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RECONSTRUCTING 3-COLORED GRIDS FROM HORIZONTAL AND VERTICAL PROJECTIONS IS NP-HARD: A SOLUTION TO THE 2-ATOM PROBLEM IN DISCRETE TOMOGRAPHY*

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Abstract. We consider the problem of coloring a grid using k colors with the restriction that each row and each column has a specific number of cells of each color. This problem has been known as the $(k-1)$-atom problem in the discrete tomography community. In an already classical result, Ryser obtained a necessary and sufficient condition for the existence of such a coloring when two colors are considered. This characterization yields a linear time algorithm for constructing such a coloring when it exists. Gardner et al. showed that for $k \geqslant 7$ the problem is NP-hard. Afterward Chrobak and Dürr improved this result by proving that it remains NP-hard for $k \geqslant 4$. We close the gap by showing that for $k = 3$ colors the problem is already NP-hard. In addition, we give some results on tiling tomography problems.

Key words. discrete tomography, 2-atom problem, tiling, NP-complete

AMS subject classifications. 52C45, 05B45, 05C70, 52C15, 05B50

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1. Introduction.

Tomography consists of reconstructing spatial objects from lower dimensional projections and has medical applications as well as nondestructive quality control applications. In the discrete variant, the objects to be reconstructed are discrete, as, for example, atoms in a crystalline structure [1].

One of the first studied problems in discrete tomography involves the coloring of a grid using a fixed number of colors with the requirement that each row and each column has a specific total number of cells of each color. In order to formalize this problem we introduce some definitions and notations.

For any positive integer n, we denote the set $\{0, \dots, n-1\}$ by $[n]$. Throughout the paper, whenever we index matrices or vectors, we let the indices start at 0. This will be very convenient in various proof constructions. The *finite grid* of size $m \times n$ corresponds to the set $[m] \times [n]$, and its elements are called *cells*. A cell (i, j) is the intersection of row i and column j; in figures we represent a cell as a unit square, and we number rows top-down and columns from left to right.

Given a finite set of colors $\mathcal{C}$, let M be an assignment of colors in $\mathcal{C}$ to each cell of the $m \times n$ grid. We denote by M_{ij} the color assigned to cell (i, j). We refer to such an assignment as a *$\mathcal{C}$-colored grid*. The *projection* of a $\mathcal{C}$-colored grid M is the

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sequence of vectors $r^c \in [n+1]^m, s^c \in [m+1]^n$, for each $c \in \mathcal{C}$, where

$$r_i^c = |\{j : M_{ij} = c\}|, \qquad s_j^c = |\{i : M_{ij} = c\}|,$$

for $i \in [m]$ and $j \in [n]$. Notice that the projection $(r^c, s^c)_{c\in\mathcal{C}}$ of a $\mathcal{C}$-colored $m \times n$ matrix satisfies

$$(1.1) \qquad \sum_{c\in\mathcal{C}} r_i^c = n, \qquad \sum_{c\in\mathcal{C}} s_j^c = m, \qquad \sum_{i\in[m]} r_i^c = \sum_{j\in[n]} s_j^c$$

for each $i \in [m]$, $j \in [n]$, and $c \in \mathcal{C}$, respectively. In the problem that we consider, the goal is to compute a $\mathcal{C}$-colored grid M that has a given projection. Formally, we consider the following problem.

k-COLOR TOMOGRAPHY PROBLEM.

- **Input:** a sequence of vectors $r^c \in [n+1]^m, s^c \in [m+1]^n$ for each $c \in \mathcal{C}$, with $|\mathcal{C}| = k$, that satisfy (1.1).
- **Output:** a $\mathcal{C}$-colored grid M with $(r^c, s^c)_{c\in\mathcal{C}}$ as projection vectors.

We refer to the problem as k-CTP.

As the projection of one of the colors is redundant by (1.1), we sometimes omit in the instance the vectors r^c, s^c for some fixed color $c \in \mathcal{C}$. In this case, this color is called the *ground color*. In the context of discrete tomography, the remaining colors can represent $k' = k-1$ different types of atoms arranged in a two-dimensional atomic structure, and the ground color represents the absence of an atom in a specific location. Due to this analogy, k-CTP was introduced in [13] as the k'-atoms consistency problem, and it has also been referred to as the *k'-atom problem* [6] or simply as the *k'-color problem* [8, 9, 21]. However, throughout this paper we will always refer to this problem as the k-CTP.

