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Viable states for monotone harvest models

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ABSTRACT

This paper deals with the control of discrete-time dynamical, monotone both in the state and in the control, in the presence of state and control monotone constraints. A state x is said to belong to the viability kernel if there exists a trajectory, of states and controls, starting from x and satisfying the constraints. Under monotonicity assumptions, we present upper and lower estimates of the viability kernel. Our motivation comes from harvest models, where some monospecies age class models, as well as specific multi-species models (with so-called technical interactions), exhibit monotonicity properties both in the state and in the control. In this context, constraints represent production and preservation requirements to be satisfied for all time, which also possess monotonicity properties. Our results help delineating domains where a viable management is possible. Numerical applications are given for two Chilean fisheries. We obtain upper bounds for production which are interesting for managers in that they only depend on the model's parameters, and not on the current stocks.

1. Introduction

This paper deals with the control of discrete-time dynamical systems of the form $x(t+1) = G\big(x(t), u(t)\big)$, $t \in \mathbb{N}$, with state $x(t) \in \mathbb{X}$ and control $u(t) \in \mathbb{U}$, in the presence of state and control constraints $\big(x(t), u(t)\big) \in \mathbb{D}$. The subset $\mathbb{D} \subset \mathbb{X} \times \mathbb{U}$ describes "acceptable configurations of the system". Such problems of dynamic control under constraints refer to viability [1] or invariance [2] frameworks. From the mathematical viewpoint, most of the viability and weak invariance results are addressed in the continuous time case. However, some mathematical works deal with the discrete-time case. This includes the study of numerical schemes for the approximation of the viability problems of continuous dynamics as in [1,3–5]. In the control theory literature, problems of constrained control have also been addressed in the discrete-time case (see the survey paper [6]); reachability of target sets or tubes for nonlinear discrete-time dynamics is examined in [7].

We consider sustainable management issues which can be formulated within such a framework as in [8–16]. The time index t is an integer and the time period $[t, t+1[$ may be a year, a month, etc. The dynamic is generally a population dynamic, with state vector $x(t)$ being either the biomass of a single species, or a couple of biomasses for a predator–prey system, or a vector of abundances at ages for one or for several species, or abundances at different spatial patches, etc. The control $u(t)$ may represent harvest levels, induced mortality or harvest effort. The "acceptable set" $\mathbb{D}$ such that $\big(x(t), u(t)\big) \in \mathbb{D}$ may include biological, ecological and economic objectives as in [10]. For instance, if the state x is a vector of abundances at ages and the control u is a harvest effort, $\mathbb{D} = \{(x, u) \mid B(x) \geq b^\flat,\ E(x, u) \geq e^\flat\}$ represents acceptable configurations where conservation is ensured by a biological indicator $B(x) \geq b^\flat$ (spawning stock biomass above a reference point, for instance) and economics is taken into account via minimal catches $E(x, u) \geq e^\flat$ (catches $E(x, u)$ above a threshold).

The viability kernel $\mathbb{V}(G, \mathbb{D})$ associated with the dynamic G and the acceptable set[1] $\mathbb{D}$ is known to play a basic role for the analysis of such problems and the design of viable control feedbacks. Unfortunately, its computation is not an easy task in general.

In [17], the authors estimated the viability kernel from below or from above under rather general monotonicity assumptions, essentially with respect to the state variable. In this paper, we deal with more specific monotonicity assumptions on the dynamic

[1] In [1], the viability kernel $\mathbb{V}_K(G)$ is defined with respect to the dynamic G and to a subset $K \subset \mathbb{X}$ of the state space $\mathbb{X}$, and the constraints on the controls are contained in the definition in G. We prefer to put together the set of state constraints with the set of admissible controls, although these sets play very different roles. Indeed, in practice, constraints are expressed *via* indicators which are functions of both variables (state and control), especially for production constraints which depend on the catches. Thus, the set $\mathbb{D}$ makes the conflicting requirements, between preservation and production, more visible than with the Aubin's formalism.

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and on the constraints. In addition to state monotonicity, we consider monotonicity in the control variable, inspired by a class of harvest models. This is why our results are more precise than, for instance, the estimation for the viability kernel provided in [17, Proposition 11].

In Section 2, we recall the viability issues in discrete-time, and we introduce monotone harvest models. Section 3 provides our main theoretical results on estimates of the viability kernel. An application to fishery management is provided in Section 4 with numerical estimates for two Chilean fisheries. We obtain upper bounds for production which are interesting for managers in that they only depend on the model's parameters, and not on the current stocks.

2. Viability issues and monotone harvest models

In this introductory section we recall the viability issues in discrete-time and afterwards we introduce monotone harvest models.

2.1. Viability in discrete-time

Let us consider a nonlinear control system described in discrete-time by the difference equation

$$\begin{cases} x(t+1) = G(x(t), u(t)), \quad t = t_0, t_0+1, \dots \\ x(t_0) \quad \text{given}, \end{cases}$$

where the *state variable* $x(t)$ belongs to the finite dimensional state space $\mathbb{X} \subset \mathbb{R}^{n_\mathbb{X}}$, the *control variable* $u(t)$ is an element of the *control set* $\mathbb{U} \subset \mathbb{R}^{n_\mathbb{U}}$ while the *dynamic* G maps $\mathbb{X} \times \mathbb{U}$ into $\mathbb{X}$. In our context, $x(t)$ will typically represent the vector of abundances per age class of a population, while $u(t)$ will be a harvest (induced mortality, harvesting effort, etc.).

