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Approximation of solutions to fractional integral equation

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ABSTRACT

In this paper we shall study a fractional integral equation in an arbitrary Banach space X. We used the analytic semigroups theory of linear operators and the fixed point method to establish the existence and uniqueness of solutions of the given problem. We also prove the existence of global solution. The existence and convergence of the Faedo–Galerkin solution to the given problem is also proved in a separable Hilbert space with some additional assumptions on the operator A. Finally we give an example to illustrate the applications of the abstract results.

1. Introduction

We consider the following fractional integral equation in a Banach space $(X, \|.\|)$:

$$u(t) = u_0 + \frac{1}{\Gamma(\beta)} \int_0^t (t-\theta)^{\beta-1}(-Au(\theta))\mathrm{d}\theta + \frac{1}{\Gamma(\beta)} \int_0^t (t-\theta)^{\beta-1} f(\theta, u(\theta))\mathrm{d}\theta, \tag{1.1}$$

where A is a closed linear operator defined on a dense set and $0 < \beta < 1$, $0 < T < \infty$. We assume $-A$ is the infinitesimal generator of an analytic semigroup $\{S(t) : t \geq 0\}$ in X and the nonlinear map f is defined from $[0, T] \times X$ into X satisfying certain conditions to be specified later.

Regarding earlier works on the existence and uniqueness of different types of solutions to fractional integral and differential equations, we refer to [1–6] and references cited in these papers.

Muslim [1] has considered the following fractional order integral equations in a Banach space X of the form

$$u(t) = u_0 + \frac{1}{\Gamma(\beta)} \int_0^t (t-\theta)^{\beta-1}(-Au(\theta))\mathrm{d}\theta + \frac{1}{\Gamma(\beta)} \int_0^t (t-\theta)^{\beta-1} f_1(\theta, u(\theta))\mathrm{d}\theta, \quad t \in (0, T] \tag{1.2}$$

and

$$u(t) = u_0 + \frac{1}{\Gamma(\beta)} \int_0^t (t-\theta)^{\beta-1}(-Au(\theta))\mathrm{d}\theta + \frac{1}{\Gamma(\beta)} \int_0^t (t-\theta)^{\beta-1} f_2(\theta, u(\theta), u(a(\theta)))\mathrm{d}\theta, \quad t \in (0, T], \tag{1.3}$$

where $-A$ is the infinitesimal generator of a compact analytic semigroup and proved the existence and uniqueness of local solutions.

Initial studies concerning the existence, uniqueness and finite-time blow-up of solutions for the following equation

$$u'(t) + Au(t) = g(u(t)), \quad t \geq 0, \tag{1.4}$$
$$u(0) = \phi,$$

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have been considered by Segal [7], Murakami [8] and Heinz and von Wahl [9]. Bazley [10,11] has considered the following semilinear wave equation

$$u''(t) + Au(t) = g(u(t)), \quad t \geq 0, \qquad (1.5)$$
$$u(0) = \phi, \qquad u'(0) = \psi,$$

and has established the uniform convergence of approximate solutions to Eq. (1.5) by using the existence results of Heinz and von Wahl [9]. Goethel [12] has proved the convergence of approximate solutions to the problem (1.4), but assumed g to be defined on the whole of H.

In this paper, we use the Banach fixed point theorem and analytic semigroup theory to prove the existence, uniqueness and approximation of solutions of the given problem (1.1). Further, we also prove certain approximation results.

The plan of the paper is as follows. In Section 3, we prove the existence and uniqueness of local solutions and in Section 4, the existence of global solution for the problem (1.1) is given. Section 5 deals with the approximation of solutions. In the last section, we have given an example.

2. Preliminaries

We note that if $-A$ is the infinitesimal generator of an analytic semigroup then for $c > 0$ large enough, $-(A + cI)$ is invertible and generates a bounded analytic semigroup. This allows us to reduce the general case in which $-A$ is the infinitesimal generator of an analytic semigroup to the case in which the semigroup is bounded and the generator is invertible. Hence, without loss of generality we suppose that

$$\|S(t)\| \leq M \quad \text{for } t \geq 0$$

and

$$0 \in \rho(-A),$$

where $\rho(-A)$ is the resolvent set of $-A$. It follows that for $0 \leq \alpha \leq 1$, A^α can be defined as a closed linear invertible operator with domain $D(A^\alpha)$ being dense in X. We have $X_\kappa \hookrightarrow X_\alpha$ for $0 < \alpha < \kappa$ and the embedding is continuous. For more details on the fractional powers of closed linear operators we refer to Pazy [13]. It can be proved easily that $X_\alpha := D(A^\alpha)$ is a Banach space with norm $\|x\|_\alpha = \|A^\alpha x\|$ and it is equivalent to the graph norm of A^α.

