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# Geometry of phase plane and radial solutions for nonlinear elliptic equations with extremal operators

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ABSTRACT

We study the existence of positive radially symmetric solutions to a class of nonlinear elliptic problems involving extremal operators and nonlinearity of exponential or polynomial type. According to the values of a parameter, we describe situations where the equation has one, finitely many, infinitely many or no solutions, by using the geometry structure of phase plane.



## 1. Introduction and main results

In this paper we study the existence of positive solutions to some fully nonlinear elliptic equations of the form

$$\begin{cases} \mathcal{M}_E(D^2u) + \mu g(u) = 0 & \text{in } B, \\ u = 0 & \text{on } \partial B, \end{cases} \tag{1.1}$$

where $B$ is the unit ball in $\mathbb{R}^N$, $\mathcal{M}_E$ is an extremal operator, $\mu > 0$ is a parameter and the nonlinearity $g$ may be of exponential type like $g(u) = e^u$, or of polynomial type like $g(u) = (1+u)^p$, $p > 1$.

The extremal operator $\mathcal{M}_E$ is of Pucci's type, which means that it is the supremum or the infimum of a class of autonomous linear second order elliptic operators. More precisely, let $0 < \lambda \leqslant \Lambda$ and $E \subset [\lambda, \Lambda]^N$ be a symmetric set, that is, a set satisfying $a = (a_1, a_2, \ldots, a_N) \in E$ if and only if $a_\pi = (a_{\pi(1)}, a_{\pi(2)}, \ldots, a_{\pi(N)}) \in E$, for all permutation $\pi$. Then we define the maximal operator

$$\mathcal{M}_E(D^2u) = \sup_{a \in E} \sum_{i=1}^{N} a_i e_i \tag{1.2}$$

where $e_i = e_i(D^2u)$, $i = 1, \ldots, N$, are the eigenvalues of the Hessian matrix $D^2u$. The extremal operator $\mathcal{M}_E$ may be alternatively defined by considering a set of symmetric matrices $\mathcal{A}$, which is invariant under orthogonal transformations. The eigenvalues of these matrices belong to a certain set $E$ as above, and we may define the operator $\mathcal{M}_E$ as

$$\mathcal{M}_E(D^2u) = \sup_{A \in \mathcal{A}} tr(AD^2u). \tag{1.3}$$

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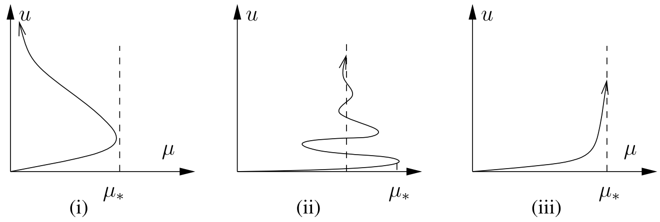


**Fig. 1.1.** Bifurcation of $u-\mu$.

These operators are also called Hamilton–Jacobi–Bellman operator. If $0<\lambda<\Lambda$, defining the set $E$ as $E=[\lambda,\Lambda]^N$ we recover the Pucci's operators as defined in Caffarelli and Cabré [3]. By considering another set $E$ we recover the operators originally defined by Pucci in [13] and [14], see also the recent work by the first two authors [11]. It is well known that Pucci's operators play a crucial role in the study of fully nonlinear equations, for more details we refer the interested reader to the book by Caffarelli and Cabré [3].

The main goal of this paper is to describe the set of positive radially symmetric solutions to Eq. (1.1) in terms of the various parameters and the geometry of the set $E$.

We notice that the fully nonlinear problems that we are considering come as the stationary version of nonlinear diffusion models, where the diffusion coefficient depends on the solution. This is motivated by the study of the classical semi-linear elliptic equation with constant diffusion coefficients as in

$$\begin{cases} \Delta u+\mu g(u)=0 & \text{in } \Omega, \\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where $\mu>0$ is a parameter, the nonlinearity $g$ is given by $g(u)=e^u$ and $\Omega$ is a smooth domain of $\mathbb{R}^N$, $N\geqslant 2$. The study of positive solutions to this kind of semi-linear elliptic equations was initiated by Emden, followed by Fowler [8], and Chandrasekar [4]. Later, Gelfand [9] investigated this problem from the theory of thermal ignition of chemically active mixture of gases, and Crandall and Rabinowitz [6,7] established the bifurcation theory and applied it to obtain some new results on positive solutions of partial differential equations to these models with $g(u)=(1+u)^p$, $p>1$. At the same time, Joseph and Lundgren [12] gave very detailed results of the radial case by the phase plane analysis method for both exponential and polynomial-type nonlinearities.

The main general result of [12] concerning Eq. (1.4) is the existence of a finite positive number $\mu_*$ such that problem (1.4) has positive solutions if $\mu\in(0,\mu_*)$ and, contrarily, it has no positive solution when $\mu>\mu_*$. When the function $g$ is given much more detailed results are obtained in [12].

In [12], in the first place the case $g(u)=e^u$ is considered:

(a) If $N\leqslant 2$ and $0<\mu<\mu_*$, problem (1.4) has exactly two positive solutions, see Fig. 1.1(i). (b) If $2<N<10$ and $\mu$ is close to and not equal to $2(N-2)$, problem (1.4) has a large finite number of positive solutions. In addition, if $\mu=2(N-2)$, there exist infinitely many positive solutions to problem (1.4), see Fig. 1.1(ii). (c) If $N\geqslant 10$ and $\mu<2(N-2)$, then there is only one positive solution, see Fig. 1.1(iii).

