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ASYMPTOTIC EQUIVALENCE AND KOBAYASHI-TYPE ESTIMATES FOR NONAUTONOMOUS MONOTONE OPERATORS IN BANACH SPACES

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Abstract. We provide a sharp generalization to the nonautonomous case of the well-known Kobayashi estimate for proximal iterates associated with maximal monotone operators. We then derive a bound for the distance between a continuous-in-time trajectory, namely the solution to the differential inclusion $\dot{x} + A(t)x \ni 0$, and the corresponding proximal iterations. We also establish continuity properties with respect to time of the nonautonomous flow under simple assumptions by revealing their link with the function $t \mapsto A(t)$. Moreover, our sharper estimations allow us to derive equivalence results which are useful to compare the asymptotic behavior of the trajectories defined by different evolution systems. We do so by extending a classical result of Passty to the nonautonomous setting.

1. **Introduction.** Motivated by either the existence or the algorithmic approximation of solutions to a differential inclusion problem of the type

$$\dot{x} + A(t)x \ni 0, \tag{1.1}$$

where $A(t)$ is a possibly time-dependent m-accretive operator with domain in a Banach space, several authors have considered some special implicit discretization schemes.

In the autonomous case where $A(t) \equiv A$, Crandall and Liggett introduced in [8] the following limit:

$$S(t)x_0 = \lim_{n\to\infty} \left(I + \frac{t}{n}A\right)^{-n} x_0 = \lim_{n\to\infty} (J^A_{t/n})^n x_0, \tag{1.2}$$

where $J^A_\lambda = (I + \lambda A)^{-1}$ is the resolvent of A. Other relevant product formulas related to existence results for evolution equations can be found in [13] and [6] (see

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also [5, 7] for nonlinear versions). Under some closedness assumptions on the operator A, they proved that this limit exists and defines a strongly continuous semigroup $\{S(t)\}_{t\geq 0}$ on X such that $x(t) := S(t-t_0)x_0$ is the strong solution to $\dot{x}+Ax \ni 0$ that satisfies $x(t_0) = x_0$. They also provided some estimates on $\|(J^A_\lambda)^n x_0 - (J^A_\mu)^m x_0\|$, and established the Lipschitz continuity of the solution. Later Kobayashi recovered in [16] similar existence results for the autonomous case, with fundamental improvements concerning certain estimates and some continuity properties. In fact, he constructed sequences of approximate solutions which converge in an appropriate sense to a solution to the differential inclusion. The key argument is an inequality that provides an estimate for the distance between arbitrary points of two independent sequences generated by the so called *proximal iterations*. More precisely, in the case where $x_k = J^A_{\lambda_k} x_{k-1}$ and $\widehat{x}_l = J^A_{\widehat{\lambda}_l}\widehat{x}_{l-1}$ with (possibly nonconstant) stepsizes $\{\lambda_k\} \subset (0,\Lambda]$ and $\{\widehat{\lambda}_l\} \subset (0,\widehat{\Lambda}]$, Kobayashi's inequality establishes that:

$$\|x_k - \widehat{x}_l\| \leq \|x_0 - u\| + \|\widehat{x}_0 - u\| + |\!|\!|Au|\!|\!| \sqrt{(\sigma_k - \widehat{\sigma}_l)^2 + \Lambda\sigma_k + \widehat{\Lambda}\widehat{\sigma}_l}, \tag{1.3}$$

where $\sigma_k = \sum_{i=1}^k \lambda_i$ (similar for $\widehat{\sigma}_l$), u is any element in the domain of A and $|\!|\!|Au|\!|\!| = \inf_{[u,v]\in A} \|v\|$. Kobayashi showed that such a remarkable inequality also holds for *inexact* proximal iterations by adding certain terms in the right-hand side of the estimate (see Remark 4.2 below).

Kobayashi's inequality is a powerful tool. Some Lipschitz continuity properties of a limit as (1.2) follow easily from such an inequality, giving explicit Lipschitz constants in terms of the data, namely

$$\|x(s) - x(t)\| \leq |\!|\!|Ax(0)|\!|\!| \; |s-t|.$$

Moreover, passing to the limit in only one of the sequences, it is possible to compare the continuous and discrete trajectories, namely, we have an estimate of the type

$$\|x_k - x(t)\| \leq \|x_0 - u\| + |\!|\!|Au|\!|\!| \sqrt{(\sigma_k - t)^2 + \Lambda\sigma_k}.$$

In [20], and still for the autonomous case, Miyadera and Kobayasi introduced the notion of *almost-orbit*, which is a kind of approximate solution to $\dot{x} + Ax \ni 0$. They used Kobayashi's inequality to prove that the continuous path constructed by linear interpolation of some proximal iterations is indeed an almost-orbit for the semigroup generated by the operator A. A converse result is given in [18]. It is known that several asymptotic properties of the orbits are inherited by the almost-orbits (see [22], [17] and [1]). More recently, Güler proves in [12] that if the operator A is the *subdifferential* of a closed, proper and convex function in a Hilbert space, both the continuous trajectory defined by the semigroup and the discrete proximal iterations either converge or diverge simultaneously; this being valid both for the strong and the weak topologies. Besides some technical difficulties, Güler's proof relies on Kobayashi's inequality together with some clever ideas borrowed from Passty, who had already obtained in [22] a similar conclusion under different but complementary assumptions. The powerful results in [22] and [12] reveal a sort of equivalence in the asymptotic behavior of the continuous and discrete trajectories.

Concerning the nonautonomous case, Crandall and Pazy provided in [9] a suitable generalization of the limiting formula (1.2). On the other hand, under some additional conditions on the operator-valued function $t \mapsto A(t)$, Kobayasi *et al.* gave in [17, Lemma 3.4] a first nonautonomous version of the original Kobayashi inequality. Then, they obtained important properties of the corresponding continuous dynamics by passing to the limit in an appropriate manner. However, their nonautonomous Kabayashi-type inequality and the resulting estimates in [17] are rather involved, and based on some extrapolations and not on optimal bounds. The results by Pavel in [23] are closely related to the latter; the author presents some Kobayashi-type estimations and uses them to derive the existence of DS-limit solutions of the differential inclusion. Their assumptions are very similar to ours but demand more precise information on the time-dependence. It is important to highlight the fact that [9, 17, 23] are concerned with existence and approximation of solutions on a *bounded* time horizon and they all contain estimations *in the spirit* of Theorems 4.1, 4.5 and 4.7 beluw, but under different hypotheses. However, we cannot see clearly whether or not their estimations could be useful to study the long-term behavior of the approximate solutions.

In this work we pretend to provide two different and independent contributions to the existing theory and also show how these two instruments can be combined in order to perform a qualitative analysis of the solutions on an infinite time horizon.

More precisely, our first goal is to give alternative nonautonomous Kobayashi-type estimates which are sharper than those provided in [17] and [23], and valid in a more general setting with respect to the properties required on the family of operators. With our approach, no sophisticated mathematical tools are needed, and the consequent estimates for discrete proximal iterates as well as continuous trajectories show explicitly and separately the different effects of the time-dependence. The fact that proving existence of the DS-limit solutions is not our main purpose allows us to obtain sharper estimations which are valid in a more general setting and require less information on the time-dependence. We do not consider here the more general setting of *quasi-dissipative* operators because the expressions become much more involved. Nevertheless, the additional difficulties posed by quasi-dissipativity are purely algebraic.

Our second goal is to extend a theory of asymptotic equivalence started by Passty in [22] to a larger class or families of operators to include the nonautonomous case. This theory allows to ensure that some “approximate trajectories” enjoy the same asymptotic properties as the orbits of the corresponding evolution systems. Finally, we exploit the sharper estimations mentioned above to obtain equivalence results for the asymptotic behavior of continuous and discrete trajectories, similar but complementary to those one can find in [22, 18, 12].

This paper is organized as follows. In section 2 we recall some definitions and results concerning monotone or accretive operators and evolution equations. We also set some notation and state the main hypotheses along with some examples. Section 3 contains an abstract asymptotic equivalence result for evolution systems that generalizes [22, Lemma 1], which we apply then to a pair of differential inclusions in a setting that includes the quasi-autonomous case (see [4]). The second application uses some new Kobayashi-type estimates, which we state in section 4 and prove in section 5.

2. **Preliminaries.**

Let $(X, \|\cdot\|)$ be a Banach space and denote by X^* its dual, which is endowed with the dual norm defined by $\|f\|_* = \sup_{\|u\|\leq 1} f(u)$. The duality product $\langle\cdot,\cdot\rangle : X \times X^* \to \mathbb{R}$ is defined by $\langle u, f\rangle = f(u)$ for all $u \in X$ and $f \in X^*$. The duality mapping $\mathcal{J} : X \rightrightarrows X^*$ is given by

$$\mathcal{J}(u) = \{\ f \in X^* \mid \|f\|_* = \|u\| \text{ and } \langle u, f\rangle = \|u\|^2\ \}.$$

Given a set-valued mapping $A : X \rightrightarrows X$, its domain is given by $D(A) = \{u \in X \mid Au \neq \emptyset\}$. For convenience of notation, sometimes we identify A with its graph by writing $[u, v] \in A$ for $v \in Au$. If $u \in D(A)$ then we set

$$|\!|\!| Au |\!|\!| = \inf_{[u,v]\in A} \|v\|.$$

A mapping $A : X \rightrightarrows X$ is said to be a *monotone operator* if for all $[u_1, v_1], [u_2, v_2] \in A$ there exists $f \in \mathcal{J}(u_1 - u_2)$ such that

$$\langle v_1 - v_2, f\rangle \geq 0. \tag{2.1}$$

A monotone operator is said to be *maximal* if its graph is not properly contained in the graph of any other monotone operator. Let I be the identity mapping in X. For $\lambda > 0$, the *resolvent* of A is defined as the mapping $J_\lambda^A = (I + \lambda A)^{-1}$. An operator A is said to be *accretive* if for all $\lambda > 0$, and $[u_1, v_1], [u_2, v_2] \in A$ one has

$$\|u_1 - u_2 + \lambda(v_1 - v_2)\| \geq \|u_1 - u_2\|.$$

This implies that J_λ^A is a single-valued nonexpansive mapping. If, in addition, the range of $I + \lambda A$ equals X for all $\lambda > 0$, the operator A is said to be *m-accretive*. A well-known consequence of general results in [19, 15] is the following:

i) A is monotone if, and only if, A is accretive.
ii) If A is m-accretive, then it is maximal monotone. The converse is true in Hilbert space but not in general Banach spaces (see counterexample in [14]).