A decision maker describes *acceptable configurations of the system* through a set $\mathbb{D} \subset \mathbb{X} \times \mathbb{U}$ termed the *acceptable set*

$$(x(t), u(t)) \in \mathbb{D}, \quad \forall t = t_0, t_0+1, \dots$$

where $\mathbb{D}$ includes both system states and controls constraints. Typical instances of such an acceptable set are given by inequalities requirements

$$\mathbb{D} = \{(x, u) \in \mathbb{X} \times \mathbb{U} \mid \forall i = 1, \dots, p, \ \mathcal{L}_i(x, u) \geq l_i\}, \tag{1}$$

where the functions $\mathcal{L}_1, \dots, \mathcal{L}_p$ may be interpreted as *indicators*, and the real numbers $l_1, \dots, l_p$ as the corresponding *thresholds* (following the ICES[2] precautionary approach terminology). For management issues, the set $\mathbb{D}$ will be the mathematical expression of preservation and/or production objectives.

Viability is defined as the ability to choose, at each time step $t = t_0, t_0+1, \dots$, a control $u(t) \in \mathbb{U}$ such that the system's configuration remains acceptable. More precisely, the system is viable if the following feasible set is not empty:

$$\mathbb{V}(G, \mathbb{D}) := \left\{ x \in \mathbb{X} \;\middle|\; \begin{array}{l} \exists (u(t_0), u(t_0+1), \dots) \text{ and} \\ \quad (x(t_0), x(t_0+1), \dots) \text{ satisfying } x(t_0) = x, \\ \quad x(t+1) = G(x(t), u(t)) \text{ and} \\ \quad (x(t), u(t)) \in \mathbb{D}, \ \forall t = t_0, t_0+1, \dots \end{array} \right\}.$$

For a decision maker, knowing the viability kernel has practical interest since it describes the set of states from which controls can be found that maintain the system in an acceptable configuration forever. However, computing this kernel is not an easy task in general.

We shall focus on estimates of viability kernels when the dynamic G and the acceptable sets have specific monotonicity properties. For this purpose, we shall introduce a generic form for dynamics and acceptable sets.

[2] International Council for the Exploration of the Sea.

2.2. Monotone harvest models

In what follows, the state space $\mathbb{X}$ and the control space $\mathbb{U}$ are subsets $\mathbb{X} \subset \mathbb{R}^{n_\mathbb{X}}$ and $\mathbb{U} \subset \mathbb{R}^{n_\mathbb{U}}$ supplied with the componentwise order: $x' \geq x$ if and only if each component of $x' = (x'_1, \dots, x'_{n_\mathbb{X}})$ is greater than or equal to the corresponding component of $x = (x_1, \dots, x_{n_\mathbb{X}})$: $x' \geq x \iff x'_i \geq x_i, \ i = 1, \dots, n_\mathbb{X}$. A mapping $f : \mathbb{R}^a \to \mathbb{R}^b$ is said to be increasing if $x \geq x' \Rightarrow f(x) \geq f(x')$. A similar definition holds for decreasing.

Dynamic

Monospecies dynamical population models generally have the following qualitative properties: (i) the higher the state abundance vector, the higher at the next period; (ii) the higher the harvest, the lower the state abundance vector at the next period. Some specific multi-species models, without ecological but with so-called technical interactions, share such properties. This motivates the following definitions (see also [17]).

We say that the dynamic $G : \mathbb{X} \times \mathbb{U} \to \mathbb{X}$ is *increasing with respect to the state* if it satisfies $\forall (x, x', u) \in \mathbb{X} \times \mathbb{X} \times \mathbb{U}, x' \geq x \Rightarrow G(x', u) \geq G(x, u)$, and is *decreasing with respect to the control* if $\forall (x, u, u') \in \mathbb{X} \times \mathbb{U} \times \mathbb{U}, u' \geq u \Rightarrow G(x, u') \leq G(x, u)$.

We shall coin $G : \mathbb{X} \times \mathbb{U} \to \mathbb{X}$ a *monotone harvest dynamic* if G is increasing with respect to the state and decreasing with respect to the control.

Bounded control set

Assuming they exist, we denote by $u^\flat, u^\sharp \in \mathbb{U}$ the lower and upper bounds of the set $\mathbb{U}$, *i.e.* $u^\flat \leq u \leq u^\sharp$ for all $u \in \mathbb{U}$.

Upper and lower dynamics without control

Let the dynamic G be a monotone harvest dynamic. Define the *upper dynamic without control* $G^\flat : \mathbb{X} \to \mathbb{X}$ by $G^\flat(x) = G(x, u^\flat)$. Notice that $G \leq G^\flat$ where, in our notation $G^\flat$, the $\flat$ refers to the control. Its t iterate ($t = t_0, t_0+1, \dots$) will be denoted by $(G^\flat)^{(t)}$. In the same way, the *lower dynamic without control* is defined by $G^\sharp(x) = G(x, u^\sharp)$. With these notations, we have that

$$G^\sharp(x) \leq G(x, u) \leq G^\flat(x), \quad \forall (x, u) \in \mathbb{X} \times \mathbb{U}.$$

Acceptable set

We say that a set $S \subset \mathbb{X}$ is an *upper set* (or is an *increasing set*) if it satisfies the following property: $\forall x \in S, \forall x' \in \mathbb{X}, x' \geq x \Rightarrow x' \in S$. In the same way, a set $K \subset \mathbb{X} \times \mathbb{U}$ is said to be an *upper set* if $\forall (x, u) \in K, \forall x' \in \mathbb{X}, x' \geq x \Rightarrow (x', u) \in K$.

