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by

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A Note on Non-Simultaneous Blow-up for a Drift-Diffusion Model

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Abstract

In this paper we consider a drift-diffusion model of parabolic-elliptic type, with three coupled equations. We prove that there exist parameter regimes for which non-simultaneous blow-up of solutions happens. This is in contrast to a two-chemotactic species model, coupled to an elliptic equation for an attractive chemical produced by the two species, where blow-up of one species implies blow-up of the other one at the same time. Also, we show that the range of parameters of the drift-diffusion model in this paper, for which blow-up happens, is larger than suggested by previous results in the literature.

1 Introduction

In this paper we consider the drift-diffusion model discussed in [3], namely

$$\begin{cases} \partial_t u_1 - \Delta u_1 + \nabla \cdot (u_1 \nabla \psi) = 0, \quad t > 0, \qquad x \in \mathbb{R}^2 \\ \partial_t u_2 - \Delta u_2 - \nabla \cdot (u_2 \nabla \psi) = 0, \quad t > 0, \qquad x \in \mathbb{R}^2 \\ -\Delta \psi = -(u_2 - u_1)\ , \qquad x \in \mathbb{R}^2 \\ u_1(0,x) = u_{1,0}(x) \geq 0, \quad u_2(0,x) = u_{2,0}(x) \geq 0, \qquad x \in \mathbb{R}^2\ , \end{cases} \tag{1}$$

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where $u_{1,0}, u_{2,0}$ are not identically zero.

This system can be interpreted as a two species system for chemotaxis like motion, where the species u_1 is moving towards higher concentrations of a chemical ψ and at the same time produces ψ, whereas the species u_2 is moving towards lower concentrations of ψ and degrades ψ. The third equation then describes the dynamics of the chemical ψ, under the assumption that its basal concentration is much larger than the changes of its concentration induced by u_1 and u_2.

Normally, the dynamics of the attractive and repelling chemical would be described by

$$\varepsilon\psi_t = \Delta\psi + u_1 - u_2\psi \ .$$

Let ψ_0 denote the basal concentration of the chemical, then

$$\psi = \psi_0 + \phi \quad , \qquad \text{where } \phi << \psi_0 \ .$$

Thus

$$\varepsilon\phi_t = \Delta\phi + u_1 - u_2\psi_0 + h.o.t. \ .$$

By rescaling we obtain

$$\Delta\phi + u_1 - u_2 = 0 \ ,$$

and the coefficients in the first two equations also change.

The model setup as given in [3] uses very particular parameters. Nevertheless, several specific qualitative features of the model can be shown. So we will stick to the given parameters for purely mathematical reasons. More complex multicomponent systems for chemotaxis for a variety of parameters have been considered by Wolansky in [6], where generalized conditions for the existence of global solutions were given.

Here we focus on the model analyzed in [3]. Kurokiba and Ogawa proved local well posedness for system (1) and the existence of blow-up in case

$$u_{1,0}, u_{2,0} \in L^2_s(\mathbb{R}^2) := \left\{ f \in L^1_{loc}(\mathbb{R}^2);\ (1+|x|^2)^{s/2} f(x) \in L^2(\mathbb{R}^2) \right\}, \text{ with } s > 1$$

and

$$\frac{\left(\int_{\mathbb{R}^2}(u_{1,0} - u_{2,0})dx\right)^2}{\int_{\mathbb{R}^2}(u_{1,0} + u_{2,0})dx} > 8\pi \ .$$

More precisely, they obtained the following result.