Let D be a nonempty subset of X and define

$$\mathcal{M}(D) = \{A : X \rightrightarrows X \mid A \text{ is } m\text{-accretive and } D(A) = D\}.$$

A collection of four sequences $(\{x_k\}, \{\lambda_k\}, \{A_k\}, \{\varepsilon_k\})$ with $\{A_k\} \subset \mathcal{M}(D)$ is said to be a *discrete proximal scheme* if $\lambda_k > 0$, and

$$(x_k - x_{k-1})/\lambda_k + A_k x_k \ni \varepsilon_k,$$

for all $k \geq 1$. The corresponding sequence $\{x_k\}$ is said to be a *discrete proximal trajectory* starting from the point $x_0 \in X$ and generated by $(\{\lambda_k\}, \{A_k\}, \{\varepsilon_k\})$.

Next, given points $x \neq \widehat{x}$ and $v, \widehat{v}$ in X, define

$$\Delta([x, v], [\widehat{x}, \widehat{v}]) = \inf_{f\in\mathcal{J}(x-\widehat{x})} \frac{\langle \widehat{v} - v, f\rangle}{\|x - \widehat{x}\|}. \tag{2.2}$$

If $x = \widehat{x}$ we set $\Delta([x, v], [\widehat{x}, \widehat{v}]) = 0$. Notice that $\Delta([x, v], [\widehat{x}, \widehat{v}]) \leq \|v - \widehat{v}\|$. If $[x, v], [\widehat{x}, \widehat{v}] \in A$ for some $A \in \mathcal{M}(D)$ then $\Delta([x, v], [\widehat{x}, \widehat{v}]) \leq 0$ by monotonicity.

Let us consider two sequences $\{A_k\}$ and $\{\widehat{A}_l\}$ in $\mathcal{M}(D)$. As in [17], we shall assume that the following condition holds:

$$\forall k, l \geq 1,\ \exists \Theta_{k,l} \geq 0,\ \forall\, [x, v] \in A_k,\ \forall\, [\widehat{x}, \widehat{v}] \in \widehat{A}_l,\ \Delta([x, v], [\widehat{x}, \widehat{v}]) \leq \Theta_{k,l}. \tag{2.3}$$

A continuous version of this condition will be presented below in (2.7).

Remark 2.1. A condition like (2.3) above was introduced in [17] to determine the existence of weak solutions for a nonautonomous differential inclusion. It can be interpreted as a smooth evolution of the geometry of the sets $A_k x$ with respect to $\widehat{A}_l \widehat{x}$. A similar assumption is made in [23]. In both cited references the authors consider a particular instance of the bi-sequence $\{\Theta_{k,l}\}$. The fact that we do not impose any symmetry assumption on the $\Theta_{k,l}$ is important when dealing with families A_k and $\widehat{A}_l$ which are different. Similar hypotheses are made by Tebbs in [26] (se also Example 2.3 below). □

Example 2.1. Let $A \in \mathcal{M}(D)$ and $B : X \rightrightarrows X$ a strongly monotone mapping such that $\|B\|_{\infty,D} := \sup_{x\in D}\sup_{v\in B(x)}\|v\| < \infty$. Set $A(r) = A + rB \in \mathcal{M}(D)$. Given two sequences $\{r_k\}$ and $\{\widehat{r}_l\}$ of positive numbers define accordingly $A_k = A(r_k)$ and $\widehat{A}_l = A(\widehat{r}_l)$ for $k, l \geq 1$. Then it is easy to verify that (2.3) is satisfied for $\Theta_{k,l} = |r_k - \widehat{r}_l|\|B\|_{\infty,D}$. □

Example 2.2. In Hilbert space, assume $f, g : X \to \mathbb{R} \cup \{\infty\}$ are proper, lower-semicontinuous and convex with g differentiable. The operator $A(r) = \partial(f + rg)$ satisfies the conditions in Example 2.1 if D is bounded or if ∇g is bounded. □

Let $A, \widehat{A} : [0,\infty) \longrightarrow \mathcal{M}(D)$. For $m \in \mathbb{N}$ and $t > t_0 \geq 0$, consider the *finite* discrete proximal trajectories $\{x_k\}_{k=0}^m$ and $\{\widehat{x}_l\}_{l=0}^m$ defined by

$$x_0 = u \text{ and } x_k = \Big(I - \tfrac{t-t_0}{m} A(t_0 + \tfrac{k(t-t_0)}{m})\Big)^{-1} x_{k-1}, \quad \text{for } k = 1, \ldots, m. \tag{2.4a}$$

$$\widehat{x}_0 = u \text{ and } \widehat{x}_l = \Big(I - \tfrac{t-t_0}{m} \widehat{A}(t_0 + \tfrac{l(t-t_0)}{m})\Big)^{-1} \widehat{x}_{l-1}, \quad \text{for } l = 1, \ldots, m. \tag{2.4b}$$

From now on, we assume that x_m and $\widehat{x}_m$ converge[1] in X as $m \to \infty$, and we respectively denote by $U(t, t_0)u$ and $\widehat{U}(t, t_0)u$ their limits, that is, for all $u \in D$ and $t > t_0 \geq 0$ we set

$$U(t, t_0)u = \lim_{m\to\infty} x_m = \lim_{m\to\infty} \prod_{k=1}^{m} \Big(I - \tfrac{t-t_0}{m} A(t_0 + \tfrac{k(t-t_0)}{m})\Big)^{-1} u, \tag{2.5a}$$

$$\widehat{U}(t, t_0)u = \lim_{m\to\infty} \widehat{x}_m = \lim_{m\to\infty} \prod_{l=1}^{m} \Big(I - \tfrac{t-t_0}{m} \widehat{A}\Big(t_0 + \tfrac{l(t-t_0)}{m}\Big)\Big)^{-1} u \tag{2.5b}$$

and $U(t_0, t_0)u = \widehat{U}(t_0, t_0)u = u$. We assume also that for each $u \in D$, the functions

$$t \mapsto \|A(t)u\| \text{ and } t \mapsto \|\widehat{A}(t)u\| \text{ are bounded and locally Riemann-integrable.} \tag{2.6}$$

Sufficient conditions on $\{A(t)\}_{t\in[0,\infty)}$ ensuring that $U(t, t_0)u$ is well defined are given in [9]. The function $U(\cdot, t_0)u$ is said to be a *weak* or *DS-limit solution*[2] of inclusion (1.1). In a time-independent domain these generalized solutions happen to coincide with *integral solutions* in the sense of Bénilan ([4]) under hypothesis (2.7) below (see Theorem 2.4 in [17]). We shall not go further on this matter here but only mention that such conditions imply in particular the continuity of $[0,\infty) \ni t \mapsto \|A(t)u\|$, hence (2.6). The trajectory $t \mapsto U(t, t_0)u$ can also be proved to satisfy (1.1) under supplementary assumptions.

Defining evolution systems, such as (2.5a) and (2.5b), by a limiting process involving (piecewise constant interpolations of) certain discretizations of the evolution equation and its variants is a common practice. Classical references are [8, 9, 16]. More recent examples can be found in the works of Oharu and Tebbs [21] (dicretization of the differential inclusion), Azuma [2] (approximations of the nonautonomous operator) or Georgescu and Oharu [11] (discretization of the weak solution equation).

[1]By virtue of the weak lower-semicontinuity of the norm, it suffices to assume weak convergence.
[2]DS for *discrete scheme*. The term was introduced in [16] in the autonomous setting $A(t) \equiv A$.

Finally, we assume the continuous version of (2.3): there exists a bounded Riemann-integrable function $\Theta : [0,\infty) \times [0,\infty) \to [0,\infty)$ satisfying

$$\forall t, s \in [0,\infty),\ \forall\, [x,v] \in A(t),\ \forall\, [\widehat{x},\widehat{v}] \in \widehat{A}(s),\ \Delta([x,v],[\widehat{x},\widehat{v}]) \leq \Theta(t,s), \tag{2.7}$$

for $\Delta(\cdot,\cdot)$ given by (2.2) (see, for instance, Example 2.1). This is precisely Hypothesis $(\mathbf{H}; C)$ in [17] but no continuity hypotheses are made here.

