



# Fundamental solutions and Liouville type theorems for nonlinear integral operators

Patricio Felmer<sup>a</sup>, Alexander Quaas<sup>b,\*</sup>

<sup>a</sup> *Departamento de Ingeniería Matemática and Centro de Modelamiento Matemático UMR2071 CNRS-UCHile, Universidad de Chile, Casilla 170 Correo 3, Santiago, Chile*

<sup>b</sup> *Departamento de Matemática, Universidad Técnica Federico Santa María Casilla: V-110, Avda. España 1680, Valparaíso, Chile*

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## Abstract

In this article we study basic properties for a class of nonlinear integral operators related to their fundamental solutions. Our goal is to establish Liouville type theorems: non-existence theorems for positive entire solutions for  $\mathcal{I}u \leq 0$  and for  $\mathcal{I}u + u^p \leq 0$ ,  $p > 1$ .

We prove the existence of fundamental solutions and use them, via comparison principle, to prove the theorems for entire solutions. The non-local nature of the operators poses various difficulties in the use of comparison techniques, since usual values of the functions at the boundary of the domain are replaced here by values in the complement of the domain. In particular, we are not able to prove the Hadamard Three Spheres Theorem, but we still obtain some of its consequences that are sufficient for the arguments.

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## 1. Introduction

During the last years there has been a renewed and increasing interest in the study of nonlinear integral operators. Motivated in part, by the important advances on the theory of nonlinear partial differential equations, a great variety of diffusion phenomena are being described using integral

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\* Corresponding author.

*E-mail addresses:* [pfelmer@dim.uchile.cl](mailto:pfelmer@dim.uchile.cl) (P. Felmer), [alexander.quaas@usm.cl](mailto:alexander.quaas@usm.cl) (A. Quaas).

operators: in Particle Models in Physics [19], in Nonlinear Reaction–Diffusion for Population Biology [3,5], in Financial Mathematics and Stochastic Control Theory [21], just to name some references.

From the mathematical point of view, given an operator one is interested in understanding the structure of the solutions sets of equations involving it. In an attempt to address an edge of this formidable problem, one tries to understand some basic questions constructing simple solutions. In this category falls the question of existence of entire solutions, fundamental solutions and the related Liouville property or Liouville type theorems.

Assuming we have an operator  $\mathcal{I}$ , the first question we are interested in addressing in this paper is the possibility of having nontrivial solutions for the equation

$$\mathcal{I}u \leq 0, \quad u \geq 0 \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

and the second question is about the possibility of having nontrivial solutions to the equation with an added power nonlinearity

$$\mathcal{I}u + u^p \leq 0, \quad u \geq 0 \quad \text{in } \mathbb{R}^N, \tag{1.2}$$

for  $p > 1$ . The study of these questions is deeply related with the existence of fundamental solutions for the operator, that is simple radially symmetric power-like solutions of the equation  $\mathcal{I}u = 0$ . In this article we consider these questions for a class of nonlinear operators introduced by Caffarelli and Silvestre [7]. We prove the existence of fundamental solutions and we use them to prove Liouville type theorems. The comparison principle is here the tool to compare the entire solutions with the fundamental solutions. At this point we have to introduce various new techniques in order to overcome the difficulties posed by the fact that the operators are non-local, and so, the values at the boundary of the functions to be compared have to be replaced by the values of the functions in the complement of the domain.

Let us be more precise about the operators we consider in this paper. Let  $K : \mathbb{R}^N \rightarrow \mathbb{R}$  be a positive even function satisfying

$$\frac{\lambda}{|y|^{N+2\alpha}} \leq K(y) \leq \frac{\Lambda}{|y|^{N+2\alpha}}, \tag{1.3}$$

where  $N \geq 2$ ,  $\Lambda \geq \lambda > 0$  and  $\alpha \in (0, 1)$ . We consider such a  $K$  as the kernel for defining the linear operator  $L_K(u)$  at  $x \in \mathbb{R}^N$  as

$$L_K(u)(x) = \int_{\mathbb{R}^N} (u(x+y) + u(x-y) - 2u(x))K(y) dy,$$

where  $u$  is such  $y \rightarrow (u(x+y) + u(x-y) - 2u(x))K(y)$  is integrable in  $\mathbb{R}^N \setminus B(0, \varepsilon)$  for all  $\varepsilon > 0$  and of class  $C^{1,1}(x)$  in the sense defined by Caffarelli and Silvestre in [7], that is, there exist  $v \in \mathbb{R}^N$  and  $M > 0$  so that

$$|u(x+y) - u(x) - v \cdot y| \leq M|y|^2,$$

for  $y$  small. In particular, the linear operator  $L_K$  is well defined at  $x$  if  $u$  is bounded, continuous and of class  $C^{1,1}(x)$ .

If we define by  $\mathcal{L}_0$  the class of all these linear operators then we define the extremal operators of  $\mathcal{L}_0$  as

$$\mathcal{M}^+u(x) = \sup_{L \in \mathcal{L}_0} L(u)(x) \quad \text{and} \quad \mathcal{M}^-u(x) = \inf_{L \in \mathcal{L}_0} L(u)(x), \tag{1.4}$$

the maximal and the minimal operator, respectively. We remark that the class  $\mathcal{L}_0$ , and a posteriori  $\mathcal{M}^+$  and  $\mathcal{M}^-$ , depends on the parameters  $\Lambda$ ,  $\lambda$  and  $\alpha$ , but we do not explicitly write them in order to avoid overcharged notation.

It is easy to see that these extremal operators can be explicitly characterized considering the functions

$$S_+(t) = \Lambda t_+ - t_- \quad \text{and} \quad S_-(t) = t_+ - \Lambda t_-$$

writing

$$\mathcal{M}^+u(x) = \int_{\mathbb{R}^N} \frac{S_+(\delta(u, x, y))}{|y|^{N+2\alpha}} dy \quad \text{and} \quad \mathcal{M}^-u(x) = \int_{\mathbb{R}^N} \frac{S_-(\delta(u, x, y))}{|y|^{N+2\alpha}} dy,$$

where  $\delta(u, x, y) = u(x + y) + u(x - y) - 2u(x)$ . Here and in the rest of the paper we will consider  $\Lambda \geq 1$  and  $\lambda = 1$  for simplicity. We observe that the operators just defined are extremal for a much larger class of operators, including nonlinear, non-autonomous operators like

$$\mathcal{F}(u)(x) = \int_{\mathbb{R}^N} \frac{G(\delta(u, x, y), x, y)}{|y|^{N+2\alpha}} dy, \tag{1.5}$$

where the nonlinear function  $G : \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}$  is continuous and it satisfies

$$S_-(t) \leq G(t, x, y) \leq S_+(t) \quad \text{for all } (t, x, y) \in \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N.$$

Our first theorem is devoted to the existence of fundamental solutions for the extremal operators  $\mathcal{M}^+$  and  $\mathcal{M}^-$ . We have:

**Theorem 1.1** (*Existence of fundamental solutions*). *Associated to the operator  $\mathcal{M}^+$ , with parameters  $(\alpha, \Lambda) \in (0, 1) \times [1, \infty)$  and dimension  $N \geq 2$ , there exist dimension-like numbers  $N^+ = N^+(\alpha, \Lambda, N)$  and  $N^- = N^-(\alpha, \Lambda, N)$  such that*

$$0 < N^+ \leq N \leq N^- < N + 2\alpha.$$

*As functions of  $\Lambda$ ,  $N^+(\alpha, \Lambda, N)$  is strictly decreasing and  $N^-(\alpha, \Lambda, N)$  is strictly increasing and they satisfy*

$$N^+(\alpha, 1, N) = N = N^-(\alpha, 1, N),$$

$$\lim_{\Lambda \rightarrow \infty} N^+(\alpha, \Lambda, N) = \begin{cases} 0 & \text{if } 2\alpha \leq 1, \\ 2\alpha - 1 & \text{if } 2\alpha > 1, \end{cases} \quad \text{and} \quad \lim_{\Lambda \rightarrow \infty} N^-(\alpha, \Lambda, N) = N + 2\alpha.$$

Moreover, these numbers are so that the functions

$$\phi_{N^+}(r) = \begin{cases} r^{-N^+ + 2\alpha} & \text{if } N^+ > 2\alpha, \\ -\log r & \text{if } N^+ = 2\alpha, \\ -r^{-N^+ + 2\alpha} & \text{if } N^+ < 2\alpha \end{cases} \tag{1.6}$$

and

$$\phi_{N^-}(r) = -r^{-N^- + 2\alpha}$$

satisfy the equation

$$\mathcal{M}^+(u(r)) = 0, \quad r > 0.$$

The functions  $\phi_{N^+}$  and  $\phi_{N^-}$  are the only power-like solutions of this equation.

The functions defined as  $\varphi_{N^+}(r) = -\phi_{N^+}(r)$  and  $\varphi_{N^-}(r) = -\phi_{N^-}(r)$  are fundamental solutions of the operator  $\mathcal{M}^-$ , respectively. They are the only power-like solutions to the equation

$$\mathcal{M}^-(u(r)) = 0, \quad r > 0.$$

**Remark 1.1.** In what follows we define, for notational convenience,  $\sigma^+ = -N^+ + 2\alpha$  and  $\sigma^- = -N^- + 2\alpha$ .

Fundamental solutions for the extremal Pucci operator ( $\alpha = 1$ ) were first defined by Labutin [17,18] and were used for the study of removability of singularities for these operators. They were used later by Cutri and Leoni [12] for the study of Liouville type theorems and later for operators involving first order terms by Capuzzo-Dolcetta and Cutri in [9]. The results in [12] were generalized by the authors in [14] for a class of extremal operators with radial symmetry. Recently Armstrong, Sirakov and Smart [2] obtained fundamental solutions for general, not necessarily radially symmetric fully nonlinear differential operators, and they were used very recently by Armstrong and Sirakov [1] to prove Liouville type theorems for these differential operators.

Now we state our main theorems on entire solutions. In these theorems and in all the paper, by solution to an integral inequality or equation we mean solution in the viscosity sense as defined in [7] as we describe in Section 2.

