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New solutions for Trudinger–Moser critical equations in $\mathbb{R}^2$

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Abstract

Let Ω be a bounded, smooth domain in $\mathbb{R}^2$. We consider critical points of the Trudinger–Moser type functional $J_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{\lambda}{2}\int_\Omega e^{u^2}$ in $H_0^1(\Omega)$, namely solutions of the boundary value problem $\Delta u + \lambda u e^{u^2} = 0$ with homogeneous Dirichlet boundary conditions, where $\lambda > 0$ is a small parameter. Given $k \geqslant 1$ we find conditions under which there exists a solution u_λ which blows up at exactly k points in Ω as $\lambda \to 0$ and $J_\lambda(u_\lambda) \to 2k\pi$. We find that at least one such solution always exists if $k = 2$ and Ω is not simply connected. If Ω has $d \geqslant 1$ holes, in addition $d + 1$ bubbling solutions with $k = 1$ exist. These results are existence counterparts of one by Druet in [O. Druet, Multibump analysis in dimension 2: Quantification of blow-up levels, Duke Math. J. 132 (2) (2006) 217–269] which classifies asymptotic bounded energy levels of blow-up solutions for a class of nonlinearities of critical exponential growth, including this one as a prototype case.

Keywords: Trudinger–Moser inequality; Blowing-up solutions; Singular perturbations

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1. Introduction and statement of main results

Let Ω be a bounded domain in $\mathbb{R}^2$ with smooth boundary and $\lambda > 0$. This paper is concerned with the analysis of solutions to the boundary value problem

$$\begin{cases} \Delta u + \lambda u e^{u^2} = 0, & u > 0 \quad \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\lambda > 0$ is a small parameter. This problem is the Euler–Lagrange equation for the functional

$$J_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{\lambda}{2}\int_\Omega e^{u^2}, \quad u \in H_0^1(\Omega), \tag{1.2}$$

which corresponds to the free energy associated to the critical Trudinger embedding (in the sense of Orlicz spaces) [18,22,23]

$$H_0^1(\Omega) \ni u \longmapsto e^{u^2} \in L^p(\Omega) \quad \forall p \geqslant 1,$$

which is connected to the critical Trudinger–Moser inequality

$$C(\Omega) = \sup\left\{ \int_\Omega e^{4\pi u^2} / u \in H_0^1(\Omega), \int_\Omega |\nabla u|^2 = 1 \right\} < +\infty,$$

[17]. Observe that, in general, critical points of the above constrained variational problem satisfy, after a simple scaling, an equation of the form (1.1). The Trudinger–Moser embedding is critical, involving loss of compactness analogous to that of the Sobolev embeddings in dimension $N \geqslant 3$,

$$H_0^1(\Omega) \ni u \longmapsto u \in L^{\frac{2N}{N-2}}(\Omega),$$

for which the problem analogous to (1.1) is

$$\begin{cases} \Delta u + \lambda u + u^{\frac{N+2}{N-2}} = 0, & u > 0 \quad \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

for $\lambda \geqslant 0$, whose associated energy is

$$I_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{\lambda}{2}\int_\Omega u^2 - \frac{N-2}{2N}\int_\Omega |u|^{\frac{2N}{N-2}}, \quad u \in H_0^1(\Omega).$$

Loss of compactness in $H_0^1(\Omega)$ for the functionals J_λ or I_λ translates into the presence of non-convergent Palais–Smale (PS) sequences. Let us consider for instance a sequence $\lambda_n \to \lambda_0 \geqslant 0$, and a sequence u_n with $\nabla I_{\lambda_n}(u_n) \to 0$, $I_{\lambda_n}(u_n) \to c$. Then, by the result in [20], u_n decomposes asymptotically into a finite sum of blowing-up standard bubbles and a critical point u_0 of I_{λ_0} yielding in particular that

$$I_{\lambda_n}(u_n) = I_{\lambda_0}(u_0) + kS_N + o(1) \quad \text{for some } k \geqslant 1,$$

where S_N is a positive constant. Existence of solutions, namely critical points of I_λ with the above property and $u_0 = 0$ is known under suitable assumptions, see [7,9,16,19]. For the Trudinger–Moser functional (1.2), a clean classification of all PS sequences for J_λ does not seem possible after the results in [3]. Actually PS holds as long as $c < 2\pi$, see [1,11]. On the other hand, for *solutions* more is known. In [2,14] a class of nonlinearities is considered for which the one in (1.1) may be regarded as the prototype. From the result in [14], we have the following fact:

Assume that u_n solves problem (1.1) *for $\lambda = \lambda_n$, with $J_{\lambda_n}(u_n)$ bounded and $\lambda_n \to 0$. Then, passing to a subsequence, there is an integer $k \geqslant 0$ such that*

$$J_{\lambda_n}(u_n) = 2k\pi + o(1). \tag{1.4}$$

When $k = 1$ a more precise answer is obtained in [2]: the solution u_n has for large n only one isolated maximum, which blows up around a point $x_0 \in \Omega$ which is characterized as follows: Let $G(x, y)$ be Green's function of the problem

$$-\Delta_x G = 4\pi\delta_y(x), \quad x \in \Omega,$$
$$G(x, y) = 0, \quad x \in \partial\Omega,$$

and H its regular part defined as

$$H(x, y) = 4\log\frac{1}{|x-y|} - G(x, y). \tag{1.5}$$

Then from [2], it follows that x_0 is a critical point of Robin's function $x \mapsto H(x, x)$.

It is natural to ask whether or not solutions satisfying (1.4) exist. In fact the existence and multiplicity question seems much more difficult than its critical Sobolev exponent counterpart. Some results are known: From the result in [3], it follows that there is a $\lambda_0 > 0$ such that a solution to (1.1) exists whenever $0 < \lambda < \lambda_0$ (this is in fact true for a larger class of nonlinearities with *critical exponential growth*). By construction this solution falls, as $\lambda \to 0$, into the bubbling category (1.4) with $k = 1$. No solution other than this one is known. Struwe in [21] built in the case of a domain with a sufficiently small hole (in the sense of Bahri and Coron [6,10]) a solution taking advantage of the presence of topology. This solution exists for a class of nonlinearities, perturbation of the Trudinger–Moser one, that also include $\lambda u e^{u^2 - u}$ for which no solution exists for small λ, in a disk, see [4,12]. It is reasonable to believe that the construction of Struwe in reality produces a second solution of Eq. (1.1), but this is so far not known.

In this paper we will address the issue of existence and multiplicity of solutions of problem (1.1) when Ω is not contractible to a point. More precisely, we provide conditions for the existence of solutions of problem (1.1) for small λ which satisfy the bubbling condition (1.4), at the same time giving a precise characterization of its bubbling location. In particular our main result implies the following: if Ω has a hole "of any size", namely Ω is not simply connected, then a solution blowing up at exactly two points and satisfying property (1.4) with $k = 2$ indeed exists. We expect this result to be true for any $k \geqslant 1$, provided that the domain is not contractible to a point.

Theorem 1. *Assume that Ω is not simply connected. Then there exists a family of solutions u_λ to problem* (1.1) *such that*

$$\frac{1}{2}\int_\Omega |\nabla u_\lambda|^2 - \frac{\lambda}{2}\int_\Omega e^{u_\lambda^2} = 4\pi + o(1)$$

where $o(1) \to 0$ as $\lambda \to 0$.

The location of the bubbling points (which are exactly two) for the solutions in this result can be thoroughly described. To this end, let us introduce the following functional of k distinct points $\xi_1, \xi_2, \dots, \xi_k \in \Omega$ and k positive numbers $m_1, m_2, \dots, m_k$,

$$\begin{aligned}\varphi_k(\xi, m) = {}& b\sum_{j=1}^k m_j^2 + 2\sum_{j=1}^k m_j^2 \log m_j^2 \\ & + \sum_{j=1}^k H(\xi_j, \xi_j) m_j^2 - \sum_{i \neq j} G(\xi_i, \xi_j) m_i m_j. \end{aligned} \tag{1.6}$$

Here G and H are the Green function for the Laplacian on Ω with Dirichlet boundary condition and its regular part, as defined above and b is an absolute constant which we will specify later. As λ approaches 0, the solution in Theorem 1 satisfies, up to subsequences,

$$u_\lambda(x) \sim \sqrt{\lambda}\sum_{j=1}^2 m_j G(x, \xi_j)$$

where (m_1, m_2, ξ_1, ξ_2) is a critical point of φ_2.

Let us consider an open set $\mathcal{D}$ compactly contained in the domain of the functional φ_k, namely

$$\bar{\mathcal{D}} \subset \{(\xi, m) \in \Omega^k \times \mathbb{R}_+^k / \xi_i \neq \xi_j \ \forall i \neq j\}.$$

We say that φ_k has a *stable critical point situation* if there exists a $\delta > 0$ such that for any $g \in C^1(\bar{\mathcal{D}})$ with $\|g\|_{C^1(\bar{\mathcal{D}})} < \delta$, the perturbed functional $\varphi_k + g$ has a critical point in $\mathcal{D}$.

Theorem 2. *Let $k \geqslant 1$ and assume that there is an open set $\mathcal{D}$ where φ_k has a stable critical point situation. Then, for all small $\lambda > 0$ there exists a solution u_λ of problem* (1.1) *such that*

$$\frac{1}{2}\int_\Omega |\nabla u_\lambda|^2 - \frac{\lambda}{2}\int_\Omega e^{u_\lambda^2} = 2k\pi + o(1)$$

where $o(1) \to 0$ as $\lambda \to 0$. Moreover, passing to a subsequence, there exists $(\xi, m) \in \mathcal{D}$ such that $\nabla\varphi_k(\xi, m) = 0$ and

$$u_\lambda(x) = \sqrt{\lambda}\left(\sum_{j=1}^{k} m_j G(x, \xi_j) + o(1)\right)$$

where $o(1) \to 0$ *on each compact subset of* $\bar{\Omega} \setminus \{\xi_1, \dots, \xi_k\}$.

As a direct consequence of the above result, we find in addition that the presence of topology of the domain induces multiplicity of solutions with a single blow-up point: when $k = 1$ a bubbling solution around a critical point of the function $H(x, x)$ exists. It is standard that $H(x, x) \to +\infty$ as $x \to \partial\Omega$, thus we always have such a solution bubbling near a global minimizer of this function. In addition, if Ω is not simply connected, Ljusternik–Schnirelmann theory yields the presence of at least $\mathrm{cat}(\Omega) = d + 1$ such solutions, where d is the number of holes of Ω.

Theorem 1 follows by showing via a topological construction that φ_2 has a stable critical point situation. We strongly believe that such a situation is in reality present for φ_k for any $k \geqslant 3$, but the construction appears to be much harder. It is reasonable to conjecture that any family of solutions satisfying (1.4) must have the concentration behavior described in the above theorem, in further precision of the result in [14].

It is interesting to mention the link of the above discovered concentration phenomena with the related Liouville equation $\Delta u + \lambda e^u = 0$ under Dirichlet boundary condition in a bounded domain Ω in $\mathbb{R}^2$, see [8,13,15] and references therein. The fine blow-up structure very close to the bubbling points is similar to that in the present problem however scalings and intermediate regimes are much more subtle here. Our choice of first approximations to bubbling solutions is inspired by the discovery of the blow-up shapes first in [5] then in [2,14]. However more accurate information is needed, in particular the discovery of the role of the distinct weights m_j, which marks a strong difference with the blow-up structure in Liouville's equation. As in other elliptic problems involving point concentration phenomena, our strategy of proof involves linearization about a first approximation, to later reduce the problem to a finite dimensional variational one of adjusting the bubbling centers and corresponding weights. The critical character of this nonlinearity is very much reflected in the delicate error terms left by the first approximation, which makes the linear elliptic theory needed fairly subtle because of the multiple-regime in the error size. While we restrict our investigation here to the nonlinearity $\lambda u e^{u^2}$, we expect that similar analysis can be carried out for a broader class of critical exponential growth nonlinearities. For simplicity in the exposition we shall only consider the prototype case.

2. A first approximation and outline of the argument

It is convenient for our purposes to rewrite problem (1.1) by replacing $u = \sqrt{\lambda}\tilde{u}$, so that the problem becomes

$$\begin{cases} \Delta\tilde{u} + \lambda\tilde{u}e^{\lambda\tilde{u}^2} = 0, & u > 0 \quad \text{in } \Omega, \\ \tilde{u} = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

Let us consider k distinct points $\xi_1, \xi_2, \dots, \xi_k$ in Ω and k positive numbers $m_1, m_2, \dots, m_k$. We choose a sufficiently small but fixed number $\delta > 0$ and assume that for $j = 1, \dots, k$,

$$\operatorname{dist}(\xi_j, \partial\Omega) \geqslant \delta, \quad |\xi_l - \xi_j| \geqslant \delta \quad \text{for } l \neq j,\ \delta < m_j < \frac{1}{\delta} \tag{2.2}$$

for some given $\delta > 0$.