Example 2.3. Similar hypotheses were used by Oharu and Tebbs in [21] for a parabolic system describing the behavior of the HIV virus in the human body and the reaction to treatment by chemotherapy (see also [26], where the author first introduces a model without treatment). In their work, the authors consider single-valued operators of the form $A(t) = L + B(t)$, where L is linear and the family $B(t)$ is proved to satisfy a more restrictive condition involving the norm of the difference $B(t)x - B(s)\widehat{x}$, namely hypothesis (H2). In their example, the coupling function Θ turns out to be a multiple of $|t-s|$. □

3. **Asymptotic equivalence and applications.** The aim of this section is to provide a tool for comparing two systems. Our "test" guarantees that some approximate trajectories (almost-orbits) have the same asymptotic properties as the true ones (orbits). This is useful for two reasons: First, if one is trying to prove convergence for a certain evolution system, one way to proceed is to show that its orbits are in fact almost-orbits of another system and use the convergence results known for that system. Second it is also useful when dealing with perturbations or discretizations. One has a model or theoretical system and wants to build an approximate trajectory which is possible to determine exactly and has the same asymptotic properties as the model. Thus the problem reduces to adjusting parameters and stopping rules in order for the approximate system to yield almost-orbits of the original one, if possible.

3.1. **Almost-orbits of contracting evolution systems.** Let D be a subset of a Banach space X. A *contracting evolution system* (*CES*) on D is a family $\{V(t,s)\ :\ t \geq s \geq 0\ \}$ of maps from D into itself satisfying:

i) $V(t,t) = I$, the identity operator in D.
ii) $V(t,s)V(s,r) = V(t,r)$.
iii) $\|V(t,s)x - V(t,s)y\| \leq \|x - y\|$

Example 3.1. Define $U_c(t,s)y$, as a DS-limit solution of (1.1) with $x(s) = y$. The family $\{U_c(t,s)\ :\ t \geq s \geq 0\ \}$ defines a contracting evolution system on D. □

Example 3.2. Let $\nu(t)$ be the greatest integer such that $\nu(t) \leq t$. Consider a family $\{A_n\}$ of m-accretive operators and a sequence $\{\lambda_n\}$ of stepsizes. Define

$$U_d(t,s) = \prod_{n=\nu(s)+1}^{\nu(t)} (I + \lambda_n A_n)^{-1},$$

where the product represents the composition of resolvents. Then V is a *CES* and $x_n = U_d(\sigma_n, \sigma_m)x_m$ for the sequence $\{x_n\}$ generated by the (exact) proximal scheme. In fact, U_d is just a piecewise constant interpolation of that sequence. □

Let V be an evolution system on D. An *orbit* of V is a function of the form $t \mapsto V(t,s)y$ for some $y \in D$. Now following [20], we shall say a locally bounded function $u : \mathbb{R}_+ \to D$ is an *almost-orbit* of V if

$$\lim_{t\to\infty} \sup_{h\geq 0} \|u(t+h) - V(t+h,t)u(t)\| = 0. \tag{3.1}$$

Two evolution systems U and V are *asymptotically equivalent* if every orbit of U is an almost orbit of V and viceversa.

Example 3.3. Let U_c be as in Example 3.1 with $A(t) \equiv A$ and let U_d be as in Example 3.2 with $A_n \equiv A$, both in Hilbert space. We have the following:

1. If the sequence $\{\lambda_n\}$ of stepsizes is in $\ell^2 \setminus \ell^1$, the systems U_c and U_d are asymptotically equivalent (see [18]).
2. If A is the subdifferential of a proper, lower-semicontinuous convex function, the same is true whenever $\{\lambda_n\} \notin \ell^1$ (see [12]). □

Lemma 3.1. *Let V be a* CES *and u_1, u_2 two almost-orbits of V. Then, the limit $\lim_{t\to\infty} \|u_1(t) - u_2(t)\|$ always exists.*

Remark 3.2. Lemma 3.1 was proved in [20] for almost-orbits of contracting semigroups, but is generalized easily to *CES*. As a consequence, if U and V are equivalent *CES*, and if **one** almost-orbit of U **or** V is bounded, then **every** almost-orbit of U **and** V is bounded. □

The following theorem and its proof are inspired by [22, Lemma 1]. By keeping only the essential, we give a shorter proof in a more general context.

Theorem 3.3. *Let V be a* CES. *If $V(t,s)x$ converges strongly (resp. weakly) as $t \to \infty$ for all x and s, then every almost-orbit of V converges strongly (resp. weakly) as $t \to \infty$.*

Proof. Let τ denote the strong or the weak topology. And suppose that the $\tau-$limit of $V(t,s)x$ as $t \to \infty$ exists for all x and s. Let u be an almost-orbit of V. Take $p \geq 0$ and set $\zeta(p) = \tau - \lim_{t\to\infty} V(t,p)u(p)$. We have

$$\zeta(p+h) - \zeta(p) = \tau - \lim_{t\to\infty} \{V(t,p+h)u(p+h) - V(t,p)u(p)\}.$$

But for all sufficiently large t we have

$$\begin{aligned} &\|V(t,p+h)u(p+h) - V(t,p)u(p)\| \\ &\qquad = \|V(t,p+h)u(p+h) - V(t,p+h)V(p+h,p)u(p)\| \\ &\qquad \leq \|u(p+h) - V(p+h,p)u(p)\| \end{aligned}$$

and so

$$\|\zeta(p+h) - \zeta(p)\| \leq \|u(p+h) - V(p+h,p)u(p)\|,$$

which tends to zero uniformly in h as $p \to \infty$. Therefore $\zeta(\cdot)$ is a Cauchy net that converges strongly to a limit ζ. Finally, we have

$$u(t+k) - \zeta \quad = \quad [u(t+k) - V(t+k,t)u(t)] + [V(t+k,t)u(t) - \zeta(t)] + [\zeta(t) - \zeta].$$

Given $\varepsilon > 0$ we can take t large enough so that the first and third terms on the right hand side are less than ε in norm, uniformly in k. Next, for such t, we let $k \to \infty$ so that the second term converges to zero for the topology τ. ■

In [22], G Passty proved this result in two special cases: when V is defined by a semigroup of contractions; and when the almost-orbits are in fact the orbits of a semigroup of contractions. This is precisely the case mentioned in Example 3.3 because U_c is defined by a semigroup of nonlinear contractions. A few historical remarks concerning the subdifferential case are in order. In [3], the author proves that there is a proper lower-semicontinuous convex function f such that the trajectories determined by the gradient method (that is, the orbits of U_c) converge weakly but not strongly to a minimizer of f. RT Rockafellar posed in [24] the question whether the proximal point algorithm could generate a sequence (that is, an orbit of U_d) converging weakly but not strongly to a minimizer of f. According to point (1) in Example 3.3, a partial answer could have been given in the case where $\{\lambda_n\} \in \ell^2 \setminus \ell^1$. However, as far as we know, nobody seems to have pointed out this fact before. Later, in [12], O Güler gave a complete positive answer by proving (2) in Example 3.3. Then he used Passty's result to conclude that the same function f found in [3] yields a proximal sequence converging weakly but not strongly.

Remark 3.4. It is possible to give versions of Theorem 3.3 for several types of convergence in average. One can also get rid of the Lipschitz-continuity assumption on the evolution system. The only price to pay is a mild hypothesis on the Banach space when dealing with the weak topology. These results and the necessary techniques will be presented in a forthcoming paper [1]. □

3.2. **Comparing the trajectories of two differential inclusions.** In this section we prove an equivalence result for trajectories defined by the differential inclusions governed by two families, $\{A(t)\}_{t\geq t_0}$ and $\{\widehat{A}(t)\}_{t\geq t_0}$, of m-accretive operators on X defined on a common t-independent domain.

Let U and $\widehat{U}$ be the evolution systems defined by A and $\widehat{A}$ as in (2.5a) and (2.5b), which we assume to exist. They are DS-limit solutions of the differential inclusion (1.1) with A and $\widehat{A}$, respectively. Notice also that U and $\widehat{U}$ are *CES*. Under (2.7), Theorem 4.7 gives

$$\|U(t,s)x - \widehat{U}(t,s)x\| \leq \sqrt{2}\left|\int_s^t \|A(\tau)x\| - \|\widehat{A}(\tau)x\| \; d\tau\right| + \int_s^t \Theta(\tau,\tau)\; d\tau \qquad (3.2)$$

for all $t \geq t_0$.