We notice that $\mathcal{C}_T = C([0, T], X)$, the set of all continuous functions from $[0, T]$ into X is a Banach space under the supremum norm given by

$$\|\psi\|_T := \sup_{0 \leq \eta \leq T} \|\psi(\eta)\|, \quad \psi \in \mathcal{C}_T.$$

It can also be proved easily that $C_T^\alpha = C([0, T]; X_\alpha)$, for all $t \in [0, T]$, is a Banach space endowed with the supremum norm

$$\|\psi\|_{T,\alpha} := \sup_{0 \leq \eta \leq T} \|\psi(\eta)\|_\alpha, \quad \psi \in \mathcal{C}_T^\alpha.$$

We assume the following conditions:

(A1) $-A$ is the infinitesimal generator of an analytic semigroup $S(t)$.
(A2) The nonlinear map $f : [0, T] \times X_\alpha \to X$ satisfies

$$\|f(t, x) - f(s, y)\| \leq L(r)[|t - s|^\nu + \|x - y\|_\alpha],$$

for all $t, s \in [0, T]$, a fixed ν, $0 < \nu \leq 1$ and $x, y \in B_r(X_\alpha)$, where $L : \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing function and for $r > 0$

$$B_r(Z) = \{z \in Z : \|z\|_Z \leq r\},$$

where $(Z, \|.\|_Z)$ is a Banach space.

We define the Riemann–Liouville integral of order $\beta > 0$ by

$$I^\beta g(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t - \theta)^{\beta - 1} g(\theta) \mathrm{d}\theta.$$

Definition 2.1. *By a mild solution of the evolution problem (1.1), we mean a continuous solution u of the following integral equation given below*

$$u(t) = \int_0^\infty \zeta_\beta(\theta) S(t^\beta \theta) u_0 \mathrm{d}\theta + \beta \int_0^t \int_0^\infty \theta (t - s)^{\beta - 1} \zeta_\beta(\theta) S((t - s)^\beta \theta) f(s, u(s), u(a(\theta))) \mathrm{d}\theta \mathrm{d}s, \qquad (2.1)$$

where $\zeta_\beta(\theta)$ is the probability density function [14,15].

For further detail on the mild solution, we refer to [1,3–5].

3. Existence of solutions

Throughout the paper we have assumed that $0 < T < \infty$, $0 < \beta < 1$ and $0 \le \alpha < 1$.
We have the following theorem regarding the existence of a local solution.

Theorem 3.1. *Suppose that $-A$ is the infinitesimal generator of an analytic semigroup $S(t)$ with $\|S(t)\| \le M$, $t \ge 0$ and that $0 \in \rho(-A)$. If the function f satisfies the condition (A2) and $u_0 \in D(A)$, then the fractional integral equation (1.1) has a unique local solution.*

Proof. We will establish the existence of a solution u of Eq. (1.1) on $[0, t_0]$ for some t_0 such that $0 < t_0 \le T$.
We take

$$\int_0^\infty \theta^{1-\alpha}\zeta_\beta(\theta)\mathrm{d}\theta = N_1, \tag{3.1}$$

where $\zeta_\beta(\theta)$ is the probability density function [4].
For any $0 < \tilde{T} \le T$, we define a mapping F from $\mathcal{C}^\alpha_{\tilde{T}}$ into $\mathcal{C}^\alpha_{\tilde{T}}$ given by,

$$(F\psi)(t) = \int_0^\infty \zeta_\beta(\theta)S(t^\beta\theta)u_0\mathrm{d}\theta + \beta\int_0^t\int_0^\infty \theta(t-s)^{\beta-1}\zeta_\beta(\theta)S((t-s)^\beta\theta)f(s,\psi(s))\mathrm{d}\theta\mathrm{d}s. \tag{3.2}$$

Clearly F is well defined. For $R > 0$, let $M\|u_0\|_\alpha \le \frac{R}{2}$ and

$$S = \{u : u \in \mathcal{C}^\alpha_{t_0}, \|u(t)\|_\alpha \le R\}.$$

Choose t_0, $0 < t_0 \le T$ such that

$$t_0 < \left[\frac{R}{2}{C_\alpha}^{-1}{N_1}^{-1}(1-\alpha)\{L(R)[T^\nu + R] + N_2\}^{-1}\right]^{\frac{1}{\beta(1-\alpha)}}, \tag{3.3}$$

where C_α is a positive constant depending on α satisfying $\|A^\alpha S(t)\| \le C_\alpha t^{-\alpha}$, for all $t > 0$ and $\|f(0,0)\| = N_2$.
To prove the theorem, first we need to show that $F : S \to S$. For any $\psi \in S$, we have $(F\psi)(0) = u_0$. If $t \in [0, t_0]$ then we have

$$\begin{aligned}\|(F\psi)(t)\|_\alpha &\le \int_0^\infty \zeta_\beta(\theta)\|S(t^\beta\theta)\|\|A^\alpha u_0\|\mathrm{d}\theta + \beta\int_0^t\int_0^\infty \theta(t-s)^{\beta-1}\zeta_\beta(\theta)\|A^\alpha S((t-s)^\beta\theta)\| \\ &\quad\times \|f(s,\psi(s)) - f(0,0)\|\mathrm{d}\theta\mathrm{d}s + \beta\int_0^t\int_0^\infty \theta(t-s)^{\beta-1}\zeta_\beta(\theta)\|A^\alpha S((t-s)^\beta\theta)\|\|f(0,0)\|\mathrm{d}\theta\mathrm{d}s. \\ &\le M\|u_0\|_\alpha + N_1C_\alpha\{L(R)[T^\nu + R] + N_2\}\frac{t_0^{\beta(1-\alpha)}}{(1-\alpha)} \\ &\le R. \end{aligned} \tag{3.4}$$