**Theorem 1.2** (*The Liouville property*). Assume that  $N^+ \leq 2\alpha$  and  $u$  is a viscosity solution of

$$\mathcal{M}^+(u) \leq 0, \quad \text{and} \quad u \geq 0, \quad \text{in } \mathbb{R}^N,$$

then  $u$  is a constant. Similarly, if  $N^+ \leq 2\alpha$  and  $u$  is a viscosity solution of

$$\mathcal{M}^-(u) \geq 0, \quad \text{and} \quad u \leq 0, \quad \text{in } \mathbb{R}^N,$$

then  $u$  is a constant.

**Remark 1.2.** The case  $N^- \leq 2\alpha$  does not occur since  $N^- > N \geq 2$ .

We observe that for a given nonlinear operator  $\mathcal{F}$  as in (1.5) we have a corresponding Liouville property, by comparison with the extremal operators. Our next result is a Liouville type theorem for the operator with a power nonlinearity. We have:

**Theorem 1.3 (Liouville type theorem).** Assume  $N^+ > 2\alpha$  and that  $u$  is a viscosity solution of

$$\mathcal{M}^+(u) + u^p \leq 0. \tag{1.7}$$

If  $p \leq \frac{N^+}{N^+ - 2\alpha}$ , then  $u \equiv 0$ . Reciprocally, if  $p > \frac{N^+}{N^+ - 2\alpha}$  then Eq. (1.7) has a nontrivial viscosity solution.

Similar statements hold with  $\mathcal{M}^-$  and  $N^-$ .

It is important to say here that the non-existence theorems of Liouville type are closely related with existence theorems in bounded domains. In the case of second order differential operators, the well-known blow-up technique introduced by Gidas and Spruck [16] allows to find a priori bounds for the positive solutions of the problem in a bounded domain, as a consequence of the non-existence theorem. Then classical degree theory is applicable to complete the existence arguments. Even though we do not investigate this line of research in this article, we believe that results of this sort are valid for non-local operators in the class considered here.

We would like to emphasize that, as far as we know, the theorems just stated are new even for the case  $\Lambda = 1$ , that corresponds to the fractional Laplacian. Related results for this linear operators are the existence and uniqueness of a positive solution for nonlinear equation

$$\Delta^\alpha(u) + u^p = 0 \tag{1.8}$$

in the Sobolev critical case  $p = (N + 2\alpha)/(N - 2\alpha)$  and the non-existence result for this nonlinear equation in the Sobolev sub-critical case, see Li [20] and Chen, Li and Ou [11]. In the case  $\alpha = 1$  and  $\Lambda = 1$ , that is for the Laplacian, Theorem 1.3 is an extension of the classical result of Gidas [15]. Concerning results of classification of solution and Liouville type result for Eq. (1.8) and  $\alpha = 1$  we mention the fundamental papers by Gidas and Spruck [16], Caffarelli, Gidas and Spruck [8] and Chen and Li [10].

Notice that a Liouville type theorem and the classification of solution for the equation

$$\mathcal{M}^+(u) + u^p = 0 \tag{1.9}$$

for  $p > \frac{N^+}{N^+ - 2\alpha}$  is a wide open problem, even in the radially symmetric case. We conjecture that there is a critical Sobolev type exponent with value between  $(N + 2\alpha)/(N - 2\alpha)$  and  $(N^+ + 2\alpha)/(N^+ - 2\alpha)$  for  $\Lambda > 1$ , that allows to classify the positive solutions, as in the case of the extremal Pucci operators, see [13].

As we have already mentioned, the proofs of Theorems 1.2 and 1.3 are based on the study of fundamental solutions for the extremal integral operators and the use of these solutions together with comparison principle. To be more specific, the proofs use two weak versions of the Hadamard Three Spheres (circles), see Lemmas 4.1 and 4.2. The difficulty of these arguments is due to the application comparison principle for non-local operator that needs the right inequality

for the function in all complement of the domain (not only on the boundary as in the case of local operators), see Theorem 2.1.

Finally, notice that the extremal operators we are considering in this article have a clear connection with nonlinear second order elliptic operators, when  $\alpha \rightarrow 1$ . Actually it is not difficult to prove that

$$\lim_{\alpha \rightarrow 1} 2(1 - \alpha)\mathcal{M}^\pm u = \int_{S^{N-1}} S_\pm \left( \sum_{i=1, N} e_i \omega_i^2 \right) d\omega,$$

where  $e_1, e_2, \dots, e_N$  are the eigenvalues of  $D^2u(x)$  and  $\omega = (\omega_1, \dots, \omega_n)$  is a generic point in  $S^{N-1}$ . For more details on other connection with nonlinear second order elliptic operators, see [7].

## 2. Preliminaries

In this section we briefly review some basic definitions and comparison theorems for integral operators. In this section  $\mathcal{I}$  will denote any linear operators or extremal operator, as defined above. The definition and comparison theorem we give here are valid for much larger class of operators as given in [7] or [4], but for this paper we do not need such generality.

**Definition 2.1.** Assume  $f : \mathbb{R} \rightarrow \mathbb{R}$  is a continuous function and  $g : \mathbb{R}^N \rightarrow \mathbb{R}$  is a function. A continuous function  $u : \mathbb{R}^N \rightarrow \mathbb{R}$  is a viscosity super-solution (sub-solution) of

$$\mathcal{I}(u) + f(u) = g(x) \quad \text{in } \mathbb{R}^N \tag{2.1}$$

at the point  $x_0 \in \mathbb{R}^N$ , if for any neighborhood  $V$  of  $x_0$  and for any  $\varphi \in C^2(\bar{V})$  such that

$$u(x_0) = \varphi(x_0) \quad \text{and} \quad u(x) > \varphi(x) \quad (\text{resp. } u(x) < \varphi(x))$$

for all  $x \in V \setminus \{x_0\}$ , then if we define

$$v(x) = u(x) \quad \text{if } x \in \mathbb{R}^N \setminus V \quad \text{and} \quad v(x) = \varphi(x) \quad \text{if } x \in V \tag{2.2}$$

we have

$$\mathcal{I}(v)(x_0) + f(v(x_0)) \leq g(x_0) \quad (\text{resp. } \mathcal{I}(v)(x_0) + f(v(x_0)) \geq g(x_0)).$$

**Remark 2.1.** In the definition we may consider inequality instead of strict inequality

$$u(x) \geq \varphi(x) \quad \text{for all } x \in V \setminus \{x_0\},$$

and in ‘some neighborhood  $V$  of  $x_0$ ’ instead of in ‘all neighborhood’. See also [4] for alternative equivalent definitions.

**Remark 2.2.** Naturally, if  $D$  is a subset of  $\mathbb{R}^N$ , we say that  $u$  is a viscosity super-solution (sub-solution) of Eq. (2.1) in  $D$  if  $u$  is a super-solution (sub-solution) of Eq. (2.1) at every point of  $D$ .

Next we recall the comparison principle, Theorem 5.2, proved in [7], that we use later to prove our theorems.

**Theorem 2.1.** *Assume  $u$  and  $v$  are super-solution and sub-solutions of the equation*

$$\mathcal{I}(u) = g,$$

*in  $\bar{\Omega}$ , where  $\Omega$  is a bounded open subset of  $\mathbb{R}^N$  and  $g$  is a continuous function in  $\bar{\Omega}$ . Moreover, assume that  $u \leq v$  in  $\mathbb{R}^N \setminus \Omega$ . Then  $u \leq v$  in  $\Omega$ .*

### 3. Fundamental solutions

In this section we study the fundamental solutions for the operators  $\mathcal{M}^+$  and  $\mathcal{M}^-$ , associated to the class of linear operators  $\mathcal{L}_0$ , as defined in (1.4). The main goal is to prove Theorem 1.1. We recall that these integral operators depend on the ellipticity parameters  $\lambda$  and  $\Lambda \geq \lambda$ , where  $\lambda$  has been normalized as  $\lambda = 1$ , and the order of the fractional parameter  $\alpha \in (0, 1)$ . We make this notational simplification for the reader convenience, since no confusion will arise.

After some basic properties we concentrate in the analysis of sign of the coefficient we get when plugging in these operators a power function. Let us start observing the simple fact that when we apply any of the extremal operators to a radial function we obtain a radial function, that is, if  $v(x) = u(r)$  with  $|x| = r$  then  $\mathcal{M}^+v(x)$  and  $\mathcal{M}^-v(x)$  are radial functions.

Now we begin the study of fundamental solutions. We define the radially symmetric functions  $v_\sigma$  as follows

$$v_\sigma(r) = \begin{cases} r^\sigma & \text{if } -N < \sigma < 0, \\ -\log r & \text{if } \sigma = 0, \\ -r^\sigma & \text{if } 0 < \sigma < 2\alpha, \end{cases} \tag{3.1}$$

our goal is to find the value of the parameter  $\sigma$  so that this function solves the equation  $\mathcal{M}^+(v_\sigma) = 0$ . Similar with  $\mathcal{M}^-(v_\sigma) = 0$ . We start describing the range of  $\sigma$  for which the evaluation of the integral operator  $\mathcal{M}^+(v_\sigma)$  and  $\mathcal{M}^-(v_\sigma)$  makes sense. In order to find when  $\mathcal{M}^+(v_\sigma)$  vanishes we need to analyze the resulting coefficient (see  $c^+$  below), in analogy with what we usually do with the Laplacian. We denote in all what follows  $e_1 = (1, 0, \dots, 0) \in \mathbb{R}^N$ . We have:

**Lemma 3.1.** *For all  $-N < \sigma < 2\alpha$ ,  $\mathcal{M}^+(v_\sigma)(x)$  is well defined for  $x \neq 0$ . Moreover,*

$$\mathcal{M}^+v_\sigma(x) = c^+(\sigma)|x|^{\sigma-2\alpha}, \quad x \in \mathbb{R}^N \setminus \{0\},$$

where

$$c^+(\sigma) = \int_{\mathbb{R}^N} \frac{S_+(\delta_\sigma(y))}{|y|^{N+2\alpha}} dy, \tag{3.2}$$

and

$$\delta_\sigma(y) = \begin{cases} |e_1 + y|^\sigma + |e_1 - y|^\sigma - 2 & \text{if } \sigma \in (-N, 0), \\ -\log |e_1 + y| - \log |e_1 - y| & \text{if } \sigma = 0, \\ -|e_1 + y|^\sigma - |e_1 - y|^\sigma + 2 & \text{if } \sigma \in (0, 2\alpha). \end{cases} \tag{3.3}$$

In addition,

$$\lim_{\sigma \rightarrow -N} \mathcal{M}^+(v_\sigma)(x) = \infty \quad \text{and} \quad \lim_{\sigma \rightarrow 2\alpha} \mathcal{M}^+(v_\sigma)(x) = -\infty.$$

Similar statement can be made for  $\mathcal{M}^-$ .