If we assume that for each $r > 0$ there is a function $F_r \in L^1(t_0,\infty;\mathbb{R})$ such that for every $x \in B(0,r)$ one has

$$\Theta(t,t) + \Big|\|A(t)x\| - \|\widehat{A}(t)x\|\Big| \leq F_r(t) \qquad (3.3)$$

almost everywhere on $[t_0,\infty)$, we get the following:

Lemma 3.5. *Under* (2.7) *and* (3.3)*, every bounded orbit of U is an almost-orbit of $\widehat{U}$.*

Proof. Let $U(\cdot,t_0)x_0$ be bounded in norm by $r > 0$. According to inequalities (3.2) and (3.3) we have

$$\|U(t+h,t_0)x_0 - \widehat{U}(t+h,t)U(t,t_0)x_0\| \leq \sqrt{2}\int_t^{t+h} F_r(\xi)\; d\xi$$

and so, $U(\cdot,t_0)x_0$ is an almost-orbit of $\widehat{U}$. ■

Example 3.4. Hypotheses (2.7) and (3.3) hold, for instance, if $A(t)x = Ax+f(t,x)$ (see, for instance, [9, 17, 25] for existence issues), $\widehat{A}(t)x = Ax + \widehat{f}(t,x)$ whenever $\|f(t,x) - \widehat{f}(s,\widehat{x})\|$ can be bounded by some function $\Phi_r(t,s)$ whenever $\|x\|, \|\widehat{x}\| \leq r$ and such that $\int_0^\infty \Phi_r(\tau,\tau)d\tau < \infty$ for each r. In this case, one can also derive the following continuity result:

$$\|U - \widehat{U}\|_\infty \leq \|x - \widehat{x}\| + 2\sqrt{2}\|f - \widehat{f}\|_{L^1}, \tag{3.4}$$

when U and $\widehat{U}$ start at x and $\widehat{x}$, respectively. This is similar to the bound given in [10, Theorem 4], where A is the gradient of a $\mathcal{C}^1$ convex function having minimizers. However, inequality (3.4) holds for any maximal monotone operator. □

Theorem 3.6. *Assume (2.7) and (3.3) hold. Then $U(t,s)x$ converges weakly (strongly) as $t \to \infty$ for all s and x if, and only if, $\widehat{U}(t,s)x$ does.*

Proof. This is an immediate consequence of Remark 3.2, Theorem 3.3 and Lemma 3.5. ■

Remark 3.7. Theorems 4.5 and 4.7 may be used to establish discrete-discrete and discrete-continuous versions of Lemma 3.5 but this will not be done here. □

4. **Kobayashi-type estimates.**

4.1. **Discrete-discrete estimate.**
The main result of this section is the following:

Theorem 4.1. *Let $(\{x_k\},\{\lambda_k\},\{A_k\},\{\varepsilon_k\})$ and $(\{\widehat{x}_l\},\{\widehat{\lambda}_l\},\{\widehat{A}_l\},\{\widehat{\varepsilon}_l\})$ be two discrete proximal schemes. If (2.3) holds then for every $u \in D$ and for all k,l we have*

$$\begin{aligned}\|x_k - \widehat{x}_l\| \quad &\leq \quad \|x_0 - u\| + \|\widehat{x}_0 - u\| + \alpha_{k,l} + \beta_{k,l} \\ &\qquad + \sqrt{(\gamma_k(u) - \widehat{\gamma}_l(u))^2 + \delta_k(u) + \widehat{\delta}_l(u) + \eta_{k,l}(u)},\end{aligned} \tag{4.1}$$

where

$$\gamma_k(u) = \sum_{i=1}^k \lambda_i |\!|\!| A_i u |\!|\!|, \quad \delta_k(u) = \sum_{i=1}^k \lambda_i^2 |\!|\!| A_i u |\!|\!|^2 \quad \textit{(similar for } \widehat{\gamma}_l \textit{ and } \widehat{\delta}_l\textit{)}, \tag{4.2}$$

and $\alpha_{k,l}$, $\beta_{k,l}$, and $\eta_{k,l}$ are defined recursively as follows:

$$\left\{\begin{array}{l} \alpha_{k,0} = e_k = \sum\limits_{i=1}^k \lambda_i \|\varepsilon_i\|,\ \alpha_{0,l} = \widehat{e}_l = \sum\limits_{j=1}^l \widehat{\lambda}_j \|\widehat{\varepsilon}_j\|,\ \textit{and} \\ \alpha_{k,l} = \frac{\widehat{\lambda}_l}{\lambda_k + \widehat{\lambda}_l}\alpha_{k-1,l} + \frac{\lambda_k}{\lambda_k+\widehat{\lambda}_l}\alpha_{k,l-1} + \frac{\lambda_k\widehat{\lambda}_l}{\lambda_k+\widehat{\lambda}_l}\|\varepsilon_k - \widehat{\varepsilon}_l\|. \end{array}\right. \tag{4.3}$$

$$\left\{\begin{array}{l} \beta_{k,0} = \beta_{0,l} = 0,\ \textit{and} \\ \beta_{k,l} = \frac{\widehat{\lambda}_l}{\lambda_k + \widehat{\lambda}_l}\beta_{k-1,l} + \frac{\lambda_k}{\lambda_k+\widehat{\lambda}_l}\beta_{k,l-1} + \frac{\lambda_k\widehat{\lambda}_l}{\lambda_k+\widehat{\lambda}_l}\Theta_{k,l}. \end{array}\right. \tag{4.4}$$

$$\left\{\begin{array}{l} \eta_{k,0}(u) = \eta_{0,l}(u) = 0,\ \textit{and} \\ \eta_{k,l}(u) = \frac{\widehat{\lambda}_l}{\lambda_k + \widehat{\lambda}_l}\eta_{k-1,l}(u) + \frac{\lambda_k}{\lambda_k+\widehat{\lambda}_l}\eta_{k,l-1}(u) \\ \qquad\qquad + \frac{2\lambda_k\widehat{\lambda}_l}{\lambda_k+\widehat{\lambda}_l}(\gamma_k - \widehat{\gamma}_l)\left[|\!|\!| \widehat{A}_l u |\!|\!| - |\!|\!| A_k u |\!|\!|\right]. \end{array}\right. \tag{4.5}$$

Remark 4.2. Notice that $\alpha_{k,l} \leq e_k + \widehat{e}_l$ for all $k,l \geq 0$. □

Remark 4.3. In the specific case where $A_k \equiv \widehat{A}_l \equiv A$, we can take $\Theta_{k,l} \equiv 0$ and get $\beta_{k,l} \equiv 0$. We also have $\eta_{k,l} \equiv 0$. Then (4.1) amounts to

$$\|x_k - \widehat{x}_l\| \leq \|x_0 - u\| + \|\widehat{x}_0 - u\| + \alpha_{k,l} + \|\!|Au|\!\|\sqrt{(\sigma_k - \widehat{\sigma}_l)^2 + \tau_k + \widehat{\tau}_l},$$

where $\sigma_k = \sum_{i=1}^k \lambda_i$ and $\tau_k = \sum_{i=1}^k \lambda_i^2$ (similar for $\widehat{\sigma}_l$ and $\widehat{\tau}_l$). We thus recover the inequality obtained by Kobayashi in [16], where $\alpha_{k,l}$ is replaced with $e_k + \widehat{e}_l$ (see Remark 4.2). □

Remark 4.4. In [17] and [23] the authors considered a similar problem in the interesting specific case where both $\{A_k\}$ and $\{\widehat{A}_l\}$ are generated by *one* continuously parameterized family $\{A(t)\}_{t\in I}$. Their estimates both resemble (4.1) but there is a fundamental difference: Our estimation keeps track of all the information contained in the sequences $\{\|\!|A_k u|\!\|\}$ and $\{\|\!|\widehat{A}_l u|\!\|\}$, while their bound involves the value $\|\!|A(t_0)u|\!\|$ for one specific t_0 and makes some extrapolations using a modulus of continuity of the function $t \mapsto \|\!|A(t)u|\!\|$. As a consequence, their estimation looks simpler. However, it is clear that using our approach we get a sharper bound in a more general setting. In this sense, the two results are complementary. Another advantage of our estimation is that, as we shall see in the next sections, it is easy to derive information about a continuous-in-time trajectory by taking limits. □

4.2. **Discrete-continuous estimate.**

Theorem 4.5. *Let $A : [0,\infty) \to \mathcal{M}(D)$ and $\widehat{A} : [0,\infty) \to \mathcal{M}(D)$ satisfying (2.5), (2.6) and (2.7). Let $t > t_0 \geq 0$ and take a discrete proximal scheme $(\{x_k\}, \{\lambda_k\}, \{A_k\}, \{\varepsilon_k\})$ with $A_k := A(s_0 + \sigma_k)$ for $\sigma_k = \sum_{i=1}^k \lambda_i$ and some $s_0 \in [0,\infty)$. For every $u \in D$ and $k \in \mathbb{N}$, we have that*

$$\|x_k - \widehat{U}(t,t_0)u\| \leq \|x_0 - u\| + \alpha_k + \beta_k + \sqrt{[\gamma_k(u) - \widehat{\mathcal{A}}_u(t,t_0)]^2 + \delta_k(u) + \eta_k(u)}, \quad (4.6)$$

where

$$\widehat{\mathcal{A}}_u(t,t_0) := \int_{t_0}^t \|\!|\widehat{A}(\xi)u|\!\| \, d\xi \quad (4.7)$$

and $\alpha_k = \sum_{i=1}^k \lambda_i \|\varepsilon_i\|$, $\beta_k = \limsup_{m\to\infty} \beta_{k,m} < \infty$ and $\eta_k(u) = \limsup_{m\to\infty} \eta_{k,m}(u) < \infty$ for the sequences given by (4.2)-(4.5) with $\widehat{A}_l = \widehat{A}\left(t_0 + \frac{l(t-t_0)}{m}\right)$ and $\Theta_{k,l} = \Theta\left(s_0 + \sigma_k, t_0 + \frac{l(t-t_0)}{m}\right)$.

Remark 4.6. In particular, $\|u - \widehat{U}(t,t_0)u\| \leq \widehat{\mathcal{A}}_u(t,t_0)$. □

4.3. **Continuous-continuous estimate.**

Theorem 4.7. *Let $u \in D$. Suppose $t - t_0 \leq s - s_0$. We have*

$$\|U(s,s_0)u - \widehat{U}(t,t_0)u\| \leq \sqrt{2\left[\mathcal{A}_u(s,s_0) - \widehat{\mathcal{A}}_u(t,t_0)\right]^2 - \left[\mathcal{A}_u(t_0 + s - t, s_0)\right]^2 + \int_0^\tau \Theta(s-\xi, t-\xi)\, d\xi}.$$

The inequality above is an equality if one takes $\widehat{A}(t)x = A(t)x \equiv c \in X$.