We shall build an approximation $\tilde{U}(x)$ which away from the points ξ_j satisfies, in agreement with the statement of Theorem 1,

$$\tilde{U}(x) = \sum_{j=1}^{k} m_j G(x, \xi_j) + o(1) \quad \text{as } \lambda \to 0. \tag{2.3}$$

Near each point ξ_j, we consider positive numbers μ_j, ε_j, to be chosen in dependence from the values of λ, ξ and m, and the function

$$\tilde{U}(x) = \sum_{j=1}^{k} m_j \left[\log \frac{1}{(\mu_j^2 \varepsilon_j^2 + |x - \xi_j|^2)^2} - H_j(x) \right] \tag{2.4}$$

where H_j is a harmonic function so that the boundary condition zero is satisfied, that is

$$\begin{cases} \Delta H_j = 0, & \text{in } \Omega, \\ H_j(x) = \log \frac{1}{(\mu_j^2 \varepsilon_j^2 + |x - \xi_j|^2)^2}, & \text{for } x \in \partial\Omega. \end{cases}$$

Let us observe that from elliptic estimates

$$H_j(x) = H(x, \xi_j) + O\big(\varepsilon_j^2 \mu_j^2\big),$$

uniformly in Ω, where H is the regular part of Green's function with zero Dirichlet boundary condition in Ω, as defined in (1.5). Hence

$$\log \frac{1}{(\mu_j^2 \varepsilon_j^2 + |x - \xi_j|^2)^2} - H_j(x) = G(x, \xi_j) + O\big(\varepsilon_j^2 \mu_j^2\big), \tag{2.5}$$

and the desired outer expansion (2.3) indeed holds for this $\tilde{U}$. Let us examine $\tilde{U}$ in a small neighborhood of a given ξ_j. We write, for $|x - \xi_j| < \delta$, with sufficiently small but fixed δ,

$$\tilde{U}(x) = m_j \big(w_j(x) + \log \varepsilon_j^{-4} + \beta_j + \theta(x) \big) \tag{2.6}$$

where

$$m_j \beta_j := -m_j \log 8\mu_j^2 - m_j H(\xi_j, \xi_j) + \sum_{i \neq j} m_i G(\xi_i, \xi_j),$$

$$m_j \theta(x) = O\big(|x - \xi_j|\big) + \sum_{i=1}^{k} O\big(\varepsilon_i^2\big)$$

and

$$w_j(x) := w_{\mu_j}\left(\frac{x-\xi_j}{\varepsilon_j}\right)$$

with

$$w_\mu(y) := \log\frac{8\mu^2}{(\mu^2+|y|^2)^2}. \tag{2.7}$$

The functions w_μ, $\mu > 0$, are the radially symmetric solutions of the Liouville equation

$$\Delta w + e^w = 0 \quad \text{in } \mathbb{R}^2.$$

The idea is to choose the numbers μ_j, ε_j in such a way that the error of approximation for $\tilde U$ is small around each point ξ_j. This error is by definition

$$R(x) = \Delta\tilde U + f(\tilde U). \tag{2.8}$$

Here and in what follows f denotes the nonlinearity

$$f(\tilde u) = \lambda\tilde u e^{\lambda\tilde u^2}. \tag{2.9}$$

Let us observe that for $|x-\xi_j| < \delta$ we have

$$-\Delta\tilde U(x) = m_j\varepsilon_j^{-2}e^{w_j} + \sum_{i=1}^k O\big(\varepsilon_i^2\big).$$

On the other hand

$$f(\tilde U) = \left(\lambda m_j\log\frac{1}{\varepsilon_j^4} + \lambda m_j\big(w_j + O(1)\big)\right)$$
$$\times e^{2m_j^2\lambda(\beta_j+\theta)\log\frac{1}{\varepsilon_j^4}} e^{2m_j^2\lambda(\log\frac{1}{\varepsilon_j^4}+\beta_j)w_j} e^{\lambda w_j^2 m_j^2} e^{\lambda m_j^2\log^2\frac{1}{\varepsilon_j^4}} e^{\lambda m_j^2(\beta_j+\theta)^2+2\lambda m_j^2\theta w_j}.$$

Let us make the following choice of ε_j,

$$2m_j^2\lambda\left(\log\frac{1}{\varepsilon_j^4} + \beta_j\right) = 1, \tag{2.10}$$

so that

$$\lambda\tilde U = \frac{1}{2m_j}\big(1 + 2m_j^2\lambda\big(w_j + O(1)\big)\big) \tag{2.11}$$

and

$$e^{\lambda\tilde U^2} = e^{\beta_j/2}\varepsilon_j^{-2}e^{w_j}e^{\lambda m_j^2 w_j^2}\big(1+O(\theta)\big)\big(1+O(\lambda)w_j\big). \tag{2.12}$$

Thus, in order to match $f(\tilde{U})$ and $-\Delta\tilde{U}$ at main order near ξ_j we must fix μ_j so that the number β_j satisfies

$$e^{\beta_j/2} = 2m_j^2, \tag{2.13}$$

namely we require that μ_j satisfies

$$\log 8\mu_j^2 = -2\log 2m_j^2 - H(\xi_j,\xi_j) + \sum_{i\neq j} m_i m_j^{-1} G(\xi_i,\xi_j). \tag{2.14}$$

Then we get

$$f(\tilde{U}) = m_j\big(1 + 2\lambda m_j^2 w_j + O(\lambda)\big)e^{w_j}\varepsilon_j^{-2}e^{\lambda m_j^2 w_j^2}\big(1+O(\theta)\big)\big(1+O(\lambda)w_j\big).$$

Now, it is easily checked that there is a $C>0$ such that for all $|x-\xi_j|<\delta$ we have

$$\big|\theta(x)\big| = O\bigg(|x-\xi_j| + \sum_i \varepsilon_i^2\bigg) \leqslant \frac{C}{\log \varepsilon_j^{-4}}\bigg(\log\bigg(1+\frac{|x-\xi_j|}{\varepsilon_j}\bigg)+1\bigg)$$

and hence

$$\big(1+O(\theta)\big)\big(1+O(\lambda w_j)\big) \leqslant \big(1 + C\lambda|w_j|\big).$$

Hence, the error of approximation is given near ξ_j by

$$R(x) = m_j\varepsilon_j^{-2}e^{w_j}\big\{1 - \big(1 + 2m_j^2\lambda w_j + O(\lambda)\big)e^{m_j^2\lambda w_j^2}\big(1+O(\lambda w_j)\big)\big\} + \sum_i O\big(\varepsilon_i^2\big).$$

Observe that for $|x-\xi_j| = O(\varepsilon)$ we have that $R(x) \sim \lambda\varepsilon_j^{-2}e^{w_j}$. On the other hand, for $|x-\xi_j|>\delta$ for all j we clearly have that $|R(x)|\leqslant C\lambda$. Hence the error of approximation satisfies the global bound

$$\big|R(x)\big| \leqslant C\lambda\rho(x),$$

where

$$\rho(x) := \sum_{j=1}^k \rho_i \chi_{B_\delta(\xi_j)}(x) + 1;$$

here χ denotes the characteristic function and

$$\rho_j := \frac{1}{2m_j^2\lambda}\big\{\big(1 + 2m_j^2\lambda(w_j+1)\big)\big(1+\lambda\big(1+|w_j|\big)\big)e^{m_j^2\lambda w_j^2} - 1\big\}\varepsilon_j^{-2}e^{w_j}.$$

For later reference let us notice that

$$\rho_j(x) = c\gamma_j\left\{\left(1+\frac{1}{\gamma_j}(w_j+1)\right)\left(1+\frac{1}{\gamma_j}\big(1+|w_j|\big)\right)e^{\frac{w_j^2}{2\gamma_j}}-1\right\}\varepsilon_j^{-2}e^{w_j}, \tag{2.15}$$

where $\gamma_j = \log \varepsilon_j^{-4}$.

This motivates us to introduce the following L^∞-weighted norm for bounded functions defined in Ω. Let us set

$$\rho(x) := \sum_{j=1}^{k} \rho_i \chi_{B_\delta(\xi_j)}(x) + 1$$

and define

$$\|h\|_* = \sup_{x\in\Omega} \rho(x)^{-1}\big|h(x)\big|, \tag{2.16}$$

so that

$$\|R\|_* \leqslant C\lambda. \tag{2.17}$$

In the rest of this paper we will look for a solution $\tilde{u}$ of problem (2.1) of the form $\tilde{u} = \tilde{U} + \phi$, where $\tilde{U}$ is defined as above in (2.4), and we aim at finding a solution for which ϕ is small, provided that the points ξ_j and scalars m_j are suitably chosen.

For small ϕ it is natural to rewrite problem (2.1) as a nonlinear perturbation of its linearization, namely,

$$\begin{cases} \Delta\phi + f'(\tilde{U})\phi = -R - [f(\tilde{U}+\phi) - f(\tilde{U}) - f'(\tilde{U})\phi] & \text{in } \Omega,\\ \phi = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.18}$$

Let us observe that

$$f'(\tilde{U}) = \lambda\big(2\lambda\tilde{U}^2+1\big)e^{\lambda\tilde{U}^2} = O(\lambda)$$

away from the points ξ_j, so the linearized operator is a small perturbation of the Laplacian away from the concentration points. Using relations (2.11), (2.12), similarly to the computation for $f(\tilde{U})$ we find that very close to the points ξ_j, at least for distances $O(\varepsilon_j)$ from ξ_j, we have

$$f'(\tilde{U}) \approx \varepsilon_j^{-2}e^{w_j}.$$

Moreover, repeating the corresponding computation for R, we readily get that

$$\left\| f'(\tilde{U}) - \sum_{j=1}^{k}\varepsilon_j^{-2}e^{w_j}\right\|_* \leqslant C\lambda. \tag{2.19}$$

For this reason, it is more convenient to rewrite problem (2.18) in the form

$$\begin{cases} L(\phi) := \Delta\phi + [\sum_{j=1}^{k} \varepsilon_j^{-2} e^{w_j(x)}]\phi = -[R + N(\phi)], & \text{in } \Omega, \\ \phi = 0, & \text{on } \partial\Omega, \end{cases} \tag{2.20}$$

where

$$N(\phi) = \left[f(\tilde{U} + \phi) - f(\tilde{U}) - f'(\tilde{U})\phi\right] + \left[f'(\tilde{U}) - \sum_{j=1}^{k} \varepsilon_j^{-2} e^{\omega_j(x)}\right]\phi. \tag{2.21}$$

What we hope for is to find a small solution ϕ to problem (2.21) which respects the size just defined for the error, namely so that $\|\phi\|_\infty \leqslant C\lambda$. Let us observe that the function $\varepsilon_j^{-2} e^{w_j}$ is actually very concentrated near ξ_j: It has size $O(\varepsilon_j^2)$ away from the point, while it globally integrates to 8π in $\mathbb{R}^2$. L is therefore a nontrivial perturbation of the Laplacian near the points while it is essentially this operator in most of the domain. Unlike the Laplacian, the operator L has an approximate kernel which in principle prevents any form of bounded invertibility. In fact L can be approximately regarded as a superposition of the linear operators

$$L_j(\phi) = \Delta\phi + \varepsilon_j^{-2} e^{w_j}\phi.$$

The problem $L_j(\phi) = 0$ has bounded solutions originating in the natural invariances of the equation $\Delta w + e^w = 0$. Let us consider the family of solutions $w_\mu(y)$ given by (2.7). Then the functions

$$z_{0j}(y) = \partial_\mu w_{\mu_j}(y), \qquad z_{lj}(y) = \partial_{y_l} w_{\mu_j}(y), \quad l = 1, 2,$$

satisfy the equation $\Delta Z + e^{w_{\mu_j}} Z = 0$. Hence the functions

$$Z_{ij}(x) := z_{ij}\left(\frac{x - \xi_j}{\varepsilon_j}\right), \quad i = 0, 1, 2, \tag{2.22}$$

are bounded solutions of $L_j(Z) = 0$ in all of $\mathbb{R}^2$. It is known that these actually span the space of *all* bounded solutions of this equation, see [8] for a proof.

We want to solve, in a uniformly bounded way, problems of the form $L(\phi) = h$. This is possible, but only for a restricted class of right-hand sides. We identify them by considering the problem projected to a suitable orthogonal of the "almost-kernel" for L. To formulate this problem, let us consider now a large but fixed number $R_0 > 0$ and a nonnegative function $\zeta(\rho)$ with $\zeta(\rho) = 1$ if $\rho < R_0$ and $\chi(\rho) = 0$ if $\rho > R_0 + 1$. We denote

$$\zeta_j(x) = \varepsilon_j^{-2} \zeta\left(\left|\frac{x - \xi_j}{\varepsilon_j}\right|\right). \tag{2.23}$$

Given $h \in L^\infty(\Omega)$, we consider the linear problem of finding a function ϕ such that for certain scalars c_{ij}, $i = 0, 1, 2$, $j = 1, \dots, k$, it satisfies

$$L(\phi) = h + \sum_{i=0}^{2}\sum_{j=1}^{k} c_{ij} Z_{ij}\zeta_j, \quad \text{in } \Omega, \tag{2.24}$$

$$\phi = 0, \quad \text{on } \partial\Omega, \tag{2.25}$$

$$\int_\Omega Z_{ij}\zeta_j\phi = 0, \quad \text{for all } i, j. \tag{2.26}$$

Consider the norm

$$\|\phi\|_\infty = \sup_{x\in\Omega}\big|\phi(x)\big|.$$

Proposition 2.1. *Let $\delta > 0$ be fixed. There exist positive numbers λ_0 and C, such that for any points ξ_j, $j = 1, \ldots, m$ in Ω, parameters m_j, $j = 1, \ldots, k$, satisfying* (2.2)*, μ_j given by* (2.14)*, and $h \in L^\infty(\Omega)$, there is a unique solution $\phi := T_\lambda(h)$ to problem* (2.24)–(2.26) *for all $\lambda < \lambda_0$. Moreover*

$$\|\phi\|_\infty \leqslant C\|h\|_*. \tag{2.27}$$

We will prove this result in the next section. It is worth mentioning that the criticality of the Moser Trudinger situation is fairly delicate compared with other problems of concentration phenomena. Not only is the form of the seeked solutions quite nonobvious, but also, even with the right ansatz, the error is not small, and the invertibility theory in which the weight ρ enters is indeed *very tight*.

Let us consider now the projected version of problem (2.20),

$$L(\phi) = -R - N(\phi) + \sum_{i=0}^{2}\sum_{j=1}^{k} c_{ij} Z_{ij}\zeta_j, \quad \text{in } \Omega, \tag{2.28}$$

$$\phi = 0, \quad \text{on } \partial\Omega, \tag{2.29}$$

$$\int_\Omega Z_{ij}\zeta_j\phi = 0, \quad \text{for all } i, j. \tag{2.30}$$

To solve this problem in $L^\infty(\Omega)$, we recast it in fixed point form

$$\phi = T_\lambda\big(-R - N(\phi)\big) := A(\phi) \tag{2.31}$$

where T_λ is the operator in Proposition 2.1. Using estimate (2.19) and the easily checked fact that $\|f''(\tilde{U})\|_* \leqslant C$ we find that

$$\big\|N(\phi)\big\|_* \leqslant C\|\phi\|_\infty^2 + C\lambda\|\phi\|_\infty.$$

This estimate, Proposition 2.1 and estimate (2.17) imply that $A(\mathcal{B}) \subset \mathcal{B}$ where $\mathcal{B} = \{\phi / \|\phi\|_\infty \leqslant M\lambda\}$ for a sufficiently large and fixed M and all small λ. Besides, it is directly checked that the operator A has a small Lipschitz constant in $\mathcal{B}$ for all small λ. Thus, the contraction mapping principle leads us to the following fact.