It has been known for a long time that for $k = 2$ colors the problem can be solved in polynomial time [23]. Around a decade ago it was shown that the k-CTP is NP-hard for $k \geqslant 7$ colors [13]. By NP-hardness, we mean that the decision variant—deciding whether a given instance is *feasible*, i.e., admits a solution—is NP-complete. Shortly after, this result was improved to show NP-hardness for $k \geqslant 4$ colors, leaving open the case when $k = 3$ [8]. The present paper closes the gap by showing that for three colors the problem is already NP-hard. The proof of this result is given in section 2.

In section 3, we present one of the consequences of our main result: The problem of finding two graphs H and G on the same set of vertices, with $H \subseteq G$ and specified degrees, is NP-hard.

In addition, in section 4 we address a more general problem, namely, tiling tomography with rectangular tiles as introduced in [6]. In this problem, we are given a set of rectangular tiles $\mathcal{T}$, an $m \times n$ grid, and projections $r^\tau \in [n+1]^m, s^\tau \in [m+1]^n$ for each tile $\tau \in \mathcal{T}$. The goal is to find a partition of the grid into translated copies of tiles from $\mathcal{T}$ (called a $\mathcal{T}$-tiling) satisfying the required projections. In accordance to our previous notation we call this problem the $\mathcal{T}$-tiling tomography problem, or $\mathcal{T}$-TTP.

For this problem, we determine the complexity of $\mathcal{T}$-TTP for almost every set $\mathcal{T}$ of tiles. More precisely, we prove the following. If $\mathcal{T}$ consists of at least three tiles, the problem is NP-hard, regardless of the sizes of the tiles in $\mathcal{T}$. It is also NP-hard if $\mathcal{T}$ consists of two tiles, one of them having width and height at least 2. These results are obtained by reductions from the 3-CTP. On the positive side, we show that if $\mathcal{T}$ consists of two tiles both of unit height (or by symmetry both of unit width), then the problem can be solved greedily in polynomial time, generalizing previous results from [12, 22, 21, 9]. We leave open the case when $\mathcal{T}$ consists of two tiles of dimensions $1 \times p$ and $q \times 1$ for $p, q \geqslant 2$. For the particular case $p = q = 2$, a clever polynomial time algorithm has been provided [25]; however, it does not seem to generalize easily.

Throughout this paper, for $|\mathcal{C}| = 2$ we denote the colors of $\mathcal{C}$ as *black* and *white* and use symbols B, W, respectively. For $|\mathcal{C}| = 3$ we denote the colors as *red*, *green*, and *yellow* and use symbols R, G, Y, respectively. We choose white and yellow to be ground colors and mostly omit their projections. For the 2-CTP, by a *colored* cell we mean a cell colored black, and use the similar convention for an uncolored cell, meaning colored white. As we previously observed, an instance (r^B, r^W, s^B, s^W) of the 2-CTP can be given just by the pair (r^B, s^B). In addition, when there is no risk of confusion, we drop the superscript and simply denote it as (r, s).

2. NP-hardness proof.

This section is devoted to the proof of the NP-hardness of the 3-CTP. We closely follow the reduction from vertex cover in [8], but with a different gadget. Vertex cover is a well-known NP-hard problem [14].

VERTEX COVER PROBLEM.

- **Input:** a graph $H = (V, E)$ and an integer k.
- **Output:** a set $S \subseteq V$ of size $|S| = k$ such that for every $uv \in E$, $u \in S$ or $v \in S$.

Given an instance (H, k) of vertex cover, the idea is to construct an instance (r^R, r^G, s^R, s^G) of the 3-CTP which has a solution if and only if the former instance has a solution. To this purpose we require some preliminary results, which are included in section 2.1. In section 2.2, we construct the gadget which is the main ingredient of this reduction. Finally, in section 2.3 we present the reduction itself and we prove that it satisfies the required conditions.

2.1. Classical lemmas.

The first result of this section is the well-known characterization of the existence of binary matrices satisfying given row and column sum vectors due to Gale and Ryser. We remark that this result characterizes the instances of 2-CTP that have a solution. Before stating the lemma, we need to introduce some notation.