Remark 4.8. Set $\widehat{A} = A$. If the function $t \mapsto |\!|\!| A(t)u |\!|\!|$ is nonincreasing then

$$\sqrt{2\left[\mathcal{A}_u(s,s_0) - \mathcal{A}_u(t,t_0)\right]^2 - \left[\mathcal{A}_u(t_0+s-t,s_0)\right]^2} \leq |\mathcal{A}_u(s,s_0) - \mathcal{A}_u(t,t_0)|.$$

□

Theorem 4.7 shows that the function U automatically inherits continuity properties from the function Θ. For instance, if $\Theta(t,s)$ is locally bounded by the difference $|t-s|$, the function U is locally Lipschitz-continuous in the pair (t_0,t).

Corollary 4.9. *Let C be a compact subset of the triangle $t \geq s \geq 0$ and assume $\Theta(t,s) \leq L|t-s|$ on C. For each $u \in D$, the function $(t,s) \mapsto U(t,s)u$ is Lipschitz-continuous with constant*[3]

$$\sqrt{2} \sup_{\delta_1 \leq \xi \leq \delta_2} |\!|\!| A(\xi)u |\!|\!| + L(\delta_2 - \delta_1),$$

where $\delta_1 = \min\{\ s \mid (t,s) \in C\ \}$, $\delta_2 = \max\{\ t \mid (t,s) \in C\ \}$.

Lipschitz-continuity of the function $t \mapsto U(t,t_0)u$ had already been proved in [9] and [17]. In the first article, even for monotone operators, their constant depends exponentially on the length of the interval (with, sometimes, a linear-affine coefficient) unless the function $t \mapsto A(t)$ is constant. In the second cited article, the authors prove Lipschitz continuity for *weak* solutions and find a constant depending on $|\!|\!| A(0)u |\!|\!|$ and a global bound for Θ. Our Proposition 4.7 shows that the constant, in fact, depends on the local (rather than global) behavior of Θ and $|\!|\!| A(\cdot)u |\!|\!|$. Moreover, if the function $t \mapsto A(t)$ is not constant, their constant grows linearly with the length of the interval. Therefore one cannot get a global Lipschitz constant even if the function has a very small variation.

We shall give a very simple example where our Lipschitz constant is global.

Example 4.1. Let $X = \mathbb{R}$. With the notation introduced in Example 2.1 take $A \equiv 0$, $B \equiv 1$ and parameterize ε by a nonincreasing positive function $\varepsilon(t)$. We have $\Theta(t,s) = |\varepsilon(t) - \varepsilon(s)|$ and $|\!|\!| A(t)u |\!|\!| = \varepsilon(t)$ for all u. For simplicity set $t_0 = s_0 = 0$ and $t \leq s$. If the function $\varepsilon(\cdot)$ is Lipschitz-continuous with constant L, we can apply the results in [17] on the interval $[0,T]$ to get $LT + 2\varepsilon(0) - \varepsilon(T)$ as a Lipschitz constant for U according to inequality (3.11) in [17] (where $\rho(r)$ is defined by $\rho(r) := \sup\{\ \Theta(s,t)\ :\ t,s \in [0,T],\ |t-s| \leq r\}$). Notice that the Lipschitz constant tends to ∞ with the length of the interval. Now let us compute the Lipschitz constant by our method. According to Proposition 4.7 and Remark 4.8 we have $\|U(s,0)u - U(t,0)u\| \leq \int_t^s \varepsilon(\xi)\ d\xi + \int_0^t [\varepsilon(t-\xi) - \varepsilon(s-\xi)]\ d\xi = \int_0^{s-t} \varepsilon(\xi)\ d\xi \leq \varepsilon(0)|s-t|$. □

5. Proofs of the estimates.

5.1. Discrete-discrete.

In order to prove Theorem 4.1 we first establish two auxiliary lemmas.

Lemma 5.1. *Let $(\{x_k\},\{\lambda_k\},\{A_k\},\{\varepsilon_k\})$ be a discrete proximal scheme. For every $u \in D$ and all $k \geq 1$ we have*

$$\|x_k - u\| \leq \|x_0 - u\| + \gamma_k(u) + e_k. \tag{5.1}$$

[3] For the ℓ^1 norm.

Proof. Let $u \in D$. We shall prove the result by induction. The estimate (5.1) is trivially satisfied for $k = 0$. Suppose (5.1) holds for some $k \geq 0$ and set

$$v_{k+1} = (x_{k+1} - x_k)/\lambda_{k+1} - \varepsilon_{k+1} \in -A_{k+1}x_{k+1}.$$

Take $y \in X$ such that $[u, -y] \in A_{k+1}$. Since $\|x_{k+1} - u\|^2 = \langle x_{k+1} - u, f\rangle$, for any $f \in \mathcal{J}(x_{k+1} - u)$, we have

$$\begin{aligned} \|x_{k+1} - u\|^2 &= \langle x_{k+1} - u + \lambda_{k+1}(y - v_{k+1}) - \lambda_{k+1}\varepsilon_{k+1}, f\rangle - \lambda_{k+1}\langle y - v_{k+1}, f\rangle \\ &\quad + \lambda_{k+1}\langle \varepsilon_{k+1}, f\rangle \\ &\leq \|x_k - u + \lambda_{k+1}y\|\|x_{k+1} - u\| - \lambda_{k+1}\langle y - v_{k+1}, f\rangle \\ &\quad + \lambda_{k+1}\|\varepsilon_{k+1}\|\|x_{k+1} - u\|. \end{aligned}$$

The monotonicity of A_{k+1} and the induction hypothesis imply

$$\begin{aligned} \|x_{k+1} - u\| &\leq \|x_k - u + \lambda_{k+1}y\| + \lambda_{k+1}\|\varepsilon_{k+1}\| \\ &\leq \|x_k - u\| + \lambda_{k+1}\|y\| + \lambda_{k+1}\|\varepsilon_{k+1}\| \\ &\leq \|x_0 - u\| + \gamma_k + e_k + \lambda_{k+1}\|y\| + \lambda_{k+1}\|\varepsilon_{k+1}\| \\ &\leq \|x_0 - u\| + \gamma_k + \lambda_{k+1}\|y\| + e_{k+1}. \end{aligned}$$

As y was arbitrarily chosen so that $[u, -y] \in A_{k+1}$, we conclude that

$$\|x_{k+1} - u\| \leq \|x_0 - u\| + \gamma_k + \lambda_{k+1}|||A_{k+1}u||| + e_{k+1} = \|x_0 - u\| + \gamma_{k+1} + e_{k+1},$$

which completes the proof of Lemma 5.1. ■

Lemma 5.2. *Let $x \neq \widehat{x}$, $v, \widehat{v}, \varepsilon, \widehat{\varepsilon} \in X$ and λ, $\widehat{\lambda} \in (0, \infty)$. Then*

$$(\lambda+\widehat{\lambda})\|x-\widehat{x}\| \leq \lambda\|\widehat{x}+\widehat{\lambda}(\widehat{v}-\widehat{\varepsilon})-x\|+\widehat{\lambda}\|x+\lambda(v-\varepsilon)-\widehat{x}\|+\lambda\widehat{\lambda}\Delta([x,v],[\widehat{x},\widehat{v}])+\lambda\widehat{\lambda}\|\varepsilon-\widehat{\varepsilon}\|.$$

Proof. Suppose $x \neq \widehat{x}$; otherwise there is nothing to prove. If $f \in \mathcal{J}(x - \widehat{x})$ then

$$\begin{aligned} (\lambda + \widehat{\lambda})\|x - \widehat{x}\|^2 &= \lambda\langle \widehat{x} - x,\ -f\rangle + \widehat{\lambda}\langle x - \widehat{x},\ f\rangle \\ &= \lambda\langle \widehat{x} + \widehat{\lambda}(\widehat{v} - \widehat{\varepsilon}) - x,\ -f\rangle + \widehat{\lambda}\langle x + \lambda(v - \varepsilon) - \widehat{x},\ f\rangle \\ &\quad + \lambda\widehat{\lambda}\langle \widehat{v} - v,\ f\rangle + \lambda\widehat{\lambda}\langle \varepsilon - \widehat{\varepsilon},\ f\rangle \\ &\leq \Big[\lambda\|\widehat{x} + \widehat{\lambda}(\widehat{v} - \widehat{\varepsilon}) - x\| + \widehat{\lambda}\|x + \lambda(v - \varepsilon) - \widehat{x}\|\Big]\,\|x - \widehat{x}\| \\ &\quad + \lambda\widehat{\lambda}\langle \widehat{v} - v,\ f\rangle + \lambda\widehat{\lambda}\|\varepsilon - \widehat{\varepsilon}\|\|x - \widehat{x}\|. \end{aligned}$$

Dividing by $\|x - \widehat{x}\|$ and taking infimum with respect to f, we conclude the proof of Lemma 5.2. ■

Proof of Theorem 4.1. In order to simplify the notation, we shall drop the dependence on u, writing γ_k for $\gamma_k(u)$, etc. It is important to remember though that the γ's, δ's and η's depend on u. We must prove that for all $k, l \geq 0$ the quantity

$$\omega_{k,l} = (\gamma_k - \widehat{\gamma}_l)^2 + \delta_k + \widehat{\delta}_l + \eta_{k,l}$$

satisfies

$$\omega_{k,l} \geq 0 \quad \text{and} \quad \|x_k - \widehat{x}_l\| \leq \|x_0 - u\| + \|\widehat{x}_0 - u\| + \alpha_{k,l} + \beta_{k,l} + \sqrt{\omega_{k,l}}. \tag{5.2}$$

We will argue by induction on the pair (k, l). First, by virtue of Lemma 5.1, we have

$$\|x_k - \widehat{x}_0\| \leq \|x_k - u\| + \|u - \widehat{x}_0\| \leq \|x_0 - u\| + \gamma_k + e_k + \|\widehat{x}_0 - u\|,$$

which proves (5.2) for any pair $(k,0)$ with $k \geq 0$ because $\widehat{\gamma}_0 = \widehat{\delta}_0 = \eta_{k,0} = \beta_{k,0} = 0$ so that $\gamma_k^2 \leq \omega_{k,0}$. Similarly, (5.2) holds with $(0,l)$ for all $l \geq 0$.