Theorem 1 (Theorem 1.1 and Proposition 1.2 in [3]) *For any $s > 1$, let*

$$(u_{1,0}, u_{2,0}) \in L^2_s(\mathbb{R}^2) \times L^2_s(\mathbb{R}^2) \ .$$

Then there exist $T = T(\|u_{1,0}\|_{L^2_s}, \|u_{2,0}\|_{L^2_s}) > 0$ and a unique solution (u_1, u_2) of (1) with initial data $(u_{1,0}, u_{2,0})$ such that
$u_1, u_2 \in C([0,T) \, ; L^2_s(\mathbb{R}^2)) \cap C((0,T); C^2(\mathbb{R}^2))$. Furthermore, the solution depends continuously on the initial data; and, in case the maximal time of existence of the solution is finite, i.e., $T_{max} < \infty$, then

$$\|u_1(t)\|_{L^2_s(\mathbb{R}^2)} + \|u_2(t)\|_{L^2_s(\mathbb{R}^2)} \to \infty, \qquad as \ t \to T_{max} \ .$$

Additionally, if the initial data of (1) satisfy $u_1(0,x) \geq 0$, $u_2(0,x) \geq 0$, then for any solution $(u_1, u_2) \in C([0,T) \, ; C^2(\mathbb{R}^2)) \times C([0,T) \, ; C^2(\mathbb{R}^2))$, we have

$$u_1(t,x) \geq 0, \quad u_2(t,x) \geq 0 \ .$$

Finally, if $u_{1,0}$, $u_{2,0} \in L^1(\mathbb{R}^2)$, then

$$\|u_1(t)\|_1 = \|u_{1,0}\|_1 \quad , \quad \|u_2(t)\|_1 = \|u_{2,0}\|_1 \, .$$

The next result shows blow-up in finite time.

Theorem 2 (Theorem 1.5 in [3]) *Let $s > 1$ and $u_{1,0}, u_{2,0} \in L^2_s(\mathbb{R}^2)$ with $u_{1,0}$, $u_{2,0} \geq 0$ everywhere and let*

$$\frac{\left(\int_{\mathbb{R}^2}(u_{1,0} - u_{2,0})dx\right)^2}{\int_{\mathbb{R}^2}(u_{1,0} + u_{2,0})dx} > 8\pi \ . \tag{2}$$

Then the solution of (1) blows up in finite time, i.e.

$$\limsup_{t \to T^-} \left(\|u_1(t,\cdot)\|_{L^\infty(\mathbb{R}^2)} + \|u_2(t,\cdot)\|_{L^\infty(\mathbb{R}^2)}\right) = +\infty \ .$$

In this paper we will prove in Section 2 that it is possible to have finite time blow-up for one species and no blow-up at that same time for the other species. This differs from the situation in the two chemotactic species model considered in [1], where both species are attracted towards the chemical. This behavior is described by two classical Keller-Segel equations for the chemotactic species and an elliptic equation for the attractive chemical. In that model, if blow-up happens for one species, the other species is blowing up too at the same time.

In Section 3 we prove that blow-up is also possible for some initial data, when inequality (2) is not satisfied, even if the difference between the initial data is very small.

2 Existence of Non-Simultaneous Blow-up

We mainly consider the radial symmetric situation. Therefore it is convenient to reformulate the problem by introducing the mass functions M_1, M_2. This allows us to reduce the number of equations of our problem.

Notation: We define

$$\begin{aligned} M_1(t,r) &= \int_{B(0,r)} u_1 dx \quad , \quad M_2(t,r) = \int_{B(0,r)} u_2 dx \ , \\ \text{thus} \quad M_2(t,r) &= 2\pi \int_0^r u_1 \rho d\rho \quad , \quad M_2(t,r) = 2\pi \int_0^r u_2 \rho d\rho \ . \end{aligned}$$

$$\text{Let} \quad \theta_1 = M_1(t,\infty) \quad , \quad \theta_2 = M_2(t,\infty) \ . \tag{3}$$

In terms of M_1 and M_2 system (1) is equivalent to

$$\left\{ \begin{array}{l} \partial_t M_1 = r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial M_1}{\partial r}\right) - \frac{M_2 - M_1}{2\pi r}\frac{\partial M_1}{\partial r} \\ \partial_t M_2 = r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial M_2}{\partial r}\right) + \frac{M_2 - M_1}{2\pi r}\frac{\partial M_2}{\partial r} \end{array} \right. . \tag{4}$$

In the following theorem we give sufficient conditions to guarantee uniform boundedness for u_2 as long as it is defined.

