_\varepsilon(V+\phi)[\phi] = 0$, a Taylor expansion gives

$$\begin{aligned} E_\varepsilon(V+\phi) - E_\varepsilon(V) &= \int_0^1 D^2 E_\varepsilon(V+t\phi)[\phi^2] t\,\mathrm{d}t \\ &= \int_0^1 \int_{-\infty}^{\infty} (N_\varepsilon(\phi) + R_\varepsilon)\phi t\,\mathrm{d}t \\ &\quad + \beta(p^*+\varepsilon) \int_0^1 \int_{-\infty}^{\infty} \mathrm{e}^{\varepsilon x} \left[V^{p^*+\varepsilon-1} - (V+t\phi)^{p^*+\varepsilon-1}\right] \phi^2 t\,\mathrm{d}t \\ &\quad + \beta p \int_0^1 \int_{-\infty}^{\infty} \mathrm{e}^{-(p^*-p)x} \left[V^{p-1} - (V+t\phi)^{p-1}\right] \phi^2 t\,\mathrm{d}t \end{aligned}$$

and since $\|\phi\|_* \le C\varepsilon^\alpha$, with $\alpha = \frac{1+\lambda}{2}$, we get

$$J_\varepsilon(\xi) - E_\varepsilon(V) = O(\varepsilon^{1+\lambda})$$

uniformly on the points satisfying (31). Differentiating now with respect to the ξ variables, we obtain

$$\begin{aligned} \partial_{\xi_l}(J_\varepsilon(\xi) - E_\varepsilon(V)) &= \int_0^1 \int_{-\infty}^{\infty} \partial_{\xi_l}[(N_\varepsilon(\phi) + R_\varepsilon)\phi] t\,\mathrm{d}t \\ &\quad + \beta(p^*+\varepsilon) \int_0^1 \int_{-\infty}^{\infty} \mathrm{e}^{\varepsilon x} \partial_{\xi_l} \left(\left[V^{p^*+\varepsilon-1} - (V+t\phi)^{p^*+\varepsilon-1}\right] \phi^2\right) t\,\mathrm{d}t \\ &\quad + \beta p \int_0^1 \int_{-\infty}^{\infty} \mathrm{e}^{-(p^*-p)x} \partial_{\xi_l} \left(\left[V^{p-1} - (V+t\phi)^{p-1}\right] \phi^2\right) t\,\mathrm{d}t. \end{aligned}$$

And from the computations made in the previous propositions we deduce

$$\partial_{\xi_l}(J_\varepsilon(\xi) - E_\varepsilon(V)) = O(\varepsilon^{1+\lambda})$$

which concludes the proof. $\square$

Proof of Theorem 1. We consider the change of variable

$$\xi_1 = -\frac{1}{p^*-p} \log\varepsilon - \log \Lambda_1$$
$$\xi_{i+1} - \xi_i = -\log\varepsilon - \log \Lambda_i \quad \forall i \ge 2$$

where the Λ_i's are positive parameters. For notational convenience, we set $\Lambda = (\Lambda_1, \dots, \Lambda_k)$. Hence it suffices to find critical points of

$$\Phi_\varepsilon(\Lambda) = \varepsilon^{-1} J_\varepsilon(\xi(\Lambda)).$$

From the previous lemma and the expansion given in Lemma 1, we get

$$\nabla \Phi_\varepsilon(\Lambda) = \nabla \Psi_k(\Lambda) + o(1)$$

where $o(1) \to 0$ uniformly on the vectors Λ satisfying $M^{-1} < \Lambda_i < M$ for any fixed large M. As we pointed out before, Ψ_k has only one critical point that we denote Λ^*. Since this critical point is nondegenerate, it follows that the local degree $\deg(\nabla \Phi_\varepsilon, \mathcal{U}, 0)$ is well defined and is nonzero. Here $\mathcal{U}$ denotes an arbitrarily small neighborhood of Λ^*. Hence for a sufficiently small ε