In the second place the case $g(u)=(1+u)^p$, $p>1$, is considered. We start by recalling the Sobolev and the Joseph–Lundgren critical exponents

$$p^*(N)=\frac{N+2}{N-2} \quad \text{and} \quad p^{**}(N):=\frac{N-2\sqrt{N-1}}{N-4-2\sqrt{N-1}}. \tag{1.5}$$

Then the results in [12] are as follows, where we simply write $p^*$ and $p^{**}$ since no confusion arises at this point:

(a) If $1<p\leqslant p^*$ and $0<\mu<\mu_*$, then problem (1.4) has exactly two positive solutions. (b) If $p^*<p<p^{**}$ and $\mu$ is close to and not equal to

$$\mu_p(N):=\tau(N-2-\tau), \tag{1.6}$$

$\tau:=2/(p-1)$, then problem (1.4) has a large finite number of positive solutions. In addition, if $\mu=\mu_p$, then there exist infinitely many positive solutions to problem (1.4). (c) If $p\geqslant p^{**}$ and $\mu<\mu_p$, then there exists only one positive solution of problem (1.4). Moreover, $\mu_*=\mu_p$.

We observe that the Joseph–Lundgren exponent is well defined and positive if $N>10$. When $2<N\leqslant 10$ we write $p^{**}=\infty$ and then case (c) is not present. Let us finally mention that additional properties are established for the minimal branch of solution and the extremal solution for the semi-linear elliptic equations with general nonlinearities, see for example [1] and the reference therein.

In this article we extend the results in [12] when the extremal operator $\mathcal{M}_E$ is considered as the diffusion term of Eq. (1.4), instead of the Laplacian. In doing so, we disclose some new phenomena for certain values of the parameters, described by some dimension like numbers which play the role of $N$.

One general observation regarding positive solutions of Eq. (1.1) is that all of them are radially symmetric, as it was proved by Da Lio and Sirakov [2], using the classical moving planes technique. This allows to reduce problem (1.1) to the study of an ordinary differential equation of the form

$$u''(r) + \frac{S(\theta)(N-1)}{\theta}\frac{u'(r)}{r} + \frac{\mu g(u(r))}{\theta} = 0, \quad u > 0 \text{ in } (0,1), \tag{1.7}$$

$$u'(0) = 0, \qquad u(1) = 0, \tag{1.8}$$

where $S(\theta)$ and $\theta$ are functions depending on the geometry of the set $E$ through $u'$ and $u''$. In the description and in the proof of our results the properties of the functions $S$ and $\theta$, which depend on the nature of set $E$, are crucial. In Section 2 we state **(D)** and **(E)**, our precise global assumption on the set $E$, that imply differentiability properties of the ingredients of Eqs. (1.7)–(1.8).

Now we state our first main results when the nonlinearity is of exponential type, that is, $g(u) = e^u$. Here we consider numbers $\theta_0$, $\theta_\infty$ and $\theta_Q$ associated to the operator $\mathcal{M}_E$ and the equation, see (2.7) and (2.8), and the corresponding dimension like numbers $N_0 = \tilde{N}(\theta_0)$, $N_\infty = \tilde{N}(\theta_\infty)$ and $N_Q = \tilde{N}(\theta_Q)$, where

$$\tilde{N}(\theta) = \frac{S(\theta)(N-1)}{\theta} + 1. \tag{1.9}$$

We mention that, as observed in the end of Section 2, if $N_\infty = 2$ then $N_Q = 2$.

**Theorem 1.1.** *Assume the set $E$ satisfies assumption* **(D)** *and* **(E)** *and that $g(u) = e^u$. Then there exists a constant $\mu_*$ depending on $N$ and the domain $E$ such that problem* (1.1) *admits at least one positive solution for $\mu < \mu_*$ and it has no solution for every $\mu > \mu_*$.*
*Moreover,*

(a) *If $N_\infty \leqslant 2$ and $0 < \mu \leqslant \mu_*$, then problem* (1.1) *has at least two positive solutions. If further $N_0 > 2$ then $0 < \mu < \mu_*$ implies problem* (1.1) *has two positive solutions and a unique solution for $\mu = \mu_*$.*
(b) *If $N_\infty > 2$, $N_Q < 10$ and $\mu$ is close to and not equal to $w_Q := 2\theta_Q(N_Q - 2)$, then problem* (1.1) *has a large finite number of positive solutions. In addition, if $\mu = w_Q$, then there exist infinitely many positive solutions to problem* (1.1)*.*
(c) *If $N_\infty > 2$ and $N_Q \geqslant 10$, then $\mu_* = w_Q$, and there exists at least one positive solution for $0 < \mu < \mu_*$. If further $N_0 > 2$ then the solution is unique.*

**Remark.** In Section 3 we prove a more general result than Theorem 1.1, replacing condition **(E)** by a local condition **(Ee)**, which weaken the differentiability properties of the coefficients in Eqs. (1.7)–(1.8). As we mentioned before, if $E = [\lambda, \Lambda]^N$ then $\mathcal{M}_E$ is Pucci's operator $M^+_{\lambda,\Lambda}$. This operator does not satisfy hypothesis **(E)**, but it satisfies the local hypothesis **(Ee)**, so we get all conclusions of Theorem 1.1 for this operator. In this case $N_\infty = N_Q = \lambda(N-1)/\Lambda + 1$.