Theorem 3 (Conditions for the boundedness of u_2) *Suppose that the initial data of (1) satisfy*

$$u_1(0,r) \geq u_2(0,r) \ ,$$

then for any $u_1, u_2 \in C([0,T]; L^2_s(\mathbb{R}^2)) \cap C([0,T]; C^2(\mathbb{R}^2))$ there exists a constant C such that $u_2(t,r) \leq C$.

Proof. Let

$$\begin{aligned} v(t,x) &= u_1(t,x) + u_2(t,x) \\ w(t,x) &= u_1(t,x) - u_2(t,x) \ . \end{aligned}$$

Then

$$\begin{aligned} \partial_t v - \Delta v + \nabla(w\nabla\psi) &= 0, \quad t>0, \quad x\in\mathbb{R}^2 \\ \partial_t w - \Delta w + \nabla(v\nabla\psi) &= 0, \quad t>0, \quad x\in\mathbb{R}^2 \\ -\Delta\psi &= w, \quad x\in\mathbb{R}^2 \\ v(0,x) = u_{1,0}(x) + u_{2,0}(x) \quad &, \quad w(0,x) = u_{1,0}(x) - u_{2,0}(x) \ , \end{aligned}$$

and in polar coordinates we obtain

$$\begin{aligned} \partial_t v - \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial v}{\partial r}) + \frac{1}{r}\frac{\partial}{\partial r}\left(wr\frac{\partial\psi}{\partial r}\right) &= 0 \\ \partial_t w - \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial w}{\partial r}) + \frac{1}{r}\frac{\partial}{\partial r}\left(vr\frac{\partial\psi}{\partial r}\right) &= 0 \\ -\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial\psi}{\partial r}) &= w \ . \end{aligned}$$

Integrating over $(0,r)$, defining $S := \int_0^r v\rho\,d\rho$, $R := \int_0^r w\rho\,d\rho$, and using that $v = \frac{1}{r}\frac{\partial}{\partial r}\int_0^r v\rho\,d\rho = \frac{1}{r}\frac{\partial S}{\partial r}$ and $w = \frac{1}{r}\frac{\partial}{\partial r}\int_0^r w\rho\,d\rho = \frac{1}{r}\frac{\partial R}{\partial r}$, we get

$$\begin{aligned} \partial_t S - r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial S}{\partial r}\right) + \frac{\partial R}{\partial r}\frac{\partial\psi}{\partial r} &= 0 \\ \partial_t R - r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial R}{\partial r}\right) + \frac{\partial S}{\partial r}\frac{\partial\psi}{\partial r} &= 0 \\ r\frac{\partial\psi}{\partial r} &= -R \ . \end{aligned}$$

Or equivalently

$$\begin{aligned} \partial_t S - r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial S}{\partial r}\right) - \frac{1}{r}\frac{\partial R}{\partial r}R &= 0 \\ \partial_t R - r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial R}{\partial r}\right) - \frac{1}{r}\frac{\partial S}{\partial r}R &= 0 \ . \end{aligned}$$

By assumption we have $R(0,r) \geq 0$ and $R(t,0) = 0$. By the maximum principle it follows that $R(t,r) \geq 0$. Using that

$$\begin{aligned} R(t,r) &= \int_0^r w\rho\, d\rho = \int_0^r (u_1 - u_2)\,\rho\, d\rho \\ &= \frac{1}{2\pi}\left(\int_{B(0,r)} u_1\rho\, d\rho - \int_{B(0,r)} u_2\rho\, d\rho\right) = \frac{1}{2\pi}(M_1 - M_2)\ , \end{aligned}$$

we conclude that

$$M_1 \geq M_2\ . \tag{5}$$

From (4) and (5) it follows that,

$$\partial_t M_2 \leq r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial M_2}{\partial r}\right)\ . \tag{6}$$