An acceptable set $\mathbb{D}$ is said to be *a production acceptable set* if $\mathbb{D}$ is *increasing with respect both to the state and to the control*, that is $\forall (x, x', u, u') \in \mathbb{X} \times \mathbb{X} \times \mathbb{U} \times \mathbb{U}, x' \geq x, u' \geq u, (x, u) \in \mathbb{D} \Rightarrow (x', u') \in \mathbb{D}$. Particular instances are given by acceptable sets of the form (1) where the indicators $\mathcal{L}_1, \dots, \mathcal{L}_p$ are increasing with respect to both variables (state and control). For instance, requiring a minimum yield may be captured by the acceptable set $\mathbb{D}_{\text{yield}} = \{(x, u) \mid Y(x, u) \geq y^\flat\}$ where $y^\flat \in \mathbb{R}$ is a minimum yield threshold and where the yield function $Y : \mathbb{X} \times \mathbb{U} \to \mathbb{R}$ is increasing with respect to both variables (state and control).

An acceptable set $\mathbb{D}$ is said to be *a preservation acceptable set* if $\mathbb{D}$ is *increasing with respect to the state and decreasing with respect to the control*, that is $\forall (x, x', u, u') \in \mathbb{X} \times \mathbb{X} \times \mathbb{U} \times \mathbb{U}, x' \geq x, u' \leq u, (x, u) \in \mathbb{D} \Rightarrow (x', u') \in \mathbb{D} >$. Particular instances are given by acceptable sets of the form (1) where the indicators $\mathcal{L}_1, \dots, \mathcal{L}_p$ are increasing with respect to the state but decreasing with respect to the control. For instance, the ICES precautionary approach may be stated in the viability framework with the following preservation acceptable set $\mathbb{D}_{\text{protect}} = \{(x, u) \in \mathbb{X} \times \mathbb{U} \mid \text{SSB}(x) \geq B_{\lim}, \ F(u) \leq F_{\lim}\}$ as in [15]. Here, $\text{SSB}(x)$ is the spawning stock's biomass, increasing with respect to the state, while the fishing mortality $F(u)$ is increasing[3] with respect to the control.

Notice that both production and preservation acceptable sets are upper sets.

For any acceptable set $\mathbb{D}$, introduce the *state constraints set*

$$\mathbb{V}_0 := \mathrm{Proj}_{\mathbb{X}}(\mathbb{D}) = \{x \in \mathbb{X} \mid \exists u \in \mathbb{U}, (x,u) \in \mathbb{D}\},$$

obtained by projecting the acceptable set $\mathbb{D}$ on to the state space $\mathbb{X}$. Introduce also

$$\begin{cases} \mathbb{V}_0{}^{\sharp} := \{x \in \mathbb{X} \mid (x, u^{\sharp}) \in \mathbb{D}\} \subset \mathbb{V}_0, & \mathbb{D}^{\sharp} := \mathbb{V}_0{}^{\sharp} \times \{u^{\sharp}\} \\ \mathbb{V}_0{}^{\flat} := \{x \in \mathbb{X} \mid (x, u^{\flat}) \in \mathbb{D}\} \subset \mathbb{V}_0, & \mathbb{D}^{\flat} := \mathbb{V}_0{}^{\flat} \times \{u^{\flat}\}. \end{cases} \tag{2}$$

Notice that if $\mathbb{D}$ is a production acceptable set, we have $\mathbb{V}_0 = \mathbb{V}_0{}^{\sharp}$, and if $\mathbb{D}$ is a preservation acceptable set, we have $\mathbb{V}_0 = \mathbb{V}_0{}^{\flat}$.

3. Viability kernel estimates for monotone harvest models

In this section, we shall provide lower and upper estimates of the viability kernel $\mathbb{V}(G,\mathbb{D})$ thanks to the following sets $\mathbb{V}(G^{\flat},\mathbb{D}^{\flat})$, $\mathbb{V}(G^{\flat},\mathbb{D}^{\sharp})$, $\mathbb{V}(G^{\sharp},\mathbb{D}^{\flat})$ and $\mathbb{V}(G^{\sharp},\mathbb{D}^{\sharp})$. These latter sets are easier to compute than the viability kernel $\mathbb{V}(G,\mathbb{D})$ because the dynamics $G^{\flat}$ and $G^{\sharp}$ have no control. Indeed, if we have a dynamic $G^{\star}$ that does not depend on the control (i.e., $G^{\star}(x,u) = G^{\star}(x)$) then, for any set $\mathbb{D}^{\star} \subset \mathbb{X} \times \mathbb{U}$ one has

$$\mathbb{V}(G^{\star},\mathbb{D}^{\star}) = \bigcap_{t=0}^{+\infty} \{x \in \mathbb{X} \mid (G^{\star})^{(t)}(x) \in \mathrm{Proj}_{\mathbb{X}}(\mathbb{D}^{\star})\}. \tag{3}$$

The trajectory generated by $G^{\star}$ and starting from an initial state in $\mathbb{V}(G^{\star},\mathbb{D}^{\star})$ remains within $\mathrm{Proj}_{\mathbb{X}}(\mathbb{D}^{\star})$ for all times.

Proposition 1. *Suppose that G is a monotone harvest dynamic and that the control set $\mathbb{U}$ has lower and upper bounds $u^{\flat}, u^{\sharp} \in \mathbb{U}$.*

1. *If $\mathbb{D}$ is a production acceptable set, then*

$$\mathbb{V}(G^{\sharp},\mathbb{D}^{\flat}) \subset \mathbb{V}(G^{\flat},\mathbb{D}^{\flat}) \cup \mathbb{V}(G^{\sharp},\mathbb{D}^{\sharp}) \subset \mathbb{V}(G,\mathbb{D}) \subset \mathbb{V}(G^{\flat},\mathbb{D}^{\sharp}). \tag{4}$$

2. *If $\mathbb{D}$ is a preservation acceptable set, then*[4]

$$\mathbb{V}(G^{\sharp},\mathbb{D}^{\sharp}) \subset \mathbb{V}(G,\mathbb{D}) = \mathbb{V}(G^{\flat},\mathbb{D}^{\flat}). \tag{5}$$

Before giving the proof, we shall make some comments on the differences between the above result and previous results under monotonicity assumptions.