Now we state our main result for polynomial case, that is, for $g(u) = (1+u)^p$, $p > 1$. First we recall the corresponding critical Sobolev exponent for extremal operators, as studied recently by Felmer and Quaas in [11]. For the nonlinear problem involving the maximal operator

$$\mathcal{M}_E\big(D^2 u\big) + u^p = 0 \quad \text{in } \mathbb{R}^N, \tag{1.10}$$

there is a critical exponent $p^*_E > 1$ satisfying $1 < p^*_E < p_\infty := (N_\infty + 2)/(N_\infty - 2)$. The last inequality holds in case the set $E$ is nontrivial, as stated in hypothesis **(E)**, otherwise the equality $p^*_E = p_\infty$ holds. The exponent $p^*_E$ plays the role of the Sobolev exponent $p^* = (N+2)/(N-2)$ for the Laplacian, in the sense that there is no positive radial solution of (1.10) if $1 < p < p^*_E$ and if $p \geqslant p^*_E$, then there is a unique positive radial solution of (1.10). This result is a generalization of the results in [10], which only covers the case of Pucci's operators. Here we consider a value $\theta_S$ as given in (4.7) and we define the dimension like number $N_S$ as in (1.9), with $\theta = \theta_S$. We still need some preliminaries, we say problem (1.1) has the property **(PI)** if:

**(PI)** *There is a closed set $\mathcal{I} \subset \mathbb{R}_+$ such that problem* (1.1) *has infinitely many solutions provided that $\mu \in \mathcal{I}$; there exist a large and finite number positive solutions if $\mu$ is close to $\mathcal{I}$ but does not belong to $\mathcal{I}$.*

Moreover, if $\mathcal{I}$ is a singleton then $\mathcal{I} = \{\mu_p(N_S)\}$, where $\mu_p$ was defined in (1.6). We say that problem (1.1) has the property **(U)** if:

**(U)** *Whenever $\mu > \mu_p(N_S)$ there is no solution and $0 < \mu < \mu_p(N_S)$, there exists at least one positive solution of problem* (1.1)*. If further $p > N_0/(N_0 - 2)$ then the solution is unique.*

**Theorem 1.2.** *Assume the set $E$ satisfies assumptions* **(D)** *and* **(E)** *and that $g(u) = (1+u)^p$. Further assume that $N_\infty > 2$. Then there exists a constant $\mu_*$ such that problem* (1.1) *admits at least one positive solution for $\mu < \mu_*$, but it has no solution for every $\mu > \mu_*$. Moreover*:

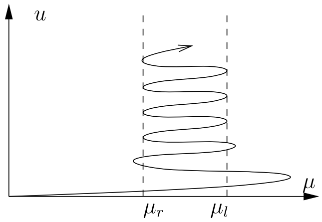


**Fig. 1.2.** Bifurcation of $u - \mu$ with interval $(\mu_r, \mu_l)$.

(a) *If $1 < p \leqslant p_E^*$ and $0 < \mu \leqslant \mu_*$, then problem* (1.1) *has at least two positive solutions. If we further assume that $N_0/(N_0 - 2) < p$ then $0 < \mu < \mu_*$ implies problem* (1.1) *has two positive solutions and a unique solution for $\mu = \mu_*$.*
(b) *If $p_E^* < p$, then problem* (1.1) *satisfies* **(PI)** *or* **(U)**.
*In addition we have*:
  (i) *If $p_\infty < p$, then* **(PI)** *does not occur with $\mathcal{I}$ different from a singleton. See Fig.* 1.2.
  (ii) *If $N_S \leqslant 10$ or if $N_S > 10$ and $p \leqslant p_S := p^{**}(N_S)$, then* **(U)** *does not occur.*
  (iii) *If $N_S > 10$ and $p_S < p$, then* **(PI)** *does not occur with $\mathcal{I}$ a singleton.*

**Remark.** We have some further properties:

(a) If $N_S > 10$ and $p > \max\{p_S, p_\infty\}$, then it follows that problem (1.1) satisfies **(U)**.
(b) If $p_E^* < p < (N_S + 2)/(N_S - 2)$, then **(PI)** holds with a nontrivial interval $\mathcal{I}$. This situation is new, when compared with the case of the Laplacian. This multiplicity phenomena is characteristic of the nontrivial extremal operators.

In Theorem 1.2 there are various situations that we cannot decide in general. When dealing with a particular operator we may obtain a more precise statement, as in the case of the Pucci extremal operator.

**Corollary 1.3.** *If $E = [\lambda, \Lambda]^N$, then $N_\infty = N_S = (\lambda(N-1))/\Lambda + 1$ and*

(a) *for $p_E^* < p < p_\infty$, there are two constants $\mu_r$ and $\mu_l$ such that problem* (1.1) *has the property* **(PI)** *with $\mathcal{I} = [\mu_r, \mu_l]$.*
(b) *If $N_\infty > 10$ then for $p_\infty < p < p_+^{**}$, problem* (1.1) *has the property* **(PI)** *with a singleton. Here and in what follows $p_+^{**} := p^{**}(N_\infty)$.*
(c) *If $p \geqslant p_+^{**}$ then property* **(U)** *holds.*

**Remark.** The Pucci operator does not satisfy hypothesis **(E)**, but it satisfies the local version **(Ep)**, which is sufficient to obtain the results of Theorem 1.2, and as a consequence the results in Corollary 1.3.

For the proof of Theorems 1.1 and 1.2 we use the method of [12], that is, we transform each equation to an equivalent two-dimensional dynamical system. By analyzing the phase plane for different values of the parameters we obtain the results. Our approach is flexible enough so that analogous results of minimal operators can be obtained, but we leave to the interested reader to obtain the precise statements.

We organize this paper as follows. In Section 2 we discuss some preliminaries about extremal operators. In Section 3 we study the problem with the exponential-type nonlinearity and we prove Theorem 1.1. In Section 4 we discuss the problem with a polynomial-type nonlinearity and we prove Theorem 1.2.