Since $u_{2,0}$ is bounded by some constant C, we have

$$M_2(0,r) = \frac{1}{2\pi}\int_{B(0,r)} u_{2,0}\rho\, d\rho \leq \frac{1}{2\pi}C\int_{B(0,r)} u_{1,0}\rho\, d\rho = Cr^2\ .$$

Introducing the transformation

$$\overline{M}(t,r) = M_2(t,r) - Cr^2\ , \tag{7}$$

it follows from (6) that

$$\begin{aligned} \partial_t\overline{M} &\leq r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\overline{M}}{\partial r}\right) \\ \overline{M}(0,r) &\leq 0 \quad , \quad \overline{M}(t,0) = 0 \quad , \quad \overline{M}(t,R) \leq 0\ , \end{aligned}$$

or equivalently

$$\begin{aligned} \partial_t\overline{M} &\leq \frac{\partial^2\overline{M}}{\partial r^2} - \frac{1}{r}\frac{\partial\overline{M}}{\partial r} \\ \overline{M}(0,r) &\leq 0 \quad , \quad \overline{M}(t,0) = 0 \quad , \quad \overline{M}(t,R) \leq 0\ . \end{aligned}$$

Thus the maximum principle yields

$$\overline{M}(t,r) = M_2(t,r) - Cr^2 \leq 0$$

and we get that $M_2(t,r) = 2\pi \int_0^r u_2 \rho \, d\rho \le Cr^2$. Using regularity theory for parabolic equations as in [1], Lemma 6, we then obtain the bound $u_2 = \frac{1}{r}\frac{\partial M_2}{\partial r} \le C$, for a suitable constant $C > 0$. ■

One of the main results of our paper now follows from Theorem 2 and Theorem 3 as a corrolary.

Corollary 4 *Suppose for system (1) and any $s > 1$, that $(u_{1,0}, u_{2,0}) \in L_s^2(\mathbb{R}^2) \times L_s^2(\mathbb{R}^2)$ with $u_{1,0} \ge u_{2,0}$. Let*

$$\frac{\left(\int_{\mathbb{R}^2} u_{1,0} - u_{2,0} dx\right)^2}{\int_{\mathbb{R}^2} (u_{1,0} + u_{2,0}) dx} > 8\pi \ .$$

Let $u_1, u_2 \in C([0,T]; L_s^2(\mathbb{R}^2)) \cap C([0,T]; C^2(\mathbb{R}^2))$ Then finite time blow-up happens for u_1, and u_2 is still bounded until that time.

The above given inequality could even be replaced by a weaker condition. But this needs knowledge about another theorem, which is formulated at a later stage in this paper.

3 Blow-up Results

Theorem 5 *For any $\varepsilon > 0$, there exist initial data $u_{1,0}$, $u_{2,0}$ satisfying the assumptions of Theorem 1, as well as*

$$u_{1,0}(r) = \frac{2(8\pi + \varepsilon)}{(r^2 + \lambda_0)^2} \quad , \quad u_{2,0}(r) \le \frac{16\pi}{(r^2+1)^2} \ ,$$

such that the corresponding solution (u_1, u_2) develops a singularity for $T < \infty$.

Proof. Assume that M_1, M_2 are defined as in (3). Consider a family of initial data $u_{1,0}^{\lambda_0}$, $u_{2,0}^{\lambda_0}$ to be specified later, and denote the corresponding solutions by $u_1^{\lambda_0}, u_2^{\lambda_0}$. Further assume that the corresponding functions $M_{1,0}^{\lambda_0}$, $M_{2,0}^{\lambda_0}$ satisfy

$$\begin{aligned} M_{1,0}^{\lambda_0}(r) &\le 9\pi \ , \\ M_{2,0}^{\lambda_0}(r) &\le \frac{8\pi r^2}{r^2+1} \ . \end{aligned} \tag{8}$$