The *conjugate* of a vector $x \in [m+1]^n$ is defined as the vector $x^* \in [n+1]^m$, where $x^*_i = |\{j : x_j > i\}|$. There is a very simple graphical interpretation of this, as shown in Figure 2.1. Let M be the $m \times n$ grid such that in column j, the first x_j cells are colored (black) and the others remain uncolored (remain white). Then the coordinates of the conjugate of x are the row sums of the colored cells of M.

Notice that this is slightly different from the usual definition of conjugate, since here the length m of x^* is fixed and then x^* can have some zero coordinates. Observe that x^* is always a nonincreasing vector. If in addition x is nonincreasing, then $x^*_i = \max\{j : x_j > i\} + 1$, and hence $x^*_i > j$ if and only if $x_j > i$. Therefore $(x^*)^* = x$, which actually motivates the term conjugate.

For integral n-vectors x, y, we say that y *majorizes*[1] x, denoted by $x \preccurlyeq y$, if $\sum_{j \in [\ell]} y_j \geqslant \sum_{j \in [\ell]} x_j$ for every $\ell \in \{1, \ldots, n-1\}$, and $\sum_{j \in [n]} y_j = \sum_{j \in [n]} x_j$. We write $x \prec y$ if in addition the inequality is strict for at least one ℓ.

[1] Sometimes it is also said that y *dominates* x.

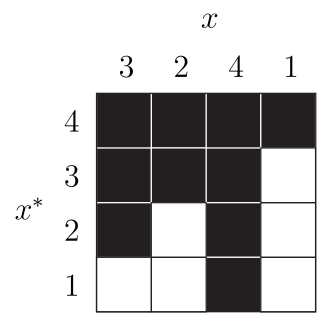


FIG. 2.1. *Example of a column vector x and its conjugate x^*.*

In the rest of this section we consider an instance (r, s) of the 2-CTP with $r \in [n+1]^m$ and $s \in [m+1]^n$. Gale and Ryser gave a useful characterization of the instances of 2-CTP which admit a solution, in terms of conjugation and majorization.

LEMMA 2.1 (Gale [12] and Ryser [22]; also see [23] for the second statement). *Let (r, s) be an instance of the 2-CTP such that r is nonincreasing. Then (r, s) has a solution if and only if $r \preccurlyeq s^*$.*

Moreover, if $r = s^*$, then there is a single solution, namely, the coloring assigning color to the first s_j cells of column j for each $j \in [n]$, and leaving the remaining cells uncolored.

Again, there is a simple graphical interpretation of this. Let M be a colored grid where in column j the first s_j cells are colored and the remaining cells are left uncolored. Then the row projection of M is the vector $r' = s^*$. If $r' = r$ we are done. Otherwise, by using $r \prec s^*$ one can show that there are rows $i < i'$ and column j such that $M_{ij} = B$ and $M_{i'j} = W$ and such that by exchanging these entries the *invariant* $r \preccurlyeq r'$ is preserved while reducing the value $\sum_i |r_i - r'_i|$. We refer to the papers cited above for a complete proof.

Now, we recall a well-known fact about the 2-CTP. Here, for a subset of rows $I \subseteq [m]$ we denote $\overline{I} = [m] \setminus I$. Similarly, for $J \subseteq [n]$ we have $\overline{J} = [n] \setminus J$.

LEMMA 2.2 (Ryser [23]). *Let (r, s) be a feasible instance of the 2-CTP. Let a row set I and a column set J be such that*

$$\sum_{i \in I} r_i - \sum_{j \in \overline{J}} s_j = |I \times J|. \tag{2.1}$$

Then every solution of (r, s) colors all the cells in $I \times J$ and none in $\overline{I} \times \overline{J}$.