## 2. Extremal operators

In this section we briefly review some basic properties of extremal operators for radially symmetric functions. Computing $D^2u(x)$, when $u(x) = u(r)$, $r = |x|$, we see that

$$D^2u(x) = \frac{u'(r)}{r} Id + \left[\frac{u''(r)}{r^2} - \frac{u'(r)}{r^3}\right] X,$$

where the prime denotes differentiation with respect to $r$, $Id$ is the $N \times N$ identity matrix and $X$ is the matrix whose entries are $x_i x_j$. Observing further that

$$D^2u(x)\frac{x}{r} = u''(r)\frac{x}{r} \quad \text{and} \quad D^2u(x)y = \frac{u'(r)}{r} y,$$

if $y$ satisfies $x \cdot y = 0$, we find that the eigenvalues of the Hessian matrix $D^2u(x)$ are $u''(r)$, which is simple, and $u'(r)/r$, which has multiplicity $N-1$. Consequently, according to definition (1.2), we see that the operator $\mathcal{M}_E$ acting on $C^2$ radially

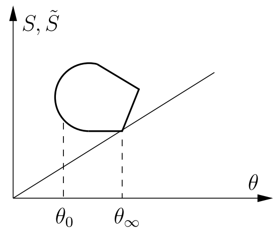


**Fig. 2.1.** Domain $E$ for $\mathcal{M}_E$ and $\mathcal{M}_E^-$.

symmetric functions is given by

$$\mathcal{M}_E\big(D^2u\big) = \sup_{(a_1,a_2)\in D} a_1u'' + \frac{a_2(N-1)u'}{r}, \tag{2.1}$$

where $D = D(E) = \{(a_1, \frac{1}{N-1}\sum_{i=2}^N a_i) \in \mathbb{R}^2\colon a \in E\}$.

The study of radially symmetric solutions of the nonlinear equations (1.1) requires some assumptions on the set $E$, which may be stated in terms of the set $D$. We assume:

**(D)** *The set $D = D(E)$ is compact, convex and its projection onto the $y$-axis is not a point.*
**(E)** *The set $\partial_- D$ does not contain any segment of a straight line. Here $\partial_- D$ denotes the lower portion of $\partial D$.*

Under the assumption **(D)** we may describe $\partial D$ by means of functions. Let $0 < \theta_- < \theta_+$ be defined as $\theta_- = \min\{\theta\colon (\theta,\cdot) \in D\}$ and $\theta_+ = \max\{\theta\colon (\theta,\cdot) \in D\}$ and define the functions $S, \widetilde{S} : [\theta_-, \theta_+] \to \mathbb{R}_+$ as

$$S(\theta) = \min\big\{y\colon (\theta, y) \in D\big\} \quad \text{and} \quad \widetilde{S}(\theta) = \max\big\{y : (\theta, y) \in D\big\}.$$

See Fig. 2.1. With these definitions we see that $S$ is convex, $\widetilde{S}$ is concave,

$$D = \big\{(\theta, y)\colon \theta \in [\theta_-, \theta_+],\ S(\theta) \leqslant y \leqslant \widetilde{S}(\theta)\big\}$$

and the graph of $S$ coincides with $\partial_- D$. In the study of positive decreasing solutions to problem (1.1) with the maximal operator, we only need to look at the side of $D$ which is described by $S$. Because $S$ is convex, it has one-sided derivatives $S'_-(\theta)$ and $S'_+(\theta)$ and it is locally Lipschitz continuous in $(\theta_-, \theta_+)$. The sub-differential of $S$ is then defined as $\partial S(\theta) = [S'_-(\theta), S'_+(\theta)]$, for $\theta \in (\theta_-, \theta_+)$, and we see that $S$ is differentiable at $\theta$ if and only if $\partial S(\theta)$ is a singleton.

The cases $\theta = \theta_-$ and $\theta = \theta_+$ are special. At $\theta_-$ we have two possibilities, either $S'_+(\theta_-)$ exists, and then we define $\partial S(\theta_-) = (-\infty, S'_+(\theta_-)]$, or

$$\lim_{\gamma\to 0^+} \frac{S(\theta_- + \gamma) - S(\theta_-)}{\gamma} = -\infty.$$

An analogous situation occurs at $\theta_+$. We observe that with these definitions, for every $Q < 0$ there is at least one solution $\theta \in [\theta_-, \theta_+]$ of the equation

$$\partial S(\theta)\theta - S(\theta) \ni Q. \tag{2.2}$$

Now we may write the equation of problem (1.1) in terms of the function $S$, solving the maximization problem. By definition of $\mathcal{M}_E$ and the description of $D$ in terms of the function $S$, while $u'(r) < 0$, we have that

$$\mathcal{M}_E\big(D^2u(r)\big) = \theta u''(r) + \frac{S(\theta)(N-1)u'(r)}{r}, \tag{2.3}$$

where $\theta \in [\theta_-, \theta_+]$ is characterized by an optimality condition

$$\frac{-u''}{(N-1)(u'/r)} \in \partial S(\theta). \tag{2.4}$$

Since $\mathcal{M}_E(D^2u(r)) = \mu g(u)$, the optimality condition may also be written as

$$\partial S(\theta)\theta - S(\theta) \ni \frac{\mu g(u)}{(N-1)(u'/r)}. \tag{2.5}$$

We observe that under the hypothesis **(D)**, this equation may have more than one solution. In this case $\theta$ may be chosen as any solution of (2.5), however for our future analysis it is convenient to have $\theta$ as a function of $r$, so we define

$$\theta(r) = \Theta\left(\frac{\mu r g(u(r))}{(N-1)u'(r)}\right), \tag{2.6}$$

where $\Theta : (-\infty, 0) \to \mathbb{R}$ is such that $\Theta(Q)$ is a solution of (2.5) with right-hand side $Q$. We observe that as $g(u) > 0$,

$$\partial S(\theta) \subset \big(-\infty, S(\theta)/\theta\big].$$

Because $S$ is convex, $\partial S(\theta)\theta - S(\theta)$ and $\Theta$ are strictly monotone increasing. We notice that under assumption **(E)** Eq. (2.5) possesses exactly one solution.