Arguing by comparison with the stationary solution

$$M_s(r) = \frac{8\pi r^2}{r^2+1}$$

and using

$$\partial_t M_2 \le r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial M_2}{\partial r}\right) + \frac{M_2}{2\pi r}\frac{\partial M_2}{\partial r}$$

it follows that

$$M_2^{\lambda_0}(r,t) \le \frac{8\pi r^2}{r^2+1} \ , \tag{9}$$

as long as the solution of (1) is defined. Notice that this inequality holds independently of the choice of $M_{1,0}^{\lambda_0}$. On the other hand, due to assumption (8), the maximum principle implies

$$M_1^{\lambda_0}(r,t) \le 9\pi \ .$$

Therefore, using regularity theory as well as the rescaling argument in [1], it follows that

$$u_2^{\lambda_0} = \frac{1}{2\pi r}\frac{\partial M_2^{\lambda_0}}{\partial r} \le C$$

for a constant C, which is independent of λ_0 and as long as the solution is defined.

Let us consider a family of initial data having the form

$$M_{1,0}^{\lambda_0}(r) = \frac{(8\pi+\varepsilon)\, r^2}{r^2+\lambda_0} \ , \tag{10}$$

where $\lambda_0 > 0$ will be specified later. It only remains to show, that the corresponding solution blows-up in finite time. To do this, an adaptation of the subsolution method in [2] is used. It follows from (9) and from the equation for M_1 in (4) that

$$\partial_t M_1 \ge r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial M_1}{\partial r}\right) + \frac{M_1}{2\pi r}\frac{\partial M_1}{\partial r} - \frac{4r}{r^2+1}\frac{\partial M_1}{\partial r} \ . \tag{11}$$

In order to prove the formation of a singularity for (11) it is thus enough to obtain a subsolution for (10), (11), which blows up in finite time. We look for a subsolution of (11) of the form

$$M(r,t) = \frac{(8\pi + \varepsilon - \alpha(t))\, r^2}{r^2 + \lambda(t)}$$

with $\lambda(t)$ and $\alpha(t)$ to be prescribed later. Notice that after some calculations we obtain

$$\begin{aligned}
&\partial_t M - r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial M}{\partial r}\right) - \frac{M_1}{2\pi r}\frac{\partial M}{\partial r} + \frac{4r}{r^2+1}\frac{\partial M}{\partial r} \\
&\quad = -\frac{(8\pi + \varepsilon - \alpha(t))\, r^2}{(r^2 + \lambda(t))^2}\left[\frac{\dot\alpha\,(r^2 + \lambda(t))}{(8\pi + \varepsilon - \alpha(t))} + \dot\lambda + \frac{(\varepsilon - \alpha(t))\,\lambda(t)}{\pi\,(r^2 + \lambda(t))} - \frac{8\lambda(t)}{r^2+1}\right] \\
&\quad \le -\frac{(8\pi + \varepsilon - \alpha(t))\, r^2}{(r^2 + \lambda(t))^2}\left[\frac{\dot\alpha\,(r^2 + \lambda(t))}{(8\pi + \varepsilon)} + \dot\lambda + \frac{\varepsilon\lambda(t)}{2\pi\,(r^2 + \lambda(t))} - 8\lambda(t)\right],
\end{aligned}$$

if we assume that $\varepsilon - \alpha(t) > \frac{\varepsilon}{2}$, $\alpha(t) > 0$. Suppose now that $\dot\lambda = -\frac{\varepsilon}{4\pi}\sqrt{\lambda(t)}$, with $\lambda(0) = \lambda_0$. Then, for $r^2 \le \sqrt{\lambda(t)}$ we obtain

$$\begin{aligned}
&\frac{\dot\alpha\,(r^2 + \lambda(t))}{(8\pi + \varepsilon)} + \dot\lambda + \frac{\varepsilon\lambda(t)}{2\pi\,(r^2 + \lambda(t))} - 8\lambda(t) \\
&\qquad \ge \frac{\dot\alpha\lambda(t)}{(8\pi + \varepsilon)} - \frac{\varepsilon}{4\pi}\sqrt{\lambda(t)} + \frac{\varepsilon\sqrt{\lambda(t)}}{4\pi} - 8\lambda(t) > 0
\end{aligned}$$