**Proof.** For  $x \neq 0$  fixed and  $\sigma \in (-N, 0)$ , the integral defining  $\mathcal{M}^+v_\sigma(x)$  has three singularities:  $y = 0$ ,  $y = x$  and  $y = -x$ . First, for  $y = 0$ , the singularity is removable by the regularity of  $v_\sigma$ . For  $y = x$  we easily see that

$$\frac{c_1\eta^{N+\sigma} - c_2\varepsilon^{N+\sigma}}{N + \sigma} \leq \int_{B(x,\eta) \setminus B(x,\varepsilon)} \frac{S_+(\delta(v_\sigma, x, y))}{|y|^{N+2\alpha}} \leq \frac{C_1\eta^{N+\sigma} - C_2\varepsilon^{N+\sigma}}{N + \sigma},$$

where  $0 < \varepsilon < \eta$  are small and for some positive constants  $c_1, c_2, C_1$  and  $C_2$ . Therefore the integral is finite since  $-N < \sigma \leq 0$ . The same holds true for the singularity  $y = -x$ , and so  $\mathcal{M}^+v_\sigma(x)$  is well defined, for  $\sigma \in (-N, 0)$ . In case  $\sigma \in (0, 2\alpha)$ , we only need to check the integral at infinity, which is well defined since  $\sigma < 2\alpha$  and

$$\frac{|S_+(\delta(v_\sigma, x, y))|}{|y|^{N+2\alpha}} \leq C_1|y|^{\sigma-2\alpha-N}.$$

In case  $\sigma = 0$  there are simultaneous singularities at  $0, x, -x$  and infinity, but the estimates are similar.

Regarding the limits, for  $\sigma < 0$ , we have that

$$\delta(v_\sigma, x, y) \geq C|x - y|^\sigma \quad \text{and} \quad \delta(v_\sigma, x, y) \geq C|x + y|^\sigma,$$

for  $y$  near  $x$  and near  $-x$ , respectively. Consequently, when integrating in balls near these singularities the result follows. In the case of  $\sigma > 0$  we have

$$\delta(v_\sigma, x, y) \leq -C|y|^\sigma,$$

for  $y$  large, so that the result follows integrating outside a large ball.

Let us consider first  $\sigma \in (-N, 0) \cup (0, 2\alpha)$ . We have that  $\delta(v_\sigma, x, y) = |x|^\sigma \delta_\sigma(y/|x|)$  and then,

$$\mathcal{M}^+v_\sigma(x) = |x|^{\sigma-2\alpha} \int_{\mathbb{R}^N} \frac{S_+(\delta_\sigma(y))}{|y|^{N+2\alpha}} dy,$$

from where (3.2) follows. When  $\sigma = 0$  we see that, according to (3.3),

$$\delta(v_\sigma, x, y) = \delta_0(y/|x|)$$

and then the result follows changing variables as above. The case of  $\mathcal{M}^-$  is similar.  $\square$



Now we see a series of lemmas towards the proof of Theorem 1.1. We start with the case of the fractional Laplacian that will serve as a reference. Using the half space representation as in [6], through an explicit computation the following lemma can be proved.

**Lemma 3.2.** *If  $\Lambda = 1$  then  $\Delta^\alpha = \mathcal{M}^+ = \mathcal{M}^-$  and  $\Delta^\alpha(r^{-(N-2\alpha)}) = 0$ .*

In the next lemma we study differentiability properties of the function  $c^+$  that we need in the proof of our Theorem 1.1 about fundamental solutions.

**Lemma 3.3.** *The function  $c^+$  is twice differentiable in  $(-N, 0) \cup (0, 2\alpha)$  and the second derivative in the case  $\sigma \in (0, 2\alpha)$  is given in (3.5). At  $\sigma = 0$  the function  $c^+$  is differentiable from the left and from the right and the derivatives are given in (3.8).*

**Proof.** We first study the differentiability of  $c^+$  at  $\sigma_0 \in (0, 2\alpha)$ . We observe that the function  $\delta_\sigma$  is of class  $C^\infty(\mathbb{R}^N \setminus \{e_1, -e_1, 0\} \times (0, 2\alpha))$ , but if  $\Lambda > 1$  the term  $S_+(\delta_\sigma(y))$  in (3.2) is not differentiable. For  $\sigma \in (0, 2\alpha)$  we see that the set

$$C_\sigma^0 = \{y/\delta_\sigma(y) = 0\}$$

is a bounded  $(N - 1)$ -dimensional smooth hyper-surface except for a singularity at  $y = 0$ . In fact, we have that

$$\nabla\delta_\sigma(y) = -\sigma|e_1 + y|^{\sigma-2}(e_1 + y) - \sigma|e_1 - y|^{\sigma-2}(e_1 - y), \tag{3.4}$$

so that  $\nabla\delta_\sigma(y) = 0$  if and only if  $y = 0$ . We define the sets  $C^0 = C_{\sigma_0}^0$ ,  $C^+ = \{y/\delta_{\sigma_0}(y) > 0\}$ ,  $C^- = \{y/\delta_{\sigma_0}(y) < 0\}$  and, for  $\varepsilon > 0$ , we further define

$$C^{0,\varepsilon} = \{y/\text{dist}(y, C^0) < \varepsilon\}, \quad C^{+,\varepsilon} = C^+ \setminus C^{0,\varepsilon} \quad \text{and} \quad C^{-,\varepsilon} = C^- \setminus C^{0,\varepsilon}.$$

By the regularity of the set  $C^0$ , there exists  $\rho > 0$  so that  $C_\sigma^0 \subset C^{0,\varepsilon}$  for all  $\sigma \in (\sigma_0 - \rho, \sigma_0 + \rho)$ . Thus

$$c^+(\sigma) = \int_{C^{+,\varepsilon} \cup C^{-,\varepsilon}} \frac{S_+(\delta_\sigma(y))}{|y|^{N+2\alpha}} dy + \int_{C^{0,\varepsilon}} \frac{S_+(\delta_\sigma(y))}{|y|^{N+2\alpha}} dy,$$

where we observe that the second term above is differentiable at  $\sigma_0$ . If we denote by  $c_\varepsilon^+(\sigma)$  the third term above, we find that

$$\left| \frac{c_\varepsilon^+(\sigma) - c_\varepsilon^+(\sigma_0)}{\sigma - \sigma_0} \right| \leq C \int_{C^{0,\varepsilon}} \left| \frac{\delta_\sigma(y) - \delta_{\sigma_0}(y)}{\sigma - \sigma_0} \right| \frac{dy}{|y|^{N+2\alpha}} \leq Cm(\varepsilon),$$

where  $m(\varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ , since the function under the integral sign is integrable in bounded sets and the measure of  $C^{0,\varepsilon}$  approaches zero as  $\varepsilon \rightarrow 0$ . From here we see that  $c^+$  is differentiable

at  $\sigma_0$  its derivative is obtained by differentiating under the integral. Using the same argument we find that  $c^+$  is twice differentiable at  $\sigma_0$  and

$$\begin{aligned} (c^+)''(\sigma_0) &= \int_{C^+} \frac{\Delta[-|e_1 + y|^{\sigma_0}(\log |e_1 + y|)^2 - |e_1 - y|^{\sigma_0}(\log |e_1 - y|)^2]}{|y|^{N+2\alpha}} dy \\ &\quad + \int_{C^-} \frac{-|e_1 + y|^{\sigma_0}(\log |e_1 + y|)^2 - |e_1 - y|^{\sigma_0}(\log |e_1 - y|)^2}{|y|^{N+2\alpha}} dy. \end{aligned} \tag{3.5}$$

To prove that  $c^+$  is twice differentiable in  $(-N, 0)$  we proceed similarly.

Next we analyze the behavior of the function  $c^+(\sigma)$  near  $\sigma = 0$  as in the case  $\sigma_0 > 0$ , but being more careful in the analysis of the set  $C_\sigma^0$ . We see that  $C_\sigma^0$ , for  $\sigma \neq 0$ , corresponds to the set of points  $y$  satisfying

$$\frac{|e_1 + y|^\sigma - 1}{\sigma} + \frac{|e_1 - y|^\sigma - 1}{\sigma} = 0. \tag{3.6}$$

The point here is that this constraint corresponds, as  $\sigma \rightarrow 0$ , to the constraint defining the set  $C_0^0$  which is

$$\log(|e_1 + y|) + \log(|e_1 - y|) = 0. \tag{3.7}$$

Moreover, if we call  $\tilde{\delta}_\sigma(y)$  and  $\tilde{\delta}_0(y)$  the left-hand side of (3.6) and (3.7), respectively, then we see from (3.4) that  $\lim_{\sigma \rightarrow 0} \nabla \tilde{\delta}_\sigma(y) = \nabla \tilde{\delta}_0(y)$ . Furthermore, for  $y \in C_0^0 \setminus \{0\}$ , we have  $\nabla \tilde{\delta}_0(y) \neq 0$ , since

$$\nabla \tilde{\delta}_0(y) = \frac{e_1 + y}{|e_1 + y|^2} - \frac{e_1 - y}{|e_1 - y|^2} = 0$$

implies that  $|e_1 + y| = |e_1 - y|$ . But, being  $y \in C_0^0$ , this further implies that  $|e_1 + y| = |e_1 - y| = 1$  and then  $y = 0$ , which is impossible.

Now we are in the same position as in the proof of differentiability of  $c^+$  at  $\sigma \neq 0$ , so we have that

$$(c^+)'(0^-) = \lim_{\sigma \uparrow 0} \frac{c^+(\sigma)}{\sigma} = - \int_{\mathbb{R}^N} \frac{S_+(\delta_0(y))}{|y|^{N+2\alpha}} dy \tag{3.8}$$

and  $(c^+)'(0^+) = -(c^+)'(0^-)$ . Here  $\delta_0$  was defined in (3.3).