In [17, Proposition 11], the authors estimated the viability kernel from below or from above under rather general monotonicity assumptions, essentially with respect to the state variable. Here, we have an additional monotonicity assumption with respect to the control variable. This is why, estimations given in the above proposition are more precise.

On the other hand, Proposition 10 in [17] establishes estimations and a way to compute the viability kernel. Nevertheless, this result needs some assumption on the dynamic (to be *saturated* at all $x \in \mathbb{V}_0$, meaning that components of the dynamic are maximized with a common control). Here, we have another type of hypothesis; this is why this previous result cannot be compared with the estimations given above in Proposition 1.

Proof. First, let us notice that whatever the acceptable set $\mathbb{D}$ and the dynamic G, we have the inclusion

$$\mathbb{V}(G^{\flat},\mathbb{D}^{\flat}) \cup \mathbb{V}(G^{\sharp},\mathbb{D}^{\sharp}) \subset \mathbb{V}(G,\mathbb{D}). \tag{6}$$

Indeed, $\mathbb{V}(G^{\flat},\mathbb{D}^{\flat}) = \bigcap_{t=0}^{+\infty}\{x \in \mathbb{X} \mid \left((G^{\flat})^{(t)}(x), u^{\flat}\right) \in \mathbb{D}\} \subset \mathbb{V}(G,\mathbb{D})$, since $x \in \mathbb{V}(G^{\flat},\mathbb{D}^{\flat})$ means that the stationary control $u(t) = u^{\flat}$ makes that the trajectory $(x(t),u(t)) = ((G^{\flat})^{(t)}(x), u^{\flat})$ belongs to $\mathbb{D}$. The same may be done with the control $u^{\sharp}$.

Second, when G is a monotone harvest dynamic and $\mathbb{D}$ is an upper set (which is the case when $\mathbb{D}$ is a production or a preservation acceptable set), we have the inclusions

$$\mathbb{V}(G^{\sharp},\mathbb{D}) \subset \mathbb{V}(G,\mathbb{D}) \subset \mathbb{V}(G^{\flat},\mathbb{D}). \tag{7}$$

This is a straightforward application of Proposition 11 in [17], because $G^{\flat} \leq G \leq G^{\sharp}$ and all these functions are increasing with respect to the state.

Now, we come to the proof.

1. On one hand, we have that $\mathbb{V}(G^{\sharp},\mathbb{D}^{\flat}) \subset \mathbb{V}(G^{\flat},\mathbb{D}^{\flat})$ by (7) with $\mathbb{D}$ replaced by $\mathbb{D}^{\flat}$ (because $\mathbb{D}^{\flat}$ is an upper set). By (6), this gives the two lower estimates of the viability kernel $\mathbb{V}(G,\mathbb{D})$ in (4).
   On the other hand, since $\mathbb{D}$ is a production acceptable set, we have $\mathbb{V}_0 = \mathbb{V}_0{}^{\sharp}$, and thus, by (2) and (3),
$$\begin{aligned}\mathbb{V}(G^{\flat},\mathbb{D}) &= \bigcap_{t=0}^{+\infty}\{x \in \mathbb{X} \mid (G^{\flat})^{(t)}(x) \in \mathbb{V}_0\} \\ &= \bigcap_{t=0}^{+\infty}\{x \in \mathbb{X} \mid (G^{\flat})^{(t)}(x) \in \mathbb{V}_0{}^{\sharp}\} \\ &= \mathbb{V}(G^{\flat},\mathbb{D}^{\sharp}).\end{aligned}$$
   As we have seen by (7) that $\mathbb{V}(G,\mathbb{D}) \subset \mathbb{V}(G^{\flat},\mathbb{D})$, this gives $\mathbb{V}(G,\mathbb{D}) \subset \mathbb{V}(G^{\flat},\mathbb{D}^{\sharp})$, hence the upper estimate of the viability kernel $\mathbb{V}(G,\mathbb{D})$ in (4).
2. The lower estimate of the viability kernel $\mathbb{V}(G,\mathbb{D})$ in (5) comes from (6).
   Now, let us prove the equality $\mathbb{V}(G^{\flat},\mathbb{D}^{\flat}) = \mathbb{V}(G,\mathbb{D})$ in (5). On one hand, by (6) we know that $\mathbb{V}(G^{\flat},\mathbb{D}^{\flat}) \subset \mathbb{V}(G,\mathbb{D})$. On the other hand, since $\mathbb{D}$ is a preservation acceptable set, we have $\mathbb{V}_0 = \mathbb{V}_0{}^{\flat}$, and thus
$$\begin{aligned}\mathbb{V}(G^{\flat},\mathbb{D}) &= \bigcap_{t=0}^{+\infty}\{x \in \mathbb{X} \mid (G^{\flat})^{(t)}(x) \in \mathbb{V}_0\} \\ &= \bigcap_{t=0}^{+\infty}\{x \in \mathbb{X} \mid (G^{\flat})^{(t)}(x) \in \mathbb{V}_0{}^{\flat}\} \\ &= \mathbb{V}(G^{\flat},\mathbb{D}^{\flat}).\end{aligned}$$
   By (7), this gives $\mathbb{V}(G^{\flat},\mathbb{D}^{\flat}) \subset \mathbb{V}(G,\mathbb{D}) \subset \mathbb{V}(G^{\flat},\mathbb{D}) = \mathbb{V}(G^{\flat},\mathbb{D}^{\flat})$. □

When the acceptable set is given by means of indicators and thresholds as in (1), and the upper dynamic $G^{\flat}$ has a steady state satisfying some requirements, we obtain the following practical conditions for nonemptyness of the viability kernel.