Hence $F : S \to S$. Our next goal is to show that F is a strict contraction mapping on S.
For all $t \in [0, t_0]$ and $\psi_1, \psi_2 \in S$, we have

$$(F\psi_1)(t) - (F\psi_2)(t) = \beta\int_0^t\int_0^\infty \theta(t-s)^{\beta-1}\zeta_\beta(\theta)S((t-s)^\beta\theta)[f(s,\psi_1(s)) - f(s,\psi_2(s))]\mathrm{d}\theta\mathrm{d}s.$$

Hence,

$$\|(F\psi_1)(t) - (F\psi_2)(t)\|_\alpha \le \beta\int_0^t\int_0^\infty \theta(t-s)^{\beta-1}\zeta_\beta(\theta)\|A^\alpha S((t-s)^\beta\theta)\|\|f(s,\psi_1(s)) - f(s,\psi_2(s))\|\mathrm{d}\theta\mathrm{d}s.$$

From condition (A2) we have

$$\begin{aligned}\|(F\psi_1)(t) - (F\psi_2)(t)\|_\alpha &\le \frac{1}{R}\frac{C_\alpha}{(1-\alpha)}N_1[L(R)(T^\nu + R) + N_2]t_0^{\beta(1-\alpha)}\|\psi_1 - \psi_2\|_{t_0,\alpha} \\ &\le \frac{1}{2}\|\psi_1 - \psi_2\|_{t_0,\alpha},\end{aligned}$$

for all $\psi_1, \psi_2 \in S$. Hence F is a strict contraction mapping on S and therefore F has a unique fixed point in S.
Hence there exists $u \in S$, such that for all $t \in [0, t_0]$, we have

$$u(t) = \int_0^\infty \zeta_\beta(\theta)S(t^\beta\theta)u_0\mathrm{d}\theta + \beta\int_0^t\int_0^\infty \theta(t-s)^{\beta-1}\zeta_\beta(\theta)S((t-s)^\beta\theta)f(s,u(s))\mathrm{d}\theta\mathrm{d}s, \tag{3.5}$$

where $u(0) = u_0$. Now we will show that the function u is Hölder continuous on $[0, t_0]$. For any $t_1,\ \ t_2 \in [0, t_0]$, where $t_1 < t_2$, we have,

$$\begin{aligned} A^\alpha[u(t_2) - u(t_1)] = & \int_0^\infty \zeta_\beta(\theta)[S(t_2^\beta\theta) - S(t_1^\beta\theta)]A^\alpha u_0 \mathrm{d}\theta \\ & + \beta \int_{t_1}^{t_2}\int_0^\infty \theta(t_2 - s)^{\beta-1}\zeta_\beta(\theta)A^\alpha S((t_2 - s)^\beta\theta)f(s, u(s))\mathrm{d}\theta\mathrm{d}s \\ & + (-\beta)\int_0^{t_1}\int_0^\infty \theta[(t_1 - s)^{\beta-1} - (t_2 - s)^{\beta-1}]\zeta_\beta(\theta)A^\alpha S((t_2 - s)^\beta\theta)f(s, u(s))\mathrm{d}\theta\mathrm{d}s \\ & + \beta\int_0^{t_1}\int_0^\infty \theta(t_1 - s)^{\beta-1}\zeta_\beta(\theta)A^\alpha[S((t_2 - s)^\beta\theta) - S((t_1 - s)^\beta\theta)]f(s, u(s))\mathrm{d}\theta\mathrm{d}s \\ = & I_1 + I_2 + I_3 + I_4. \end{aligned} \tag{3.6}$$

Hence,

$$\|u(t_2) - u(t_1)\|_\alpha \le \|I_1\| + \|I_2\| + \|I_3\| + \|I_4\|. \tag{3.7}$$

We have

$$\begin{aligned} I_1 &= \int_0^\infty \zeta_\beta(\theta)[S(t_2^\beta\theta) - S(t_1^\beta\theta)]A^\alpha u_0\mathrm{d}\theta \\ &= \int_0^\infty \zeta_\beta(\theta)\left[\int_{t_1}^{t_2}\beta\theta t^{\beta-1}A^\alpha S(t^\beta\theta)Au_0\mathrm{d}t\right]\mathrm{d}\theta. \end{aligned}$$

Therefore

$$\begin{aligned} \|I_1\| &\le \int_0^\infty \zeta_\beta(\theta)\int_{t_1}^{t_2}\beta\theta t^{\beta-1}\|A^\alpha S(t^\beta\theta)\|\,\|Au_0\|\mathrm{d}t\mathrm{d}\theta \\ &\le C_\alpha\beta\int_0^\infty \theta^{1-\alpha}\zeta_\beta(\theta)\int_{t_1}^{t_2} t^{\beta(1-\alpha)-1}\|Au_0\|\mathrm{d}t\mathrm{d}\theta. \end{aligned}$$

Hence

$$\begin{aligned} \|I_1\| &\le \frac{N_1}{(1-\alpha)}C_\alpha\|Au_0\|(t_2^{\beta(1-\alpha)} - t_1^{\beta(1-\alpha)}) \\ &\le C_\alpha\|Au_0\|N_1\beta(t_1 + \delta(t_2 - t_1))^{\beta(1-\alpha)-1}(t_2 - t_1) \\ &\le C_\alpha\|Au_0\|N_1\beta\delta^{\beta(1-\alpha)-1}(t_2 - t_1)^{\beta(1-\alpha)}, \end{aligned} \tag{3.8}$$

where C_α is some positive constant satisfying $\|A^\alpha S(t)\| \le C_\alpha t^{-\alpha}$ for all $t \ge 0$ and $0 < \delta < 1$.