**Corollary 3.1.** *The function  $c^+$  is strictly convex in  $(-N, 0)$  and strictly concave in  $(0, 2\alpha)$ .*

**Proof.** In the interval  $(0, 2\alpha)$  the second derivative of  $c^+$  is given by (3.5) which negative, finishing the proof. In the interval  $(-N, 0)$  we could proceed similarly, however here  $c^+$  is strictly convex simply because the function  $S_+(\delta_\sigma(y))$  is strictly convex as a function of  $\sigma$ , for all  $y$ .  $\square$

Now we are prepared to prove the existence of fundamental solutions.

**Proof of Theorem 1.1.** As a first step we find an increasing surjective function  $\sigma^+ : [1, \infty) \rightarrow [-N + 2\alpha, 2\alpha)$ , such that for  $\sigma = \sigma^+(\Lambda)$  we have  $\mathcal{M}^+(v_\sigma) = 0$ . This means that for each  $\Lambda$ ,  $\mathcal{M}^+$  (that depends on  $\Lambda$ ) has  $v_{\sigma^+(\Lambda)}$  as a fundamental solution. According to Lemma 3.1 we have to analyze the zeroes of the coefficient  $c^+$ . At this point it is convenient to consider  $\Lambda$  as an explicit variable, so we write  $c^+(\sigma, \Lambda)$ . We observe that  $c^+(\sigma, \Lambda)$  is strictly increasing in  $\Lambda$ , that  $c^+(0, \Lambda) = 0$  for all  $\Lambda$  and we recall that  $c^+(\sigma, \Lambda)$  is strictly convex in  $\sigma$ . For  $\Lambda = 1$  the function  $c^+(\sigma, 1)$  is associated to the linear operator  $-(-\Delta)^\alpha$  and it satisfies  $c^+(-N + 2\alpha, 1) = 0$ , according to Lemma 3.2. Clearly  $c^+(\sigma, 1) < 0$  for  $\sigma \in (-N + 2\alpha, 0)$  and we can define

$$\Lambda^* = \sup\{\Lambda > 1/c^+(\sigma, \Lambda) \text{ is negative at some point in } (-N, 0)\},$$

which is bounded, since for large  $\Lambda$  we have  $(c^+)'(0^-, \Lambda) < 0$  as can be seen from (3.8). By the above properties it is now clear that for all  $\Lambda \in (1, \Lambda^*)$  there is a unique  $\sigma^+(\Lambda) \in (-N + 2\alpha, 0)$  such that  $c^+(\sigma^+(\Lambda), \Lambda) = 0$ . We observe  $\sigma^+(\Lambda)$  defines an increasing function, that further satisfies  $\lim_{\Lambda \uparrow \Lambda^*} \sigma^+(\Lambda) = 0$ .

Since  $c^+(\sigma^+(\Lambda), \Lambda) = 0$ , from the left differentiability of  $c^+$  at  $\sigma = 0$  and its continuity, we find that  $(c^+)'(0^-, \Lambda^*) = 0$ . A consequence of this is that  $(c^+)'(0^+, \Lambda^*) = 0$  as we can see from (3.8). Then, by monotonicity in  $\Lambda$ , we see that  $(c^+)'(0^-, \Lambda) < 0$  and  $(c^+)'(0^+, \Lambda) > 0$  if  $\Lambda > \Lambda^*$ . Thus, by the concavity of  $c^+$  in  $(0, 2\alpha)$  and its limit property at  $2\alpha$  given in Lemma 3.1, we find that, for every  $\Lambda > \Lambda^*$  there exists  $\sigma^+(\Lambda) \in (0, 2\alpha)$  such that  $c^+(\sigma^+(\Lambda), \Lambda) = 0$ . Notice that  $c^+(1, \Lambda) < 0$  for all  $\Lambda \geq 1$ , therefore  $\sigma^+(\Lambda) \in (0, \min\{1, 2\alpha\})$ . Since the set  $\{\delta_\sigma > 0\}$  has positive measure for fix  $\sigma \in (0, \min\{1, 2\alpha\})$  we have that  $\lim_{\Lambda \rightarrow \infty} c^+(\sigma, \Lambda) = \infty$ . From last fact it follows that  $\lim_{\Lambda \rightarrow \infty} \sigma^+(\Lambda) = \min\{1, 2\alpha\}$ . This completes the construction of the function  $\sigma^+$ .

If we define

$$N_{\Lambda, \alpha}^+ = -\sigma^+(\Lambda) + 2\alpha,$$

we finish the part of the proof concerning  $N^+$  and the functions  $\phi_{N^+}$ .

As a second step we prove the results related to  $N^-$ , that is, we find  $\sigma \in (-N, -N + 2\alpha)$  such that  $\mathcal{M}^+(-v_\sigma) = 0$ , or equivalently  $\mathcal{M}^-(v_\sigma) = 0$ . We notice that the function  $v_\sigma(r) = r^\sigma$  is a convex function of  $r$  and a convex function of  $\sigma$ . Here we can find a coefficient analogous to  $c^+$  in Lemma 3.1. We have that for any  $-N < \sigma < 0$ ,

$$\mathcal{M}^- v_\sigma(x) = c^-(\sigma)|x|^{\sigma-2\alpha}$$

where

$$c^-(\sigma) = \int_{\mathbb{R}^N} \frac{S_-(\delta_\sigma(y))}{|y|^{N+2\alpha}} dy. \tag{3.9}$$

The goal is to find a decreasing surjective function  $\sigma^- : [1, \infty) \rightarrow (-N, -N + 2\alpha]$  such that, for  $\sigma = \sigma^-(\Lambda)$ ,

$$\mathcal{M}^-(v_\sigma) = 0.$$

This means that for each  $\Lambda$ ,  $\mathcal{M}^-$  (that depends on  $\Lambda$ ) has  $v_{\sigma^-(\Lambda)}$  as a fundamental solution. As before we explicit the parameter  $\Lambda$  in  $c^-$ .

Using the arguments of Lemma 3.3 we can prove that  $c^-$  is twice differentiable and that its second derivative is positive, so that  $c^-$  is strictly convex. Then, since  $\lim_{\sigma \rightarrow -N} c^-(\sigma, \Lambda) = \infty$  and  $c^-(\sigma, \Lambda) < c^-(\sigma, 1) \equiv c^+(\sigma, 1)$ , we find that for every  $\Lambda > 1$  there exists a unique  $\sigma^-(\Lambda)$  such that  $c^-(\sigma^-(\Lambda), \Lambda) = 0$ . Since the set  $\{\delta_\sigma < 0\}$  has positive measure for fix  $\sigma \in (-N, 0)$  we have that  $\lim_{\Lambda \rightarrow \infty} c^-(\sigma, \Lambda) = -\infty$ . From last fact it follows that  $\lim_{\Lambda \rightarrow \infty} c^-(\Lambda) = -N$ . So, function  $\sigma^-$  has the required properties.

If we define

$$N_{\Lambda, \alpha}^- = -\sigma^-(\Lambda) + 2\alpha$$

and the function  $\phi_{N^-} = -r^{-N^-+2\alpha}$  we complete the proof of the theorem regarding  $N^-$  and  $\phi_{N^-}$ .

The corresponding functions  $\varphi_{N^+}$  and  $\varphi_{N^-}$  obviously satisfy the required properties. The theorem is now proved.  $\square$

**Remark 3.1.** One may think that there are still other fundamental solutions. One could try to find some  $\sigma \in (0, 2\alpha)$  such that  $\mathcal{M}^-(-r^\sigma) = 0$ . However, as we see next, such a  $\sigma$  does not exist.

First, by properties of the extremal operators, the equation is equivalent to  $\mathcal{M}^+(r^\sigma) = 0$ . Next we see that  $\mathcal{M}^+(r^\sigma) > -(-\Delta)^\alpha(r^\sigma)$ , for all  $\Lambda > 1$ . Then we observe that, as a function of  $\sigma$ ,  $-(-\Delta)^\alpha(r^\sigma)$  is convex and it vanishes at  $-N + 2\alpha$  and 0. Thus,  $-(-\Delta)^\alpha(r^\sigma) > 0$  for all  $\sigma \in (0, 2\alpha)$ , from where the conclusion follows.

#### 4. Hadamard property for Liouville type theorems

A key ingredient in the study of Liouville type theorems for Pucci’s operators has been the Hadamard Three Spheres Theorem, see [12] and [14]. The proof of this theorem in the case of the Laplacian and Pucci’s operator requires a comparison with fundamental solutions through the use of the maximum principle: the maximum of a sub-harmonic function is achieved at the boundary. In the case of integral operators we do not have such a maximum principle, since values of the function at the boundary of a domain have a weaker meaning than in the differential case. However, for the analysis needed for proving the Liouville type theorems less information is needed. In this section we prove two properties of super-harmonic functions, that is functions satisfying  $\mathcal{M}^+u \leq 0$  or  $\mathcal{M}^-u \leq 0$  in  $\mathbb{R}^N$ , in the case when  $N^+ > 2\alpha$  or  $N^- > 2\alpha$ . We call them Hadamard properties, since both of them are consequences of Hadamard Three Spheres Theorem in the second order differential case and they will be sufficient for our purposes.

In proving our lemmas we use comparison techniques that require the modification of the fundamental solution near the origin, in order to put it below the super-harmonic function near the origin. This is necessary since  $u$  is bounded and the fundamental solutions are singular at the origin.

We recall that the operators we are working with depend on the parameters  $\lambda = 1, 1 < \Lambda$  and  $\alpha$ , but we do not write them explicitly. We define

$$m(r) = \min_{|x| \leq r} u(x),$$

where  $u$  is a non-negative function.

**Lemma 4.1.** *Given  $N \geq 2$ ,  $\Lambda > 1$  and  $\alpha \in (0, 1)$  and assume that  $N^+ > 2\alpha$ . Then, for all  $\sigma \in (-N, -N^+ + 2\alpha)$ , there exists  $c > 0$  such that for every non-negative viscosity solution  $u \neq 0$  of*

$$\mathcal{M}^+u(x) \leq 0 \quad \text{in } \mathbb{R}^N \tag{4.1}$$

we have

$$m(r) \geq cm(r_1)r^\sigma, \quad \text{for all } r \geq r_1 \geq 1. \tag{4.2}$$

Similarly, assume that  $N^- > 2\alpha$ , then for all  $\sigma \in (-N, -N^- + 2\alpha)$  there exists  $c > 0$  such that for every non-negative viscosity solution  $u \neq 0$  of

$$\mathcal{M}^-u(x) \leq 0 \quad \text{in } \mathbb{R}^N \tag{4.3}$$

we have (4.2).