Proof. Sets I and J divide the grid into four parts, $I \times J$, $I \times \overline{J}$, $\overline{I} \times J$, and $\overline{I} \times \overline{J}$. The value $\sum_{i \in I} r_i$ equals the number of colored cells in the first two parts and $\sum_{j \in \overline{J}} s_j$ the number of colored cells in the second and last parts. So the difference is the number of colored cells in $I \times J$ minus the number of colored cells in $\overline{I} \times \overline{J}$. So when (2.1) holds, the first part must have only colored cells and the last part none. $\square$

To introduce the last result of this section we need some extra notation. For $0 \leqslant k \leqslant t$, we denote by $\binom{[t]}{k}$ the set of all vectors x in $\{0,1\}^t$ for which the sum $\sum_{i \in [t]} x_i$ equals k. Observe that $\binom{[t]}{k}$ are the characteristic vectors of the k-sets (also called k-combinations) of $[t]$.

It is not difficult to see that majorization $\preccurlyeq$ defines a partial order in $\binom{[t]}{k}$. We denote by $L_{t,k}$ the lattice generated by this partial order in $\binom{[t]}{k}$. (See Figure 2.2 for

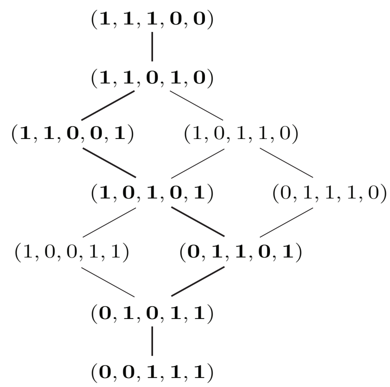


FIG. 2.2. *The lattice $L_{5,3}$ of the set $\binom{[5]}{3}$ ordered by majorization upward. Only nontransitive arcs are shown. Bold indicates a chain of maximum length.*

an example.) Observe that $L_{t,k}$ has unique maximum and minimum elements which are, respectively,

$$\overline{x} = (\underbrace{1, \ldots, 1}_{k}, 0, \ldots, 0) \quad \text{and} \quad \underline{x} = (0, \ldots, 0, \underbrace{1, \ldots, 1}_{k}).$$

As for any partial order, a sequence of elements satisfying $x^0 \prec x^1 \prec \cdots \prec x^q$ is called a chain, and the length of the longest chain is called the *depth* of the partial order.

The following is the key lemma in the construction used in [7]. It shows that the depth of $L_{t,k}$ is polynomially bounded in terms of k and t. For sake of completeness, we include the proof presented in [7] with only some slight modifications, mainly in notation.

LEMMA 2.3. *For $0 \leqslant k \leqslant t$, the depth of $L_{t,k}$ is $k(t-k)+1$.*

Proof. We define the function φ that assigns $\varphi(x) = \sum_{\ell\in[t],\ell\neq 0} \sum_{i\in[\ell]} x_i - \frac{1}{2}k(k+1)$ to each $x \in L_{t,k}$. Observe that for $x \prec y$ we have $\sum_{j\in[\ell]} x_j \leqslant \sum_{j\in[\ell]} y_j$ for every $\ell \in \{1, \ldots, t\}$, and the inequality is strict for at least one ℓ. Therefore $x \prec y$ implies $\varphi(x) < \varphi(y)$. Then any chain $x^1 \prec x^2 \prec \ldots \prec x^q$ in $L_{t,k}$ satisfies $\varphi(x^1)+q-1 \leqslant \varphi(x^q)$.

Furthermore, straightforward calculation shows that $\varphi(\overline{x}) = k(t-k)$ and $\varphi(\underline{x}) = 0$. Then the length of any chain in $L_{t,k}$ is at most $\varphi(\overline{x}) - \varphi(\underline{x}) + 1 = k(t-k)+1$. ∎

There is a simple way to visualize this lemma. Figure 2.2 contains the lattice $L_{5,3}$ and shows only nontransitive arcs. Observe that every edge in the path corresponds to shift a 1 exactly one position. Then in a path from $\overline{x}$ to $\underline{x}$, we shift each of the k 1's of $\overline{x}$ exactly $t-k$ positions to the right. Then, any maximum size chain has exactly $k(t-k)+1$ elements. We remark that the value $\varphi(x)$ as defined in the proof corresponds with the level of x in $L_{t,k}$.

2.2. The gadget. The gadget depends on some integers t, k, u, v with $1 \leqslant k \leqslant t$, $u, v \in [t]$, $u \neq v$, and two vectors $x, y \in \binom{[t]}{k}$.