It is convenient to isolate $u''$ in (1.7) and write

$$u'' = \tilde{g}\left(u, \frac{u'}{r}\right)$$

where $\tilde{g}$ is defined as the function

$$\tilde{g}(u, w) = -\frac{S(\theta)}{\theta}(N-1)w - \frac{\mu g(u)}{\theta},$$

with $\theta = \Theta(\mu g(u)/(N-1)w)$ and $(u, w) \in \Omega := (-\infty, 0] \times (0, \infty)$. The main properties of $\tilde{g}$ were studied in [11] and we recall them briefly. Under the general hypothesis **(D)** the function $\tilde{g}$ is Lipschitz continuous and it is differentiable away from the curves of the form

$$Q = \frac{\mu g(u)}{(N-1)w}$$

where $Q$ is such that the graph of $S$ coincides with the straight line $s = d\theta + Q$ in some interval with non-empty interior. If we further assume that **(E)** holds, then the graph of $S$ does not have flat segments and thus $\tilde{g}$ is differentiable everywhere in $\Omega$. See Lemma 2.2 and Proposition 2.3 in [11].

Associated to the behavior of a solution $u$ of (1.7) at the origin $r = 0$ and at infinity $r = \infty$, we have special values of $\theta$, as we discuss in further detail in Sections 3 and 4. At this point we just define them as follows: $\theta_0$ and $\theta_\infty$ are such that

$$-\frac{1}{N-1} \in \partial S(\theta_0) \quad \text{and} \quad \frac{S(\theta_\infty)}{\theta_\infty} \in \partial S(\theta_\infty). \tag{2.7}$$

If the first (second) equation has more than one solution, then we choose the smallest (largest) solution. Given a solution of Eq. (1.7) we write

$$N_0 = \tilde{N}(\theta_0), \qquad N_\infty = \tilde{N}(\theta_\infty),$$

according to definition (1.9) and abusing notation we write $\tilde{N}(r) = \tilde{N}(\theta(r))$. Then we can prove, following the proof of Lemma 3.5 of [11] with slight changes, that

$$N_0 \geqslant \tilde{N}(r) \geqslant N_\infty,$$

for all $r$ such that $u(r) > 0$. In general, the functions $\theta(r)$ and $\tilde{N}(r)$ are measurable functions, having discontinuities and both $\theta(r)$ and $\tilde{N}(r)$ are bounded and bounded away from 0, see Fig. 2.1.

Another important value of $\theta$ is the one associated to the radially symmetric function $u = \ln r$. In this case, we define $\theta_Q$ as a solution of the optimality condition (2.4)

$$\frac{1}{N-1} \in \partial S(\theta_Q) \tag{2.8}$$

and we write $N_Q = \tilde{N}(\theta_Q)$.

**Remark.** By looking at the definition of $\theta_\infty$ and $\theta_Q$ in (2.7) and (2.8), respectively, we see that whenever $N_\infty = 2$ necessarily $N_Q = 2$.

## 3. Exponential nonlinearity

We devote this section to prove Theorem 1.1. For this purpose we transform our problem (1.7) into an autonomous dynamical system in the phase plane, by using a classical change of variables. Then we obtain the extremal parameters and extremal solutions by studying the precise structure of the solutions to this system through a phase plane analysis.

Let us start by transforming (1.7), for $g(u) = e^u$, through the following change of variables:

$$t = \ln r, \qquad v(t) = -\frac{du}{dt}, \qquad w(t) = \mu e^{2t} e^u. \tag{3.1}$$

Letting $\dot{v}$ and $\dot{w}$ denote differentiation with respect to $t$, a direct computation gives

$$\dot{w} = 2\mu e^{2t} e^u + \mu e^{2t} e^u \dot{u} = w(2 - v).$$

Then, by multiplying (1.7) by $r^2 = e^{2t}$ we obtain that

$$e^{2t}u''(r) + \frac{S(\theta)(N-1)}{\theta}e^t u'(r) + \frac{\mu e^{2t}e^u}{\theta} = 0.$$

By using the above equation and

$$\dot v = -\frac{d}{dt}\big(e^t u'(r)\big) = -e^{2t}u''(r) - e^t u'(r) = -e^{2t}u''(r) + v,$$

we get

$$\dot v = (2-\tilde N)v + \frac{w}{\theta},$$

where we have used $\tilde N$ as defined in (1.9) and where $\theta$, as obtained in (2.6), is given by

$$\theta = \Theta\left(\frac{-w}{(N-1)v}\right).$$

If $u$ satisfies boundary conditions (1.8) we observe that $w > 0$ and $w(-\infty) = \lim_{t\to-\infty}\mu e^{2t}e^{u(e^t)} = 0$ since $u(1) = 0$ and that $v(-\infty) = \lim_{t\to-\infty} -u'(e^t)e^t = 0$ because of $u'(0) = 0$. Hence, problem (1.7)–(1.8) reduces to a first order autonomous system in the plane $(w, v)$

$$\begin{cases} \dot w = w(2-v), \\ \dot v = (2-\tilde N)v + \frac{w}{\theta}, \end{cases} \tag{3.2}$$

with the additional conditions

$$w(-\infty) = 0, \qquad v(-\infty) = 0. \tag{3.3}$$

Here the optimality equation (2.4) is given by

$$\frac{v - \dot v}{(N-1)v} \in \partial S(\theta). \tag{3.4}$$

Assuming that $N_\infty > 2$, we have $\tilde N(\theta) > 2$ for all $\theta \in [\theta_0, \theta_\infty]$, as we discussed before. Thus we see that our system (3.2) possesses a critical point $(w_Q, v_Q)$, which is given by $v_Q = 2$ and

$$w_Q = 2S(\theta_Q)(N-1) - 2\theta_Q, \tag{3.5}$$

where $\theta_Q$ in given by (2.8). Further we see that this is the only critical point of the system in the first quadrant.