Also, we have

$$\|I_2\| \le \frac{L_f(R)}{(1-\alpha)}C_\alpha N_1(t_2 - t_1)^{\beta(1-\alpha)}, \tag{3.9}$$

and

$$\|I_3\| \le \beta N_1 L_f(R)C_\alpha\int_0^{t_1}(t_1 - s)^{\lambda-1}[(t_1 - s)^{-\lambda\mu} - (t_2 - s)^{-\lambda\mu}]\mathrm{d}s, \tag{3.10}$$

where $L_f(R) = \{L(R)[T^\nu + R] + N_2\},\ \lambda = 1 - \beta\alpha$ and $\mu = \frac{1-\beta}{1-\beta\alpha}$.

Hence, after some calculation [similar to Theorem 3.2 [4]] we get

$$\|I_3\| \le \beta N_1 L_f(R)C_\alpha\mu{\delta_1}^{\mu-1}(1-c)^{-\lambda(1-\mu)-1}(t_2 - t_1)^{\lambda(1-\mu)}, \tag{3.11}$$

where $c = (1 - (\frac{\mu}{\lambda})^{\frac{1}{\lambda\mu}})$ and $0 < \delta_1 \le 1$.

Similarly we get,

$$\begin{aligned} \|I_4\| &\le \beta N_1 L_f(R)\frac{C_{1+\alpha}}{\alpha}\int_0^{t_1}(t_1 - s)^{\beta-1}[(t_1 - s)^{-\beta\alpha} - (t_2 - s)^{-\beta\alpha}]\mathrm{d}s \\ &\le \beta N_1 L_f(R)\frac{C_{1+\alpha}}{\alpha}{\delta_2}^{\alpha-1}(1-c_1)^{-\beta(1-\alpha)-1}(t_2 - t_1)^{\beta(1-\alpha)}, \end{aligned}$$

where $c_1 = (1 - (\frac{\alpha}{\beta})^{\frac{1}{\alpha\beta}}),\ 0 < \delta_2 \le 1$ and $C_{1+\alpha}$ is some positive constant satisfying $\|A^{\alpha+1}S(t)\| \le C_{1+\alpha}t^{-1-\alpha}$ for all $t \ge 0$.

Thus the function u satisfies a Hölder condition on $[0, t_0]$. With the help of the condition (A2), we can prove easily that the map $t \longmapsto f(t, u(t))$ is Hölder continuous on $[0, t_0]$. This completes the proof of the theorem. □

4. Global existence

Theorem 4.1. *Suppose that $0 \in \rho(-A)$ and $-A$ generates the analytic semigroup $S(t)$ with $\|S(t)\| \leq M$, for $t \geq 0$, $u_0 \in D(A)$ and the function $f : [0, \infty) \times X_\alpha \to X$ satisfies the condition* (A2) *. If there is a continuous nondecreasing real valued function $k(t)$ such that*

$$\|f(t, x)\| \leq k(t)(1 + \|x\|_\alpha) \quad \text{for } t \geq 0,\ x \in X_\alpha, \tag{4.1}$$

then the fractional integral equation (1.1) has a unique solution u which exists for all $t \geq 0$.

Proof. By Theorem 3.1, we can continue the solution of Eq. (1.1) as long as $\|u(t)\|_\alpha$ stays bounded. It is therefore sufficient to show that if u exists on $[0, T)$, then $\|u(t)\|_\alpha$ is bounded as $t \uparrow T$.

For $t \in [0, T)$, we have

$$A^\alpha u(t) = \int_0^\infty \zeta_\beta(\theta) A^\alpha S(t^\beta \theta) u_0 \mathrm{d}\theta + \beta \int_0^t \int_0^\infty \theta (t-s)^{\beta-1} \zeta_\beta(\theta) A^\alpha S((t-s)^\beta \theta) f(s, u(s)) \mathrm{d}\theta \mathrm{d}s. \tag{4.2}$$

From the above equation we get

$$\|u(t)\|_\alpha \leq M\|u_0\|_\alpha + \beta \int_0^t \int_0^\infty \theta (t-s)^{\beta-1} \zeta_\beta(\theta) \|A^\alpha S((t-s)^\beta \theta)\| \|f(s, u(s))\| \mathrm{d}\theta \mathrm{d}s.$$

Hence

$$\|u(t)\|_\alpha \leq C_1 + C_2 \int_0^t (t-s)^{\beta(1-\alpha)-1} \|u(s)\|_\alpha \mathrm{d}s, \tag{4.3}$$

where $C_1 = M\|u_0\|_\alpha + \frac{k(T) N_1 C_\alpha T^{\beta(1-\alpha)}}{(1-\alpha)}$ and $C_2 = k(T)\beta N_1 C_\alpha$. Hence from Lemma 6.7 [Chapter 5 in Pazy [13]], u is a global solution.