The interpretation of these parameters is that $[t]$ represents the vertex set of a graph on t vertices, x, y the characteristic vectors of two vertex sets of size k, and uv an edge of the graph. The gadget is defined as the instance of the 3-CTP of t rows

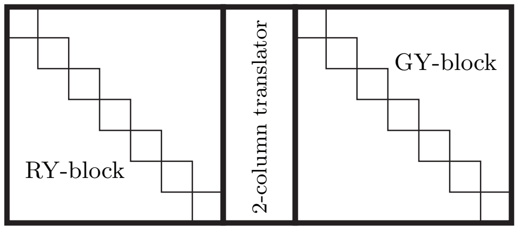


FIG. 2.3. *The structure of the gadget for $t = 7$.*

and $2t + 2$ columns with the following projections for $i, j \in [t]$:

$$r_i^R = \begin{cases} i+2 & \text{if } i \in \{u, v\}, \\ i+1 & \text{otherwise,} \end{cases} \qquad r_i^G = \begin{cases} i+1 & \text{if } i \in \{u, v\}, \\ i+2 & \text{otherwise,} \end{cases}$$

$$s_j^R = t - j + x_j - 1, \qquad s_j^G = 0,$$

$$s_t^R = 1, \qquad s_t^G = t - 1,$$

$$s_{t+1}^R = t - k + 1, \qquad s_{t+1}^G = k - 1,$$

$$s_{t+2+j}^R = 0, \qquad s_{t+2+j}^G = t - j - y_j.$$

We recall that R, G, Y refer to the colors red, green, and yellow. A cell colored with color red will be called a *red cell*. A *green cell* and a *yellow cell* are defined analogously.

Observe that any solution of the gadget has no green cell in the first t columns, no yellow cell in the columns $t, t+1$, and no red cells in the last t columns. Hence, the $t \times (2t + 2)$ grid can be divided into three parts (see Figure 2.3): a $t \times t$ block having colors red and yellow (called *RY-block*), a $t \times 2$ rectangle having colors red and green (called 2-*column translator*), and another $t \times t$ block having colors green and yellow (called *GY-block*). Again, each of the previous $t \times t$ blocks is subdivided into an upper triangle, a diagonal, and a lower triangle.

We will use the main diagonal of the RY-block (and GY-block) to encode the characteristic vector of a set of size k. The 2-column translator plays a double role: on the one hand, it allows us to translate a red-yellow code in the RY-block to a green-yellow code in the GY-block; on the other hand, if the two main diagonals codify the same vector, it allows us to test whether the edge uv is covered by that vector.

LEMMA 2.4. *If the instance (r^R, r^G, s^R, s^G) of* 3-CTP *defined above has a solution, then $x \preccurlyeq y$. Moreover, if $x = y$, then it has a solution if and only if $x_u + x_v \geqslant 1$. See* Figure 2.4 *for an example of a coloring of the gadget when $x = y$.*

Proof. Assume the instance has a solution; we will show that this implies $x \preccurlyeq y$. Consider the yellow projection vectors $r^Y = 2t+2-r^R-r^G$ and $s^Y = t-s^R-s^G$. We have that $r_i^Y = 2(t-i)-1$ for $i \in [t]$. Note that r^Y is a nonincreasing vector. Similarly, we obtain that $s_j^Y = j + 1 - x_j$ and $s_{t+2+j}^Y = j + y_j$ for $j \in [t]$ and $s_t^Y = s_{t+1}^Y = 0$. The conjugate of the yellow column projections is the vector $(s^Y)^*$ with coordinates

$$(s^Y)_i^* = 2(t-i) - 1 - x_i + y_i.$$

Then it is clear that $r^Y \preccurlyeq (s^Y)^*$ if and only if $x \preccurlyeq y$. By assumption, the instance (r^R, r^G, s^R, s^G) has a solution, and then the instance (r^Y, s^Y) of 2-CTP has a solution as well. By Lemma 2.1, it follows that $r^Y \preccurlyeq (s^Y)^*$ and therefore $x \preccurlyeq y$. This shows the first part of the lemma.