Proposition 2.2. *Under the assumptions of Proposition* 2.1 *there exist positive numbers C and λ_0, such that for all $0 < \lambda < \lambda_0$ problem* (2.28)–(2.30) *has a solution $\phi = \phi(\xi, m)$ which defines a continuous map into $L^\infty(\Omega)$ and satisfies*

$$\|\phi\|_\infty \leqslant C\lambda.$$

The constant C is uniform on all (ξ, m) satisfying the constraints (2.2).

As a function of points and parameters, this ϕ is actually smooth. We will check this in Section 4. Evaluating at this ϕ in problem (2.28)–(2.30), the constants c_{ij} define functions $c_{ij} = c_{ij}(\xi, m)$. Thus we need to find a solution (ξ, m) of the $3k \times 3k$ system

$$c_{ij}(\xi, m) = 0 \quad \text{for all } i = 0, 1, 2,\ j = 1, \dots, k. \tag{2.32}$$

To solve this problem, we formulate it in variational form. Let J_λ be the energy functional of problem (1.1) defined in (1.2). As we will see in Lemma 5.1, if (ξ, m) is a critical point of the functional

$$(\xi, m) \longmapsto J_\lambda\big(\sqrt{\lambda}\big(\tilde{U}(\xi, m) + \phi(\xi, m)\big)\big) =: \mathcal{I}_\lambda(\xi, m), \tag{2.33}$$

then it automatically satisfies system (2.32). In Lemma 6.1 we will show that at main order we have

$$\mathcal{I}_\lambda(\xi, m) = 2k\pi + a\lambda + \lambda 4\pi \varphi_k(\xi, m) + o(\lambda)$$

where a is a fixed constant and φ_k is the functional introduced in (1.6) and $o(1) \to 0$ in the C^1-sense as $\lambda \to 0$, uniformly on (ξ, m) satisfying (2.2). From here, using the definition of nontrivial critical point situation, the result of Theorem 1 immediately follows. Theorem 2 will be established as a special case of this result in Section 7.

In the remainder of this paper we will carry out the above outlined construction.

A main step in solving problem (2.20) for small ϕ under a suitable choice of the points ξ_j and the parameters m_j is that of a solvability theory for the linear operator L. This is the content of next section.

3. Analysis of the linearized operator

We will prove here Proposition 2.1. At the very core of the proof is the following estimate for the Laplacian. Let us consider fixed positive numbers R and M and for $\varepsilon > 0$ the annular region

$$A_\varepsilon = \left\{x/R < |y| < \frac{M}{\varepsilon}\right\}$$

and the function

$$\rho_\varepsilon\big(|y|\big) = \gamma e^w\left(e^{\frac{w^2}{2\gamma}}\left(1 + \frac{w+1}{\gamma}\right)\left(1 + \frac{|w|}{\gamma}\right) - 1\right)$$

where

$$w(|y|)=\log\frac{8\mu^2}{(\mu^2+|y|^2)^2},\qquad \gamma=\log\varepsilon^{-4}.$$

Let us consider the problem

$$\begin{cases}-\Delta\Psi=\rho_\varepsilon & \text{in } A_\varepsilon,\\ \Psi=0 & \text{on } \partial A_\varepsilon.\end{cases}\tag{3.1}$$

Lemma 3.1. *There exist constants C, ε_0 depending only on uniform upper and away from zero lower bounds for R, M, such that for all $\varepsilon<\varepsilon_0$ the solution Ψ_ε to problem* (3.1) *satisfies $\|\Psi_\varepsilon\|_\infty\leqslant C$.*

Proof. The solution Ψ is radial, say $\Psi=\Psi(r)$, $r=|y|$. Let us consider the change of variables $\psi(t)=\Psi(e^t)$. Then it is straightforward to check that ψ satisfies the two-point boundary value problem

$$-\psi''(t)=e^{2t}\rho_\varepsilon(e^t),\quad t\in[\log R,\log M+\gamma/4],\qquad \psi(\log R)=\psi(\log M+\gamma/4)=0.$$

Since $w(r)=-4\log r+O(1)$ in all the considered range, we find that we can estimate

$$e^{2t}\rho_\varepsilon(e^t)\leqslant Ce^{-2t}\left\{\left(1+\frac{t}{\gamma}+\frac{M}{\gamma}\right)\left(1+\frac{t}{\gamma}\right)e^{\frac{2t^2}{\gamma}}-1\right\}=:g(t).$$

Thus, in order to prove the desired result it suffices to show that the solution of the problem

$$-\tilde\psi''(t)=g(t),\quad t\in[\log R,\log M+\gamma/4],\qquad \tilde\psi(\log R)=\tilde\psi(\log M+\gamma/4)=0$$

is uniformly bounded. Here and in the rest of the proof, C denotes a generic constant independent of large γ.

Since $\tilde\psi$ is concave and positive, it suffices to show that the quantity $\tilde\psi(a)$ is uniformly bounded at some point a distant of order γ from the boundary, let us say $a=\frac{\gamma}{8}$. Let $G(t,s)$ be the Green's function of $-\psi''$ with Dirichlet boundary conditions in the interval. Then

$$G\left(t,\frac{\gamma}{8}\right)\leqslant C\min\left\{t-\log R,\log M+\frac{\gamma}{4}-t\right\}.$$

Hence, since

$$\tilde\psi\left(\frac{\gamma}{8}\right)=\int_{\log R}^{\log M+\frac{\gamma}{4}}G\left(t,\frac{\gamma}{8}\right)g(t)\,dt$$

we get

$$\left|\tilde{\psi}\left(\frac{\gamma}{8}\right)\right| \leqslant C \int_{\log R}^{\frac{\gamma}{8}} (t - \log R)\left|g(t)\right| dt + C \int_{\frac{\gamma}{8}}^{\log M + \frac{\gamma}{4}} \left(\log M + \frac{\gamma}{4} - t\right)\left|g(t)\right| dt. \tag{3.2}$$

The first integral in (3.2) can be estimated as follows

$$\int_{\log R}^{\frac{\gamma}{8}} (t - \log R)\left|g(t)\right| dt \leqslant C \int_{\log R}^{\frac{\gamma}{8}} e^{-2t} e^{\frac{2t^2}{\gamma}} \left(1 + t^4\right) dt \leqslant C \int_{\log R}^{\frac{\gamma}{8}} e^{-t}\left(1 + t^4\right) dt \leqslant C,$$

since in this region $\frac{2t^2}{\gamma} \leqslant t$. Concerning the second integral in (3.2), we observe that in that region $|g(t)| \sim \gamma e^{-2t} e^{\frac{2t^2}{\gamma}} (1 - \frac{t}{\gamma})$, so we get

$$\int_{\frac{\gamma}{8}}^{\log M + \frac{\gamma}{4}} \left(\log M + \frac{\gamma}{4} - t\right)\left|g(t)\right| dt \leqslant C \int_{\frac{\gamma}{8}}^{\frac{\gamma}{4} + O(1)} e^{-2t + \frac{2t^2}{\gamma}} \left(\gamma - t + O(1)\right)^2 \leqslant C \int_{O(1)}^{\frac{\gamma}{4} + O(1)} e^{-s}\left(s + O(1)\right)^2 ds \leqslant C,$$

where we have used the change of variable $s = \gamma - t$.

We have thus established the existence of a positive constant C independent of ε such that

$$\left|\Psi(y)\right| \leqslant C, \quad \text{for } R < |y| < \frac{M}{\varepsilon},$$

and the proof of the lemma is concluded. □

An immediate consequence of the above estimate is the following.

Corollary 1. *Let us consider now the weight $\rho_j(x)$ defined in* (2.15)*, and the solution Ψ_j to the problem*

$$-\Delta\Psi_j(x) = \rho_j, \quad \text{in } R\varepsilon_j < |x - \xi_j| < M,$$
$$\Psi_j = 0 \quad \text{on } \partial A_\varepsilon. \tag{3.3}$$

Then there exists a $C > 0$, independent of all small ε_j such that

$$\left|\Psi_j(x)\right| \leqslant C, \quad \text{for all } R\varepsilon_j < |x - \xi_j| < M.$$

The proof of Proposition 2.1 makes use of an a priori estimate.

Lemma 3.2. *Under the assumptions of Proposition* 2.1*, there exist positive numbers λ_0 and C, such that for any points ξ_j, $j = 1, \ldots, m$ in Ω, and positive numbers m_j, μ_j, for $j = 1, \ldots, k$, which satisfy relations* (2.2) *and* (2.14)*, and any solution ϕ to* (2.24)–(2.26)*, one has*

$$\|\phi\|_\infty \leqslant C\|h\|_*, \tag{3.4}$$

for all $\lambda < \lambda_0$.

Proof. We will carry out the proof of the a priori estimate (3.4) by contradiction. We assume then the existence of sequences $\lambda_n \to 0$, points $\xi_j^n \in \Omega$ and numbers m_j^n, μ_j^n which satisfy relations (2.2) and (2.14), functions h_n with $\|h_n\|_* \to 0$, ϕ_n with $\|\phi_n\|_\infty = 1$, and constants c_{ijn} such that

$$L(\phi_n) = h_n + \sum_{i=0}^{2}\sum_{j=1}^{k} c_{ijn} Z_{ij}\zeta_j, \quad \text{in } \Omega, \tag{3.5}$$

$$\phi_n = 0, \quad \text{on } \partial\Omega, \tag{3.6}$$

$$\int_\Omega Z_{ij}\zeta_j\phi_n = 0, \quad \text{for all } i, j. \tag{3.7}$$

We will prove that in reality under the above assumption we must have that $\phi_n \to 0$ uniformly in Ω, which is a contradiction that concludes the result of the lemma.

Passing to a subsequence we may assume that the points ξ_j^n approach limiting, distinct points ξ_j^* in Ω. We claim that $\phi_n \to 0$ in C^1 local sense on compacts of $\bar{\Omega} \setminus \{\xi_1^*, \ldots, \xi_k^*\}$. Indeed, let us observe that $h_n \to 0$ locally uniformly, away from the points ξ_j. Away from the ξ_j's we have then $\Delta\phi_n \to 0$ uniformly on compact subsets on $\bar{\Omega} \setminus \{\xi_1^*, \ldots, \xi_k^*\}$. Since ϕ_n is bounded it follows also that passing to a further subsequence, ϕ_n approaches in C^1 local sense on compacts of $\bar{\Omega} \setminus \{\xi_1^*, \ldots, \xi_k^*\}$ a limit ϕ^* which is bounded and harmonic in $\Omega \setminus \{\xi_1, \ldots, \xi_k\}$ and $\phi^* = 0$ on $\partial\Omega$. Hence ϕ^* extends smoothly to a harmonic function in Ω, so that $\phi^* = 0$, and the claim follows.

For notational convenience, we shall omit the explicit dependence on n in the rest of the proof.

Next we claim that $c_{ij} \to 0$ for all $i = 0, 1, 2$ and $j = 1, \ldots, k$. Fix j and multiply Eq. (3.5) against $Z_{ij}(x) = z_{ij}(\frac{x-\xi_j}{\varepsilon_j})$, with z_{ij} given by (2.22). We integrate the resulting relation in $B(\xi_j, \delta)$ and obtain

$$\int_{B(\xi_j,\delta)} hZ_{ij} + \sum_{l=0}^{2} c_{lj}\int Z_{ij}Z_{lj}\zeta_j = \int_{B(\xi_j,\delta)} L(\phi)Z_{ij}$$

$$= \int_{B(\xi_j,\delta)} \phi L(Z_{ij}) + \int_{\partial B(\xi_j,\delta)} [Z_{ij}\nabla\phi\cdot\nu - \phi\nabla Z_{ij}\cdot\nu].$$

Let us observe that $L(Z_{ij}) = \sum_{l\neq j} \varepsilon_l^2 = o(1)$ and that the C^1 convergence to zero of ϕ on $\partial B(\xi_j, \delta)$ imply that the boundary integral in the above relation also approaches zero. On the other hand, we have

$$\left| \int\limits_{B(\xi_j,\delta)} hZ_{ij} \right| \leqslant C\|h\|_*, \quad \text{and} \quad \int Z_{ij} Z_{lj}\zeta_j = \kappa_i \delta_{lj}$$

where κ_i is a positive universal constant. Then we get $c_{ij} \to 0$, as desired. Observe then that $\tilde{h}_n := h_n + \sum c_{ij}^n Z_{ij}\zeta_j$ satisfies $\|\tilde{h}_n\|_* \to 0$.

Our next claim is that ϕ_n approaches zero uniformly, close to the concentration points. More precisely, we have that

$$\sup_{|x-\xi_j|<R\varepsilon_j} \left|\phi(x)\right| \to 0 \quad \text{for all } R > 0.$$

Let us assume the opposite, so that there exists an index j and $R > 0$ such that, for all n,

$$\sup_{|x-\xi_j^n|<R\varepsilon_j} \left|\phi_n(x)\right| \geqslant \kappa > 0.$$

Let us set $\hat{\phi}_n(z) = \phi_n(\xi_j^n + \varepsilon_j z)$. Elliptic estimates imply that $\hat{\phi}_n$ converges uniformly over compacts to a bounded solution $\hat{\phi} \neq 0$ of the problem in $\mathbb{R}^2$

$$\Delta\phi + \frac{8\mu_j^2}{(\mu_j^2 + |z|^2)^2}\phi = 0.$$

According to the nondegeneracy result in [8], $\hat{\phi}$ is therefore a linear combination of the functions z_{ij}, $i = 0, 1, 2$. However, our assumed orthogonality conditions on ϕ_n pass to the limit and yield $\int \zeta(|z|)z_{ij}\hat{\phi} = 0$, for $i = 0, 1, 2$, thus a contradiction from which the claim follows.