Now suppose (5.2) is true for $(k-1,l)$ and $(k,l-1)$. Take v_k and $\widehat{v}_l$ such that $x_{k-1} = x_k + \lambda_k(v_k - \varepsilon_k)$ and $\widehat{x}_{l-1} = \widehat{x}_l + \widehat{\lambda}_l(\widehat{v}_l - \widehat{\varepsilon}_l)$. We have $[x_k, v_k] \in A_k$ and $[\widehat{x}_l, \widehat{v}_l] \in \widehat{A}_l$. By Lemma 5.2 together with (2.3), we get

$$(\lambda_k + \widehat{\lambda}_l)\|x_k - \widehat{x}_l\| \leq \lambda_k\|\widehat{x}_{l-1} - x_k\| + \widehat{\lambda}_l\|x_{k-1} - \widehat{x}_l\| + \lambda_k\widehat{\lambda}_l\Theta_{k,l} + \lambda_k\widehat{\lambda}_l\|\varepsilon_k - \widehat{\varepsilon}_l\|.$$

Therefore,

$$\begin{aligned}
\|x_k - \widehat{x}_l\| &\leq \tfrac{\widehat{\lambda}_l}{\lambda_k+\widehat{\lambda}_l}\|x_{k-1} - \widehat{x}_l\| + \tfrac{\lambda_k}{\lambda_k+\widehat{\lambda}_l}\|\widehat{x}_{l-1} - x_k\| + \tfrac{\lambda_k\widehat{\lambda}_l}{\lambda_k+\widehat{\lambda}_l}\Theta_{k,l} + \tfrac{\lambda_k\widehat{\lambda}_l}{\lambda_k+\widehat{\lambda}_l}\|\varepsilon_k - \widehat{\varepsilon}_l\| \\
&\leq \tfrac{\widehat{\lambda}_l}{\lambda_k+\widehat{\lambda}_l}\left[\|x_0 - u\| + \|\widehat{x}_0 - u\| + \alpha_{k-1,l} + \beta_{k-1,l} + \sqrt{\omega_{k-1,l}}\right] \\
&\qquad + \tfrac{\lambda_k}{\lambda_k+\widehat{\lambda}_l}\left[\|x_0 - u\| + \|\widehat{x}_0 - u\| + \alpha_{k,l-1} + \beta_{k,l-1} + \sqrt{\omega_{k,l-1}}\right] \\
&\qquad\quad + \tfrac{\lambda_k\widehat{\lambda}_l}{\lambda_k+\widehat{\lambda}_l}\Theta_{k,l} + \tfrac{\lambda_k\widehat{\lambda}_l}{\lambda_k+\widehat{\lambda}_l}\|\varepsilon_k - \widehat{\varepsilon}_l\|.
\end{aligned}$$

Using (4.3) and (4.4), we conclude that

$$\|x_k-\widehat{x}_l\| \leq \|x_0-u\|+\|\widehat{x}_0-u\|+\alpha_{k,l}+\beta_{k,l}+\left(\tfrac{\widehat{\lambda}_l}{\lambda_k+\widehat{\lambda}_l}\sqrt{\omega_{k-1,l}} + \tfrac{\lambda_k}{\lambda_k+\widehat{\lambda}_l}\sqrt{\omega_{k,l-1}}\right). \quad (5.3)$$

We claim that $\omega_{k,l} \geq 0$ and

$$\tfrac{\widehat{\lambda}_l}{\lambda_k+\widehat{\lambda}_l}\sqrt{\omega_{k-1,l}} + \tfrac{\lambda_k}{\lambda_k+\widehat{\lambda}_l}\sqrt{\omega_{k,l-1}} \leq \sqrt{\omega_{k,l}}. \quad (5.4)$$

Indeed, we have

$$\left(\tfrac{\widehat{\lambda}_l}{\lambda_k+\widehat{\lambda}_l}\sqrt{\omega_{k-1,l}} + \tfrac{\lambda_k}{\lambda_k+\widehat{\lambda}_l}\sqrt{\omega_{k,l-1}}\right)^2 \leq \tfrac{\widehat{\lambda}_l}{\lambda_k+\widehat{\lambda}_l}\omega_{k-1,l} + \tfrac{\lambda_k}{\lambda_k+\widehat{\lambda}_l}\omega_{k,l-1}.$$

Direct computations show that

$$(\gamma_{k-1} - \widehat{\gamma}_l)^2 + \delta_{k-1} = (\gamma_k - \widehat{\gamma}_l)^2 + \delta_k - 2(\gamma_k - \widehat{\gamma}_l)\lambda_k|\!|\!|A_k u|\!|\!|,$$

and

$$(\gamma_k - \widehat{\gamma}_{l-1})^2 + \widehat{\delta}_{l-1} = (\gamma_k - \widehat{\gamma}_l)^2 + \widehat{\delta}_l + 2(\gamma_k - \widehat{\gamma}_l)\widehat{\lambda}_l|\!|\!|\widehat{A}_l u|\!|\!|.$$

It follows from this and (4.5) that

$$\begin{aligned}
\tfrac{\widehat{\lambda}_l}{\lambda_k+\widehat{\lambda}_l}\omega_{k-1,l} + \tfrac{\lambda_k}{\lambda_k+\widehat{\lambda}_l}\omega_{k,l-1} &= (\gamma_k - \widehat{\gamma}_l)^2 + \delta_k + \widehat{\delta}_l + \tfrac{\widehat{\lambda}_l}{\lambda_k+\widehat{\lambda}_l}\eta_{k-1,l} + \tfrac{\lambda_k}{\lambda_k+\widehat{\lambda}_l}\eta_{k,l-1} \\
&\qquad + \tfrac{2\lambda_k\widehat{\lambda}_l}{\lambda_k+\widehat{\lambda}_l}(\gamma_k - \widehat{\gamma}_l)\left[|\!|\!|\widehat{A}_l u|\!|\!| - |\!|\!|A_k u|\!|\!|\right] \\
&= (\gamma_k - \widehat{\gamma}_l)^2 + \delta_k + \widehat{\delta}_l + \eta_{k,l} \\
&= \omega_{k,l},
\end{aligned}$$

which proves our claim and completes the proof of Theorem 4.1. ∎

5.2. **Discrete-continuous.** In order to prove Theorem 4.5 we consider the points $\{\widehat{x}_l\}_{l=0}^m$ defined by (2.4b) and pass to the limit in inequality (4.1). Inequality (4.6) follows almost immediately, but the fact that β_k and $\eta_k(u)$ are finite is a more delicate issue.

First observe that using the recurrence formulae (4.4) and (4.5) repeatedly, along with the initial conditions, we can obtain closed expressions for $\beta_{k,l}$ and $\eta_{k,l}(u)$, respectively. Several summation and multiplication operations are involved in such expressions, making them unpractical for useful computations. Since no more generality is needed for proving our estimates we will assume that $\widehat{\lambda}_l \equiv \widehat{\lambda}$. Moreover, and just to simplify the notation and arguments, we will also assume that the sequence $\{\lambda_k\}$ is nonincreasing. Set $\mu_n = \frac{\widehat{\lambda}}{\widehat{\lambda}+\lambda_n}$, $\nu_n = \frac{\lambda_n}{\widehat{\lambda}+\lambda_n}$ and $M_{n,i} = \prod_{k=n-i}^{n} \mu_k$. With this notation, set

$$B_{n,m} = \lambda_1 \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \binom{i+j}{j} M_{n,i} \nu_1^j \Theta_{n-i,m-j} \tag{5.5}$$

and

$$\begin{aligned} H_{n,m}(u) &= 2\lambda_1 \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \binom{i+j}{j} M_{n,i} \nu_1^j \gamma_{n-i}(u) \| A_{n-i} u \| \\ &\quad -2\lambda_n \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \binom{i+j}{j} M_{n,i} \nu_n^j \gamma_{n-i}(u) \| \widehat{A}_{m-j} u \| \\ &\quad -2\lambda_n \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \binom{i+j}{j} M_{n,i} \nu_n^j \widehat{\gamma}_{m-j}(u) \| A_{n-i} u \| \\ &\quad +2\lambda_1 \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \binom{i+j}{j} M_{n,i} \nu_1^j \widehat{\gamma}_{m-j}(u) \| \widehat{A}_{m-j} u \|. \end{aligned} \tag{5.6}$$

The following estimations are fundamental in the proof of Theorem 4.5.