To complete the proof of the theorem, we only need to show that u is unique for the whole interval.

Let u_1 and u_2 be two solutions of the given fractional integral equation (1.1). Then for any $t > 0$, we have

$$\begin{aligned} \|u_1(t) - u_2(t)\|_\alpha &\leq \beta \int_0^t \int_0^\infty \theta (t-s)^{\beta-1} \zeta_\beta(\theta) \|A^\alpha S((t-s)^\beta \theta)\| \|f(s, u_1(s)) - f(s, u_2(s))\| \mathrm{d}\theta \mathrm{d}s \\ &\leq \beta N_1 C_\alpha \int_0^t (t-s)^{\beta(1-\alpha)-1} \|f(s, u_1(s)) - f(s, u_2(s))\| \mathrm{d}s \\ &\leq L(R)\beta N_1 C_\alpha \int_0^t (t-s)^{\beta(1-\alpha)-1} \|u_1(s) - u_2(s)\|_\alpha \mathrm{d}s. \end{aligned}$$

Hence from Lemma 6.7 [Chapter 5 in Pazy [13]], the solution u is unique. This completes the proof of the theorem. □

5. Approximate solutions

To prove the existence and convergence of approximate solutions to fractional integral equation we need additional conditions on X and A.

We assume the following assumptions:

(A3) X is a separable Hilbert space.

(A4) The operator A is a closed, positive definite, self-adjoint linear operator from the domain $D(A) \subset X$ into X such that $D(A)$ is dense in X. We assume A has the pure point spectrum

$$0 < \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m \leq \cdots,$$

where $\lambda_m \to \infty$ as $m \to \infty$ and a corresponding complete orthonormal system of eigenfunctions $\{\phi_i\}$, i.e.,

$$A\phi_i = \lambda_i \phi_i \quad \text{and} \quad \langle \phi_i, \phi_j \rangle = \delta_{ij},$$

where $\delta_{ij} = 1$ if $i = j$ and zero otherwise.

If the conditions (A4) is satisfied then $-A$ is the infinitesimal generator of an analytic semigroup $\{S(t) : t \geq 0\}$ in X.

Let X_n denote the subspace of X generated by $\{\phi_0, \phi_1, \ldots, \phi_n\}$ and $P^n : X \to X_n$ be the associated projections operators.

For $n = 0, 1, 2, \ldots$, we define the maps F_n on S as follows: for $u \in S$ and $t \in [0, T]$

$$(F_n u)(t) = \int_0^\infty \zeta_\beta(\theta) S(t^\beta \theta) P^n u_0 \mathrm{d}\theta + \beta \int_0^t \int_0^\infty \theta (t-s)^{\beta-1} \zeta_\beta(\theta) S((t-s)^\beta \theta) P^n f(s, P^n u(s)) \mathrm{d}\theta \mathrm{d}s. \tag{5.1}$$

Theorem 5.1. *Let $n \geq n_0$, where n_0 is large enough and $n, n_0 \in \mathbb{N}$. If the conditions* (A2)–(A4) *are satisfied then there exists a unique $u_n \in S$ such that $F_n u_n = u_n$ for each $n = 0, 1, 2, 3, \ldots$, i.e. u_n satisfies the approximate integral equation*

$$u_n(t) = \int_0^\infty \zeta_\beta(\theta) S(t^\beta\theta) P^n u_0 \mathrm{d}\theta + \beta \int_0^t \int_0^\infty \theta(t-s)^{\beta-1} \zeta_\beta(\theta) S((t-s)^\beta\theta) P^n f_n(s, P^n u_n(s)) \mathrm{d}\theta \mathrm{d}s. \tag{5.2}$$

Moreover the solution u_n is uniformly Hölder continuous on $[0, t_0]$.

Proof. With the help of Theorem 3.1, we can show easily that the mapping F_n has a unique fixed point given by

$$u_n(t) = \int_0^\infty \zeta_\beta(\theta) S(t^\beta\theta) P^n u_0 \mathrm{d}\theta + \beta \int_0^t \int_0^\infty \theta(t-s)^{\beta-1} \zeta_\beta(\theta) S((t-s)^\beta\theta) P^n f(s, P^n u_n(s)) \mathrm{d}\theta \mathrm{d}s \tag{5.3}$$

which is uniformly Hölder continuous on $[0, t_0]$. □

Corollary 5.2. *If $u_0 \in D(A)$ then $u_n(t) \in D(A^\eta)$ for all $t \in [0, t_0]$, where $0 \leq \eta < 1$.*

Proof. As the function, u_n is Hölder continuous on $[0, t_0]$, hence with the help of Theorem 3.1, we can see that the map $t \mapsto P^n f(t, P^n u_n(t))$ is also Hölder continuous on $[0, t_0]$, hence $u_n \in D(A)$. Since $D(A) \subset D(A^\eta)$ and $u_n \in D(A)$ hence our Corollary is proved. □

Corollary 5.3. *If $u_0 \in D(A)$ then there exists a constant M_0 independent of n such that*

$$\|A^\eta u_n\| \leq M_0,$$

for all $t \in [0, t_0]$ and $0 \leq \eta < 1$.