Let us fix such a number $R > 0$ which we may take larger whenever it is needed and consider the quantity

$$\|\phi\|_i = \sup_{\bigcup_{j=1}^k B(\xi_j, R\varepsilon_j)} |\phi|.$$

The desired result, $\|\phi_n\|_\infty \to 0$ is a consequence of the following fact: there is a uniform constant $C > 0$ such that

$$\|\phi\|_\infty \leqslant C\big[\|\phi\|_i + \|h\|_*\big]. \tag{3.8}$$

We will establish this with the use of barriers. Indeed, a crucial fact is that the operator L satisfies the maximum principle in Ω outside balls centered at the points ξ_j of radius $R\varepsilon_j$, for some fixed $R > 0$. Let us check this. Given $a > 0$, consider the function

$$Z(x) = \sum_{j=1}^{k} z_0\left(a\frac{|x-\xi_j|}{\varepsilon_j}\right), \quad x \in \Omega, \tag{3.9}$$

where $z_0(r) = \frac{r^2-1}{1+r^2}$ is the radial solution in $\mathbb{R}^2$ of $\Delta z_0 + \frac{8}{(1+r^2)^2} z_0 = 0$. Then we have

$$-\Delta Z = \sum_{j=1}^{k} \varepsilon_j^{-2} \frac{8a^2(a^2\varepsilon_j^{-2}|x-\xi_j|^2 - 1)}{(1+a^2\varepsilon_j^{-2}|x-\xi_j|^2)^3}.$$

So that for $|x-\xi_j|^2 > 100a^{-2}\varepsilon_j^2$ for all j,

$$-\Delta Z \geqslant 2\sum_{j=1}^{k} \frac{a^2\varepsilon_j^{-2}}{(1+a^2\varepsilon_j^{-2}|x-\xi_j|^2)^2} \geqslant \sum_{j=1}^{m} \frac{\varepsilon_j^2}{a^2|x-\xi_j|^4}.$$

On the other hand, in the same region,

$$\left(\sum_{j=1}^{k} \varepsilon_j^{-2} e^{\tilde{\omega}_j}\right) Z \leqslant C \sum_{j=1}^{m} \frac{\varepsilon_j^2}{|x-\xi_j|^4}.$$

Hence if a is taken small and fixed, and $R > 0$ is chosen sufficiently large depending on this a, then we have that $L(Z) < 0$ in $\tilde{\Omega} := \Omega \setminus \bigcup_{j=1}^{m} B(\xi_j, R\varepsilon_j)$ and $Z > 0$ in $\tilde{\Omega}$. Thus in this region L satisfies the maximum principle, namely if $L(\psi) \leqslant 0$ in $\tilde{\Omega}$ and $\psi \geqslant 0$ on $\partial\tilde{\Omega}$ then $\psi \geqslant 0$ in $\tilde{\Omega}$.

Let us consider the function Ψ_j in Corollary 1. Since Ψ_j is uniformly bounded, it is easy to see that if R is chosen sufficiently large then the uniformly bounded function $\psi_j(x) := \Psi_j(4x)$ satisfies $-L(\psi_j) > \rho_j$ for $R\varepsilon_j < |x-\xi_j| < M$, where we fix $M > 0$ such that $\Omega \subset B(\xi_j, M)$.

Let us set

$$\tilde{\phi}(y) = 2\|\phi\|_i Z(x) + \|h\|_* \sum_{j=1}^{k} \psi_j(x),$$

where Z is the function defined above in (3.9). Then, it is easily checked that, choosing R larger if necessary, $-L(\tilde{\phi}) \leqslant h$, $\tilde{\phi} \geqslant \phi$ on $\partial\tilde{\Omega}$, where $\tilde{\Omega} = \Omega \setminus \bigcup_{j=1}^{m} B(\xi_j, R\varepsilon_j)$. Hence $|\phi| \leqslant \tilde{\phi}$ on $\tilde{\Omega}$ and estimate (3.8) follows. The proof of the lemma is concluded. □

We have now the main ingredient to prove Proposition 2.1.

Proof of Proposition 2.1. It only remains to prove the solvability assertion. To this purpose we consider the space

$$H = \left\{\phi \in H_0^1(\Omega)\colon \int_{\Omega} \zeta_j Z_{ij}\phi = 0 \text{ for } i = 0, 1, 2,\ j = 1, \dots, k\right\},$$

endowed with the usual inner product $[\phi, \psi] = \int_\Omega \nabla\phi\nabla\psi$. Problem (2.24)–(2.26) expressed in weak form is equivalent to that of finding a $\phi \in H$, such that

$$[\phi, \psi] = \int_\Omega [W\phi - h]\psi\, dx, \quad \text{for all } \psi \in H,$$

where $W = \sum_{j=1}^k \varepsilon_j{}^{-2} e^{w_j}$. With the aid of Riesz's representation theorem, this equation gets rewritten in H in the operator form $\phi = K(\phi) + \tilde{h}$, for certain $\tilde{h} \in H$, where K is a compact operator in H. Fredholm's alternative guarantees unique solvability of this problem for any h provided that the homogeneous equation $\phi = K(\phi)$ has only the zero solution in H. This last equation is equivalent to (2.24)–(2.26) with $h \equiv 0$. Thus existence of a unique solution follows from the a priori estimate (2.27). This finishes the proof. □

Proposition 2.1 says in particular that the unique solution $\phi = T(h)$ of (2.24)–(2.26) defines a continuous linear map from the Banach space $\mathcal{C}_*$ of all functions h in L^∞ for which $\|h\|_* < \infty$, into L^∞, with norm bounded uniformly in λ.

4. Differentiability with respect to parameters

Let ϕ be the solution to the nonlinear projected problem (2.28)–(2.30), whose existence is guaranteed by Proposition 2.2. This section is devoted to study the dependence of ϕ on $(\xi, m) = (\xi_1, \dots, \xi_k, m_1, \dots, m_k)$, where the points ξ_j and the parameters m_j satisfy the constraints (2.2). A direct consequence of the fixed point characterization of ϕ given by Proposition 2.2 together with the fact that the error term R in the right-hand side of Eq. (2.28) depends continuously (in the $*$-norm) on (ξ, m), is that the map $(\xi, m) \to \phi$ into the space $C(\bar{\Omega})$ is continuous (in the ∞-norm).

We analyze next the differentiability of this map, say with respect to ξ_{11}.

We start with the following fact: Fix $h \in \mathcal{C}_*$ and let $\phi = T_\lambda(h)$ be the solution to the linear projected problem (2.24)–(2.26) whose existence is guaranteed by Proposition 2.1. Then, under the same assumptions of Proposition 2.1, there exist positive constants λ_0 and C such that, for all $\lambda < \lambda_0$,

$$\left\| \partial_{\xi_{11}} T_\lambda(h) \right\|_\infty \leqslant C\|h\|_*. \tag{4.1}$$

Indeed, we have in general

$$\left\| \partial_{\xi_{hl}} T_\lambda(h) \right\|_\infty \leqslant C\|h\|_*, \qquad \left\| \partial_{m_j} T_\lambda(h) \right\|_\infty \leqslant C\|h\|_*. \tag{4.2}$$

Differentiating Eq. (2.24) and the orthogonality condition (2.26), we get that $Z := \partial_{\xi_{11}}\phi$ satisfies

$$L(Z) = -\partial_{\xi_{11}}\left(\sum_{j=1}^k \varepsilon_j^{-2} e^{\tilde{\omega}_j(x)}\right)\phi + \sum_{i,j} c_{ij}\partial_{\xi_{11}}(Z_{ij}\zeta_j) + \sum_{i,j} d_{ij} Z_{ij}\zeta_j,$$

with $d_{ij} = \partial_{\xi_{11}}(c_{ij})$, and

$$\int_\Omega Z_{ij}\zeta_j Z = -\int \partial_{\xi_{11}}(Z_{ij}\zeta_j)\phi.$$

Define

$$\alpha_{ab} = \frac{\int \phi \partial_{\xi_{11}}(Z_{ab}\zeta_b)}{\int (Z_{ab}\zeta_b)^2}, \qquad \tilde{Z} = Z + \sum_{a,b} \alpha_{ab} Z_{ab}\zeta_b$$

for $a = 0, 1, 2$ and $b = 1, \ldots, k$. We have $\int \tilde{Z} Z_{ij}\zeta_j = 0$ for all i, j. Furthermore

$$\tilde{Z} = T_\lambda(f), \quad \text{where}$$
$$f = -\partial_{\xi_{11}}\left(\sum_{j=1}^k \varepsilon_j^{-2} e^{\tilde{\omega}_j(x)}\right)\phi + \sum_{i,j} c_{ij}\partial_{\xi_{11}}(Z_{ij}\zeta_j) + \sum_{ab} \alpha_{ab} L(Z_{ab}\zeta_b).$$

Using the result of Proposition 2.1, we get $\|\tilde{Z}\|_\infty \leqslant C\|h\|_*$ and hence the validity of (4.1).

Let ϕ be now the solution to (2.28)–(2.30). Since $\phi = T_\lambda(-(N(\phi) + R))$, we have formally that

$$\partial_{\xi_{11}}\phi = (\partial_{\xi_{11}} T_\lambda)\big(-\big(N(\phi) + R\big)\big) + T_\lambda\big(-\big(\partial_{\xi_{11}} N(\phi) + \partial_{\xi_{11}} R\big)\big).$$

From (4.1) we get

$$\big\|(\partial_{\xi_{11}} T_\lambda)\big(-\big(N(\phi) + R\big)\big)\big\|_\infty \leqslant C\big\|N(\phi) + R\big\|_* \leqslant C\lambda.$$

On the other hand, a direct computation gives

$$\partial_{\xi_{11}} N(\phi) = \big[f'(U+\phi) - f'(U) - f''(U)\phi\big]\partial_{\xi_{11}} U + \partial_{\xi_{11}}\left[f'(U) - \sum_{j=1}^k \varepsilon_j^{-2} e^{\tilde{\omega}_j(x)}\right]\phi$$
$$+ \big[f'(U+\phi) - f'(U)\big]\partial_{\xi_{11}}\phi + \left[f'(U) - \sum_{j=1}^k \varepsilon_j^{-2} e^{\tilde{\omega}_j(x)}\right]\partial_{\xi_{11}}\phi.$$

Thus, using (2.19)

$$\big\|\partial_{\xi_{11}} N(\phi)\big\|_* \leqslant C\big[\lambda^{-1}\|\phi\|_\infty^2 + \lambda\|\phi\|_\infty + \lambda\|\partial_{\xi_{11}}\phi\|_\infty + \lambda\|\partial_{\xi_{11}}\phi\|_\infty\big].$$

Since $\|\partial_{\xi_{11}} R\|_* \leqslant C\lambda$, we can conclude that

$$\|\partial_{\xi_{11}}\phi\|_\infty \leqslant C\lambda.$$

Analogous computation holds true if we differentiate with respect to m_j.

The above computation can be made rigorous by using the implicit function theorem and the fixed point representation (2.31) which guarantees C^1 regularity in (ξ, m). Thus we have the validity of the following:

Lemma 4.1. *Consider the map $(\xi, m) \mapsto \phi$ into the space $C(\bar{\Omega})$, where ϕ is the unique solution to the nonlinear projected problem* (2.28)–(2.30)*, whose existence is guaranteed by Proposition* 2.2*. Under the assumptions of Proposition* 2.1 *the derivative $D_\xi \phi$ (or $D_m \phi$) exists and defines a continuous function of (ξ, m). Besides, there is a constant $C > 0$, such that*

$$\|D_\xi \phi\|_* \leqslant C\lambda, \qquad \|D_m \phi\|_* \leqslant C\lambda.$$

After problem (2.28)–(2.30) has been solved, we will find solutions to the full problem (2.20) (or equivalently (1.1)) if we manage to adjust (ξ, m) in such a way that $c_{ij}(\xi, m) = 0$ for all i, j. A nice feature of this system of equations is that it turns out to be equivalent to finding critical points of a functional of (ξ, m) which is close, in an appropriate sense, to the energy of the first approximation U. We make this precise in the next sections.

5. Variational reduction

As we have said, after problem (2.28)–(2.30) has been solved, we find a solution to problem (2.20) and hence to the original problem if ξ and m is such that

$$c_{ij}(\xi, m) = 0 \quad \text{for all } i, j. \tag{5.1}$$

This problem is indeed variational: it is equivalent to finding critical points of a function of ξ and m. To see this let us recall the energy functional J_λ associated to problem (1.1), namely

$$J_\lambda(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx - \frac{\lambda}{2} \int_\Omega e^{u^2} \, dx. \tag{5.2}$$

We define, as in (2.33),

$$\mathcal{I}_\lambda(\xi, m) \equiv J_\lambda\big(\sqrt{\lambda}\big(\tilde{U}(\xi, m) + \phi(\xi, m)\big)\big), \tag{5.3}$$

where ϕ is the solution of problem (2.28)–(2.30) given by Proposition 2.1. Critical points of $\mathcal{I}_\lambda$ correspond to solutions of (5.1) for small λ, as the following result states.

Lemma 5.1. *Under the assumptions of Proposition* 2.1*, the functional $\mathcal{I}_\lambda(\xi, m)$ is of class C^1. Moreover, for all $\lambda > 0$ sufficiently small,*

$$D_{\xi,m} \mathcal{I}_\lambda(\xi, m) = 0 \quad \Longrightarrow \quad c_{ij}(\xi, m) = 0 \quad \textit{for all } i, j.$$

Proof. A direct consequence of the results obtained in Section 4 and of the definition of the function $\tilde{U}$ is the fact that the map $(\xi, m) \to \mathcal{I}_\lambda(\xi, m)$ is of class C^1.