**Remark 4.1.** Notice that  $u(x) > 0$  for all  $x \in \mathbb{R}^N$ . In fact, since  $u$  is non-negative and  $u \neq 0$ , if  $u$  attains the minimum value 0 at a point  $x$ , just by computing  $\mathcal{M}^+$  at  $x$  we get  $\mathcal{M}^+u(x) > 0$ , a contradiction. The same holds for  $\mathcal{M}^-$ .

**Remark 4.2.** We observe that (4.2) is a bit weaker than what is usually achieved in the second order differential case, since  $\sigma < \sigma^+$ , the decay rate of the fundamental solutions.

**Proof.** We only do the proof for  $\mathcal{M}^+$ , since the other case is analogous. Let  $R > r_1 \geq 1$ ,  $\varepsilon > 0$  and for  $\sigma \in (-N, -N^+ + 2\alpha)$  define the function

$$w(r) = \begin{cases} \varepsilon^\sigma & \text{if } 0 < r \leq \varepsilon, \\ r^\sigma & \text{if } \varepsilon \leq r. \end{cases}$$

We claim that for  $\varepsilon$  small  $\mathcal{M}^+w(|x|) \geq 0$  for all  $r_1 < |x| < R$ . Postponing the proof claim, we define

$$\phi(x) = m(r_1) \frac{w(|x|) - w(R)}{w(\varepsilon) - w(R)}, \quad \text{for } |x| \leq R$$

and  $\phi(x) = 0$  for  $|x| \geq R$ . Now, using the claim, we have that

$$\mathcal{M}^+\phi \geq 0, \quad \text{for all } r_1 < |x| < R$$

and, since  $u(x) \geq \phi(|x|)$  if  $|x| \leq r_1$  or  $|x| \geq R$ , using Theorem 2.1 we obtain  $u(x) \geq \phi(|x|)$ , for all  $r_1 \leq |x| \leq R$ . Thus, taking the limit as  $R \rightarrow \infty$  and we obtain (4.2) with  $c = \varepsilon^{-\sigma}$ . Now we prove the claim: we have

$$\mathcal{M}^+r^\sigma = c^+(\sigma)r^{\sigma-2\alpha},$$

with  $c^+(\sigma) > 0$ , since  $\sigma \in (-N, -N^+ + 2\alpha)$ , and we see that

$$\mathcal{M}^+ w(r) = r^{\sigma-2\alpha} (c^+(\sigma) - I(r, \varepsilon)),$$

with

$$I(r, \varepsilon) = r^{-\sigma+2\alpha} \int_{B_\varepsilon(x) \cup B_\varepsilon(-x)} \frac{S_+(\delta(x, y, r^\sigma)) - S_+\delta(x, y, w)}{|y|^{N+2\alpha}} dy,$$

here  $r = |x|$ . Making a change of variables and using the fact that for all  $a, b \in \mathbb{R}$  we have

$$S_-(a - b) \leq S_+(a) - S_+(b) \leq S_+(a - b), \tag{4.4}$$

we find

$$\begin{aligned} I(r, \varepsilon) &\leq \int_{B_{\varepsilon/r}(e_1)} \frac{S_+(|e_1 - y|^\sigma - |\varepsilon/r|^\sigma)}{|y|^{N+2\alpha}} dy \\ &\quad + \int_{B_{\varepsilon/r}(-e_1)} \frac{S_+(|e_1 + y|^\sigma - |\varepsilon/r|^\sigma)}{|y|^{N+2\alpha}} dy. \end{aligned}$$

Then we consider that  $r = |x| \geq r_1 \geq 1$  and take  $\varepsilon > 0$  small enough to obtain  $I(r, \varepsilon) < c^+(\sigma)$ , for all  $r \geq r_1$ , completing the proof.  $\square$

The next lemma is more delicate since, for comparison purposes, the fundamental solution needs to be cut near the origin in a big portion. Then it is necessary to cut it also in the complement of a large ball, so we produce an appropriate balance in the errors occurring both sides.

**Lemma 4.2.** *Given  $N \geq 2$ ,  $\Lambda > 1$  and  $\alpha \in (0, 1)$  and assume that  $N^+ > 2\alpha$ . Then, there is  $r_1 > 0$  and a constant  $c$  such that for every non-negative viscosity solution of  $u \neq 0$  of (4.1) we have*

$$m(R/2) \leq cm(R), \quad \text{for all } R \geq r_1. \tag{4.5}$$

Similarly, assuming that  $N^- > 2\alpha$ , there is  $r_1 > 0$  and a constant  $c$  such that for every non-negative viscosity solution of  $u \neq 0$  of (4.3) we have (4.5).

**Proof.** Given  $\varepsilon > 0$  and  $R > 0$ , we define

$$R_0 = R \left[ \frac{\varepsilon}{1 + \varepsilon 2^{\sigma^+}} \right]^{-1/\sigma^+}$$

and assume that  $\varepsilon$  is such that  $R_0 < R/2$ . We consider the functions

$$w_R(r) = \begin{cases} (R_0)^{\sigma^+} & \text{if } 0 < r \leq R_0, \\ r^{\sigma^+} & \text{if } R_0 \leq r \leq 2R, \\ (2R)^{\sigma^+} & \text{if } r \geq 2R \end{cases} \quad \text{and} \quad w(r) = \begin{cases} (R_0)^{\sigma^+} & \text{if } 0 < r \leq R_0, \\ r^{\sigma^+} & \text{if } R_0 \leq r, \end{cases}$$

and define

$$\phi(r) = m(R/2) \frac{w_R(r) - w(2R)}{w(R_0) - w(2R)}.$$

We observe that  $u(x) \geq \phi(|x|)$  for all  $|x| \leq R/2$  or  $|x| \geq 2R$ . Next we claim that

$$\mathcal{M}^+ \phi(|x|) \geq 0, \quad \text{for all } R/2 < |x| < 2R. \tag{4.6}$$

Assuming the claim for the moment, we may apply the comparison principle Theorem 2.1 to obtain that  $u(x) \geq \phi(|x|)$  for all  $R/2 < |x| < 2R$ , from where we obtain, by taking the minimum of  $u$  in  $0 < |x| \leq R$ , that

$$m(R) \geq \varepsilon m(R/2)(1 - 2^{\sigma^+}).$$

The result follows taking  $c = \varepsilon(1 - 2^{\sigma^+})$ . Next we show that the claim (4.6) holds if we choose  $\varepsilon > 0$  small enough. We just need to see that  $\mathcal{M}^+ w_R(|x|) \geq 0$  if  $R/2 < |x| < 2R$ . By definition of  $\sigma^+$  we have that

$$0 = \mathcal{M}^+(r^{\sigma^+}) = \mathcal{M}^+ w(r) + I(\varepsilon, r), \tag{4.7}$$

where  $r = |x|$  and

$$\begin{aligned} I &= \int_{B_{R_0}(x) \cup B_{R_0}(-x)} \frac{S_+ \delta(x, y, r^{\sigma^+}) - S_+ \delta(x, y, w)}{|y|^{N+2\alpha}} dy \\ &\leq \int_{B_{R_0}(x)} \Lambda \frac{|x - y|^{\sigma^+} - (R_0)^{\sigma^+}}{|y|^{N+2\alpha}} dy + \int_{B_{R_0}(-x)} \Lambda \frac{|x + y|^{\sigma^+} - (R_0)^{\sigma^+}}{|y|^{N+2\alpha}} dy, \end{aligned}$$

by (4.4) and since the balls  $B_{R_0}(x)$  and  $B_{R_0}(-x)$  are disjoint. We only need to estimate one of these integrals, since they are equal. By definition of  $R_0$ , for every  $y \in B_{R_0}(x)$ , we have that  $|y| \geq R/3$  if we take  $\varepsilon$  small enough. Then we obtain

$$\begin{aligned} \int_{B_{R_0}(x)} \frac{|x - y|^{\sigma^+} - (R_0)^{\sigma^+}}{|y|^{N+2\alpha}} dy &\leq \left(\frac{3}{R}\right)^{N+2\alpha} \int_0^{R_0} (r^{\sigma^+ + N - 1} - (R_0)^{\sigma^+} r^{N-1}) dr \\ &\leq C R^{-N^+} \varepsilon^{-(\sigma^+ + N)/\sigma^+}, \end{aligned} \tag{4.8}$$

where we have used the definition of  $R_0$  and the fact that  $\sigma^+ + N > 0$ . The constant  $C$  does not depend on  $\varepsilon$  nor  $R$ .

On the other hand we consider  $E(\varepsilon, r)$  such that  $\mathcal{M}^+ w = \mathcal{M}^+ w_R - E(\varepsilon, r)$  and we estimate its value from below. We recall that  $w_R(r) \geq w(r)$  for all  $r$  and we observe that for  $y \notin B_{5R}(0)$

we have  $w_R(x + y) = w_R(x - y) = (2R)^{\sigma^+}$  and  $\delta(x, y, w_R), \delta(x, y, w) \geq 0$ , thus

$$\begin{aligned} E(\varepsilon, r) &\geq \int_{B_{5R}^c(0)} \frac{S_+\delta(x, y, w_R) - S_+\delta(x, y, w)}{|y|^{N+2\alpha}} dy \\ &\geq \int_{B_{5R}^c(0)} \frac{2(2R)^{\sigma^+} - |x - y|^{\sigma^+} - |x + y|^{\sigma^+}}{|y|^{N+2\alpha}} dy. \end{aligned} \tag{4.9}$$

We see that, for  $x, y$  such that  $R/2 \leq |x| \leq 2R$  and  $|y| \geq 5R$ , we have  $|x - y| \geq 3|y|/5$  and  $|x + y| \geq 3|y|/5$ . Consequently

$$E(\varepsilon, r) \geq 2 \int_{5R}^{\infty} \left[ (2R)^{\sigma^+} r^{-2\alpha-1} - \left(\frac{3}{5}\right)^{\sigma^+} r^{\sigma^+-2\alpha-1} \right] dr \geq C R^{-N^+}, \tag{4.10}$$

where the generic constant  $C$  is positive and does not depend on  $R$  nor  $\varepsilon$ . We recall that, by definition  $N^+ = -\sigma^+ + 2\alpha$ . Since

$$\mathcal{M}^+ w_R = E(\varepsilon, r) - I(\varepsilon, r),$$

for all  $R/2 \leq r \leq 2R$ , from (4.8) and (4.10) the result follows if we choose  $\varepsilon$  small enough. The case  $\mathcal{M}^-$  is similar.  $\square$

### 5. The Liouville property

We devote this section to the proof of Theorem 1.2, which is obtained by comparing the super-harmonic function with fundamental solutions. The goal is to reach a contradiction by proving that the super-harmonic function possesses a global minimum. As in Section 4, we also need to adapt the fundamental solutions, cutting them near the origin, in order to have proper comparison with the given super-harmonic function outside the domain where the equation holds.