Corollary 1. *Suppose that G is a monotone harvest dynamic, that the control set $\mathbb{U}$ has lower and upper bounds $u^{\flat}, u^{\sharp} \in \mathbb{U}$ and that the acceptable set $\mathbb{D}$ is given by (1) with indicators $\mathcal{L}_1, \ldots, \mathcal{L}_p$ being upper semi-continuous functions in the first (state) variable. Assume also that the upper dynamic $G^{\flat}$ has a steady state $\bar{x}(u^{\flat})$ and there exists $L < 1$ such that*

$$\|G^{\flat}(x) - \bar{x}(u^{\flat})\| \leq L\|x - \bar{x}(u^{\flat})\|, \quad \forall x \in \mathbb{V}_0 \tag{8}$$

for some norm $\|\cdot\|$ in $\mathbb{X} \subset \mathbb{R}^{n_{\mathbb{X}}}$.

[3] Hence $-F(u)$ is decreasing with respect to the control. To be consistent with the notation in (1), it suffice to rewrite $\mathbb{D}_{\mathrm{protect}} = \{(x,u) \in \mathbb{X} \times \mathbb{U} \mid \mathrm{SSB}(x) \geq B_{\lim},\ -F(u) \geq -F_{\lim}\}$.

[4] In this case, our result also provides a viable strategy: it consists in applying the lower control $u^{\flat}$. This is an open-loop control, which may be interpreted as *precautionary*. However, our emphasis is not on exhibiting viable strategies.

1. *If $\mathbb{D}$ is a production acceptable set, one has*

$$\exists i = 1, \dots, p, \quad \mathcal{L}_i(\bar{x}(u^\flat), u^\sharp) < l_i \Rightarrow \mathbb{V}(G, \mathbb{D}) = \emptyset. \tag{9}$$

2. *If $\mathbb{D}$ is a preservation acceptable set, one has*

$$\mathbb{V}(G, \mathbb{D}) \neq \emptyset \Leftrightarrow \mathcal{L}_i(\bar{x}(u^\flat), u^\flat) \geq l_i \quad \forall i = 1, \dots, p. \tag{10}$$

Proof. We proceed to prove Statement 1 by a contra-reciprocal argument. Let us suppose that $\mathbb{V}(G, \mathbb{D}) \neq \emptyset$ and take x in this set (which is included in $\mathbb{V}_0$). From Proposition 1, x belongs to $\mathbb{V}(G^\flat, \mathbb{D}^\sharp)$ or, equivalently

$$(G^\flat)^{(t)}(x) \in \mathbb{V}_0{}^\sharp = \mathbb{V}_0, \quad \forall t \geq t_0 \Leftrightarrow \mathcal{L}_i\big((G^\flat)^{(t)}(x), u^\sharp\big) \geq l_i \\ \forall i = 1, \dots, p \, \forall t \geq t_0.$$

Since $(G^\flat)^{(t)}(x) \in \mathbb{V}_0$ for all $t \geq t_0$, condition (8) implies $(G^\flat)^{(t)}(x) \to \bar{x}(u^\flat)$. Then, from the upper semi-continuity property of functions $\mathcal{L}_i(\cdot, u^\sharp)$, we obtain the desired inequalities $\mathcal{L}_i(\bar{x}(u^\flat), u^\sharp) \geq l_i$, $\forall i = 1, \dots, p$. The proof of necessary condition ($\Rightarrow$) in Statement 2 is analogous. For the sufficient condition ($\Leftarrow$) directly one can prove that $\bar{x}(u^\flat) \in \mathbb{V}(G, \mathbb{D})$ by taking the stationary control $u(t) = u^\flat$. □

The previous Corollary 1 provides necessary conditions (in the case of a production acceptable set) and necessary and sufficient conditions (in the case of a preservation acceptable set) to assure the non-emptiness of the viability kernel. The quantities $\mathcal{L}_i(\bar{x}(u^\flat), u^\sharp)$ (for production acceptable sets) and $\mathcal{L}_i(\bar{x}(u^\flat), u^\flat)$ (for preservation acceptable sets) can be interpreted as *maximal thresholds* for the acceptable configurations. That is, no trajectory $x(\cdot) = (x(t_0), x(t_0 + 1), \dots)$ can generate values $\mathcal{L}_i(x(t), u(t))$ above these values for all periods of time t, whatever the initial state x_0 and the control trajectory $u(\cdot) = (u(t_0), u(t_0 + 1), \dots)$ be.

An alternative (non-equivalent) condition to (8) in the above Corollary 1 would be supposing that the steady state $\bar{x}(u^\flat)$ is globally asymptotic stable on $\mathbb{V}_0$ for the dynamic $G^\flat$. However, this is an assumption which is difficult to verify. A weaker one would restrict global asymptotic stability to the subset $\mathbb{V}(G, \mathbb{D}) \subset \mathbb{V}_0$ (see the proof of Corollary 1). Nevertheless, it is neither elegant nor practical to make any assumption on the viability kernel $\mathbb{V}(G, \mathbb{D})$, which is an object of study and which might be empty.

4. Application to fishery management

In this section we apply and specify the previous results in the case of an age-structured abundance population model, especially with a Beverton–Holt stock-recruitment relationship. With this, we provide numerical estimates for two Chilean fisheries.

4.1. An age class dynamical model

We consider an age-structured abundance population model with a possibly nonlinear stock-recruitment relationship, derived from fish stock management (see [18], and also [15] for more details).