$$\deg(J_\varepsilon, \mathcal{U}, 0) \neq 0.$$

We conclude that there exists a critical point Λ_ε^* of Φ_ε such that

$$\Lambda_\varepsilon^* = \Lambda^* + o(1).$$

Then for $\xi^* = \xi(\Lambda^*)$ we obtain that

$$v^* = \sum_{i=1}^{k} U(x - \xi_i(\Lambda_\varepsilon^*)) + \phi(\xi(\Lambda_\varepsilon^*)) = \sum_{i=1}^{k} U(x - \xi_i^*)(1 + o(1))$$

is a solution of (9), and going back in the transformation (8) we obtain that

$$u_\varepsilon^*(r) = \gamma_N \sum_{i=1}^{k} \mathrm{e}^{\xi_i^*} \left(\frac{1}{1 + \mathrm{e}^{(p^*-1)\xi_i^*} r^2} \right)^{\frac{N-2}{2}} (1 + o(1))$$

is a solution of (5), where

$$\mathrm{e}^{\xi_i^*} = \varepsilon^{-(i-1)-\frac{1}{p^*-p}} \prod_{j=1}^{i} (\Lambda_i^*)^{-1}$$

and setting $\alpha_i = \Pi_{j=1}^{i} (\Lambda_i^*)^{-1}$, then

$$\alpha_i = \left[\frac{a_5(p^* - p)}{a_3 k} \right]^{\frac{1}{p^*-p}} \left(\frac{a_2}{a_3} \right)^{i-1} \frac{(k-i)!}{(k-1)!}$$

where the constants a_2, a_3, a_5, are given by (23). □

5. Proof of Theorem 2

In this section we consider the transformation

$$v(x) = r^{\frac{2}{p^*-1}} u(r) \big|_{r=\mathrm{e}^{\frac{p^*-1}{2}x}} \tag{36}$$

and then problem (6) turns out to be equivalent to

$$\begin{cases} v''(x) + \beta[\mathrm{e}^{\varepsilon x} v^{p^*-\varepsilon}(x) + \mathrm{e}^{-(q-p^*)x} v^q(x)] - v = 0 & \text{in } \mathbb{R} \\ 0 < v(x) \to 0 & \text{as } x \to \pm\infty. \end{cases} \tag{37}$$

The associated functional reads

$$\begin{aligned} \hat{E}_\varepsilon(\psi) = {} & \frac{1}{2} \int_{-\infty}^{\infty} |\psi'|^2 \mathrm{d}x + \frac{1}{2} \int_{-\infty}^{\infty} |\psi|^2 \mathrm{d}x - \frac{\beta}{p^* - \varepsilon + 1} \int_{-\infty}^{\infty} \mathrm{e}^{\varepsilon x} |\psi|^{p^*-\varepsilon+1} \mathrm{d}x \\ & - \frac{\beta}{q+1} \int_{-\infty}^{\infty} \mathrm{e}^{-(q-p^*)x} |\psi|^{q+1} \mathrm{d}x. \end{aligned} \tag{38}$$

We define $\hat{U}$ as the transformation via (36) of w, and for small $\varepsilon > 0$ we define

$$\hat{\xi}_1 = -\frac{1}{q-p^*}\log\varepsilon - \log\hat{\Lambda}_1$$
$$\hat{\xi}_{i+1} - \hat{\xi}_i = -\log\varepsilon - \log\hat{\Lambda}_{i+1} \quad i = 1, \dots, k-1 \tag{39}$$

where the points $\hat{\Lambda}_i$ are positive parameters. We look for a solution of (37) of the form

$$v(x) = \sum_{i=1}^{k} \hat{U}(x - \hat{\xi}_i) + \phi$$

where ϕ is small. In a similar way to the proof of Lemma 1, we can prove that for $N \geq 3$, $k \in \mathbb{N}$, $q > p^*$ and $\delta > 0$ fixed, if