**Proof of Theorem 1.2.** By Remark 4.1, we may assume that  $u(x) > 0$  for all  $x$ . Let us consider first the case  $N^+ < 2\alpha$ , that is,  $\sigma^+ = -N^+ + 2\alpha \in (0, 2\alpha)$ , and consider the function

$$w(r) = \begin{cases} -\varepsilon^\sigma & \text{if } 0 < r \leq \varepsilon, \\ -r^\sigma & \text{if } \varepsilon \leq r, \end{cases} \tag{5.1}$$

where  $\varepsilon > 0$  and  $\sigma \in (0, \sigma^+)$ . We recall that  $-r^{\sigma^+}$  is a fundamental solution for  $\mathcal{M}^+$ , where as usual,  $|x| = r$ . According to the analysis in Section 3, we have that

$$\mathcal{M}^+(-r^\sigma) = c^+(\sigma)r^{\sigma-2\alpha},$$

with  $c^+(\sigma) > 0$ . Now we choose  $r_1 > 1 \geq 2\varepsilon$  and consider

$$\mathcal{M}^+ w(r) = c^+(\sigma)r^{\sigma-2\alpha} + \int_{B_\varepsilon(x) \cup B_\varepsilon(-x)} \frac{S_+(\delta(x, y, w)) - S_+(\delta(x, y, -r^\sigma))}{|y|^{N+2\alpha}} dy$$



for  $|x| > r_1$ . In order to estimate this integral, by (4.4) and symmetry, we just need to estimate

$$\int_{B_\varepsilon(x)} \frac{|x - y|^\sigma - \varepsilon^\sigma}{|y|^{N+2\alpha}} dy = r^{\sigma-2\alpha} \int_{B_{\varepsilon/|x|}(e_1)} \frac{|e_1 - y|^\sigma - (\varepsilon/|x|)^\sigma}{|y|^{N+2\alpha}} dy = e(\varepsilon, r)r^{\sigma-2\alpha}.$$

It is not difficult to see that  $e(\varepsilon, r) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ , uniformly since  $|x| \geq r_1$ . Thus, choosing  $\varepsilon$  small enough we have that

$$\mathcal{M}^+ w \geq 0, \quad \text{for all } |x| \geq r_1.$$

We define now the function

$$\phi(x) = m(r_1) \frac{w(|x|) - w(r_2)}{w(\varepsilon) - w(r_2)}, \quad \text{for } |x| \leq r_2,$$

and  $\phi(x) = 0$  for  $|x| \geq r_2$  and we see that

$$\mathcal{M}^+ \phi \geq 0, \quad \text{for all } r_1 < |x| < r_2,$$

and  $u(x) \geq \phi(x)$  for all  $r_1 \leq |x|$  or  $|x| \geq r_2$ . Then we use comparison Theorem 2.1 to obtain that  $u(x) \geq \phi(x)$  for  $r_2 \geq |x| \geq r_1$ . If we take limit when  $r_2 \rightarrow \infty$ , noticing that  $w(r_2) \rightarrow -\infty$ , we obtain that

$$u(x) \geq m(r_1), \quad \text{for all } r_1 < |x|. \tag{5.2}$$

But then  $u$  has a global minimum point in  $B(0, r_1)$ , contradicting  $\mathcal{M}^+ u \leq 0$ .

To conclude we analyze the case  $\sigma^+ = -N^+ + 2\alpha = 0$ . Take  $\sigma$  such that  $-N < \sigma < -N^- - 2\alpha < -N^+ - 2\alpha = 0$ . Thus  $\mathcal{M}^-(r^\sigma) = c^-(\sigma)r^{\sigma-2\alpha}$ , with  $c^-(\sigma) > 0$ . Notice that

$$\mathcal{M}^+(r^\sigma - \log r) \geq \mathcal{M}^+(-\log r) + \mathcal{M}^-(r^\sigma) = c^-(\sigma)r^{\sigma-2\alpha}.$$

Define now the function

$$w(r) = \begin{cases} \varepsilon^\sigma - \log \varepsilon & \text{if } 0 < r \leq \varepsilon, \\ r^\sigma - \log r & \text{if } \varepsilon \leq r. \end{cases} \tag{5.3}$$

As in the cases discussed above, we can choose  $\varepsilon$  small enough such that

$$\mathcal{M}^+ w \geq 0, \quad \text{for all } |x| \geq r_1$$

and from here we conclude as before.  $\square$

### 6. Nonlinear Liouville type theorem

In this section we provide a proof of our main theorem on the Liouville type non-existence result. In the proof we use appropriate estimates derived directly from the scaling property of the integral operator and an adequate test function on the equation. This estimated is put together with the Hadamard properties that we proved in Section 4.

The critical case requires an extra work, since the usual fundamental solution does not provide a sharp enough estimate.

**Proof of Theorem 1.3.** (The sub-critical case.) Let  $\eta : [0, \infty) \rightarrow \mathbb{R}$  such that  $0 \leq \eta(r) \leq 1$ ,  $\eta \in C^\infty$ ,  $\eta$  non-increasing,  $\eta(r) = 1$  if  $0 \leq r \leq 1/2$  and  $\eta(r) = 0$  if  $r \geq 1$ . It is obvious that there exists  $C > 0$  such that

$$-\mathcal{M}^+(\eta|x|) \leq C.$$

Define now  $\xi(x) = m(R/2)\eta(|x|/R)$ , where  $m(r)$  was defined in Section 4. Then by scaling property of  $\mathcal{M}^+$  we have

$$\mathcal{M}^+(\xi|x|) \geq -\frac{m(R/2)C}{R^{2\alpha}}.$$

In addition,  $\xi(x) = 0 \leq u(x)$  if  $|x| > R$  and  $\xi(x) = m(R/2) \leq u(x)$  if  $|x| \leq R/2$ . Thus there exists a global minimum of  $u(x) - \xi(x)$  achieved in a point  $x_R$  with  $|x_R| < R$ .

Let now  $\varphi(x) := \xi(x) - \xi(x_R) + u(x_R)$  and  $N = B(0, R)$ , then  $\varphi(x_R) = u(x_R)$  and  $u(x) \geq \varphi(x)$  for all  $x \in N$ . If  $v$  is defined as in (2.2), since  $u$  is a viscosity super-solution of (1.7), we have

$$\mathcal{M}^+(v) + u(x_R)^p \leq 0. \tag{6.1}$$

We claim that

$$\mathcal{M}^\pm(v)(x_R) \geq \mathcal{M}^\pm(\xi)(x_R).$$

In fact  $w(x) := v(x) - \xi(x) \geq 0$  for all  $x \in \mathbb{R}^N$ , and  $x_R$  is a global minimum of  $w$ . Thus,  $\mathcal{M}^-(w)(x_R) > 0$  and then the claim follows by the fact that  $\mathcal{M}^-(v - \xi) \leq \mathcal{M}^+(v) - \mathcal{M}^+(\xi)$ . Therefore from (6.1) we get

$$m(R)^p \leq u(x_R)^p \leq \frac{m(R/2)C}{R^{2\alpha}}.$$

Using now Lemma 4.2, from the above inequality we obtain

$$m(R) \leq \frac{C}{R^{\frac{2\alpha}{p-1}}}. \tag{6.2}$$

Now, let us assume that  $p < \frac{N^+}{N^+ - 2\alpha}$ . Then we choose  $\sigma < -N^+ + 2\alpha$  such that  $p - 1 < -2\alpha / \sigma < \frac{2\alpha}{N^+ - 2\alpha}$  and we apply Lemma 4.1 to obtain (4.2). Combining (4.2) with (6.2) we reach a con-

tradition, unless  $u \equiv 0$ . In a completely similar way, replacing  $\mathcal{M}^+$  by  $\mathcal{M}^-$  and  $N^+$  by  $N^-$  we obtain the proof for  $\mathcal{M}^-$ . This finishes the proof of the theorem in the sub-critical case.  $\square$

**Remark 6.1.** We would like to mention that in studying Liouville type theorems for general fully nonlinear second order operators, in a very recent paper Armstrong and Sirakov [1] avoided the use of an estimate like in Lemma 4.2 by using an estimate for the first half eigenvalue for the operator in an annulus. We do not have such an eigenvalue theory here, but it would be interesting to explore this idea in the future.

In the case  $p = \frac{N^\pm}{N^\pm - 2\alpha}$  this argument cannot be applied directly and we need some extra work. We define the function  $\Gamma^\pm(x) = \eta(x)h^\pm(x)$  for  $x \neq 0$ , where

$$\eta(x) = \log(1 + |x|) \quad \text{and} \quad h^\pm(x) = |x|^{-N^\pm + 2\alpha},$$

then the following lemma allows to use  $\Gamma^\pm$  as a good comparison function:

**Lemma 6.1.** *There exists a constant  $C > 0$  such that*

$$\mathcal{M}^-(\Gamma^\pm)(x) \geq -C|x|^{-N^\pm}, \quad x \neq 0. \tag{6.3}$$

**Proof.** We have

$$\begin{aligned} \mathcal{M}^-(\Gamma^-)(x) &= \mathcal{M}^-(\eta h^-)(x) - \eta(x)\mathcal{M}^-(h^-)(x) \\ &\geq \mathcal{M}^-(\eta h^- - \eta(x)h^-)(x), \end{aligned} \tag{6.4}$$

where  $\eta(x)$  is considered constant regarding the integral defining  $\mathcal{M}^-$ . Similarly we have

$$\begin{aligned} \mathcal{M}^+(\Gamma^+)(x) &= \mathcal{M}^+(\eta h^+)(x) - \eta(x)\mathcal{M}^+(h^+)(x) \\ &\geq \mathcal{M}^-(\eta h^+ - \eta(x)h^+)(x). \end{aligned} \tag{6.5}$$