Time is measured in years, and the time index $t \in \mathbb{N}$ represents the beginning of year t and of yearly period $[t, t + 1[$. Let $A \in \mathbb{N}^*$ denote a maximum age, and $a \in \{1, \dots, A\}$ an age class index, all expressed in years. The state is the vector $x = (x_a)_{a=1,\dots,A} \in \mathbb{R}_+^A$, the *abundances* at age: for $a = 1, \dots, A - 1$, $x_a(t)$ is the number of individuals of age between $a - 1$ and a at the beginning of yearly period $[t, t + 1[$; $x_A(t)$ is the number of individuals of age greater than $A - 1$. The control $u(t)$ is the *fishing effort (multiplier)*, supposed to be applied in the middle of period $[t, t+1[$. The control dynamical model is

$$x(t + 1) = G\big(x(t), u(t)\big), \quad t = t_0, t_0 + 1, \dots, \ x(t_0) \text{ given},$$

where the vector function $G = (G_a)_{a=1,\dots,A}$ is defined for any $x \in \mathbb{R}_+^A$ and $u \in \mathbb{R}_+$ by

$$\begin{cases} G_1(x, u) = \varphi\big(\mathrm{SSB}(x)\big), \\ G_a(x, u) = \mathrm{e}^{-(M_{a-1} + uF_{a-1})} x_{a-1}, \quad a = 2, \dots, A - 1, \\ G_A(x, u) = \mathrm{e}^{-(M_{A-1} + uF_{A-1})} x_{A-1} + \pi \times \mathrm{e}^{-(M_A + uF_A)} x_A. \end{cases} \tag{11}$$

In the above formulas, M_a is the natural *mortality rate* of individuals of age a, F_a is the mortality rate of individuals of age a due to harvesting between t and $t + 1$, supposed to remain constant during period $[t, t + 1[$ (the vector $(F_a)_{a=1,\dots,A}$ is termed the *exploitation pattern*), and the parameter $\pi \in \{0, 1\}$ is related to the existence of a so-called *plus-group* (if we neglect the survivors older than age A then $\pi = 0$, otherwise $\pi = 1$ and the last age class is a plus group). The function φ describes a *stock-recruitment relationship*. The *spawning stock biomass* SSB is defined by

$$\mathrm{SSB}(x) := \sum_{a=1}^{A} \gamma_a w_a x_a, \tag{12}$$

that is summing the contributions of individuals to reproduction, where $(\gamma_a)_{a=1,\dots,A}$ are the *proportions of mature individuals* (some may be zero) at age and $(w_a)_{a=1,\dots,A}$ are the *weights at age* (all positive).

4.2. An acceptable set reflecting conflicting preservation and production objectives

We shall consider an acceptable set $\mathbb{D}$ which reflects conflicting objectives of *preservation* – measured by the spawning stock biomass being high enough – and of *production*, measured by the following yield indicator.

The production in term of biomass at the beginning of period $[t, t + 1[$ is (see [18])

$$Y\big(x, u\big) = \sum_{a=1}^{A} w_a \frac{uF_a}{uF_a + M_a} \left(1 - \mathrm{e}^{-(M_a + uF_a)}\right) x_a. \tag{13}$$

We focus our analysis on the acceptable set

$$\mathbb{D}_{\mathrm{yield}}(y_{\min}, B_{\lim}) := \{(x, u) \mid Y(x, u) \geq y_{\min}, \ \mathrm{SSB}(x) \geq B_{\lim}\}, \tag{14}$$

where the yield function Y is given by (13) and SSB by (12). Contrarily to the ICES precautionary approach as analyzed in [15], we do not focus only on *preservation issues* ($\mathrm{SSB}(x) \geq B_{\lim}$) but also on *production issues* by asking for a minimal yield ($Y(x, u) \geq y_{\min}$).

4.3. Monotonicity properties

The set $\mathbb{D}_{\mathrm{yield}}(y_{\min}, B_{\min})$ is a production acceptable set. Indeed, on one hand, the yield Y is increasing with respect both to the state and to the control. On the other hand, the spawning stock's biomass SSB is increasing with respect to the state and does not depend on the control.

The dynamic (11) is a monotone harvest one whenever the recruitment function φ in (11) is non-decreasing.

We now focus on the existence of equilibrium points. For this (see [18]), we consider the following proportions of equilibrium recruits which survive up to age a:

$$\begin{cases} s_1(u) := 1 \\ s_a(u) := \exp\Big(-\big(M_1 + \dots + M_{a-1} + u(F_1 + \dots + F_{a-1})\big)\Big), \\ \quad a = 2, \dots, A - 1 \\ s_A(u) := \dfrac{1}{1 - \pi \mathrm{e}^{-(M_A + uF_A)}} \exp\Big(-\big(M_1 + \dots + M_{A-1} \\ \quad + u(F_1 + \dots + F_{A-1})\big)\Big). \end{cases}$$

Let also $\mathrm{spr}(u) := \sum_{a=1}^{A} \gamma_a w_a s_a(u)$ be the *spawning per recruit* at equilibrium. When the recruitment function is Beverton–Holt $\varphi(B) = \frac{B}{\alpha+\beta B}$ (which includes the constant case, taking $\alpha = 0$, and the linear case, taking $\beta = 0$), there exists an equilibrium point for any control $u \geq 0$. It is given by $\bar{x}(u) = (\bar{x}_a(u))_{a=1,\ldots,A}$, where $\bar{x}_a(u) = Z(u)s_a(u)$ and $Z(u) = \max\left\{0, \frac{\mathrm{spr}(u)-\alpha}{\beta \mathrm{spr}(u)}\right\}$ if $\beta > 0$, $Z(u) = 0$ if $\beta = 0$.