Lemma 5.3. *For each, n, m we have $\beta_{n,m} \leq B_{n,m}$ and $\eta_{n,m}(u) \leq H_{n,m}(u)$.*

Proof. We prove the estimation for $\beta_{n,m}$ and leave the other to the reader. From the definition (4.4) of $\beta_{k,l}$ we see that

$$\beta_{k,l} \leq \mu_k \beta_{k-1,l} + \nu_1 \beta_{k,l-1} + \lambda_1 \mu_k \Theta_{k,l} \qquad \text{for} \quad k, l \geq 1,$$

while $\beta_{k,0} = \beta_{0,l} = 0$. Therefore, $\beta_{n,m}$ can be bounded by a combination of the values $\Theta_{k,l}$ for $1 \leq k \leq n$, $1 \leq l \leq m$. Observe that the factor $\Theta_{n-i,m-j}$ will appear once for each monotonic path Γ from $(n-i, m-j)$ to (n, m) on an integer-node graph (see Figure 1) where the horizontal arcs leading to column k are weighted μ_k and the vertical arcs all weigh ν_1.

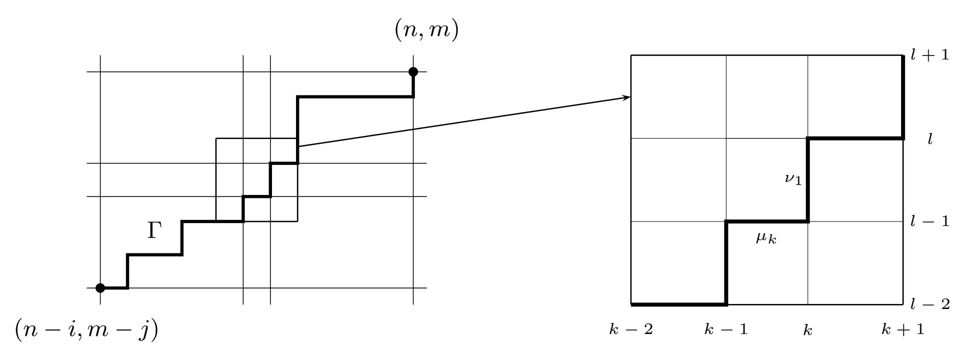

Figure 1

By an elementary combinatorial argument, there are $\binom{i+j}{j}$ such paths. ■

Next, we shall prove that $B_{n,m}$ and $H_{n,m}(u)$ converge as $m \to \infty$, from which we deduce that β_n and $\eta_n(u)$ are finite. To simplify the notation let $\Lambda_{n,i} = \prod_{p=n-i}^{n} \lambda_p$.

Lemma 5.4. *Let $\tau = t - t_0$. With the notation introduced above, the following holds for each $n \geq 1$:*

$$\lim_{m\to\infty} B_{n,m} = \lambda_1 \int_0^\tau \sum_{i=0}^{n-1} \frac{1}{i!} \Theta(s_0 + \sigma_n - \sigma_i, t - \xi) \left(\frac{\xi^i e^{-\frac{\xi}{\lambda_1}}}{\Lambda_{n,i}} \right) d\xi$$

and

$$\begin{aligned} \lim_{m\to\infty} H_{n,m}(u) &= \lambda_1 \int_0^\tau \sum_{i=0}^{n-1} \frac{2}{i!} \gamma_{n-i} \| A_{n-i} u \| \left(\frac{\xi^i e^{-\frac{\xi}{\lambda_1}}}{\Lambda_{n,i}} \right) d\xi \\ &- \lambda_n \int_0^\tau \sum_{i=0}^{n-1} \frac{2}{i!} \gamma_{n-i} \| A(t-\xi) u \| \left(\frac{\xi^i e^{-\frac{\xi}{\lambda_n}}}{\Lambda_{n,i}} \right) d\xi \\ &- \lambda_n \int_0^\tau \sum_{i=0}^{n-1} \frac{2}{i!} \mathcal{A}_u(t-\xi) \ \| A_{n-i} u \| \left(\frac{\xi^i e^{-\frac{\xi}{\lambda_n}}}{\Lambda_{n,i}} \right) d\xi \\ &+ \lambda_1 \int_0^\tau \sum_{i=0}^{n-1} \frac{2}{i!} \mathcal{A}_u(t-\xi) \ \| A(t-\xi) u \| \left(\frac{\xi^i e^{-\frac{\xi}{\lambda_1}}}{\Lambda_{n,i}} \right) d\xi. \end{aligned}$$

Proof. Fix $n \geq 1$. The idea is to express $B_{n,m}$ and $H_{n,m}$ as Riemann sums of certain step functions and apply the Dominated Convergence Theorem as $m \to \infty$. First set $\widehat{\lambda} = \frac{\tau}{m}$ and take $h \in \{1, n\}$. We have

$$\sum_{j=0}^{m-1} \binom{i+j}{j} M_{n,i} \nu_h^j = \int_0^\tau \psi_{h,m}(\xi) \ d\xi,$$

where $\psi_{h,m}$ is a step function defined as follows: if $\frac{j\tau}{m} \leq \xi < \frac{(j+1)\tau}{m}$ then

$$\begin{aligned} \psi_{h,m}(\xi) &= \frac{m}{\tau} \binom{i+j}{j} M_{n,i} \nu_h^j \\ &= \tfrac{1}{i!} \left[\prod_{p=n-i}^{n} \left(\tfrac{1}{\lambda_p + \frac{\tau}{m}} \right) \right] \left[\prod_{q=1}^{i} \left(\tfrac{j\tau}{m} + \tfrac{q\tau}{m} \right) \right] \left[\left(1 + \tfrac{\tau}{m\lambda_h} \right)^{\frac{m}{\tau}} \right]^{-\frac{j\tau}{m}}. \quad (5.7) \end{aligned}$$

Notice that for $\frac{j\tau}{m} \leq \xi < \frac{(j+1)\tau}{m}$ we have

$$\psi_{h,m}(\xi) \leq \frac{m}{\tau} \binom{i+j}{j} M_{n,i} = \frac{m}{\tau} \frac{(j+i)!}{i! \ j!} \prod_{p=n-i}^{n} \left(\frac{\frac{\tau}{m}}{\lambda_p + \frac{\tau}{m}} \right)$$

because $\nu_h^j \leq 1$. Thus

$$\psi_{h,m}(\xi) \leq \frac{\tau^i}{m^i} \frac{(j+1)(j+2)\cdots(j+i)}{i!} \prod_{p=n-i}^{n} \left(\frac{1}{\lambda_p} \right) \leq \frac{\tau^i \ (2m)^i}{i! \ m^i \ \Lambda_{n,i}} = \frac{(2\tau)^i}{i! \ \Lambda_{n,i}}.$$

and so, the sequence $\{\psi_{h,m}\}_m$ is uniformly bounded. Moreover,

$$\lim_{m\to\infty} \psi_{h,m}(\xi) = \frac{\xi^i e^{-\frac{\xi}{\lambda_h}}}{i!\ \Lambda_{n,i}}$$

on $[0,\tau]$. To see this we use representation (5.7). The only difficult part is the middle bracket. But $\prod_{q=1}^{i} \left(\frac{j\tau}{m} + \frac{q\tau}{m}\right)$ is a polynomial in $\frac{j\tau}{m}$ of degree i. The leading coefficient is 1, while the rest are bounded by a constant (depending only on i and τ) times $\frac{1}{m}$ and so they vanish as $m \to \infty$. We have proved that the sequence $\{\psi_{h,m}\}$ is uniformly bounded and pointwise convergent on $[0,\tau]$. But the same is true for the sequences

$$\Theta_{n-i,m-j}, \qquad \|\widehat{A}_{m-j}u\| \qquad \text{and} \qquad \widehat{\gamma}_{m-j}(u),$$

which converge, almost everywhere, to

$$\Theta(s_0 + \sigma_n - \sigma_i, t-\xi), \qquad \|\widehat{A}(t-\xi)u\| \qquad \text{and} \qquad \widehat{\mathcal{A}}_u(t-\xi),$$

respectively[4] due to the hypothesized Riemann-integrability. The result follows from the Dominated Convergence Theorem. ■

Proof of Theorem 4.5. Let $\{\widehat{x}_l\}_{l=0}^m$ be the points defined by (2.4b). By virtue of Theorem 4.1, we have

$$\begin{aligned}\left\| x_k - \prod_{l=1}^{m} \left(I - \frac{\tau}{m}\widehat{A}_l\right)^{-1} u \right\| \quad \leq \quad & \|x_0 - u\| + \alpha_{k,m} + \beta_{k,m} \\ & + \sqrt{(\gamma_k(u) - \widehat{\gamma}_m(u))^2 + \delta_k(u) + \widehat{\delta}_m(u) + \eta_{k,m}(u)}.\end{aligned}$$

Since the subjacent proximal scheme is exact, we have $\alpha_{k,m} = e_k$ for all m. It is easy to see that $\lim_{m\to\infty} \widehat{\delta}_m(u) = 0$, while $\lim_{m\to\infty} \widehat{\gamma}_m(u) = \mathcal{A}_u(t,t_0)$. Letting $m \to \infty$ in the previous inequality, we obtain (4.6). Finally, β_n and $\eta_n(u)$ are finite by virtue of (5.5), (5.6) and Lemma 5.4. ■