Furthermore, thanks to Lemma 4.1, we have that

$$\begin{aligned} D_{\xi,m}\mathcal{I}_\lambda(\xi,m) &= DJ_\lambda\big(\sqrt{\lambda}\big(U(\xi,m)+\phi(\xi,m)\big)\big)\big[\sqrt{\lambda}D_{\xi,m}U(\xi,m)\big] \\ &\quad + DJ_\lambda\big(\sqrt{\lambda}\big(U(\xi,m)+\phi(\xi,m)\big)\big)\big[\sqrt{\lambda}D_{\xi,m}\phi(\xi,m)\big] \\ &= DJ_\lambda\big(\sqrt{\lambda}\big(U(\xi,m)+\phi(\xi,m)\big)\big)\big[\sqrt{\lambda}D_{\xi,m}U(\xi,m)\big]\big(1+o(1)\big). \end{aligned} \tag{5.4}$$

Let $\tilde{u}(x) = U(\xi,m)(x)+\phi(\xi,m)(x)$. For any l define

$$I_l(v) = \frac{m_l^2}{2}\int_{\Omega_l}|Dv|^2 - \int_{\Omega_l} e^v e^{\lambda m_l^2 v^2}, \tag{5.5}$$

where $\Omega_l = \frac{\Omega-\xi_l}{\varepsilon_l}$. If we perform the change of variables $\tilde{u}(x) = m_l v_l(\frac{x-\xi_l}{\varepsilon_l}) + \frac{1}{2\lambda m_l}$, we get

$$\tilde{J}_\lambda(\tilde{u}) := J_\lambda(\sqrt{\lambda}u) = I_l(v)$$

and, as a direct consequence of (2.6), (2.10) and (2.13),

$$v_l(y) = \omega_{\mu_l}(y) + \sum_j \big(O\big(|\varepsilon_l y + \xi_l - \xi_j|\big) + O\big(\varepsilon_j^2\big)\big) \quad \text{for } |y| \leqslant \frac{\delta}{\varepsilon_l}. \tag{5.6}$$

Furthermore, we have

$$\Delta\tilde{u} + \lambda\tilde{u}e^{\lambda\tilde{u}^2} = \sum_{ij} c_{ij}\zeta_j Z_{ij}(x), \quad x\in\Omega,$$

and that $\tilde{v}_l(y)$ solves in Ω_l

$$m_l\varepsilon_l^{-2}\big[\Delta\tilde{v}_l + e^v\big(1+2\lambda m_l^2\tilde{v}_l\big)e^{\lambda m_l^2\tilde{v}_l^2}\big] = \sum_{ij} c_{ij}\zeta\left(\frac{\varepsilon_l y+\xi_l-\xi_j}{\varepsilon_j}\right)\varepsilon_j^{-2}Z_{ij}\left(\frac{\varepsilon_l y+\xi_l-\xi_j}{\varepsilon_j}\right).$$

Thus we start with the computation of $D_{m_1}\mathcal{I}_\lambda(\xi,m)$. From (5.4), we get

$$\begin{aligned} D_{m_1}\mathcal{I}_\lambda(\xi,m) &= D_{m_1}I_l(\tilde{v}_l) = DI_l(\tilde{v}_l)[D_{m_1}\tilde{v}_l] \\ &= \sum_{ij}\left(\int_{\Omega_l}\zeta\left(\frac{\varepsilon_l y+\xi_l-\xi_j}{\varepsilon_j}\right)\varepsilon_j^{-2}Z_{ij}\left(\frac{\varepsilon_l y+\xi_l-\xi_j}{\varepsilon_j}\right)D_{m_1}v_l\,dy\right)c_{ij}. \end{aligned}$$

Fix now i and j. To compute the coefficient in front of c_{ij} in the above expression, we choose $l=j$ and obtain

$$\int_{\Omega_l}\zeta\left(\frac{\varepsilon_l y+\xi_l-\xi_j}{\varepsilon_j}\right)\varepsilon_j^{-2}Z_{ij}\left(\frac{\varepsilon_l y+\xi_l-\xi_j}{\varepsilon_j}\right)D_{m_1}v_l\,dy = \frac{\partial\mu_j}{\partial m_1}\int_{\mathbb{R}^2} z_{0j}^2(y)\,dy\,\big(1+o(1)\big).$$

Thus we conclude that, for any $h = 1, \dots, k$,

$$D_{m_h}\mathcal{I}_\lambda(\xi, m) = \sum_j \frac{\partial \mu_j}{\partial m_h} \int_{\mathbb{R}^2} Z_{0j}^2(y)\, dy\, c_{0j}\big(1 + o(1)\big).$$

A direct argument shows on the other hand that, for $a = 1, 2$, $b = 1, \dots, k$,

$$D_{\xi_{ab}}\mathcal{I}_\lambda(\xi, m) = \sum_j \left(\frac{\partial \mu_j}{\partial \xi_{ab}} \int_{\mathbb{R}^2} Z_{0j}^2(y)\, dy\, c_{0j} + \int_{\mathbb{R}^2} Z_{1j}^2(y)\, dy \right) c_{ab}\big(1 + o(1)\big).$$

We can conclude that $D_{\xi,m}\mathcal{I}_\lambda(\xi, m) = 0$ implies the validity of a system of equations of the form

$$\left[\sum_j \frac{\partial \mu_j}{\partial m_h} c_{0j}\right]\big(1 + o(1)\big) = 0, \quad h = 1, \dots, k, \tag{5.7}$$

$$\left[A \sum_j \frac{\partial \mu_j}{\partial \xi_{ab}} c_{0j} + c_{ab}\right]\big(1 + o(1)\big) = 0, \quad a = 1, 2,\ b = 1, \dots, k, \tag{5.8}$$

for some fixed constant A, with $o(1)$ small in the sense of the L^∞ norm as $\lambda \to 0$. The conclusion of the lemma follows if we show that the matrix $(\frac{\partial \mu_j}{\partial m_h})$ of dimension $k \times k$ is invertible in the range of the points ξ_j and parameters m_j we are considering. Indeed, this fact implies unique solvability of (5.7). Inserting this in (5.8) we get unique solvability of (5.8).

Consider the definition of the μ_j's, in terms of m_j's and points ξ_j given in (2.14). These relations correspond to the gradient $D_m F(m, \xi)$ of the function F defined as follows

$$F(m, \xi) = \frac{1}{2} \sum_{j=1}^k m_j^2\big[-2\log 2m_j^2 - \log 8\mu_j^2 - 1 - H(\xi_j, \xi_j)\big] + \sum_{i \neq j} G(\xi_i, \xi_j) m_i m_j.$$

It is natural to perform the change of variable $s_j = m_j^2$. With abuse of notation, the above function now reads as follows

$$F(s, \xi) = \frac{1}{2} \sum_{j=1}^k s_j\big[-2\log 2s_j - \log 8\mu_j^2 - 1 - H(\xi_j, \xi_j)\big] + \sum_{i \neq j} G(\xi_i, \xi_j) \sqrt{s_i s_j}.$$

This is a strictly convex function of the parameters s_j, for parameters s_j uniformly bounded and uniformly bounded away from 0 and for points ξ_j in Ω uniformly far away from each other and from the boundary. For this reason, the matrix $(\frac{\partial^2 F}{\partial s_i \partial s_j})$ is invertible in the range of parameters and points we are considering. Thus, by the implicit function theorem, relation (2.14) defines a diffeomorphism between μ_j and m_j. This fact gives the invertibility of $(\frac{\partial \mu_j}{\partial m_l})$ we were aiming at.

This concludes the proof of the lemma. □

In order to solve for critical points of the functional $\mathcal{I}_\lambda$, a key step is its expected closeness to the functional $J_\lambda(\sqrt{\lambda}\tilde{U})$, which we will analyze in the next section.

Lemma 5.2. *The following expansion holds*

$$\mathcal{I}_\lambda(\xi, m) = J_\lambda(\sqrt{\lambda}\tilde{U}) + \theta_\lambda(\xi, m),$$

where

$$|\theta_\lambda| + |\nabla\theta_\lambda| = O\big(\lambda^3\big),$$

uniformly on points and parameters satisfying the constraints in Proposition 2.1.

Proof. Taking into account $DJ_\lambda(\sqrt{\lambda}(\tilde{U}+\phi))[\phi] = 0$, a Taylor expansion gives

$$\begin{aligned} &J_\lambda\big(\sqrt{\lambda}(\tilde{U}+\phi)\big) - J_\lambda(\sqrt{\lambda}\tilde{U}) \\ &\quad = \lambda \int_0^1 D^2 J_\lambda\big(\sqrt{\lambda}(\tilde{U}+t\phi)\big)[\phi]^2\,(1-t)\,dt \qquad (5.9)\\ &\quad = \lambda \int_0^1 \bigg(\int_\Omega \big[N(\phi)+R\big]\phi + \int_\Omega \big[f'(\tilde{U}) - f'\big((\tilde{U}+t\phi)\big)\big]\phi^2\bigg)(1-t)\,dt. \qquad (5.10)\end{aligned}$$

Since $\|\phi\|_\infty \leqslant C\lambda$, we get

$$J_\lambda\big(\sqrt{\lambda}(\tilde{U}+\phi)\big) - J_\lambda(\sqrt{\lambda}\tilde{U}) = \tilde{\theta}_\lambda = O\big(\lambda^3\big).$$

Let us differentiate with respect to ξ. We use the representation (5.9) and differentiate directly under the integral sign, thus obtaining, for each $j = 1, 2$, $l = 1, \dots, k$,

$$\begin{aligned} &\partial_{\xi_{jl}}\big[J_\lambda\big(\sqrt{\lambda}(\tilde{U}+\phi)\big) - J_\lambda(\sqrt{\lambda}\tilde{U})\big] \\ &\quad = \lambda \int_0^1 \bigg(\int_\Omega \partial_{\xi_{jl}}\big[\big(N(\phi)+R\big)\phi\big] + \int_\Omega \partial_{\xi_{jl}}\big[f'(\tilde{U}) - f'(\tilde{U}+t\phi)\big]\phi^2\bigg)(1-t)\,dt.\end{aligned}$$

Using the fact that $\|\partial_\xi\phi\|_\infty \leqslant C\lambda$ and the computations in the proof of Lemma 4.1 we get

$$\partial_{\xi_{jl}}\big[J_\lambda\big(\sqrt{\lambda}(\tilde{U}+\phi)\big) - J_\lambda(\sqrt{\lambda}\tilde{U})\big] = \partial_{\xi_{jl}}\tilde{\theta}_\lambda = O\big(\lambda^3\big).$$

In a very analogous way one gets

$$\partial_{m_j}\big[J_\lambda\big(\sqrt{\lambda}(\tilde{U}+\phi)\big) - J_\lambda(\sqrt{\lambda}\tilde{U})\big] = O\big(\lambda^3\big).$$

The continuity in ξ and m of all these expressions is inherited from that of ϕ and its derivatives in ξ and in m in the L^∞ norm. The proof is complete. □

6. Asymptotics of energy of approximate solution

The purpose of this section is to give an asymptotic estimate of $J_\lambda(U)$ where $U(x)=\sqrt{\lambda}\tilde{U}$. The function U is defined as

$$U(x)=\sqrt{\lambda}\sum_{j=1}^{k} m_j\left[\log\frac{1}{(\mu_j^2\varepsilon_j^2+|x-\xi_j|^2)^2}-H_j(x)\right] \tag{6.1}$$

(see (2.4)) and J_λ is the energy functional associated to (1.1), whose definition we recall below

$$J_\lambda(u)=\frac{1}{2}\int_\Omega |\nabla u|^2\,dx-\frac{\lambda}{2}\int_\Omega e^{u^2}\,dx.$$

We have the following result.

Lemma 6.1. *Let $\delta>0$ be a fixed small number and U be the function defined in* (6.1). *With the choice* (2.14) *for the parameters μ_j, the following expansion holds*

$$J_\lambda(U)=2k\pi+a\lambda+4\pi\lambda\varphi_k(\xi,m)+\lambda^2\Theta_\lambda(\xi,m) \tag{6.2}$$

where the function $\varphi_k(\xi,m)=\varphi_k(\xi_1,\dots,\xi_k,m_1,\dots,m_k)$ is defined by

$$\varphi_k(\xi,m)=b\sum_{j=1}^{k}m_j^2+2\sum_{j=1}^{k}m_j^2\log m_j^2 + \sum_{j=1}^{k}H(\xi_j,\xi_j)m_j^2-\sum_{i\neq j}G(\xi_i,\xi_j)m_im_j. \tag{6.3}$$

Here G and H are the Green function for the Laplacian on Ω with Dirichlet boundary condition and its regular part, as defined in Section 1*, and a, b are absolute constants. In* (6.2)*, Θ_λ is a smooth function of $(\xi,m)=(\xi_1,\dots,\xi_k,m_1,\dots,m_k)$, bounded together with its derivatives, as $\lambda\to 0$, uniformly on points $\xi_1,\dots,\xi_m\in\Omega$ and parameters $(m_1,\dots,m_k)\in(\mathbb{R}^+)^k$ satisfying* (2.2).

Remark 6.1. In the sequel, by θ_λ, Θ_λ we will denote generic functions of ξ and m that are bounded, together with its derivatives, in the region $\operatorname{dist}(\xi_i,\partial\Omega)>\delta$ and $|\xi_i-\xi_j|>\delta$, and $\delta<m_j<\frac{1}{\delta}$.