if $\dot\alpha$ is sufficiently large ($\dot\alpha \ge 80\pi$ is enough). On the other hand, if $r^2 \ge \sqrt{\lambda(t)}$ we obtain

$$\begin{aligned}
&\frac{\dot\alpha\,(r^2 + \lambda(t))}{(8\pi + \varepsilon)} + \dot\lambda + \frac{\varepsilon}{2}\frac{\lambda(t)}{\pi\,(r^2 + \lambda(t))} - 8\lambda(t) \\
&\qquad \ge \frac{\dot\alpha\sqrt{\lambda(t)}}{(8\pi + \varepsilon)} - \frac{\varepsilon}{4\pi}\sqrt{\lambda(t)} + 0 - 8\lambda(t) > 0
\end{aligned}$$

assuming that $\lambda(t)$ is sufficiently small, i.e. λ_0 is sufficiently small, and $\dot\alpha$ is sufficiently large (again $\dot\alpha \ge 80\pi$ is enough, if ε is small).
For fixed $\varepsilon > 0$ then choosing λ_0 sufficiently small, we obtain a subsolution for M_1 developing a singularity at a time t, which is of order $\frac{\lambda_0^2}{\varepsilon}$. In particular, $|\alpha(t) - \alpha(0)| \le \frac{80\pi\lambda_0^2}{\varepsilon}$, and $\frac{80\pi\lambda_0^2}{\varepsilon} < \frac{\varepsilon}{2}$ if λ_0 is small. This provides the desired

subsolution for M_1 and implies the formation of a singularity in finite time for u_1. ■

The proof of Theorem 5 shows that blow-up for solutions of (1) is possible, even if the difference of masses for the species u_1, u_2 is arbitrarily small. More precisely, we have $\int_{\mathbb{R}^2} u_{1,0}\, dx = 8\pi+\varepsilon$. Now choosing $u_{2,0} = \frac{16\pi}{(r^2+1)^2}$, we obtain that $\int_{\mathbb{R}^2} u_{2,0}\, dx = 8\pi$. Therefore

$$\frac{\left(\int_{\mathbb{R}^2}(u_{1,0} - u_{2,0})dx\right)^2}{\int_{\mathbb{R}^2}(u_{1,0} + u_{2,0})dx}$$

can be arbitrarily small, and still blow-up happens. Further, our Theorem provides sufficient conditions for blow-up. However, these are very dependent on the specific form of the initial data.

Now we derive some sufficient conditions for blow-up of solutions of system (1), that only depend on the masses of the two species involved. So we prove blow-up for a different class of initial data.

Theorem 6 (Blow-up for u_2) *Consider system (1), where the initial data $u_{1,0}, u_{2,0} \in C_0^\infty(\mathbb{R}^2)$ are radially symmetric, smooth and have compact support. Let θ_1, θ_2 be given as in (3). If*

$$\theta_2 > 8\pi + 2\theta_1 \ ,$$

we have $T_{\max} < \infty$, where $T_{\max}$ is the maximal time of existence of the variable u_2.

Proof. To prove this result we follow some of the techniques given in [5]. In the following arguments we will assume for the moment that all occurring integrals are well defined. At the end of the proof we will check this in detail.

Multiplying the second equation of (1) by $|x|^2$ and integrating the resulting relation over $\mathbb{R}^2$ we obtain

$$\partial_t \int_{\mathbb{R}^2} u_2 \left|x\right|^2 dx = \int_{\mathbb{R}^2} \left|x\right|^2 \Delta u_2 dx + \int_{\mathbb{R}^2} \left|x\right|^2 \nabla \cdot (u_2 \nabla \psi) dx. \qquad (12)$$