$$\delta < \Lambda_i < \delta^{-1} \quad \forall i = 1, \dots, k \tag{40}$$

then there are constants $b_1, \dots, b_n$ depending only on N and q, such that

$$\hat{E}_\varepsilon(\hat{V}) = kb_1 + \varepsilon\hat{\Psi}_k(\hat{\Lambda}) - kb_4\varepsilon + \varepsilon\hat{\Theta}_\varepsilon(\hat{\Lambda}) - \frac{b_3 k}{2(q-p^*)}((1-k)(q-p^*)-2)\varepsilon\log\varepsilon \tag{41}$$

where $\hat{\Lambda} = (\hat{\Lambda}_1, \dots, \hat{\Lambda}_k)$ and

$$\hat{\Psi}_k(\hat{\Lambda}) = b_3 k \log\hat{\Lambda}_1 - b_5\hat{\Lambda}_1^{(q-p^*)} + \sum_{i=2}^{k}[(k-i+1)b_3\log\hat{\Lambda}_i - b_2\hat{\Lambda}_i] \tag{42}$$

with $\Theta_\varepsilon(\Lambda) \to 0$ as $\varepsilon \to 0$, uniformly in the C^1-sense on the points $\hat{\Lambda}_i$ that satisfy (40). Besides the constants are given by

$$\begin{cases} b_1 = \dfrac{1}{2}\displaystyle\int_{-\infty}^{\infty}|\hat{U}'(x)|^2\mathrm{d}x + \dfrac{1}{2}\int_{-\infty}^{\infty}\hat{U}^2(x)\mathrm{d}x - \dfrac{\beta}{p^*+1}\int_{-\infty}^{\infty}\hat{U}^{p^*+1}(x)\mathrm{d}x \\ b_2 = \beta C_N\displaystyle\int_{-\infty}^{\infty}\mathrm{e}^x\hat{U}^{p^*}(x)\mathrm{d}x \\ b_3 = \dfrac{\beta}{p^*+1}\displaystyle\int_{-\infty}^{\infty}\hat{U}^{p^*+1}(x)\mathrm{d}x \\ b_4 = \dfrac{1}{(p^*+1)^2}\displaystyle\int_{-\infty}^{\infty}\hat{U}^{p^*+1}(x)\mathrm{d}x - \dfrac{1}{p^*+1}\int_{-\infty}^{\infty}\hat{U}^{p^*+1}(x)\log\hat{U}(x)\mathrm{d}x \\ b_5 = \dfrac{\beta}{q+1}\displaystyle\int_{-\infty}^{\infty}\mathrm{e}^{-(q-p^*)x}\hat{U}^{q+1}(x)\mathrm{d}x. \end{cases} \tag{43}$$

It follows that the only critical point of $\hat{\Psi}_k$ is nondegenerate and is given by

$$\hat{\Lambda}^* = \left(\left[\frac{b_3 k}{b_5(q-p^*)}\right]^{\frac{1}{q-p^*}}, \frac{(k-1)b_3}{b_2}, \frac{(k-2)b_3}{b_2}, \dots, \frac{b_3}{b_2}\right).$$

The finite-dimensional reduction can be worked in a way similar to the Section 3, except for (32) and (33), that get replaced by

$$\|N(\phi)\|_* \leq C(\|\phi\|_*^{\min\{p^*,2\}} + \|\phi\|_*^{\min\{\frac{p^*+1}{2},\frac{3}{2}\}}) \tag{44}$$
$$\left\|\frac{\partial N_\varepsilon}{\partial\phi}\right\|_* \leq C(\|\phi\|_*^{\min\{p^*-1,2\}} + \|\phi\|_*^{\min\{\frac{p^*-1}{2},2\}}). \tag{45}$$

The finite-dimensional variational problem and the conclusion of the theorem can be derived in a way analogous to the Section 4. □

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