Now we can state our local assumption on the set $E$.

**(Ee)** *Eq.* (2.8) *has a unique solution.*

We see then, that assuming the set $E$ satisfies **(Ee)**, as a consequence of Proposition 2.3 in [11], the field in (3.2) is differentiable at the critical point $(w_Q, v_Q)$.

Now we state a theorem, more general than Theorem 1.1.

**Theorem 3.1.** *The conclusions of Theorem* 1.1 *remain valid, if we assume the same hypotheses, except for* **(E)** *which is replaced by* **(Ee)**.

**Proof.** The beginning of the analysis is the existence and uniqueness of a solution to the initial value problem

$$\begin{cases} u''(r) + \dfrac{S(\theta)(N-1)}{\theta}\dfrac{u'(r)}{r} + \dfrac{\mu g(u(r))}{\theta} = 0, \quad r > 0, \\ u'(0) = 0, \quad u(0) = 1. \end{cases} \tag{3.6}$$

As it is done in [11], the existence of a local solution near $r = 0$ follows from the Schauder fixed point theorem. From there, global existence follows from the fact that the system is Lipschitz continuous away from the origin. However, the uniqueness of the initial value problem at the origin is more delicate.

We proceed as in [11], analyzing the phase plane. First, all possible solutions approach the origin $O = (0,0)$, $\theta(t) \to \theta_0$ and $\tilde N(t) \to N_0$ as $t \to -\infty$. Then the nonlinear field approaches a linear one, whose representing matrix is

$$A = \begin{pmatrix} 2 & 0 \\ \frac{1}{\theta_0} & 2 - N_0 \end{pmatrix}.$$

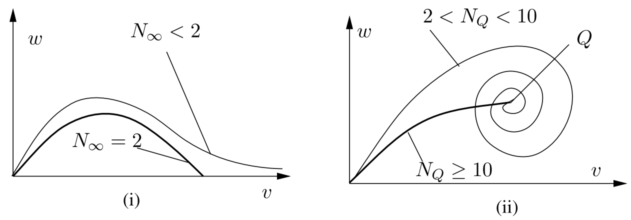


**Fig. 3.1.** Phase plane analysis for $w - v$.

If we assume that $N_0 > 2$, then the linearized system represents a saddle. Moreover, by properly modifying the field, as in Proposition 3.11 in [11], we see that the nonlinear system approaches the linear system well enough to use the Stable–Unstable Manifold Theorem 4.1 in [5]. This implies that there is only one orbit going out of the origin $O$ and that it enters into the first quadrant of the phase plane $w - v$. This in turn implies that (3.6) has a unique solution.

Thus, we have proved that there exists a solution of (3.6) which corresponds to an orbit which starts from the origin $O : (0, 0)$ and enters the first quadrant. When $N_0 > 2$ this solution is unique. The rest of the proof of the theorem relies on a detailed analysis of this orbit of (3.2), (3.3), call basic orbit, depending on the various parameters of the problem. From now on we fix one of these orbits and denote it by $(w, v)$. This orbit remains in the first quadrant permanently. In fact, the first equation in (3.2) says that the $v$-axis is invariant for the flow. On the other hand, while $v > 0$ the maximum and minimum values of $w$ are achieved on the line $v = 2$. Thus, by analyzing the equations for $w$ and $v$, while $v < 2$ the function $w$ remains bounded and then it is bounded for all $t$. For any given point $P = (w(t_0), v(t_0))$ in the orbit, the new orbit $(w(t - t_0), v(t - t_0))$ reaches the point $P$ at time zero, since system (3.2) is autonomous.

Let us denote by $\mu^*$ the supremum of $w$, which we know exists since $w$ is bounded. In case this value is achieved, we assume without loss of generality, that our basic orbit $(w, v)$ is such that $w(0) = \mu^*$. If $0 < \mu < \mu^*$, then the line $w \equiv \mu$ intersects in at least one point with the basic orbit. By time translation we obtain that $w(0)$ arrives at this point, that is, $w(0) = \mu$. We observe that this implies that for the original equation we have a solution $u$ such that $u(e^0) = u(1) = 0$. In general, we see that for each point of the (translated) basic orbit with $w(0) = \mu$ it corresponds a solution $u$ of problem (1.7), with parameter $\mu$, through the change of variables (3.1). This solution is given explicitly as $u(r) = \ln(w(t)e^{-2t}/\mu)$. When $\mu^*$ is achieved, the corresponding solution is called extremal solution and, when the basic orbit is unique, the extremal solution is also unique.

Let us analyze now the other critical point of the dynamical system. We first look at the case $N_\infty < 2$. Here we have $\frac{S(\theta_\infty)}{\theta_\infty} < \frac{1}{N-1}$ so that, for every $\theta \leqslant \theta_\infty$ and $p \in \partial S(\theta)$ we have $p \leqslant \frac{S(\theta_\infty)}{\theta_\infty} < \frac{1}{N-1}$, since $\partial S$ is monotone and then Eq. (2.8) does not have a solution. Then we conclude that $\dot{v}$ never vanishes, since the optimality condition (3.4) at such a point would imply that (2.8) has a solution. See Fig. 3.1(i).

In case $N_\infty = 2$, then $\frac{S(\theta_\infty)}{\theta_\infty} = \frac{1}{N-1}$ and so $\theta_\infty = \theta_Q$. Then there is no critical point in the first quadrant, but all points on the $v$-axis are critical points. We can further prove that $v$ is bounded. See Fig. 3.1(i). We have proved part (a) of the theorem.