From Green's identity we get

$$
\begin{aligned}
\partial_t \int_{\mathbb{R}^2} u_2 |x|^2 \, dx &= \int_{\mathbb{R}^2} \left(\Delta |x|^2\right) u_2 dx - \int_{\partial\mathbb{R}^2} u_2 \nabla |x|^2 \cdot dS \\
&\quad + \int_{\mathbb{R}^2} |x|^2 \, \nabla \cdot (u_2 \nabla \psi) dx \\
&\leq 4 \int_{\mathbb{R}^2} u_2 dx - \int_{\mathbb{R}^2} \nabla |x|^2 \cdot (u_2 \nabla \psi) dx \\
&= 4 \int_{\mathbb{R}^2} u_2 dx - 2 \int_{\mathbb{R}^2} u_2 (x \cdot \nabla \psi) dx.
\end{aligned}
$$

Using $\frac{\partial \psi}{\partial r} = \frac{M_2 - M_1}{2\pi r}$ and the identity $x \cdot \nabla \psi = r \frac{\partial \psi}{\partial r}$ we get

$$
\begin{aligned}
\int_{\mathbb{R}^2} u_2 (x \cdot \nabla \psi) dx &= 2\pi \int_0^\infty u_2 r \frac{\partial \psi}{\partial r} r \, dr \\
&= 2\pi \int_0^\infty u_2 \left(\frac{M_2 - M_1}{2\pi} \right) r \, dr \\
&= - \int_0^\infty M_1 u_2 r \, dr + \frac{1}{2\pi} \int_0^\infty M_2 \frac{\partial M_2}{\partial r} dr \\
&\geq -\theta_1 \int_0^\infty u_2 r \, dr + \frac{1}{4\pi} \int_0^\infty \frac{\partial M_2^2}{\partial r} dr \\
&= -\frac{1}{2\pi} \theta_1 \theta_2 + \frac{1}{4\pi} \theta_2^2.
\end{aligned}
$$

Let $m(t) = \int_{\mathbb{R}^2} u_2 |x|^2 \, dx$. From (12) it follows that

$$
\frac{d}{dt} m(t) \leq 4\theta_2 - 2 \left(-\frac{1}{2\pi} \theta_1 \theta_2 + \frac{1}{4\pi} \theta_2^2 \right) = \frac{1}{2\pi} \theta_2 (8\pi + 2\theta_1 - \theta_2) \; .
$$

Since $\theta_2 > 8\pi + 2\theta_1$, we obtain

$$
0 \leq m(t) < m(0) + \frac{1}{2\pi} \theta_2 (8\pi + 2\theta_1 - \theta_2) t \; .
$$

Thus there exists a $T_0 \in (0, \infty)$ such that

$$
m(t) \to 0 \quad \text{as } t \to T_0.
$$

Therefore $T_{\max} \leq T_0 < \infty$.

It only remains to prove, that u_1, u_2 and their derivatives decay sufficiently fast. Since we have assumed compact support for the initial data this follows easily. Let

$$\psi_1 = M_1 - \int_{\mathbb{R}^2} u_{1,0}\, dx \quad , \quad \psi_2 = M_2 - \int_{\mathbb{R}^2} u_{2,0}\, dx \ .$$

Then

$$\partial_t \psi_1 = r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \psi_1}{\partial r}\right) - \frac{M_1 - M_2}{r}\frac{\partial \psi_1}{\partial r} \ , \tag{13}$$

$$\partial_t \psi_2 = r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \psi_2}{\partial r}\right) + \frac{M_1 - M_2}{r}\frac{\partial \psi_2}{\partial r} \ . \tag{14}$$

So, due to the boundedness of $M_1,\ M_2$ we obtain that

$$\partial_t \psi_1 \le \frac{\partial^2 \psi_1}{\partial r^2} + C\frac{\partial \psi_1}{\partial r} \quad \text{and}$$
$$\partial_t \psi_2 \le \frac{\partial^2 \psi_2}{\partial r^2} + C\frac{\partial \psi_2}{\partial r} \quad \text{for} \quad r \ge 1 \ .$$

By comparison with the fundamental solution of the heat equation with a constant convective term it follows that

$$|\psi_1| + |\psi_2| \le Ce^{-\alpha(T)r^2} \quad \text{for} \quad r \ge 1 \ , \ \ 0 \le t \le T \ , \quad \text{and some} \quad \alpha(t) > 0 \ .$$