Proof. From Eq. (5.3) we have,

$$u_n(t) = \int_0^\infty \zeta_\beta(\theta) S(t^\beta\theta) P^n u_0 \mathrm{d}\theta + \beta \int_0^t \int_0^\infty \theta(t-s)^{\beta-1} \zeta_\beta(\theta) S((t-s)^\beta\theta) P^n f(s, P^n u_n(s)) \mathrm{d}\theta \mathrm{d}s. \tag{5.4}$$

Hence

$$\begin{aligned}
\|A^\eta u_n(t)\| &\leq M\|u_0\|_\eta + \beta \int_0^t \int_0^\infty \theta(t-s)^{\beta-1} \zeta_\beta(\theta) \|A^\eta S((t-s)^\beta\theta)\| \|P^n f(s, P^n u_n(s))\| \mathrm{d}\theta \mathrm{d}s. \\
&\leq M\|u_0\|_\eta + N_1 C_\eta \{L(R)[T^\nu + R] + N_2\} \frac{t_0^{\beta(1-\eta)}}{(1-\eta)} \\
&\leq M_0.
\end{aligned} \tag{5.5}$$

This completes the proof of the Corollary. □

In order to prove the convergence, we need the following stronger assumption on the nonlinear map f than (A2).

(A2′) The map f is defined from $[0, \infty) \times X_\alpha$ into $D(A^\eta)$ for $0 < \alpha < \eta < 1$ and there exist a nondecreasing function $\tilde{L}$ from $[0, \infty)$ into $[0, \infty)$ such that

$$\|f(t, u) - f(s, v)\|_\eta \leq \tilde{L}(r)\{|t-s|^\nu + \|u-v\|_\alpha\},$$

for all $t, s \in [0, T]$, $\nu \in (0, 1]$ and $u, v \in B_r(X_\alpha)$, where $r > 0$.

Theorem 5.4. *We choose η such that $0 \leq \alpha + \eta < 1$. If $u_0 \in D(A)$ and the assumptions* (A2)–(A4) *and* (A2′) *are satisfied then $\{u_n\} \subset S$ is a Cauchy sequence and therefore converges to a unique function $u \in S$.*

Proof. Let $n \geq m \geq n_0$, where n_0 is large enough and $n, m, n_0 \in \mathbb{N}$. Hence from Theorem 5.1 we have

$$u_n(t) = \int_0^\infty \zeta_\beta(\theta) S(t^\beta\theta) P^n u_0 \mathrm{d}\theta + \beta \int_0^t \int_0^\infty \theta(t-s)^{\beta-1} \zeta_\beta(\theta) S((t-s)^\beta\theta) P^n f(s, P^n u_n(s)) \mathrm{d}\theta \mathrm{d}s. \tag{5.6}$$

For $t \in [0, t_0]$, we have

$$\|A^\alpha(u_n(t) - u_m(t))\| \leq \beta \int_0^t \int_0^\infty \theta(t-s)^{\beta-1} \zeta_\beta(\theta) \|A^\alpha S((t-s)^\beta\theta)\| \|P^n f(s, P^n u_n(s)) - P^m f(s, P^m u_m(s))\| \mathrm{d}\theta \mathrm{d}s. \tag{5.7}$$

We have

$$\begin{aligned}\|P^nf(s,P^nu_n(s))-P^mf(s,P^mu_m(s))\| &\le \|P^n[f(s,P^nu_n(s))-P^nf(s,P^mu_m(s))]\|+\|(P^n-P^m)f(s,P^mu_m(s))\|\\ &\le \|P^n[f(s,P^nu_n(s))-f(s,P^nu_m(s))]\|+\|P^n[f(s,P^nu_m(s))-f(s,P^mu_m(s))]\|\\ &\quad+\|(P^n-P^m)f(s,P^mu_m(s))\|\\ &\le L(R)\|u_n(s)-u_m(s)\|_\alpha+L(R)\|A^{\alpha-\eta}(P^n-P^m)A^\eta u_m(s)\|\\ &\quad+\|A^{-\eta}(P^n-P^m)A^\eta f(s,P^mu_m(s))\|. \end{aligned} \tag{5.8}$$

Let $m<n$ then $X_m\subset X_n$. Let $X_m^\perp$ be the orthogonal complement of X_m for all $m=0,1,2,\dots$, then $X_m^\perp\supset X_n^\perp$. We can write $X=X_m\oplus X_m^\perp=X_n\oplus X_n^\perp$.

Let $z\in X$ be an arbitrary element. Then, we can write $z=z_m+y_m$, where $z_m\in X_m$ and $y_m\in X_m^\perp$. Then, $P^mz=z_m\in X_m$. We can see easily that $y_m\in X_m^\perp\Rightarrow y_m=\Sigma_{i=m+1}^n a_i\phi_i+y_m'$, where $y_m'\in X_n^\perp$. Let, $z_m'=\Sigma_{i=m+1}^n a_i\phi_i$.