Next we assume that $N_\infty > 2$, that implies the existence of the critical point $Q = (w_Q, v_Q)$, with $v_Q = 2$, which is the only one in the first quadrant. When $v > 2$, we consider the second equation in (3.2) and we see that $\dot{v} = 0$ if $(\tilde{N} - 2)v - w/\theta = 0$. If $\dot{v}$ stays positive, since $w$ is decreasing we see that eventually $\dot{v} = 0$. As a conclusion we find that our basic orbit is bounded.

Next, Dulac's criterion tells that there exists no periodic orbit. In fact, suppose that there is a periodic orbit, which surrounds the region $\Gamma$ and consider

$$\int_\Gamma \left( \frac{\partial}{\partial w} \frac{\dot{w}}{w} + \frac{\partial}{\partial v} \frac{\dot{v}}{w} \right) dw\, dv = \int_\Gamma \frac{2 - \tilde{N}}{w} dw\, dv < 0.$$

Here, in the computation of the derivative we used the optimality condition (3.4) and the fact that the system is almost everywhere differentiable. Thus, by the Poincaré–Bendixon theorem, we have that the basic orbit goes to the critical point $Q$. See Fig. 3.1(ii).

Finally we analyze the nature of the critical point $Q$. By our hypothesis **(Ee)** we know that the field in (3.2) is differentiable at $Q$. Then, near $Q$, we may write the system as

$$\dot{z} = Az + o(z),$$

where $z = (w, v)$ and the matrix $A$ is given by

$$A = \begin{pmatrix} 0 & 2\theta_Q(N_Q - 2) \\ \frac{1}{\theta_Q} & 2 - N_Q \end{pmatrix}.$$

It follows that the matrix $A$ has complex eigenvalues when

$$[N_Q - 2][N_Q - 10] < 0.$$

Thus, for $2 < N_Q < 10$, the critical point $Q$ is a spiral sink of the basic orbit. And for $N_Q \geqslant 10$, the critical point $Q$ is a nodal sink. See Theorems 2.1 and 2.2 on page 376 of [5]. From here we complete the proof of part (b) of the theorem. By observing the places on the phase plane $w - v$ where $v$ and $w$ attain their local maximum and minimum values, we can deduce part (c). □

**Remark.** We used hypothesis $N_0 > 2$ to obtain that the initial value problem (3.6) has a unique solution. We do not know if this assumption is really necessary.

If we assume that (3.6) has a unique solution, then the conclusions of Theorem 1.1(a) and (c) (and Theorem 3.1) regarding uniqueness and exact multiplicity are true without constraint on $N_0$.

**Remark.** We notice that under hypothesis **(E)** the function $\tilde{g}$ is differentiable, however without further assumption the function $\Theta(Q)$ given by Eq. (2.6) may not be Lipschitz continuous, and then we are not able to prove uniqueness of (3.6). See operator (3.9) and (3.10) in [11].

**Remark.** In the case of Pucci's operator, in the phase space we find that the region where $u'' = 0$ corresponds to

$$\lambda(N-1)v = w.$$

On the other hand the critical point $Q = (w_Q, v_Q)$ with $v_Q = 2$ satisfies

$$w_Q = 2\lambda(N-1) - 2\Lambda.$$

Thus, the critical point $Q$ is located in the interior of a region where $\tilde{N}$ and $\theta$ are constant and the local assumption **(Ee)** is naturally valid.

## 4. Polynomial type nonlinearity

In the study of our problem with a polynomial type nonlinearity an approach similar to the one in the previous section can be used. We transform the problem to an equivalent one, where the parameter $\mu$ does not appear in the equation but we see it in the initial condition. Then we make a second change of variables transforming the problem to the phase plane, where we can analyze the system to establish the existence results as in the previous case.

We start with our first change of variables. If $u$ is a solution to problem (1.7) with $g(u) = (1+u)^p$, we define $\alpha := u(0)$,

$$\tau = \frac{2}{p-1} \quad \text{and} \quad \varepsilon = \left(\frac{1}{\mu(1+\alpha)^{p-1}}\right)^{1/2},$$

and we consider the change of variables

$$u(r) + 1 = (1+\alpha)v(s), \qquad r = s\varepsilon. \tag{4.1}$$

Then problem (1.7)–(1.8) reduces to

$$\begin{cases} v'' + \dfrac{S(\theta)(N-1)}{\theta}\dfrac{v'}{s} + \dfrac{v^p}{\theta} = 0, \\ v(1/\varepsilon) = (1+\alpha)^{-1}, \quad v'(0) = 0, \quad v(0) = 1. \end{cases} \tag{4.2}$$

In order to use the phase plane analysis we transform (4.2) through a second change of variables

$$x(t) = s^{\tau} v(s), \qquad s = e^t. \tag{4.3}$$

By direct computation, we find that (4.2) is equivalent to the first order system

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -(\tilde{N} - 2\tau - 2)y + \tau(\tilde{N} - 2 - \tau)x - \dfrac{x^p}{\theta}, \end{cases} \tag{4.4}$$

where $\tilde{N}$ is defined by (1.9) and the dot denotes differentiation with respect to $t$. This system is complemented with the initial conditions

$$x(-\infty) = 0, \qquad y(-\infty) = 0. \tag{4.5}$$

Notice that by using the same change of variables the optimality condition becomes

$$\partial S(\theta)\theta - S(\theta) \ni \frac{x^p}{(N-1)(y - \tau x)} \tag{4.6}$$

and we have

$$\theta(t) = \Theta\left(\frac{x^p}{(N-1)(y-\tau x)}\right).$$

The existence and uniqueness theory for Eq. (4.4) with the initial conditions (4.5), or equivalently for (4.2) is well studied in [11]. If we assume that $p > N_0/(N_0 - 2)$ then there is a unique solution as proved in Proposition 3.11 of [11]. We remark that we do not know if this hypothesis is really necessary in order to have uniqueness.