Using regularity estimates for (13), (14) in the set $\{L \le r \le L+1 \ , \ 0 \le t \le T\}$ we then obtain

$$\left|\frac{\partial^\ell \psi_1}{\partial r^\ell}\right| + \left|\frac{\partial^\ell \psi_2}{\partial r^\ell}\right| \le C_l e^{-a(T)r^2} \quad \text{for} \quad r \ge 2 \ , \ \ 0 \le t \le T \ ,$$

where $a(T)$ in general is different from $\alpha(T)$ for any $\ell = 1, 2, \ldots$. Thus all the integrals in (12) and the formulas which follow, are convergent. ■

By symmetry we also get the following result

Theorem 7 *Let the initial data $u_{1,0}, u_{2,0} \in C_0^\infty(\mathbb{R}^2)$ of our system (1) be radially symmetric, smooth and have compact support. Let*

$$\theta_1 > 8\pi + 2\theta_2 \tag{15}$$

then $T_{\max} < \infty$, where $T_{\max}$ is maximal time of existence for the variable u_1.

As mentioned already, Theorem 5 shows, that it is possible to obtain blow-up for (1) even if the difference between the masses $|\theta_1 - \theta_2|$ is very small, as long as one of the masses is supercritical and the corresponding initial density is suitably concentrated. The conditions for blow-up in Theorem 6 require stronger assumptions on the differences of the masses than Theorem 5, but do not depend on the given densities.

Now we can compare the derived conditions with the ones in the paper by Kurokiba and Ogawa (cf. [3]). From Theorems 1 and 6 we conclude that the assumption

$$\frac{(\theta_1 - \theta_2)^2}{\theta_1 + \theta_2} < 8\pi \tag{16}$$

as suggested in [3] is not sufficient to have boundedness of the variables. To see this, it is enough to find θ_1 and θ_2 such that

$$(i) \quad \frac{(\theta_1 - \theta_2)^2}{\theta_1 + \theta_2} < 8\pi \quad \text{and} \quad (ii) \quad \theta_1 > 8\pi + 2\theta_2 \; .$$

The first inequality requires $\theta_2 \in (\theta_1 + 4\pi - 4\sqrt{\pi^2 + \pi\theta_1}, \theta_1 + 4\pi + 4\sqrt{\pi^2 + \pi\theta_1})$. This region intersects with the region resulting from (ii), namely $\theta_2 < \frac{\theta_1}{2} - 4\pi$, since

$$\theta_1 + 4\pi - 4\sqrt{\pi^2 + \pi\theta_1} < \frac{\theta_1}{2} - 4\pi \Leftrightarrow 8\pi + \frac{\theta_1}{2} < 4\sqrt{\pi^2 + \pi\theta_1}$$
$$\Leftrightarrow \quad 64\pi^2 + 8\pi\theta_1 + \frac{\theta_1^2}{4} < 16(\pi^2 + \pi\theta_1) \Leftrightarrow 48\pi^2 - 8\pi\theta_1 + \frac{\theta_1^2}{4} < 0 \; .$$

This inequality is fulfilled for $\theta_1 \in (8\pi, 24\pi)$. Therefore, the conditions (15) and (16) can be satisfied at the same time. Thus blow-up for u_1 happens.

4 Conclusion

For the drift-diffusion model considered in this paper it was proved, that blow-up for one variable and boundedness for the other one can be obtained for certain parameter regimes. This is different in case one deals with two chemotactic species, both producing the attractive chemical, and both being attracted to it. In this case, if finite time blow-up happens for one species, the same is true for the other species at the same time, cf. [1].

In this paper it was shown that blow-up for the drift-diffusion model is also possible for a range of parameters, which was not considered in [3]. More precisely it was proved, that solutions can blow-up, even if the difference between the masses of the two involved species is very small. For this to hold, one of the masses has to be supercritical and the corresponding initial density has to be suitably concentrated. On the other hand, with stronger assumptions on the differences of the masses, a blow-up result without direct dependence on the masses themselves was be obtained.

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