5.3. **Continuous-continuous.** The idea is to use Theorem 4.5 and pass to the limit once more. To do this, we shall compute $\lim_{n\to\infty} \beta_n$ and $\lim_{n\to\infty} \eta_n(u)$. It is not difficult to verify that $\beta_n = \lim_{m\to\infty} B_{n,m}$ while $\eta_n(u) = \lim_{m\to\infty} H_{n.m}(u)$ because $\lambda_k \equiv \frac{\sigma}{n}$. On the other hand, for $\xi \in [0,\tau]$, $\zeta \in [0,\sigma]$ and $0 \leq i \leq n-1$ define

$$f_n(\xi,\zeta) = \frac{2}{i!\ \xi}\left(\frac{n\xi}{\sigma}\right)^{i+1} e^{-\frac{n\xi}{\sigma}} \qquad \text{with} \quad \frac{i\sigma}{n} \leq \zeta < \frac{(i+1)\sigma}{n}.$$

The expressions in Lemma 5.4 become, respectively,

$$\beta_n = \int_0^\tau \left\{ \sum_{i=0}^{n-1} \frac{1}{2}\Theta(s_0 + \sigma_n - \sigma_i, t-\xi) f_n\left(\xi, \tfrac{i\sigma}{n}\right)\frac{\sigma}{n} \right\} d\xi \tag{5.8}$$

[4] Again, j is related to ξ via $\frac{j\tau}{m} \leq \xi < \frac{(j+1)\tau}{m}$.

and

$$
\begin{aligned}
\eta_n(u) \quad = \quad & \int_0^\tau \left\{ \sum_{i=0}^{n-1} \gamma_{n-i} \|A_{n-i}u\| \; f_n\left(\xi, \tfrac{i\sigma}{n}\right) \frac{\sigma}{n} \right\} \, d\xi \\
& - \int_0^\tau \left\{ \sum_{i=0}^{n-1} \gamma_{n-i} \; \|A(t-\xi)u\| f_n\left(\xi, \tfrac{i\sigma}{n}\right) \frac{\sigma}{n} \right\} \, d\xi \\
& - \int_0^\tau \left\{ \sum_{i=0}^{n-1} \; \mathcal{A}_u(t-\xi, t_0) \|A_{n-i}u\| \; f_n\left(\xi, \tfrac{i\sigma}{n}\right) \frac{\sigma}{n} \right\} \, d\xi \\
& + \int_0^\tau \left\{ \sum_{i=0}^{n-1} \; \mathcal{A}_u(t-\xi, t_0) \; \|A(t-\xi)u\| f_n\left(\xi, \tfrac{i\sigma}{n}\right) \frac{\sigma}{n} \right\} \, d\xi. \qquad (5.9)
\end{aligned}
$$

Lemma 5.5. *Fix $\xi \in [0, \tau]$. The sequence $\{f_n(\xi, \cdot\,)\}$ converges uniformly to zero on every closed subset of $[0, \sigma]$ not containing ξ.*

Proof. Take $\zeta \in [0, \sigma]$ and define $i_n = \left\lfloor \frac{n\zeta}{\sigma} \right\rfloor$ so that

$$
\frac{n\zeta}{\sigma} - 1 < i_n \leq \frac{n\zeta}{\sigma} \qquad (5.10)
$$

and

$$
f_n(\xi, \zeta) = \frac{1}{i_n! \; \xi} \left(\frac{n\xi}{\sigma} \right)^{i_n+1} e^{-\frac{n\xi}{\sigma}}
$$

for each n. Stirling's Formula states that

$$
\lim_{m\to\infty} \frac{\sqrt{2\pi} \; m^{m+1/2}}{e^m \; m!} = 1.
$$

The sequence being convergent, there exists a constant $M > 0$ such that

$$
f_n(\xi, \zeta) \leq \frac{M}{\xi \; (i_n)^{i_n+1/2}} \left(\frac{n\xi}{\sigma} \right)^{i_n+1} e^{i_n - \frac{n\xi}{\sigma}}
$$

for all n. Denote $\frac{\xi}{\sigma}$ by a and $\frac{\zeta}{\sigma}$ by b. Since the convergence around zero is straightforward, we may assume $b \geq 2/n$ and so $bn - 1 \geq 1$. By virtue of the double inequality (5.10), we have

$$
\begin{aligned}
f_n(\xi, \zeta) \quad & \leq \quad \frac{M(an)^{bn+1} e^{bn-an}}{\xi \; (bn-1)^{bn-1/2}} \\
& = \quad \frac{M}{\xi} \; an\sqrt{bn} \left(\frac{an}{bn-1} \right)^{bn} e^{n(b-a)} \\
& = \quad \frac{M}{\sigma} \left(\frac{bn}{bn-1} \right)^{bn} \sqrt{b} n^{3/2} \left[\left(\frac{a}{b} \right)^{bn} e^{n(b-a)} \right] \\
& = \quad \frac{4M\sqrt{b}}{\sigma} n^{3/2} \left[e^{\left(b-a+b\ln\left(\frac{a}{b}\right)\right)} \right]^n .
\end{aligned}
$$

For the second equality we used the fact that $(\frac{z}{z-1})^z \leq 4$ for all $z \geq 2$. Now, if ζ is in a closed set not containing ξ then $|b - a| \geq c$ for some $c > 0$. By continuity, $b - a + b\ln\left(\frac{a}{b}\right) \leq -d$ for some $d > 0$ and the result follows. ∎

Remark 5.6. Each of the sums in braces on the right-hand sides of equations (5.8) and (5.9) can be interpreted as integrals of the form

$$\int_0^\sigma \phi_n(\xi,\zeta) f_n(\xi,\zeta)\ d\zeta,$$

where the sequences $\{\phi_n\}$ and $\{f_n\}$ have the following properties:

i) For each $\xi \in [0,\tau]$, and each n, $\phi_n(\xi,\cdot\ )$ is a step function and the sequence $\{\phi_n(\xi,\cdot\ )\}$ converges uniformly to a continuous function $\phi(\xi,\cdot)$ as $n \to \infty$.
ii) For each $\xi \in [0,\tau]$, and each n, $f_n(\xi,\cdot\ )$ is a step function and the sequence $\{f_n(\xi,\cdot\ )\}$ converges uniformly (and thus in L^1) to zero on every closed subset of $[0,\sigma]$ not containing ξ.
iii) For $\xi \in [0,\tau]$ set

$$I_n(\xi) := \int_0^\sigma f_n(\xi,\zeta)\ d\zeta = \sum_{i=0}^{n-1} \frac{2}{i!}\left(\frac{n\xi}{\sigma}\right)^i e^{-\frac{n\xi}{\sigma}}.$$

In order to compute $\lim_{n\to\infty} I_n(\xi)$ consider the normalized partial averages of a sequence of independent Poisson-distributed random variables with parameter $a = \xi/\sigma$. According to the Central Limit Theorem their distribution functions, say F_n, converge uniformly to the normal distribution function. The sum above corresponds to $2F_n(\xi_n)$ where $\xi_n = \sqrt{\frac{n}{a}}(1-a)$. Hence

$$\lim_{n\to\infty} I_n(\xi) = \begin{cases} 2 & \text{if} \quad \xi < \sigma \\ 1 & \text{if} \quad \xi = \sigma \\ 0 & \text{if} \quad \xi > \sigma. \end{cases}$$

iv) The integrals are bounded functions of ξ. □

A direct consequence of the Dominated Convergence Theorem is the following:

Lemma 5.7. *Let $\{\phi_n\}$ and $\{f_n\}$ satisfy i), ii), iii) and iv). Then*

$$\lim_{n\to\infty} \int_0^\tau \int_0^\sigma \phi_n(\xi,\zeta) f_n(\xi,\zeta)\ d\zeta\ d\xi = 2\int_0^\delta \phi(\xi,\xi)\ d\xi,$$

where $\delta = \min\{\tau,\sigma\}$.

Proof of Theorem 4.7. According to Remark 5.6 and Lemma 5.7, we have

$$\begin{aligned} \lim_{n\to\infty} \eta_n(u) &= 2\int_0^\tau \mathcal{A}_u(s-\xi,s_0)\|A(s-\xi)u\| d\xi - 2\int_0^\tau \mathcal{A}_u(s-\xi,s_0)\|\widehat{A}(t-\xi)u\| d\xi \\ &\quad -2\int_0^\tau \widehat{\mathcal{A}}_u(t-\xi,t_0)\|A(s-\xi)u\| d\xi + 2\int_0^\tau \widehat{\mathcal{A}}_u(t-\xi,t_0)\|\widehat{A}(t-\xi)u\| d\xi \\ &= 2\int_0^\tau \Big(\mathcal{A}_u(s-\xi,s_0) - \widehat{\mathcal{A}}_u(t-\xi,t_0)\Big)\Big(\|A(s-\xi)u\| - \|\widehat{A}(t-\xi)u\|\Big) d\xi \\ &= \Big[\mathcal{A}_u(s,s_0) - \widehat{\mathcal{A}}_u(t,t_0)\Big]^2 - [\mathcal{A}_u(t_0+s-t,s_0)]^2 \end{aligned}$$

while

$$\lim_{n\to\infty} \beta_n = \int_0^\tau \Theta(s-\xi,t-\xi)\ d\xi.$$

The result follows from Theorem 4.5. ■

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