From now on we consider the orbit denoted by $(x(t), y(t))$, which emanates from the origin (in case there is more than one of these orbits we just pick any one). We claim that each point of this trajectory corresponds to one solution of problem (4.2). In fact, given such a point $Q_0 = (x(t_0), y(t_0))$, by translating time we may achieve that the orbit arrives $Q_0$ at time zero. From our changes of variables (4.1) and (4.3), it is easy to recover $u$ out of $x$ as follows

$$x(t) = \left(\frac{r}{\varepsilon}\right)^\tau v\left(\frac{r}{\varepsilon}\right) = \left(\frac{r}{\varepsilon}\right)^\tau \frac{1+u(r)}{1+\alpha} = \mu^{\tau/2}\big(1+u(r)\big)r^\tau.$$

Hence $x(0) = \mu^{1/(p-1)}$ and then $u(1) = 0$. Since the $x(t)$ is bounded, see [10], we may define

$$x_* = \sup\big\{x(t) \bigm| t > 0 \text{ such that } x(s) > 0 \text{ for all } s \leqslant t\big\}$$

and the maximum existence value of $\mu > 0$ is given by

$$\mu_* = x_*^{p-1}.$$

Conversely, given $0 < \mu \leqslant \mu_*$ there is a point $(x, y)$ on the basic orbit so that $x(0) = \mu^{1/(p-1)}$, after proper time translation. This orbit determines a solution to problem (1.7)–(1.8) with $g(u) = (1+u)^p$ by

$$u(r) = \mu^{-1/(p-1)}x(t)e^{-2t/(p-1)} - 1.$$

Summarizing the above facts, we have the following lemma.

**Lemma 4.1.** *Each point on the basic orbit of system* (4.4) *with initial conditions* (4.5) *corresponds to one solution of problem* (1.7)–(1.8) *with* $g(u) = (1+u)^p$.

Letting $\dot{x} = \dot{y} = 0$, we see that the system (4.4) possesses two critical points $O = (0, 0)$ and $S = (x_S, 0)$, where

$$x_S = \big(\mu_p(N_S)\big)^{1/(p-1)}$$

and $\mu_p(\cdot)$ was defined in (1.6). Notice that the optimality condition (2.6) at the critical point $S$ becomes

$$\frac{\tau+1}{(N-1)} = \frac{p+1}{(p-1)(N-1)} \in \partial S(\theta_S). \tag{4.7}$$

Now we can state our local assumption on the set $E$.

**(Ep)** *Eq.* (4.7) *has a unique solution.*

We see then, that assuming the set $E$ satisfies **(Ep)**, as a consequence of Proposition 2.3 in [11], the field in (3.2) is differentiable at the critical point $(w_S, v_S)$.

Let us now analyze the stability of $S$ assuming **(Ep)** holds. This implies that the field in (4.4) is differentiable at $S$ and then, near $S$, we may write the nonlinear system as

$$\dot{z} = Az + o(z),$$

where $z = (w, v)$ and the matrix $A$ is given by

$$A = \begin{pmatrix} 0 & 1 \\ 2(N_S - \tau - 2) & 2\tau + 2 - N_S \end{pmatrix}.$$

Then it is easy to see that the eigenvalues of this matrix are complex if

$$\tau^2 - [N_S - 4]\tau + [N_S - 2]^2/4 - 2(N_S - 2) < 0.$$

If we define the numbers

$$\tau_\pm = \frac{N_S - 4}{2} \pm \sqrt{N_S - 1},$$

we may conclude that whenever $\tau \in (\tau_-, \tau_+)$, then the linearized orbits, and consequently the basic orbit, spiral towards the critical point $S$. See Theorems 2.1 and 2.2 on page 376 of [5].

Now we state a theorem, more general than Theorem 1.2.

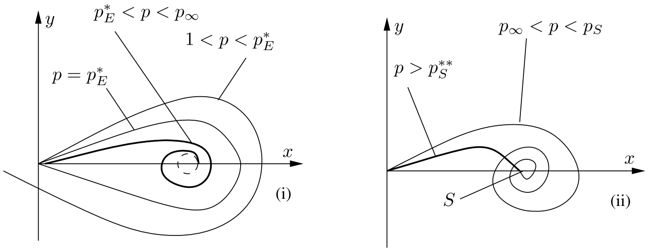


**Fig. 4.1.** Phase plane analysis for $y - x$.

**Theorem 4.2.** *The conclusions of Theorem* 1.2 *remain valid, if we assume the same hypotheses, except for* **(E)** *which is replaced by* **(Ep)**.

**Proof.** By Lemma 4.1, we only need to discuss the behavior of the basic orbit, which is a solution of system (4.4) with initial conditions (4.5). It is known from [11] the existence of a critical exponent $p_E^*$ such that when $1 < p < p_E^*$ the basic orbit crosses exactly one time the positive $x$-axis and then the negative $y$-axis, while if $p = p_E^*$ the basic orbit is a homoclinic orbit, see Fig. 4.1(i). If further $p > N_0/(N_0 - 2)$, then we obtain two positive solutions for $1 < p \leqslant p_E^*$ and $\mu < \mu_*$, and a unique solution for $\mu = \mu_*$. In the general case, we obtain the existence part of (a), completing the proof of (a).

If $p_E^* < p$ the solutions of the systems are bounded, as shown in [11]. Thus, by the Poincaré–Bendixon theorem **(PI)** or **(U)** holds. If $p > p_\infty$ then by Dulac's criterion **(PI)** does not occur with $\mathcal{I}$ a singleton, see [11, Proposition 4.4].

The other cases are consequences of the stability of the critical point $S$ that we have established. □

**Remark.** We may state remarks analogous to the three remarks given at the end of Section 3. We leave the detail to the reader.

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