Here we have used that for any  $u, v$  such that  $\mathcal{M}^-(u)$  and  $\mathcal{M}^+(v)$  are well defined we have

$$\mathcal{M}^-(u) \geq \mathcal{M}^-(u - v) + \mathcal{M}^-(v) \quad \text{and} \quad \mathcal{M}^-(u) \leq \mathcal{M}^+(u + v) - \mathcal{M}^+(v).$$

Our purpose is to find a lower estimate for  $\mathcal{M}^-(\eta h - \eta(x)h)(x)$ , when  $x$  is large. We have

$$\mathcal{M}^-(\eta h - \eta(x)h)(x) = \int_{\mathbb{R}^N} S_-(\hat{\delta}(x, y)) \frac{dy}{|y|^{N+2\alpha}}, \tag{6.6}$$

where

$$\hat{\delta}(x, y) = (\eta(x + y) - \eta(x))h(x + y) + (\eta(x - y) - \eta(x))h(x - y).$$

At this point it is convenient to write  $h(x) = |x|^\sigma$ , where  $\sigma = -N^- + 2$  or  $\sigma = -N^+ + 2$ . Thus, we get

$$\begin{aligned} \hat{\delta}(x, y) &= |x + y|^\sigma \log\left(\frac{1 + |x + y|}{1 + |x|}\right) + |x - y|^\sigma \log\left(\frac{1 + |x - y|}{1 + |x|}\right) \\ &= r^\sigma \delta_1(r, z) \end{aligned}$$

where  $x = re_1, z = y/r$  and

$$\delta_1(r, z) = |e_1 + z|^\sigma \log\left(\frac{1 + r|e_1 + z|}{1 + r}\right) + |e_1 - z|^\sigma \log\left(\frac{1 + r|e_1 - z|}{1 + r}\right).$$

Since  $\eta$  and  $h$  are radially symmetric, there is no loss of generality in considering  $x = re_1$ . Now we introduce this expression back into (6.6) and make the change of variables  $z = y/r$  to obtain

$$\mathcal{M}^-(\eta h - \eta(x)h)(x) = r^{\sigma-2\alpha} I(r) \tag{6.7}$$

where

$$I(r) = \int_{\mathbb{R}^N} S_-(\delta_1(r, z)) \frac{dz}{|z|^{N+2\alpha}}. \tag{6.8}$$

In order to complete the proof we just need to find a constant  $C \geq 0$  such that  $I(r) \geq -C$  for large  $r$ . For this purpose we study the integral (6.8) at the singularities  $e_1, -e_1, 0$  and at infinity. It is convenient to write

$$\delta_1(r, z) = g(|e_1 + z|, \theta) + g(|e_1 - z|, \theta), \tag{6.9}$$

where

$$g(t, \theta) = t^\sigma \log(1 + \theta(t - 1)) \quad \text{and} \quad \theta = \frac{r}{1 + r},$$

for  $t > 0, \theta \in [0, 1)$  and  $r \geq 0$ . First consider  $B_1 = \{z/|z + e_1| \leq 1/2\}$  and observe that  $g(|e_1 - z|, \theta)$  is bounded in  $B_1$  while  $g(|e_1 + z|, \theta)$  has a singularity at  $-e_1 \in B_1$ . Then we see that, for a generic constant  $C$ ,

$$\begin{aligned} \int_{B_1} |g(|e_1 + z|, \theta)| \frac{dz}{|z|^{N+2\alpha}} &\leq -C \int_0^{1/2} g(t, \theta) t^{N-1} dt \\ &\leq -C \int_0^{1/2} t^{\sigma+N-1} \log(t) dt \leq C. \end{aligned} \tag{6.10}$$

Notice that  $1 + \theta(t - 1) \geq t$ , as  $\theta \in [0, 1)$ . We have used that  $\sigma + N > 0$ , that is,  $-N^- + 2\alpha + N > 0$  and  $-N^+ + 2\alpha + N > 0$ , which hold by Theorem 1.1. Thus, the integral in (6.8) when consider over  $B_1$  is bounded below by a constant independent of  $r$ . Similar result is obtained in  $B_2 = \{z/|z - e_1| \leq 1/2\}$ .

On the set  $B_3 = \{z \mid |z| \geq 2\}$  we have

$$|\delta_1(r, z)| \leq C|z|^{\sigma-N-2\alpha} \log |z|.$$

Since  $\sigma - N - 2\alpha = -N^\pm - N < -N$  the integral in (6.8) when consider over  $B_3$  is also bounded by a constant independent of  $r$ .

We finally analyze the behavior of the integral over  $B_4 = \{z \mid |z| \leq 1/2\}$ . By the fundamental theorem of calculus we have

$$\delta_1(r, z) = z^t \cdot \int_0^1 (1-t) D^2 \delta_1(r, tz) dt \cdot z, \tag{6.11}$$

where we used that  $\delta_1(r, 0) = 0$ ,  $D\delta_1(r, 0) = 0$  and derivatives are considered only with respect to  $z$ . Thus, to estimate the integral (6.8) over  $B_4$  we just need to prove

$$|D^2 \delta_1(r, z)| \leq C, \quad \text{for all } |z| \leq 1/2, \tag{6.12}$$

for a constant  $C$  independent of  $r$ . It is a direct computation to obtain

$$g'(t, \theta) = \sigma t^{\sigma-1} \log(1 + \theta(t-1)) + \frac{\theta t^\sigma}{1 + \theta(t-1)},$$

$$g''(t, \theta) = \sigma(\sigma-1)t^{\sigma-2} \log(1 + \theta(t-1)) + \frac{2\theta\sigma t^\sigma}{1 + \theta(t-1)} - \frac{\theta^2 t^\sigma}{(1 + \theta(t-1))^2}$$

and then we see that there is a constant so that

$$|g'(t, \theta)|, |g''(t, \theta)| \leq C, \quad \text{for all } \theta \in [0, 1), 1/2 \leq t \leq 3/2.$$

Then, computing the derivatives of  $\delta_1$  using formula (6.9),

$$\frac{\partial^2 \delta_1(r, z)}{\partial z_i \partial z_j} = g''(|e_1 + z|, \theta) D_{ij} + g'(|e_1 + z|, \theta) d_{ij},$$

for certain functions  $D_{ij}$  and  $d_{ij}$ , which are bounded since  $|z| \leq 1/2$ . From here we obtain (6.12), completing the proof of the lemma.  $\square$

**Proof of Theorem 1.3 continued.** (The critical case.) We do the proof only for  $\mathcal{M}^+$ , since the other case is similar. We start by proving that for certain  $r_1 > 0$  and  $c > 0$  we have

$$u(x) \geq cm(r_1)r^{\sigma^+}, \quad \text{for } r \geq r_1. \tag{6.13}$$

From Eq. (1.7) and Lemma 4.1 we have that, for any  $\sigma < \sigma^+$ ,

$$\mathcal{M}^+ u(x) = -u^p \leq c(m(r_1))^p r^{p\sigma}, \quad \text{for } r \geq r_1. \tag{6.14}$$

On the other hand we consider the function  $w$  defined as

$$w(r) = \begin{cases} \varepsilon^{\sigma^+} & \text{if } 0 < r \leq \varepsilon, \\ r^{\sigma^+} & \text{if } \varepsilon \leq r, \end{cases} \tag{6.15}$$

where  $\varepsilon > 0$  and  $\varepsilon < r_1/2$ . Since  $r^{\sigma^+}$  is a fundamental solution for  $\mathcal{M}^+$  we have that

$$\mathcal{M}^+ w(r) = \int_{B_\varepsilon(x) \cup B_\varepsilon(-x)} \frac{S_+(\delta(x, y, w)) - S_+(\delta(x, y, r^{\sigma^+}))}{|y|^{N+2\alpha}} dy.$$

We easily see that  $\delta(x, y, w) - \delta(x, y, r^{\sigma^+}) = \varepsilon^{\sigma^+} - |x - y|^{\sigma^+}$  and  $|y| \geq |x|/2$  if  $y \in B_\varepsilon(x)$  and  $r \geq r_1$ . Consequently, by (4.4)

$$\int_{B_\varepsilon(x)} \frac{S_+(\delta(x, y, w)) - S_+(\delta(x, y, r^{\sigma^+}))}{|y|^{N+2\alpha}} dy \geq -c \frac{\varepsilon^{\sigma^+ + N}}{|x|^{N+2\alpha}}$$

for some constant  $c$  and then, by symmetry of the integrals, we obtain that

$$\mathcal{M}^+ w(r) \geq -c \frac{\varepsilon^{\sigma^+ + N}}{|x|^{N+2\alpha}}. \tag{6.16}$$

If we define

$$\phi(r) = m(r_1) \frac{w(r) - w(r_2)}{w(\varepsilon) - w(r_2)},$$

we have that

$$\mathcal{M}^+ \phi(r) \geq \frac{m(r_1)}{w(\varepsilon) - w(r_2)} \mathcal{M}^+ w(r) \geq -\frac{c}{|x|^{N+2\alpha}}, \tag{6.17}$$

for all  $r \geq r_1$ . On the other hand, we recall that  $\sigma^+ + N > 0$  and we choose  $\sigma < \sigma^+$  such that  $-\sigma p < N + 2\alpha$ . Then, using (6.14), (6.17) and taking  $r_1$  large enough, by the choice of  $\sigma$ , we find that

$$\mathcal{M}^+ u \leq -\frac{c}{|x|^{-p\sigma}} \leq -\frac{c}{|x|^{-N+2\alpha}} \leq \mathcal{M}^+ \phi$$

and  $u(x) \geq \phi(x)$  for all  $r = |x|$  such that  $0 \leq r \leq r_1$  or  $r \geq r_2$ . Thus, by comparison principle Theorem 2.1 we have that  $u(x) \geq \phi(r)$  for all  $r_1 \leq r = |x| \leq r_2$ . Taking the limit as  $r_2 \rightarrow \infty$ , we find (6.13).