4.4. Minimal viable production issues

The following statement establishes *maximum sustainable thresholds* for the indicators SSB and Y. It is an application of Corollary 1.

Proposition 2. *Assume that the stock-recruitment relationship φ is Beverton–Holt $\varphi(B) = \frac{B}{\alpha+\beta B}$ (allowing the cases $\alpha = 0$ or $\beta = 0$), that the fishing effort u is bounded from below and above by $0 \leq u^\flat \leq u \leq u^\sharp$. Suppose that*

$$\phi_G(u^\flat) := \varphi'\left(\mathrm{SSB}(\bar{x}(u^\flat))\right) \max_{a=1,\ldots,A} \gamma_a w_a + \max_{a=1,\ldots,A} \mathrm{e}^{-(M_a+u^\flat F_a)} < 1. \tag{15}$$

Then, ensuring a minimal viable production and spawning stock biomass requires that the production and preservation thresholds $y_{\min}$ and $B_{\lim}$ are not too high:

$$\left.\begin{array}{ll} & y_{\min} > Y(\bar{x}(u^\flat), u^\sharp) \\ \text{or} & B_{\lim} > \mathrm{SSB}(\bar{x}(u^\flat)) \end{array}\right\} \Rightarrow \mathbb{V}\big(G, \mathbb{D}_{\mathrm{yield}}(y_{\min}, B_{\lim})\big) = \emptyset.$$

Proof. In order to apply Corollary 1, let us prove that, for $B_{\lim} > \mathrm{SSB}(\bar{x}(u^\flat))$, one has the following property

$$\|G(x, u^\flat) - \bar{x}(u^\flat)\|_1 \leq \phi_G(u^\flat)\|x - \bar{x}(u^\flat)\|_1, \tag{16}$$

for all x in $\mathbb{V}_0$ (projection on $\mathbb{R}_+^A$ of the acceptable set $\mathbb{D}_{\mathrm{yield}}(y_{\min}, B_{\lim})$) where $\|x\|_1$ is the norm $\sum_{a=1}^{A} |x_a|$ in $\mathbb{R}^A$. For any x one has

$$\|G(x, u^\flat) - \bar{x}(u^\flat)\|_1 \leq \left|\varphi(\mathrm{SSB}(x)) - \varphi(\mathrm{SSB}(\bar{x}(u^\flat)))\right| + \sum_{a=1}^{A} \mathrm{e}^{-(M_a+u^\flat F_a)}|x_a - \bar{x}(u^\flat)_a|. \tag{17}$$

If $x \in \mathbb{V}_0$ then $\mathrm{SSB}(x) \geq B_{\lim} > \mathrm{SSB}(\bar{x}(u^\flat))$ and therefore one obtains

$$\left|\varphi(\mathrm{SSB}(x)) - \varphi(\mathrm{SSB}(\bar{x}(u^\flat)))\right| \leq \max_{B\in\left[\mathrm{SSB}(\bar{x}(u^\flat)),\mathrm{SSB}(x)\right]} |\varphi'(B)| \sum_{a=1}^{A} \gamma_a w_a |x_a - \bar{x}(u^\flat)_a|.$$

By the concavity of φ (which implies that φ' is decreasing), this gives

$$\begin{aligned} &\left|\varphi(\mathrm{SSB}(x)) - \varphi(\mathrm{SSB}(\bar{x}(u^\flat)))\right| \\ &\leq \left|\varphi'(\mathrm{SSB}(\bar{x}(u^\flat)))\right| \sum_{a=1}^{A} \gamma_a w_a |x_a - \bar{x}(u^\flat)_a| \\ &\leq \left(\varphi'\left(\mathrm{SSB}(\bar{x}(u^\flat))\right) \max_{a=1,\ldots,A} \gamma_a w_a\right) \|x - \bar{x}(u^\flat)\|_1. \end{aligned}$$

The above inequality together with (17) and the definition of (15) make it possible to obtain (16) and then, the condition (8) of Corollary 1. □

The above result can be interpreted as follows:

- There is no vector of abundance which allows one to obtain, starting from it, catches greater than the *maximal production threshold* $Y(\bar{x}(u^\flat), u^\sharp)$, during all the periods.
- Starting from any vector of abundance, whatever the harvest, the minimum level of spawning stock biomass (SSB) observed during all the periods will be lower than (or equal to) the *maximal preservation threshold* $\mathrm{SSB}(\bar{x}(u^\flat))$.

Table 1
Maximal sustainable thresholds for *Chilean sea bass* and for *Alfonsino*.

Definition	Notation	*Chilean sea bass*	*Alfonsino*
Maximal threshold for a sustainable catch (tons)	$Y(\bar{x}(u^\flat), u^\sharp)$	15 166	16 158
Maximal threshold for a sustainable SSB (tons)	$\mathrm{SSB}(\bar{x}(u^\flat))$	56 521	52 373
Constant defined by (15)	$\phi_G(u^\flat)$	0.852	0.818

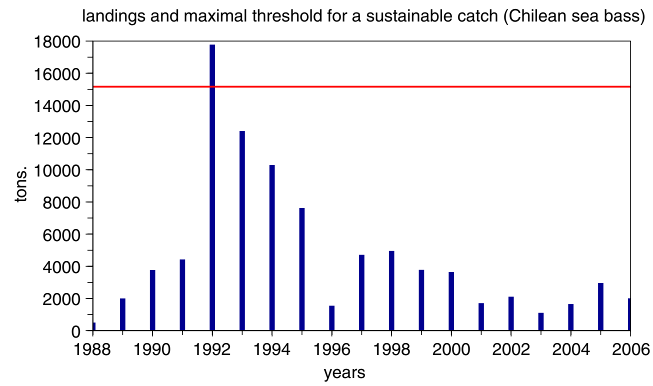

Fig. 1. Chilean sea bass: landings (1988–2006) (tons) and $Y(\bar{x}(u^\flat), u^\sharp)$.