Hence, $z=z_m+z_m'+y_m'$ and $P^nz=z_m+z_m'$.

Therefore,

$$P^nz-P^mz=z_m'=\Sigma_{i=m+1}^n a_i\phi_i.$$

If, $z=\Sigma_{i=1}^\infty a_i\phi_i$ then $\|z\|^2=\Sigma_{i=1}^\infty|a_i|^2$.

Since, $A^{\alpha-\eta}\phi_i=\lambda_i^{\alpha-\eta}\phi_i$ [16]. Hence, we have

$$\begin{aligned}\|A^{\alpha-\eta}(P^n-P^m)z\|^2 &= \langle A^{\alpha-\eta}(P^n-P^m)z,\ A^{\alpha-\eta}(P^n-P^m)z\rangle\\ &= \langle \Sigma_{i=m+1}^n a_iA^{\alpha-\eta}\phi_i,\ \Sigma_{j=m+1}^n a_jA^{\alpha-\eta}\phi_j\rangle\\ &= \langle \Sigma_{i=m+1}^n a_i\lambda_i^{\alpha-\eta}\phi_i,\ \Sigma_{j=m+1}^n a_j\lambda_j^{\alpha-\eta}\phi_j\rangle\\ &= \Sigma_{i,j=m+1}^n a_ia_j\lambda_i^{\alpha-\eta}\lambda_j^{\alpha-\eta}\langle\phi_i,\phi_j\rangle\\ &\le \lambda_{m+1}^{2(\alpha-\eta)}(\Sigma_{i=m+1}^n|a_i|^2)\\ &\le \frac{1}{\lambda_m^{2(\eta-\alpha)}}\|z\|^2. \end{aligned}$$

Therefore,

$$\begin{aligned}\|A^{\alpha-\eta}(P^n-P^m)A^\eta u_m(s)\| &\le \frac{1}{\lambda_m^{(\eta-\alpha)}}\|A^\eta u_m(s)\|\\ &\le \frac{1}{\lambda_m^{(\eta-\alpha)}}M_0, \end{aligned} \tag{5.9}$$

where M_0 is same as in Corollary 5.3.

Now we use the inequality (5.9) and assumption (A2′) in inequality (5.8) and get the following inequality

$$\|P^nf(s,P^nu_n(s))-P^mf(s,P^mu_m(s))\|\le L(R)\|u_n(s)-u_m(s)\|_\alpha+L(R)\frac{M_0}{\lambda_m^{\eta-\alpha}}+\frac{C_1}{\lambda_m^\eta}, \tag{5.10}$$

where $C_1=\tilde{L}(R)\{T^\nu+R\}+\|f(0,0)\|_\eta$ and $\alpha<\eta<1$.

From inequality (5.10) and inequality (5.7), we get the following inequality

$$\begin{aligned}\|A^\alpha[u_n(t)-u_m(t)]\| \le{}& L(R)\beta\int_0^t\int_0^\infty\theta(t-s)^{\beta-1}\zeta_\beta(\theta)\|A^\alpha S((t-s)^\beta\theta)\|\|u_n(s)-u_m(s)\|_\alpha\mathrm{d}\theta\mathrm{d}s\\ &+L(R)\frac{M_0}{\lambda_m^{\eta-\alpha}}\beta\int_0^t\int_0^\infty\theta(t-s)^{\beta-1}\zeta_\beta(\theta)\|A^\alpha S((t-s)^\beta\theta)\|\mathrm{d}\theta\mathrm{d}s\\ &+\frac{C_1}{\lambda_m^\eta}\beta\int_0^t\int_0^\infty\theta(t-s)^{\beta-1}\zeta_\beta(\theta)\|A^\alpha S((t-s)^\beta\theta)\|\mathrm{d}\theta\mathrm{d}s. \end{aligned} \tag{5.11}$$

The first integral of the above inequality (5.11) can be estimated as follows

$$\begin{aligned}&L(R)\beta\int_0^t\int_0^\infty\theta(t-s)^{\beta-1}\zeta_\beta(\theta)\|A^\alpha S((t-s)^\beta\theta)\|\|u_n(s)-u_m(s)\|_\alpha\mathrm{d}\theta\mathrm{d}s\\ &\quad\le\beta C_\alpha L(R)N_1\int_0^t(t-s)^{\beta(1-\alpha)-1}\|u_n(s)-u_m(s)\|_\alpha\mathrm{d}s. \end{aligned} \tag{5.12}$$

Second integral of the above inequality (5.11) is estimated as follows

$$L(R)\frac{M_0}{\lambda_m^{\eta-\alpha}}\beta\int_0^t\int_0^\infty \theta(t-s)^{\beta-1}\zeta_\beta(\theta)\|A^\alpha S((t-s)^\beta\theta)\|\mathrm{d}\theta\mathrm{d}s \le \frac{C_\alpha L(R)M_0N_1}{\lambda_m^{\eta-\alpha}(1-\alpha)}t_0^{\beta(1-\alpha)}. \tag{5.13}$$