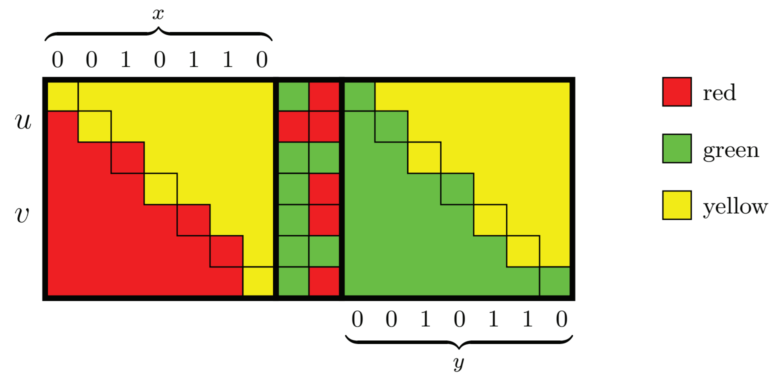


FIG. 2.4. *A solution of the gadget for $t = 7$, $k = 3$, $u = 1$, $v = 4$, and $x = y$. Observe that $x_v = 1$.*

Now assume that for $x = y$ the instance of the 3-CTP has a solution. Since $(s^Y)^*_i = r^Y_i - x_i + y_i$ for each i, we have $r^Y = (s^Y)^*$. By Lemma 2.1, any solution must color in yellow the s^Y_j first cells of column j for each j and no other cell. In particular this means that the lower triangle of the RY-block must be red, the lower triangle of the GY-block must be green, and both upper triangles have to be yellow. Also on the first diagonal, the cell (i, i) has to be red if $x_i = 1$ and yellow otherwise. On the second diagonal, the cell $(i, t + 2 + i)$ must be yellow if $x_i = 1$ and green otherwise.

Let us assume by contradiction that $x_u = x_v = 0$. Then the cells $(u, t), (u, t + 1), (v, t), (v, t + 1)$ of the 2-column translator have to be all red to satisfy the row projections. This contradicts the column projection $s^R_t = 1$, and hence the instance is not realizable.

Conversely, assume $x_u + x_v \geqslant 1$. We color the RY-block and the GY-block as above. Notice that this immediately shows that for each $j \in [t]$, the number of red and green cells is $t-j-(1-x_j)$ and 0, respectively, for column j, and 0 and $t-j-x_j$, respectively, for column $t + 2 + j$.

We will color the cells of the 2-column translator in a manner that respects the required row and column projections. If $i \notin \{u, v\}$ and $x_i = 1$, that is, (i, i) is red and $(i, t+2+i)$ is yellow, we color the cells (i, t) and $(i, t+1)$ in green. Then the number of red and green cells in row i is $i+1$ and $i+2$, respectively. If $i \notin \{u, v\}$ and $x_i = 0$, that is, (i, i) is yellow and $(i, t + 2 + i)$ is green, we color the cell (i, t) in green and $(i, t + 1)$ in red. Again, the number of red and green cells in row i is $i + 1$ and $i + 2$, respectively.

Without loss of generality assume that $x_v = 1$. Then we color (v, t) in green and $(v, t + 1)$ in red. Since (v, v) is red and $(v, t + 2 + v)$ is yellow, the number of red and green cells in row v is $i + 2$ and $i + 1$, respectively. For row u, we color (u, t) in red and the cell $(u, t + 1)$ in red if $x_u = 0$ and in green otherwise. It is not difficult to check that in both cases the number of red and green cells in row u is $i + 2$ and $i + 1$, respectively.

Finally, since the only red cell in column t is (u, t), the number of green cells is $t - 1$. Also, the number of green cells in column $t + 1$ is $\sum_{i \neq v} x_i = \sum_i x_i - 1 = k - 1$, since $x_v = 1$. Therefore the coloring (or $\{R, G, Y\}$-assignment) defined above is a solution of the instance, which concludes the proof of the lemma. ▯

2.3. The reduction. Let us consider an instance (H, k) of vertex cover. Let $t = |V(H)|$ and $\mu = |E(H)|$. Thus, we will assume that $V(H) = [t]$ and that

$\{e_0, e_1, \dots, e_{\mu-1}\}$ is a list of the edges of H. Observe that without loss of generality, we can assume that $k \leqslant t-2$.