4.5. Numerical applications to Chilean fisheries

We provide numerical estimates obtained for the species Chilean sea bass (*Dissostichus eleginoides*), harvested in the south of Chile, and Alfonsino (*Beryx splendens*), harvested in the Juan Fernández archipelago. The dynamic of the Chilean sea bass can be described by the model (11) with a Beverton–Holt stock-recruitment relationship φ. For the Alfonsino, females and males are distinguished, each following a dynamic (11) with a Beverton–Holt stock-recruitment relationship φ. Thus, for this species, the state is the abundances at age for females and males and the resulting dynamic is a monotone harvest one. For both species the mortality is supposed to be the same at all ages. Numerical data have been provided by the *Centro de Estudios Pesqueros—Chile (CEPES)*.

Table 1 sums up the maximal production and preservation thresholds (in tons) obtained from Proposition 2 for both species and the values of $\phi_G(u^\flat)$ defined by (15).

Chilean sea bass

Fig. 1 displays the Chilean sea bass landings, between 1988 and 2006. The horizontal line represents the maximal threshold $Y(\bar{x}(u^\flat), u^\sharp)$. Hence, it may be seen that the catches obtained in 1992 were not sustainable: even if the species were abundant, such landings could not be maintained forever.

Fig. 2 displays the Chilean sea bass spawning stock's biomass (SSB), between 1988 and 2006. The horizontal line represents the maximal threshold $\mathrm{SSB}(\bar{x}(u^\flat))$. The SSB observed during the first six years could not have been sustained forever.

Alfonsino

For the Alfonsino, Figs. 3 and 4 show that both spawning stock biomasses and landings are below the maximal threshold. Thus, we cannot conclude that these levels indicate a non-viable fishery management.

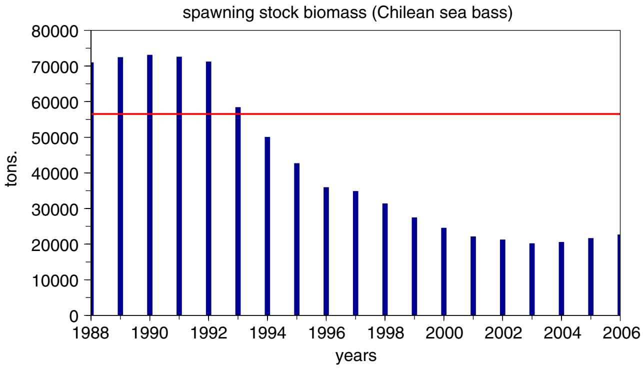

Fig. 2. Chilean sea bass: SSB (1988–2006) (tons) and SSB($\bar{x}(u^\flat)$).

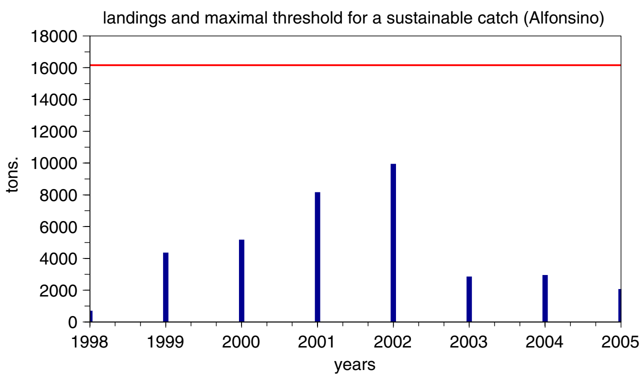

Fig. 3. Alfonsino: landings (1998–2005) (tons) and $Y(\bar{x}(u^\flat), u^\sharp)$.

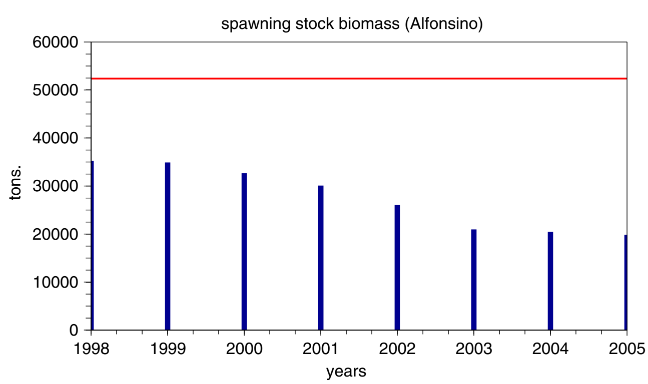

Fig. 4. Alfonsino: SSB (1998–2005) (tons) and SSB($\bar{x}(u^\flat)$).

5. Conclusion

We have introduced monotone harvest models, characterized by monotonicity properties. We have shown how these latter may help in providing estimates of the viability kernel for so-called production and preservation acceptable sets. When the acceptable set is defined by inequalities requirements given by indicator functions and thresholds, we provide conditions on these thresholds to test whether the viability kernel is empty or not.

This theoretical framework is applied to fishery management analysis. We obtain upper bounds for production which are interesting for managers in that they only depend on the model's parameters, and not on the current stocks. Our formulas for so-called maximal sustainable thresholds give sensible values: Chilean sea bass data violate these bounds, while Alfonsino data are within.

We have thus provided a general method to analyze up to what points can conflicting production and preservation objectives be sustainably achieved for a class of models including monospecies age class and multi-species with technical interactions.

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