Proof. We write

$$U(x)=\sum_{j=1}^{k}U_j(x),\quad \text{with } U_j(x)=\sqrt{\lambda}m_j\big[u_j(x)-H_j(x)\big]$$

and

$$u_j(x) = \log \frac{1}{(\mu_j^2 \varepsilon_j^2 + |x - \xi_j|^2)^2}.$$

We start with

$$\frac{1}{2} \int_\Omega |\nabla U|^2 \, dx = \frac{1}{2} \left(\sum_{j=1}^m \int_\Omega |\nabla U_j|^2 \, dx + \sum_{j \neq i} \int_\Omega \nabla U_j \nabla U_i \, dx \right). \tag{6.4}$$

Fix j. We have

$$\begin{aligned} \frac{1}{2} \int_\Omega |\nabla U_j|^2 \, dx &= \lambda m_j^2 \left[\frac{1}{2} \int_\Omega |\nabla u_j|^2 \, dx - \int_\Omega \nabla u_j \nabla H_j \, dx + \frac{1}{2} \int_\Omega |\nabla H_j|^2 \, dx \right] \\ &= \lambda m_j^2 \left[\frac{1}{2} \int_\Omega |\nabla u_j|^2 \, dx - \int_{\partial\Omega} u_j \frac{\partial H_j}{\partial \nu} \, d\sigma + \frac{1}{2} \int_{\partial\Omega} H_j \frac{\partial H_j}{\partial \nu} \, d\sigma \right] \\ &= \lambda m_j^2 \left[\frac{1}{2} \int_\Omega |\nabla u_j|^2 \, dx - \frac{1}{2} \int_{\partial\Omega} H_j \frac{\partial H_j}{\partial \nu} \, d\sigma \right] \end{aligned} \tag{6.5}$$

where ν denotes the unitary outer normal of $\partial\Omega$. In the above equation we used the facts that H_j is harmonic in Ω and U_j is zero on the boundary $\partial\Omega$.

We will now evaluate $\int_\Omega |\nabla u_j|^2 \, dx$. Let $\tilde{\delta} > 0$ be small and fixed. We split the previous integral into two pieces, namely

$$\int_\Omega |\nabla u_j|^2 \, dx = \int_{B(\xi_j, \tilde{\delta})} |\nabla u_j|^2 \, dx + \int_{\Omega \setminus B(\xi_j, \tilde{\delta})} |\nabla u_j|^2 \, dx. \tag{6.6}$$

Direct computations show, using (2.10), (2.13) and (2.14)

$$\begin{aligned} &\int_{B(\xi_j, \tilde{\delta})} |\nabla u_j|^2 \, dx \\ &= 16 \int_{B(\xi_j, \tilde{\delta})} \frac{|x - \xi_j|^2}{(\mu_j^2 \varepsilon_j^2 + |x - \xi_j|^2)^2} \, dx \\ &= 16 \int_{B(0, \frac{\tilde{\delta}}{\mu_j \varepsilon_j})} \frac{|y|^2}{(1 + |y|^2)^2} \, dy \quad \left(y = \frac{x - \xi_j}{\varepsilon_j \mu_j} \right) \\ &= 16\pi \left[-2 \log \varepsilon_j \mu_j - 1 + \log\left[(\varepsilon_j \mu_j)^2 + \tilde{\delta}^2 \right] + \frac{(\varepsilon_j \mu_j)^2}{(\varepsilon_j \mu_j)^2 + \tilde{\delta}^2} \right] \end{aligned}$$

$$
\begin{aligned}
&= 16\pi\Big[-2\log\varepsilon_j - \log 8\mu_j^2 + \log 8 - 1 + \log\big[(\varepsilon_j\mu_j)^2 + \tilde\delta^2\big] + \frac{(\varepsilon_j\mu_j)^2}{(\varepsilon_j\mu_j)^2 + \tilde\delta^2}\Big]\\
&= 16\pi\Big[\frac{1}{4m_j^2\lambda} - \frac{\beta_j}{2} + 2\log 2m_j^2 + H(\xi_j,\xi_j) - \sum_{i\neq j} m_i m_j^{-1} G(\xi_i,\xi_j) + \log 8 - 1\\
&\quad + \log\big[(\varepsilon_j\mu_j)^2 + \tilde\delta^2\big] + \frac{(\varepsilon_j\mu_j)^2}{(\varepsilon_j\mu_j)^2 + \tilde\delta^2}\Big]\\
&= 16\pi\Big[\frac{1}{4m_j^2\lambda} + \log 2m_j^2 + H(\xi_j,\xi_j) - \sum_{i\neq j} m_i m_j^{-1} G(\xi_i,\xi_j) + \log 8 - 1\\
&\quad + \log\big[(\varepsilon_j\mu_j)^2 + \tilde\delta^2\big] + \frac{(\varepsilon_j\mu_j)^2}{(\varepsilon_j\mu_j)^2 + \tilde\delta^2}\Big].
\end{aligned}
\tag{6.7}
$$

On the other hand, taking into account that for the fundamental solution Γ we have $\nabla\Gamma(x,\xi) = \frac{4}{|x-\xi|}$ and that $H = \Gamma$ on $\partial\Omega$, we have

$$
\begin{aligned}
\int\limits_{\Omega\setminus B(\xi_j,\tilde\delta)} |\nabla u_j|^2\,dx &= 16 \int\limits_{\Omega\setminus B(\xi_j,\tilde\delta)} \frac{|x-\xi_j|^2}{(\mu_j^2\varepsilon_j^2 + |x-\xi_j|^2)^2}\,dx\\
&= \int\limits_{\Omega\setminus B(\xi_j,\tilde\delta)} \big|\nabla\Gamma(x,\xi_j)\big|^2\,dx + (\varepsilon_j\mu_j)^2\Theta_{\tilde\delta}(\xi_j),\\
&= \int\limits_{\partial\Omega} H(x,\xi_j)\frac{\partial\Gamma}{\partial\nu}(x,\xi_j)\,d\sigma - 32\pi\log\frac{1}{\tilde\delta} + (\varepsilon_j\mu_j)^2\Theta_{\tilde\delta}\\
&= \int\limits_{\Omega} H(x,\xi_j)\Delta\Gamma(x,\xi_j)\,dx + \int\limits_{\partial\Omega} \Gamma(x,\xi_j)\frac{\partial H}{\partial\nu}(x,\xi_j)\,d\sigma\\
&\quad - 32\pi\log\frac{1}{\tilde\delta} + (\varepsilon_j\mu_j)^2\Theta_{\tilde\delta}\\
&= -8\pi H(\xi_j,\xi_j) + \int\limits_{\partial\Omega} H(x,\xi_j)\frac{\partial H}{\partial\nu}(x,\xi_j)\,d\sigma\\
&\quad - 32\pi\log\frac{1}{\tilde\delta} + (\varepsilon_j\mu_j)^2\Theta_{\tilde\delta}.
\end{aligned}
\tag{6.8}
$$

In the above formula $\Theta_{\tilde\delta}(\xi_j)$ is a function dependent on $\tilde\delta$ which is uniformly bounded, together with its derivatives, in the region $\operatorname{dist}(\xi_j,\partial\Omega) > \delta$.

Noticing that the integral on the left-hand side in (6.6) is independent from $\tilde\delta$ and that

$$
\int\limits_{\partial\Omega} H(x,\xi_j)\frac{\partial H}{\partial\nu}(x,\xi_j)\,d\sigma - \int\limits_{\partial\Omega} H_j(x)\frac{\partial H_j}{\partial\nu}(x)\,d\sigma = O\big((\varepsilon_j\mu_j)^2\big),
$$

from (6.5)–(6.8) we thus conclude that, for $j = 1, \dots, k$,

$$\int_\Omega |\nabla U_j|^2\,dx = \lambda m_j^2 16\pi\left[\frac{1}{4m_j^2\lambda} + \log 2m_j^2 + \frac{1}{2}H(\xi_i,\xi_i) - \sum_{i\neq j} m_i m_j^{-1} G(\xi_i,\xi_j) + \log 8 - 1 + \varepsilon_j^2 \log\frac{1}{\varepsilon_j}\,O(1)\right]. \tag{6.9}$$

We next deal with the mixed term in (6.4). Fix $i \neq j$.

Notice that

$$\Delta U_i(x) = -\sqrt{\lambda} m_i \frac{8\mu_i^2\varepsilon_i^2}{((\varepsilon_i\mu_i)^2 + |x-\xi_i|^2)^2}.$$

Moreover $U_i = 0$ on $\partial\Omega$. Hence we can write

$$\begin{aligned}\int_\Omega \nabla U_i \nabla U_j\,dx &= \sqrt{\lambda} m_i \int_\Omega \frac{8\mu_i^2\varepsilon_i^2}{((\varepsilon_i\mu_i)^2+|x-\xi_i|^2)^2} U_j(x)\,dx \\ &= \lambda m_i m_j \int_{\frac{1}{\varepsilon_i\mu_i}(\Omega-\xi_i)} \frac{8}{(1+|y|^2)^2}\log\frac{1}{(\varepsilon_j\mu_j)^2 + |\varepsilon_i\mu_i y + \xi_i - \xi_j|^2}\,dy \\ &\quad - \lambda m_i m_j \int_{\frac{1}{\varepsilon_i\mu_i}(\Omega-\xi_i)} \frac{8}{(1+|y|^2)^2} H_j(\varepsilon_i\mu_i y + \xi_i + \varepsilon)\,dy \\ &= 8\pi\lambda m_i m_j G(\xi_i,\xi_j) + O\left(\varepsilon_j^2\log\frac{1}{\varepsilon_j} + \varepsilon_i^2\log\frac{1}{\varepsilon_i}\right) + O\big(\varepsilon_i^2+\varepsilon_j^2\big), \end{aligned} \tag{6.10}$$

where the $O(\cdot)$ terms have uniform bounds in ξ in the region considered.

Summing up all the previous information contained in (6.9)–(6.10) and using the definition (2.14) for μ_j, we finally get the estimate for (6.4), namely

$$\frac{1}{2}\int_\Omega |\nabla U|^2\,dx = 2\pi k + 4\pi\lambda\left[(2\log 8 - 2)\sum_{j=1}^k m_j^2 + 2\sum_{j=1}^k m_j^2 \log 2m_j^2 + \sum_{j=1}^k m_j^2 H(\xi_j,\xi_j) - \sum_{i\neq j} m_i m_j G(\xi_i,\xi_j) + \sum_{j=1}^k \varepsilon_j^2\log\frac{1}{\varepsilon_j}\,O(1)\right]. \tag{6.11}$$

Let us now evaluate the second term in the energy. We have

$$\lambda\int_\Omega e^{U^2}\,dx = \lambda\left[\sum_{j=1}^k \int_{B(\xi_j,\delta\sqrt{\varepsilon_j})} e^{U^2}\,dx\right] + A_\lambda. \tag{6.12}$$

Taking into account (2.3), we first observe that

$$A_\lambda = \lambda\big[|\Omega| + \lambda\Theta_\lambda(\xi, m)\big] \tag{6.13}$$

with Θ_λ a function, uniformly bounded together with its derivatives, as $\lambda \to 0$. Now, we write

$$\int\limits_{B(\xi_j,\delta\sqrt{\varepsilon_j})} e^{U^2}\,dx = \int\limits_{B(\xi_j,\delta\varepsilon_j|\log\varepsilon_j|)} e^{U^2}\,dx + \int\limits_{B(\xi_j,\delta\sqrt{\varepsilon_j})\setminus B(\xi_j,\delta\varepsilon_j|\log\varepsilon_j|)} e^{U^2}\,dx$$
$$= I_1 + I_2.$$

We will show next that

$$I_1 = 16\pi m_j^2 + \lambda\Theta_\lambda(\xi, m), \qquad I_2 = \lambda\Theta_\lambda(\xi, m) \tag{6.14}$$

for some function Θ_λ, uniformly bounded together with its derivatives, as $\lambda \to 0$. Indeed, performing the change of variables $y = \frac{x-\xi_j}{\varepsilon_j}$ and using the notations $V_j(y) = 2\gamma_j U(\varepsilon_j y + \xi_j) - 2\gamma_j^2$ and $\gamma_j = \log\varepsilon_j^{-4}$, we have

$$I_1 = \varepsilon_j^2 e^{\gamma_j^2} \int\limits_{B(0,\delta|\log\varepsilon_j|)} e^{V_j(y) + \frac{V_j^2(y)}{4\gamma_j^2}}\,dy$$
$$= 2m_j^2 \int\limits_{\mathbb{R}^2} \frac{8}{(1+|y|^2)^2}\,dy + \lambda\Theta_\lambda(\xi, m) = 16\pi m_j^2\big[1 + \lambda\Theta_\lambda(\xi, m)\big].$$

On the other hand

$$|I_2| \leqslant C \int\limits_{\delta|\log\varepsilon_j|}^{\delta\varepsilon_j^{-\frac{1}{2}}} \frac{1}{r^4} e^{\frac{\log^2 r}{\gamma_j^2}}\, r\,dr$$

$(t = \log r)$

$$= C \int\limits_{R_1 + \log\gamma_j^2}^{R_2 + \frac{\gamma_j^2}{4}} e^{-2t + \frac{4t^2}{\gamma_j^2}}\,dt \leqslant C \int\limits_{R_1 + \log\gamma_j^2}^{R_2 + \frac{\gamma_j^2}{4}} e^{-t}\,dt = O(\lambda).$$

We can thus conclude that estimate (6.2) holds true in C^0-sense. The C^1-closeness is a direct consequence of the fact that $\Theta_\lambda(\xi, m)$ is bounded together with its derivatives in the considered region. □

7. Proofs of the theorems

In this section we carry out the proofs of our main results.

Proof of Theorem 2. Let $\mathcal{D}$ be the open set such that

$$\bar{\mathcal{D}} \subset \{(\xi, m) \in \Omega^k \times \mathbb{R}_+^k\colon\ \xi_i \neq \xi_j,\ \forall i \neq j\},$$

where φ_k has a stable critical point situation. Then any C^1-perturbation of φ_k has a critical point in $\mathcal{D}$. Thanks to the results contained in Lemma 5.2 and Lemma 6.1, we thus conclude that the function $\mathcal{I}_\lambda(\xi, m)$, which is C^1-close to $\varphi_k(\xi, m)$ when λ is small enough, has a critical point $(\bar{\xi}_\lambda, \bar{m}_\lambda)$ in $\mathcal{D}$, for all such λ. From Lemma 5.1 we have then that

$$c_{ij}(\bar{\xi}_\lambda, \bar{m}_\lambda) = 0 \quad \text{for all } i, j,$$

and therefore

$$u_\lambda(x) = \sqrt{\lambda}\big(\tilde{U}(\bar{\xi}_\lambda, \bar{m}_\lambda)\big)(x) + \phi(\bar{\xi}_\lambda, \bar{m}_\lambda)(x)$$

is a solution to our problem (1.1). The qualitative properties of this solution predicted by Theorem 2 are a direct consequence of our construction. This concludes the proof. □

Proof of Theorem 1. We shall apply the result of Theorem 2 for the case $k = 2$. Thus we want to prove that the function φ_2 has a stable critical point situation in some open set $\mathcal{D}$, compactly contained in $\Omega^2 \times \mathbb{R}_+^2$.