Let $N = (k(t-k)+1)\mu + 1$ be the number of gadgets that we will use in the reduction. We define the instance (r^R, r^G, s^R, s^G) with $N(t+1)+1$ rows and $N(t+2)+t$ columns as follows.

For row $p = 0, \dots, N(t+1)$, let

$$a = \left\lfloor \frac{p}{t+1} \right\rfloor \quad \text{and} \quad i = p \bmod (t+1).$$

We view the set of rows as divided into N blocks of $t+1$ rows each and a last block with a single row. With this convention, a is the block index and $0 \leqslant i \leqslant t$ corresponds to the cell index inside the row block of p.

Let us consider the edge $e_{a \bmod \mu} = uv$, where $u < v$. We define the row projections as

$$r_p^R = a(t+2) + \begin{cases} t-k & \text{if } i = 0 \text{ and } a < N, \\ 0 & \text{if } i = 0 \text{ and } a = N, \\ i+1 & \text{if } i-1 \in \{u,v\}, \\ i & \text{if } i-1 \in [t] \setminus \{u,v\}, \end{cases}$$

$$r_p^G = (N-1-a)(t+2) + \begin{cases} t+2 & \text{if } i = 0 \text{ and } a = 0, \\ t+2+k & \text{if } i = 0 \text{ and } a > 0, \\ i & \text{if } i-1 \in \{u,v\}, \\ i+1 & \text{if } i-1 \in [t] \setminus \{u,v\}. \end{cases}$$

In the same manner, for column $q = 0, \dots, N(t+2)+t-1$, let

$$b = \left\lfloor \frac{q}{t+2} \right\rfloor \quad \text{and} \quad j = q \bmod (t+2).$$

Similarly as for the rows, we view the set of columns as divided into N blocks with $t+2$ columns each and a last block with only t columns. Again, b corresponds to the block index and $0 \leqslant j \leqslant t+1$ to the cell index inside each column block. For block index $b < N$, we define

$$s_q^R = (N-1-b)(t+1) + 1 + \begin{cases} t-j & \text{if } j \in [t], \\ 1 & \text{if } j = t, \\ t-k+1 & \text{if } j = t+1. \end{cases}$$

For $b = N$ we set $s_q^R = 0$, for each $q = N(t+2)+j$ with $j \in [t]$. Finally, in order to define s^G we need to distinguish two cases. First, for block index $b = 0$ we have

$$s_q^G = \begin{cases} 0 & \text{if } j \in [t], \\ t & \text{if } j = t, \\ k & \text{if } j = t+1. \end{cases}$$

Moreover, for $0 < b \leqslant N$ the requirements are

$$s_q^G = (b-1)(t+1) + 1 + \begin{cases} t-j & \text{if } j \in [t], \\ t-1 & \text{if } j = t, \\ k-1 & \text{if } j = t+1. \end{cases}$$

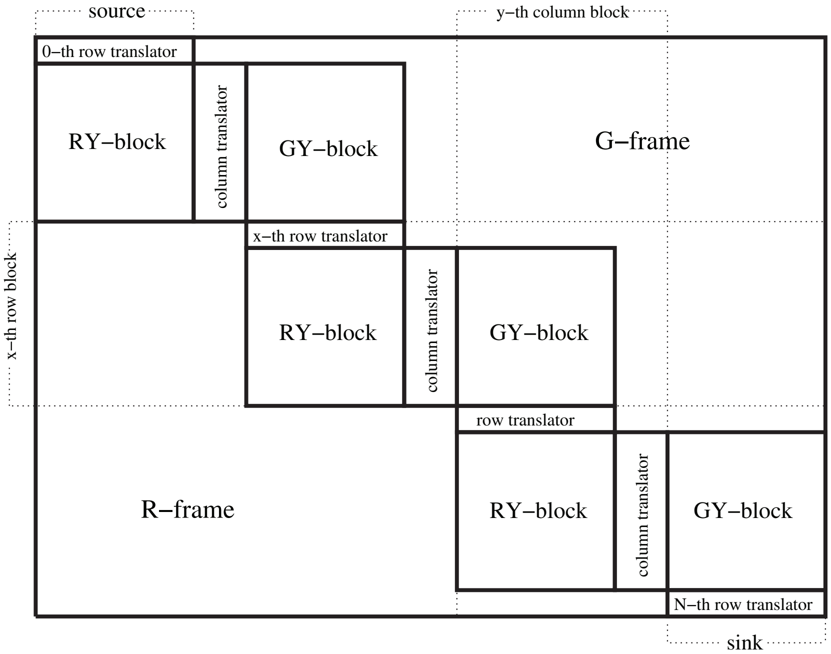


FIG. 2.5. *The general structure of our reduction.*

LEMMA 2.5. *The instance* (r^R, r^G, s^R, s^G) *has a solution if and only if there exists a* k*-vertex cover in* H*.*