At this point we have to distinguish two cases, depending on the value of  $\sigma^+$ . The first case corresponds to  $\sigma^+ \in (-N, -1]$ . Here we observe that the function  $\Gamma$  is decreasing for all  $r > 0$ , with a singularity at the origin if  $\sigma^+ \in (-N, -1)$  and bounded if  $\sigma^+ = -1$ . We consider  $\varepsilon > 0$

and define the function

$$w(r) = \begin{cases} \Gamma(\varepsilon) & \text{if } 0 < r \leq \varepsilon, \\ \Gamma(r) & \text{if } \varepsilon \leq r. \end{cases} \tag{6.18}$$

We have

$$\begin{aligned} \mathcal{M}^+ w(x) &= \int_{\mathbb{R}^N} \frac{S_+(\delta(x, y, \Gamma))}{|y|^{N+2\alpha}} dy \\ &+ \int_{B_\varepsilon(x) \cup B_\varepsilon(-x)} \frac{S^+(\delta(x, y, w)) - S^+(\delta(x, y, \Gamma))}{|y|^{N+2\alpha}} dy. \end{aligned}$$

The first integral can be estimated using Lemma 6.1. If we assume that  $\varepsilon < r_1/2$ , then for every  $y \in B_\varepsilon(x) \cup B_\varepsilon(-x)$  we have  $|y| \geq |x|/2$  thus, using (4.4) the second integral can be estimated as

$$\int_{B_\varepsilon(x) \cup B_\varepsilon(-x)} \left| \frac{\delta(x, y, w) - \delta(x, y, \Gamma)}{|y|^{N+2\alpha}} dy \right| \leq \frac{c}{|x|^{N+2\alpha}} \left| \int_{B_\varepsilon(x)} (\Gamma(\varepsilon) - \Gamma(|x - y|)) dy \right|$$

Using the definition of  $\Gamma$  and the fact that  $\sigma^+ + N > 0$  we see that this integral is bounded by a term of the form  $o(1)|x|^{-N-2\alpha}$ , where  $o(1) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ . Putting together this and the estimate in Lemma 6.1 we find that

$$\mathcal{M}^+ w(x) \geq -\frac{c}{|x|^{N^+}} - \frac{o(1)}{|x|^{N+2\alpha}} \geq -\frac{c}{|x|^{N^+}}, \quad \text{for all } |x| \geq r_1, \tag{6.19}$$

where we used the fact that  $N^+ < N + 2\alpha$ . Then we define

$$\phi(x) = m(r_1) \frac{w(r) - w(r_2)}{w(\varepsilon) - w(r_2)}, \quad |x| < r_2,$$

and  $\phi(x) = 0$ , for  $|x| \geq r_2$ , where  $r_2 > r_1$ . We observe that  $\phi(x) \leq u(x)$  for all  $x$  such that  $|x| \leq r_1$  or  $|x| \geq r_2$ . Moreover

$$\mathcal{M}^+ \phi(x) \geq -\frac{c}{|x|^{N^+}}, \quad \text{for all } r_1 \leq |x| \leq r_2.$$

From here, the equation for  $u$  and (6.13) we can use the comparison Theorem 2.1 to obtain  $u(x) \geq \phi(x)$  for all  $r_1 < |x| < r_2$ . Taking limit as  $r_2 \rightarrow \infty$  we find that

$$u(x) \geq c \frac{\log(1 + |x|)}{|x|^{N^+-2\alpha}}, \quad \text{for all } r_1 < |x|.$$

From here and estimate (6.2) we find that

$$\frac{C}{|x|^{N^+-2\alpha}} \geq m(r) \geq c \frac{\log(1 + |x|)}{|x|^{N^+-2\alpha}}$$

for all  $|x|$  large, a contradiction.

We still need to analyze the remaining case, when  $\sigma^+ \in (-1, 0)$ . In this case the function  $\Gamma(r)$  is increasing near the origin and decreasing for  $r$  large, with exactly one maximum point, say at  $\tilde{r}_1 > 0$ . We define the function

$$w(r) = \begin{cases} \Gamma(r_1) & \text{if } 0 < r \leq \tilde{r}_1, \\ \Gamma(r) & \text{if } r_1 \leq r \end{cases} \tag{6.20}$$

and then the comparison function

$$\phi(x) = m(r_1) \frac{w(r) - w(r_2)}{w(r_1) - w(r_2)}, \quad |x| < r_2,$$

with  $\phi(x) = 0$ , for  $|x| \geq r_2$ , where  $r_2 > r_1$ . We observe that  $\phi(x) \leq u(x)$  for all  $x$  such that  $|x| \leq r_1$  or  $|x| \geq r_2$ . Moreover

$$\mathcal{M}^+ \phi(x) \geq -\frac{c}{|x|^{N^+}}, \quad \text{for all } r_1 \leq |x| \leq r_2.$$

Here we used Lemma 6.1 and the fact that  $\Gamma$  is increasing in  $(0, r_1)$ . From here we proceed as before, completing the proof in the critical case.  $\square$

We still need to prove the existence statement in the super-critical case. We start with a lemma on a general inequality we use later.

**Lemma 6.2.** *Let  $\alpha \in (0, 1)$  and consider  $q$  be such that*

$$\frac{1}{p-1} < q < \frac{N^\pm - 2\alpha}{2\alpha}, \tag{6.21}$$

which exists by our assumption. Then, for all  $s \in [0, 1)$ ,  $t \geq 0$  and  $u \geq 0$  the following inequality holds:

$$\begin{aligned} & (1 - s + ((s+t)^2 + u^2)^{1/2})^{-2\alpha q} + (1 - s + ((s-t)^2 + u^2)^{1/2})^{-2\alpha q} \\ & \leq ((1+t)^2 + u^2)^{-\alpha q} + ((1-t)^2 + u^2)^{-\alpha q}. \end{aligned} \tag{6.22}$$

**Proof.** We define a function  $f(s, t, u)$  as the left-hand side minus the right-hand side of (6.22). Given  $t \geq 0$  and  $u \geq 0$ , we see that

$$f(0, t, u) = 2(1 + (t^2 + u^2)^{1/2})^{-2\alpha q} - ((1+t)^2 + u^2)^{-\alpha q} - ((1-t)^2 + u^2)^{-\alpha q} \leq 0,$$

since

$$(1 + (t^2 + u^2)^{1/2})^2 \geq (1+t)^2 + u^2 \geq (1-t)^2 + u^2,$$

where the first inequality can easily be seen by direct computation. Next, taking  $s = 1$  we easily see that  $f(1, t, u) = 0$ . Finally, we compute the partial derivative with respect to  $s$ ,



$$\begin{aligned} \frac{\partial f}{\partial s}(s, t, u) &= \frac{-2\alpha q}{(1 - s + ((s + t)^2 + u^2)^{1/2})^{2\alpha q + 1}} \left( -1 + \frac{s + t}{((s + t)^2 + u^2)^{1/2}} \right) \\ &\quad - \frac{2\alpha q}{(1 - s + ((s - t)^2 + u^2)^{1/2})^{2\alpha q + 1}} \left( -1 + \frac{s - t}{((s - t)^2 + u^2)^{1/2}} \right) \end{aligned}$$

which is easily seen non-negative. With this we complete the proof of the lemma.  $\square$

**Proof of Theorem 1.3.** (The super-critical case.) We define the function

$$v(x) = \frac{1}{(1 + |x|)^{2\alpha q}}, \tag{6.23}$$

with  $q$  as in (6.21) and we prove next that  $v$  satisfies (1.7). As a direct consequence of this lemma we have the following inequality:

$$\frac{1}{(1 - s + |se_1 + y|)^{2\alpha q}} + \frac{1}{(1 - s + |se_1 - y|)^{2\alpha q}} \leq \frac{1}{|e_1 + y|^{2\alpha q}} + \frac{1}{|e_1 - y|^{2\alpha q}}.$$

Now we consider  $r = |x|$ ,  $\hat{x} = x/r$ ,  $s = r/(1 + r)$  and we write

$$\begin{aligned} \delta(v, x, y) &= \frac{1}{(1 + |x + y|)^{2\alpha q}} + \frac{1}{(1 + |x - y|)^{2\alpha q}} - \frac{2}{(1 + |x|)^{2\alpha q}} \\ &= \frac{1}{(1 + |x|)^{2\alpha q}} \left\{ \frac{1}{(1 - s + |se_1 + \tilde{y}|)^{2\alpha q}} + \frac{1}{(1 - s + |se_1 - \tilde{y}|)^{2\alpha q}} - 2 \right\} \\ &\leq \frac{1}{(1 + |x|)^{2\alpha q}} \left\{ \frac{1}{|e_1 + \tilde{y}|^{2\alpha q}} + \frac{1}{|e_1 - \tilde{y}|^{2\alpha q}} - 2 \right\}, \end{aligned} \tag{6.24}$$

where  $\tilde{y} = Ry/(1 + r)$ , with  $R$  an appropriate rotation matrix. Here we used the inequality proved above. Now, in the case of  $\mathcal{M}^+$ , we use that  $S_+$  is an increasing function to compose it with this inequality, recalling the definition of  $c^+$  in (3.2), multiplying by  $1/|y|^{N+2\alpha}$  and then integrating in  $\mathbb{R}^N$  we obtain

$$\begin{aligned} \mathcal{M}^+(v) &= \int_{\mathbb{R}^N} \frac{S_+(\delta(v, x, y))}{|y|^{N+2\alpha}} dy \\ &\leq \frac{1}{(1 + |x|)^{2\alpha q}} \int_{\mathbb{R}^N} \frac{S_+(\delta_{2\alpha q}(\tilde{y}))}{|y|^{N+2\alpha}} dy \\ &= \frac{1}{(1 + |x|)^{2\alpha(q+1)}} c^+(-2\alpha q) = \frac{-C}{(1 + |x|)^{2\alpha(q+1)}}. \end{aligned}$$

Since  $q$  was chosen to satisfy (6.21) we see, from the definition of  $N^+$  and the properties of  $c^+$ , that  $-C = c^+(-2\alpha q) < 0$ . Then we have that

$$\mathcal{M}^+(cv) + (cv)^p \leq \frac{-cC}{(1 + |x|)^{2\alpha(q+1)}} + \frac{c^p}{(1 + |x|)^{2\alpha qp}}.$$

Now we use (6.21) and we choose  $c$  small enough to finally obtain

$$\mathcal{M}^+(cv) + (cv)^p \leq 0,$$

completing the proof. With a similar argument we obtain a function  $v$  that serves as solution in the case of  $\mathcal{M}^-$  such that

$$\mathcal{M}^-(cv) + (cv)^p \leq 0,$$

whenever  $p > N^-(N^- - 2\alpha)$ . Here we just notice that after (6.24) we could use  $S_-$  and then argue analogously. The proof of Theorem 1.2 is now complete.  $\square$

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