We make the change of variables $s_j = m_j^2$. With slight abuse of notation we write

$$\varphi_2(\xi, s) = \sum_{j=1,2} \big(bs_j + 2s_j \log s_j + H(\xi_j, \xi_j)s_j\big) - 2G(\xi_1, \xi_2)\sqrt{s_1 s_2}.$$

To establish Theorem 1 we need to show the existence of a stable critical point situation for $\varphi_2(\xi, s)$. To do so we shall show the existence of a critical point for φ_2 obtained through a min–max characterization, which is in fact preserved for small C^1 perturbations of the functional. The rest of the section is devoted to carry out this construction.

Let us fix a small number $\delta > 0$ to be chosen later. We define $\mathcal{D}$ to be

$$\mathcal{D} = \mathbb{R}_+^2 \times \Omega_\delta^2, \quad \text{where } \Omega_\delta^2 = \{y \in \Omega^2 / \operatorname{dist}\big(y, \partial\Omega^2\big) > \delta\}.$$

Denote by Ω_1 a bounded nonempty component of $\mathbb{R}^2 \setminus \bar{\Omega}$ and assume that $0 \in \Omega_1$. Consider a closed, smooth Jordan curve γ contained in Ω which encloses Ω_1. We let S be the image of γ and $B = [\delta, \delta^{-1}]^2 \times S \times S$. Thus B is a closed and connected subset of $\mathcal{D}$.

Let Γ be the class of all maps $\Phi \in C(B, \mathcal{D})$ with the property that there exists a function $\Psi \in C([0, 1] \times B, \mathcal{D})$ such that

$$\Psi(0, \cdot) = \mathrm{Id}_B, \qquad \Psi(1, \cdot) = \Phi. \tag{7.1}$$

Then we define

$$\mathcal{C} = \inf_{\Phi \in \Gamma} \sup_{z \in B} \varphi_2\big(\Phi(z)\big). \tag{7.2}$$

We will show that

$$\mathcal{C} \geqslant -K \tag{7.3}$$

for some fixed constant K independent of δ, and also that if $\delta > 0$ is chosen sufficiently small, then for all $(\xi, s) \in \partial\mathcal{D}$ such that $\varphi_2(\xi, s) = \mathcal{C}$ there exists a vector τ tangent to $\partial\mathcal{D}$ at (ξ, s) such that

$$\nabla\varphi_2(\xi, s) \cdot \tau \neq 0. \tag{7.4}$$

Under the conditions (7.3) and (7.4), a critical point $(\bar{\xi}, \bar{s})$ for φ_2 with $\varphi_2(\bar{\xi}, \bar{s}) = \mathcal{C}$ exists, as a standard deformation argument involving the negative gradient flow of φ_2 shows. This structure is clearly preserved for small $C^1(\bar{\mathcal{D}})$-perturbations of φ_2 and hence a stable critical point situation for this functional is established.

We begin with proving inequality (7.3).

Lemma 7.1. *There exists $K > 0$, independent of the small number δ used to define $\mathcal{D}$ such that $\mathcal{C} \geqslant -K$.*

Proof. We need to prove the existence of $K > 0$ independent of small δ such that if $\Phi \in \Gamma$, then there exists a point $\bar{z} \in B$ for which

$$\varphi_2\big(\Phi(\bar{z})\big) \geqslant -K. \tag{7.5}$$

We write

$$z = (z_1, z_2, z_3, z_4), \quad \Phi(z) = \big(\Phi_1(z), \Phi_2(z), \Phi_3(z), \Phi_4(z)\big),$$

with

$$(z_1, z_2), \big(\Phi_1(z), \Phi_2(z)\big) \in \mathbb{R}^2_+, \qquad (z_3, z_4), \big(\Phi_3(z), \Phi_4(z)\big) \in \Omega^2.$$

We claim that for any $(z_1, z_2) \in \mathbb{R}^2_+$ there exists a $\hat{z} \in S \times S$ such that $\Phi_3(z_1, z_2, \hat{z})$ and $\Phi_4(z_1, z_2, \hat{z})$ have antipodal directions, more precisely

$$\frac{\Phi_3(z_1, z_2, \hat{z})}{|\Phi_3(z_1, z_2, \hat{z})|} = R_\pi \frac{\Phi_4(z_1, z_2, \hat{z})}{|\Phi_4(z_1, z_2, \hat{z})|}, \tag{7.6}$$

where R_π denotes a rotation in the plane of an angle π. This fact clearly implies that the existence of a number $M > 0$ depending only on Ω such that $G(\Phi_3(z_1, z_2, \hat{z}), \Phi_4(z_1, z_2, \hat{z})) \leqslant M$. Thus

$$\begin{aligned}\varphi_2\big(\Phi(z_1,z_2,\hat z)\big) &\geqslant 2\sum_{j=1,2}\Phi_j(z_1,z_2,\hat z)\log\Phi_j(z_1,z_2,\hat z)\\ &\quad-2M\sqrt{\Phi_1(z_1,z_2,\hat z)\Phi_2(z_1,z_2,\hat z)}\\ &\geqslant \min_{r>0,s>0}\big[2(r\log r+s\log s)-2M\sqrt{rs}\,\big]\geqslant -K,\end{aligned}$$

for some explicit number K, which depends on M, but it is independent of δ. This gives the validity of estimate (7.5).

We will prove (7.6) by means of a degree argument. Fix (z_1,z_2). Let us consider an orientation-preserving homeomorphism $h: S^1\to S$ and the map $f: S^1\times S^1\to S^1\times S^1$ defined as $f(\zeta)=(f_1(\zeta),f_2(\zeta))$ with

$$f_1(\zeta_1,\zeta_2)=\frac{\Phi_3(z_1,z_2,h(\zeta_1),h(\zeta_2))}{|\Phi_3(z_1,z_2,h(\zeta_1),h(\zeta_2))|},\qquad f_2(\zeta_1,\zeta_2)=\frac{\Phi_4(z_1,z_2,h(\zeta_1),h(\zeta_2))}{|\Phi_4(z_1,z_2,h(\zeta_1),h(\zeta_2))|}.$$

If we show that f is onto, we get in particular the validity of (7.6).

By definition, there exists a $\Psi\in\Gamma$ such that $\Psi(1,\cdot)=\Phi$. If we denote by $\Psi_i(t,\cdot)$ the components of the map Ψ and $\tilde\Psi_i(t,\xi_1,\xi_2)=\Psi_i(t,z_1,z_2,\xi_2,\xi_2)$, it follows that, for $i=3,4$, $\tilde\Psi_i\in C([0,1]\times S^1\times S^1,\Omega_\delta^2)$, $\tilde\Psi_i(0,\cdot)=\mathrm{Id}_{S^1\times S^1}$ and $\tilde\Psi_i(1,\cdot)=\Phi_i$. Define a homotopy $F:[0,1]\times S^1\times S^1\to S^1\times S^1$ by

$$F_1(t,\zeta)=\frac{\tilde\Psi_3(t,h(\zeta_1),h(\zeta_2))}{|\tilde\Psi_3(t,h(\zeta_1),h(\zeta_2))|}\quad\text{and}\quad F_2(t,\zeta)=\frac{\tilde\Psi_4(t,h(\zeta_1),h(\zeta_2))}{|\tilde\Psi_4(t,h(\zeta_1),h(\zeta_2))|}.$$

Let us notice that $F(1,\zeta)=f(\zeta)$ and

$$F(0,\zeta)=\left(\frac{h(\zeta_1)}{|h(\zeta_1)|},\frac{h(\zeta_2)}{|h(\zeta_2)|}\right).$$

This function defines a homeomorphism of $S^1\times S^1$, which we regard as embedded in $\mathbb{R}^3$, parametrized as follows:

$$\zeta:(\theta_1,\theta_2)\in[0,2\pi)^2\mapsto(\rho_1\cos\theta_1,\rho_1\sin\theta_1,0)+(0,\rho_2\cos\theta_2,\rho_2\sin\theta_2),$$

for $0<\rho_2<\rho_1$. The map f defined above can be read in the introduced variables as

$$f(\zeta)=\big(\rho_1 f_1(\zeta),0\big)+\big(0,\rho_2 f_2(\zeta)\big).$$

The function f can be extended to a continuous map $\tilde f: T\to T$, where T is the solid torus described by

$$(\theta_1,\theta_2,\rho)\in[0,2\pi)^2\times[0,\rho_2]\mapsto(\rho_1\cos\theta_1,\rho_1\sin\theta_1,0)+(0,\rho\cos\theta_2,\rho\sin\theta_2)$$

and

$$\tilde f(\zeta,\rho)=\big(\rho_1 f_1(\zeta),0\big)+\big(0,\rho f_2(\zeta)\big).$$

The function $\tilde{f}$ is homotopic to a homeomorphism of T, along a deformation which maps $\partial T = S^1 \times S^1$ into itself. Thus $\deg(\tilde{f}, T, P) \neq 0$ for all P in the interior of T. We see next how this fact implies that f is onto. Take $(\theta_1^*, \theta_2^*) \in [0, 2\pi)^2$ and $\rho^* \in (0, \rho_2)$ then there exist $\zeta^{**} \in S^1 \times S^1$ and $\rho^{**} \in (0, \rho_2)$ such that

$$\big(\rho_1 f_1\big(\zeta^{**}\big), 0\big) + \big(0, \rho^{**} f_2\big(\zeta^{**}\big)\big) = \big(\rho_1 \cos\theta_1^*, \rho_1 \sin\theta_1^*, 0\big) + \big(0, \rho^* \cos\theta_2^*, \rho^* \sin\theta_2^*\big).$$

Thus we get $f_1(\zeta^{**}) = (\cos\theta_1^*, \sin\theta_1^*)$, $f_2(\zeta^{**}) = (\cos\theta_2^*, \sin\theta_2^*)$ and $\rho^* = \rho^{**}$. It then follows that f is onto. This concludes the proof. □

We now prove (7.4).

Lemma 7.2. *There exists a sufficiently small $\delta > 0$ with the following properties: If $(\bar{\xi}, \bar{s}) \in \partial\mathcal{D}_\delta$ is such that $\varphi_2(\bar{\xi}, \bar{s}) = \mathcal{C}$, then there exists a vector τ, tangent to $\partial\mathcal{D}_\delta$ at the point $(\bar{\xi}, \bar{s})$, so that*

$$\nabla\varphi_2(\bar{\xi}, \bar{s}) \cdot \tau \neq 0.$$

Proof. Let us assume there exist a sequence $\delta = \delta_n \to 0$ and points $(\xi_n, s_n) \in \mathcal{D}_\delta$ such that (omitting the subscript n), $(\xi, s) \to (\bar{\xi}, \bar{s}) \in \bar{\Omega}^2 \times \mathbb{R}_+^2$ and $\varphi_2(\xi, s) \to \mathcal{C} < 0$. We shall show that there exists a tangent vector τ, tangent to $\bar{\Omega}^2 \times \mathbb{R}_+^2$, such that $\nabla\varphi_2(\bar{\xi}, \bar{s}) \cdot \tau \neq 0$.

Assume first that $\bar{\xi} \in \Omega^2$. If $|s| \to \infty$, then $\varphi_2(\xi, s) \to \infty$. Thus we may assume that $|s|$ is bounded.

Let us observe now the following fact: for any points $\xi = (\xi_1, \xi_2)$ fixed and far from each other and from the boundary, the function

$$\varphi_2(\xi, s) = \sum_{j=1,2} \big(bs_j + 2s_j \log s_j + H(\xi_, \xi_j)s_j\big) - 2G(\xi_1, \xi_2)\sqrt{s_1 s_2}$$

is strictly convex as a function of s, and it is bounded below. Hence it has a unique minimum point, which we denote by $(\bar{s}_1, \bar{s}_2)$. Then each component $\bar{s}_i$ of $\bar{s}$ is a function of ξ_1 and ξ_2, namely $\bar{s}_i = \bar{s}_i(\xi_1, \xi_2)$ for $i = 1, 2$. Furthermore, a direct computation shows that

$$\varphi_2(\xi, \bar{s}) = -2(\bar{s}_1 + \bar{s}_2) \tag{7.7}$$

and

$$\varphi_2(\xi, \bar{s}) \leqslant \min_{i=1,2}\Big(\min_{s_i=0} \varphi_2(\xi, s)\Big) \leqslant -2e^{-\frac{b+2}{2}} \min_{i=1,2} e^{-\frac{H(\xi_i, \xi_i)}{2}}. \tag{7.8}$$

Assuming again that $\bar{\xi} \in \Omega^2$, if $s = (s_1, s_2) \to (0, 0)$, then we would get that $\mathcal{C} = 0$, which is impossible. On the other hand, if say s_1 is far away from 0 and $s_2 \to 0$, then $\partial_{s_2}\varphi_2(\xi, s_1, s_2) \to -\infty$, and then we can take $\tau = \partial_{s_2}\varphi_2$.

Let us consider now the case in which $\operatorname{dist}(\xi_2, \partial\Omega) = \delta$. As $\delta \to 0$, this fact implies that $H(\xi_2, \xi_2) \to \infty$, then we must also have that $|\xi_1 - \xi_2| \to 0$ to keep the value of φ_2 bounded. By construction we have $\operatorname{dist}(\xi_1, \partial\Omega) \geqslant \delta$. Two cases arise: if $\nabla_s\varphi_2(\xi, s) \neq 0$, then we can chose τ parallel to $\nabla_s\varphi_2(\xi, s)$. Otherwise, we are in the case in which $\nabla_s\varphi_2(\xi, s) = 0$. It remains to analyze this case.