Similarly third integral can be calculated as follows

$$\frac{C_1}{\lambda_m^\eta}\beta\int_0^t\int_0^\infty \theta(t-s)^{\beta-1}\zeta_\beta(\theta)\|A^\alpha S((t-s)^\beta\theta)\|\mathrm{d}\theta\mathrm{d}s \le \frac{C_\alpha C_1 N_1}{(1-\alpha)\lambda_m^\eta}t_0^{\beta(1-\alpha)}. \tag{5.14}$$

Finally, we deduce from the inequality (5.11) that

$$\begin{aligned}\|A^\alpha[u_n(t)-u_m(t)]\| &\le \beta C_\alpha L(R)N_1\int_0^t (t-s)^{\beta(1-\alpha)-1}\|u_n(s)-u_m(s)\|_\alpha \mathrm{d}s\\ &\quad + \frac{C_\alpha L(R)M_0N_1}{\lambda_m^{\eta-\alpha}(1-\alpha)}t_0^{\beta(1-\alpha)} + \frac{C_\alpha C_1N_1}{\lambda_m^\eta(1-\alpha)}t_0^{\beta(1-\alpha)}.\end{aligned} \tag{5.15}$$

The application of Gronwall's inequality and adding $m\to\infty$ to the above inequality gives the required result. This completes the proof the theorem. □

With the help of the Theorems 5.1, 5.4 and 4.1 we can state the following existence, uniqueness and convergence result.

Theorem 5.5. *Let $n\ge n_0$, where n_0 is large enough and $n, n_0\in\mathbb{N}$. If the conditions* (A2)–(A4) *and* (A2′) *are satisfying then there exist a unique $u\in S$ such that $u_n\to u$ in S and u is given by Eq. (3.5).*

Proof. We only need to prove that the limit u obtained above satisfies the integral equation (3.5). We have

$$\|(P^n-I)u_0\|\to 0,\quad \text{as } n\to\infty. \tag{5.16}$$

We also have

$$\|P^nf(t,P^nu_n)-f(t,u(t))\| \le \|(P^n-I)f(t,P^nu_n(t))\| + \|f(t,P^nu_n(t))-f(t,u(t))\|, \tag{5.17}$$

where $\|(P^n-I)f(t,P^nu_n(t))\|\to 0$ and $\|f(t,P^nu_n(t))-f(t,u(t))\|\to 0$, as $n\to\infty$.

Therefore,

$$\begin{aligned}&\int_0^\infty \zeta_\beta(\theta)S(t^\beta\theta)\|(P^n-I)u_0\|\mathrm{d}\theta + \beta\int_0^t\int_0^\infty \theta(t-s)^{\beta-1}\zeta_\beta(\theta)S((t-s)^\beta\theta)\\ &\quad\times\|P^nf(s,P^nu_n(s))-f(s,u(s))\|\mathrm{d}\theta\mathrm{d}s\to 0\quad \text{when } n\to\infty.\end{aligned} \tag{5.18}$$

Hence from inequalities (5.16)–(5.18) we can see that u_n converges to u where u is given by Eq. (3.5). This completes the proof of the theorem. □

6. Example

Let $X=L^2((0,1);\mathbb{R})$. We consider the following fractional order integral equation,

$$\begin{aligned}&w(t,x)=w(0,x)+\frac{1}{\Gamma(\beta)}\int_0^t(t-\theta)^{\beta-1}(\partial_x^2 w(\theta,x))\mathrm{d}\theta+\frac{1}{\Gamma(\beta)}\int_0^t(t-\theta)^{\beta-1}F(\theta,\partial_x w(\theta,x))\mathrm{d}\theta,\\ &w(t,0)=w(t,1)=0,\quad t\in[0,T],\ 0<T<\infty,\end{aligned} \tag{6.1}$$

where F is a given sufficiently smooth function which satisfies the Hölder condition.

We define an operator A.

$$Au=-u''\quad \text{with } u\in D(A)=H_0^1(0,1). \tag{6.2}$$

Here, clearly the operator A is self-adjoint, with compact resolvent and is the infinitesimal generator of an analytic semigroup $S(t)$. We take $\alpha=1/2$, $D(A^{1/2})$ is a Banach space with norm

$$\|x\|_{1/2} := \|A^{1/2}x\|,\quad x\in D(A^{1/2}),$$

and we denote this space by $X_{1/2}$.

The Eq. (6.1) can be reformulated as the following abstract equation in $X=L^2((0,1);\mathbb{R})$:

$$u(t)=u_0+\frac{1}{\Gamma(\beta)}\int_0^t(t-\theta)^{\beta-1}(-Au(\theta))\mathrm{d}\theta+\frac{1}{\Gamma(\beta)}\int_0^t(t-\theta)^{\beta-1}f(\theta,u(\theta))\mathrm{d}\theta, \tag{6.3}$$

where $u(t) = w(t, .)$ that is $u(t)(x) = w(t, x)$, $t \in [0, T]$, $x \in (0, 1)$ and the function $f : [0, T] \times X_{1/2} \to X$ is given by

$$f(t, u(t))(x) = F(t, \partial_x w(t, x)). \tag{6.4}$$

We can take $f(t, u) = h(t)g(u')$, where h is Hölder continuous and $g : X \to X$ is Lipschitz continuous on X. In particular we can take $g(u) = sinu$, $g(u) = \xi u$, $g(u) = arctan(u)$, where ξ is constant.

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