Formula (7.7) leads us to change variables as

$$r = s_1 + s_2, \quad rt = s_1 \quad \text{with } 0 < r < \infty,\ 0 < t < 1.$$

In these new variables the function φ_2 gets rewritten as

$$\varphi_2(\xi, r, t) = r\big[b + 2t\log t + 2(1-t)\log(1-t)\big] + 2r\log r + r\big[H(\xi_1,\xi_1)t + H(\xi_2,\xi_2)(1-t) - 2G(\xi_1,\xi_2)\sqrt{t(1-t)}\big].$$

The relation $\frac{\partial\varphi_2(r,t)}{\partial r} = 0$ gives $r = e^{-\frac{C+h(\xi,t)}{2}}$, where C is an explicit positive number and $h(\xi,t)$, for $0 < t < 1$ is given by

$$h(\xi,t) = H(\xi_1,\xi_1)t + H(\xi_2,\xi_2)(1-t) - 2G(\xi_1,\xi_2)\sqrt{t(1-t)} + 2t\log t + 2(1-t)\log(1-t). \tag{7.9}$$

To get the minimum value of φ_2 in the variable s is thus equivalent to get the minimum of the function h as a function of t in the interval (0, 1). Differentiating $h(\xi,t)$ with respect to t we get

$$\frac{H(\xi_2,\xi_2) - H(\xi_1,\xi_1)}{2G(\xi_1,\xi_2)} = \frac{t-\frac{1}{2}}{\sqrt{t(1-t)}} - \frac{1}{G(\xi_1,\xi_2)}\log\frac{t}{1-t}. \tag{7.10}$$

This relation defines uniquely the value of t. Thus the relation $\nabla_s\varphi_2(\xi,s) = 0$ implies that

$$\varphi_2(\xi,s) = -2r = -2e^{-\frac{C+h(\xi,t)}{2}} \tag{7.11}$$

with t uniquely defined by (7.10) and h given by (7.9). Next we want to analyze the dependence of this t on the points ξ_1 and ξ_2. Our first claim is that t is away both from 0 and 1. This fact is a direct consequence of the following statement: there exists a positive number C such that

$$\left|\frac{H(\xi_2,\xi_2) - H(\xi_1,\xi_1)}{2G(\xi_1,\xi_2)}\right| \leqslant C. \tag{7.12}$$

We show the validity of (7.12). We assume by contradiction that

$$\left|\frac{H(\xi_2,\xi_2) - H(\xi_1,\xi_1)}{2G(\xi_1,\xi_2)}\right| \to +\infty. \tag{7.13}$$

We have $\delta = \operatorname{dist}(\xi_2, \partial\Omega)$. Let us denote $d_1 = \operatorname{dist}(\xi_1, \partial\Omega)$, and $d = |\xi_1 - \xi_2|$. Condition (7.13) implies that d_1 and $d \to 0$, with $\delta \geqslant d_1$ and $\delta = o(d)$. Let us consider the expanded domain $\tilde\Omega = \delta^{-1}\Omega$ and observe that for this domain its associated Green's function and regular part are given by

$$\tilde H(x_1,x_2) = 4\log\delta + H(\delta x_1, \delta x_2), \qquad \tilde G(x_1,x_2) = G(\delta x_1, \delta x_2). \tag{7.14}$$

Furthermore, $\operatorname{dist}(\xi_2, \partial\Omega) = \delta$ implies $\operatorname{dist}(\frac{\xi_2}{\delta}, \partial\tilde\Omega) = 1$. After a rotation and translation, we assume that $\frac{\xi_2}{\delta} = (0,1)$ and as $\delta \to 0$ the domain $\tilde\Omega$ becomes the half-plane $x_2 > 0$. We denote respectively by G_0 and H_0 Green's function and its regular part, associated to the half plane $x_2 > 0$. The expressions for G_0 and H_0 are explicit:

$$H_0(x, y) = 4\log\frac{1}{|x-\bar{y}|}, \quad \bar{y} = (y_1, -y_2)$$

where $y = (y_1, y_2)$, and

$$G_0(x, y) = 4\log\frac{1}{|x-y|} - 4\log\frac{1}{|x-\bar{y}|}.$$

We thus compute the expression in (7.13)

$$\begin{aligned}\frac{H(\xi_2,\xi_2)-H(\xi_1,\xi_1)}{2G(\xi_1,\xi_2)} &= \frac{\tilde{H}(\frac{\xi_2}{\delta},\frac{\xi_2}{\delta})-\tilde{H}(\frac{\xi_1}{\delta},\frac{\xi_1}{\delta})}{2\tilde{G}(\frac{\xi_1}{\delta},\frac{\xi_2}{\delta})}\\ &= \frac{H_0((0,1),(0,1)) - 4\log\frac{\delta}{|\xi_1-\bar{\xi}_1|} + o(1)}{4\log\frac{\delta}{|\xi_1-\delta(0,1)|}} = O(1),\end{aligned}$$

but this is in contradiction with (7.13).

The next step is to study the dependence of t on the points ξ_i. Let us call

$$\mu = \frac{H(\xi_2,\xi_2)-H(\xi_1,\xi_1)}{2G(\xi_1,\xi_2)}, \qquad \lambda = \frac{1}{G(\xi_1,\xi_2)}.$$

Let us analyze how t depends on μ and λ. Let us set $z = t - \frac{1}{2}$. Then Eq. (7.10) defines uniquely $z = z(\mu, \lambda)$

$$\mu = \frac{z}{\sqrt{\frac{1}{4} - z^2}} + \lambda f(z). \tag{7.15}$$

We observe that

$$z(\mu, 0) = \frac{1}{2}\frac{H(\xi_2,\xi_2)-H(\xi_1,\xi_1)}{\sqrt{G(\xi_1,\xi_2)^2 + (H(\xi_2,\xi_2)-H(\xi_1,\xi_1))^2}}.$$

Differentiating expression (7.15) with respect to μ and to λ we get that $|z_\mu| + |z_\lambda|$ is bounded if μ and λ are bounded. Now we replace the values of $t = t(\mu, \lambda)$, defined by the relation $t = z + \frac{1}{2}$, in (7.9). We thus get a function $h = h(H(\xi_1,\xi_1), H(\xi_2,\xi_2), G(\xi_1,\xi_2), t(\mu,\lambda))$.

Our next claim is that the derivatives of h with respect to $H(\xi_1,\xi_1)$, to $H(\xi_2,\xi_2)$, and to $G(\xi_1,\xi_2)$ are bounded above and below away from 0. We show this fact for $\frac{\partial}{\partial G(\xi_1,\xi_2)}h$. We have

$$\frac{\partial}{\partial G(\xi_1,\xi_2)} h\big(H(\xi_1,\xi_1), H(\xi_2,\xi_2), G(\xi_1,\xi_2), t(\mu,\lambda)\big)$$

$$
\begin{aligned}
&= \frac{\partial}{\partial G(\xi_1, \xi_2)} h + \frac{\partial h}{\partial t}\left[\frac{\partial t}{\partial \mu}\frac{\partial \mu}{\partial G(\xi_1, \xi_2)} + \frac{\partial t}{\partial \lambda}\frac{\partial \lambda}{\partial G(\xi_1, \xi_2)}\right] \\
&= -\sqrt{t(1-t)} + O\left(\frac{1}{G(\xi_1, \xi_2)}\right) = O(1) + O\left(\frac{1}{G(\xi_1, \xi_2)}\right).
\end{aligned}
$$

The above conclusion holds true since we have that $|t_\mu| + |t_\lambda|$ is bounded. Furthermore,

$$
\frac{\partial \lambda}{\partial G(\xi_1, \xi_2)} = -\frac{1}{G(\xi_1, \xi_2)^2} = o(1), \qquad \frac{\partial \mu}{\partial G(\xi_1, \xi_2)} = -\frac{H(\xi_1, \xi_1) - H(\xi_2, \xi_2)}{G(\xi_1, \xi_2)^2} = o(1).
$$

Finally, we Taylor expand

$$
\begin{aligned}
&\frac{\partial h}{\partial t}\big(H(\xi_1, \xi_1), H(\xi_2, \xi_2), G(\xi_1, \xi_2), t(\mu, \lambda)\big) \\
&\quad = \frac{\partial h}{\partial t}\big(H(\xi_1, \xi_1), H(\xi_2, \xi_2), G(\xi_1, \xi_2), t(\mu, 0)\big) \\
&\quad\quad + \frac{\partial^2 h}{\partial^2 t}\big(H(\xi_1, \xi_1), H(\xi_2, \xi_2), G(\xi_1, \xi_2), \tilde{t}(\mu, \lambda)\big)\big(t(\mu, \lambda) - t(\mu, 0)\big) \\
&\quad = O(1),
\end{aligned}
$$

since

$$
\frac{\partial^2 h}{\partial^2 t}\big(H(\xi_1, \xi_1), H(\xi_2, \xi_2), G(\xi_1, \xi_2), \tilde{t}(\mu, \lambda)\big) = O(1) G(\xi_1, \xi_2)
$$

and

$$
\big(t(\mu, \lambda) - t(\mu, 0)\big) \sim \lambda \sim \frac{1}{G(\xi_1, \xi_2)}.
$$

In a similar fashion we get that the quantities

$$
\frac{\partial h}{\partial H(\xi_1, \xi_1)}, \qquad \frac{\partial h}{\partial H(\xi_2, \xi_2)}
$$

are bounded above and below away from 0 in the considered region.

We have now the tools to conclude the proof of our lemma. We recall that the case we are discussing is the following: $\mathrm{dist}(\xi_2, \partial\Omega) = \delta$, $\xi_1 \to \xi_2$, with $\mathrm{dist}(\xi_1, \partial\Omega) \geqslant \delta$, $\nabla_s \varphi_2(\xi, s) = 0$ which implies the validity of (7.11), namely

$$
\varphi_2(\xi, s) = -2e^{-\frac{C + h(\xi, t)}{2}}
$$

with t uniquely defined by (7.10) and h given by (7.9). We argue by contradiction, assuming that (7.4) does not hold. Then we have in particular that

$$
\nabla_{\xi_2} \varphi_2(\xi, s) \cdot \tau = 0 \tag{7.16}
$$

for any vector τ tangent to $\partial\Omega_\delta$ at ξ_2, where $\Omega_\delta = \{x \in \Omega\colon \operatorname{dist}(x, \partial\Omega) > \delta\}$. Note that

$$\nabla_{\xi_2}\varphi_2(\xi, s) = e^{-\frac{C+h(\xi,t)}{2}}\left[\frac{\partial h}{\partial G(\xi_1,\xi_2)}\nabla_{\xi_2}G(\xi_1,\xi_2) + \frac{\partial h}{\partial H(\xi_2,\xi_2)}\nabla_{\xi_2}H(\xi_2,\xi_2)\right].$$

We recall that $\frac{\partial h}{\partial G(\xi_1,\xi_2)}$ and $\frac{\partial h}{\partial H(\xi_2,\xi_2)}$ are bounded above and below away from 0. We denote $\rho = |\xi_1 - \xi_2| \to 0$. Only two cases may occur, namely $\frac{\delta}{\rho} \to \infty$ or $\frac{\delta}{\rho} \leqslant c_0$, for some constant c_0. We shall show that in both cases relation (7.16) is impossible.

Let us assume first that $\frac{\delta}{\rho} \to \infty$ and define

$$x_j = \frac{\xi_j - \xi_1}{\rho} \quad \text{for } j = 1, 2,$$

and $\tilde{x}_j = \lim_{\delta\to 0} x_j$. Let us define

$$\tilde{\varphi}(x_1, x_2) = \varphi_2(\xi_1 + \rho x_1, \xi_1 + \rho x_2, s).$$

After rotation we may assume that in (7.16) we have $\tau = (0, 1)$, and hence (writing $\xi_2 = (\xi_2^1, \xi_2^2)$)

$$\lim_{\delta\to 0}\partial_{x_2^2}\tilde{\varphi}(x_1, x_2) = \lim_{\delta\to 0}\rho\,\partial_{\xi_2^2}\varphi_2(\xi_1 + \rho x_1, \xi_1 + \rho x_2, s) = 0.$$

On the other hand, since away from the boundary the function $H(x, x)$ is bounded, we get

$$\lim_{\delta\to 0}\partial_{x_2^2}\tilde{\varphi}(x_1, x_2) = -C\partial_{x_2^2}\log\frac{1}{|\tilde{x}_1 - \tilde{x}_2|} \neq 0,$$

a contradiction. Thus, we necessarily have that $\frac{\delta}{\rho}$ is bounded. The interesting case is when $\xi_1 \in \partial\Omega_\delta$. If not, we can reproduce the argument above to reach a contradiction. Let us assume first that $\delta = o(\rho)$. In this case we find that (7.13) holds true, which leads us to a contradiction. Let us assume then that $\frac{\delta}{\rho} \to c$. We consider the scaled domain $\tilde{\Omega} = \delta^{-1}\Omega$, whose associated Green's function $\tilde{G}$ and regular part $\tilde{H}$ are given by (7.14). Furthermore, in this scaled domain the number t defined by relation (7.10) remains away from 0 and 1, since the quantity

$$\frac{\tilde{H}(\xi_2,\xi_2) - \tilde{H}(\xi_1,\xi_1)}{2\tilde{G}(\xi_1,\xi_2)}$$

remains bounded. Furthermore, after a rotation and translation, we may assume that $\tilde{\xi}_2 := \frac{\xi_2}{\delta} \to (0, 1)$, $\tilde{\xi}_1 := \frac{\xi_1}{\delta} \to (a, 1)$, for some $a > 0$, as $\delta \to 0$ and the domain $\tilde{\Omega}$ becomes the half-plane $x_2 > 0$. Under this condition, we see that the derivative of φ_2 in the direction $\mathsf{e} = (0, 1)$ is not 0, reaching again a contradiction with (7.16), and the proof is concluded. □

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