Proof. For one direction of the statement, assume that the vertex cover instance has a solution, and let $z \in \binom{[t]}{k}$ be the characteristic vector of a vertex cover of size k, i.e., $z_i = 1$ if and only if i belongs to the vertex cover.

We now construct a solution to the instance of the 3-CTP. Consider the partitioning of the grid, as in Figure 2.5. For convenience we sometimes refer to the source as the 0*th row translator* and to the sink as the N*th row translator*. Moreover, we refer to the cell at column $a(t+2)+j$ and row $a(t+1)$ as the j*th cell of the* a*th row translator*. We color the R-frame in red and the G-frame in green. Let j be in $[t]$. We color the jth cell of the source in yellow if $z_j = 1$ and in red otherwise. For $a = 1, \dots, N-1$ we color the jth cell of the ath row translator in green if $z_j = 1$ and in red otherwise. In the sink we color the jth cell in green if $z_j = 1$ and in yellow otherwise.

Now for each block index $a = 0, \dots, N-1$, consider the instance for the gadget defined by $x = y = z$, and u, v, with $u < v$ and such that $uv = e_{a \bmod \mu}$. Since z is a vertex cover, we have that $z_u + z_v \geqslant 1$ and therefore by Lemma 2.4 the instance for gadget a is feasible. Then we color the $(t+1) \times (2t+2)$ cells starting at $(a(t+1)+1, a(t+2))$ exactly as in the solution for the gadget. It is straightforward to check that this grid coloring satisfies the required projections, and therefore the instance of the 3-CTP has a solution.

For the converse, assume that (r^R, r^G, s^R, s^G) has a solution, namely, a grid M colored with colors in $\{R, G, Y\}$. For every $a = 1, \dots, N$ we apply Lemma 2.2 for the red color and intervals $I = [a(t+1), N(t+1)]$ and $J = [0, a(t+2)-1]$, which shows that in M, $I \times J$ is completely red and $\overline{I} \times \overline{J}$ is free of any red. This implies that in M the R-frame must be all red, and all GY-blocks must be free of any red. Using an analogous argument, we have that every cell of the G-frame must be green, and all RY-blocks must be free of any green.

Together with the red and green row projections, this implies that in the source, k cells are yellow and $t-k$ are red, and in the sink k cells are green and $t-k$ are yellow. Moreover, in each row translator k cells are green and $t-k$ are red. We define vectors $x^0, x^1, \ldots, x^N \in \binom{[t]}{k}$ such that for all $j \in [t]$,

- $x_j^0 = 1$ if and only if the jth cell in the source is yellow,
- $x_j^a = 1$ if and only if the jth cell in the ath row translator is green for all $1 \leqslant a \leqslant N$.

For $a = 0, \ldots, N$, consider the colored subgrid M_a, corresponding to gadget a (that is, the part of the solution given by the intersection of rows $[a(t+1)+1, a(t+1)+t]$ and columns $[a(t+2), a(t+2)+2t+1]$). We number the rows of M_a from 0 to $t-1$ and the columns from 0 to $2t+1$. Let $uv = e_{a \bmod \mu}$. By subtracting from the row projections the number of red and green cells in the frames, we deduce that for each $i \in [t]$, row i in M_a contains $i+2$ red cells and $i+1$ green cells if $i \in \{u, v\}$ and $i+1$ red cells and $i+2$ green cells if $i \notin \{u, v\}$.

We proceed similarly for the columns t and $t+1$. By subtracting from the column projections the quantities that are in the frames, we deduce that column t of M_a contains one red cell and $t-1$ green cells, and column $t+1$ contains $t-k+1$ red cells and $k-1$ green cells. In addition, column $a(t+2)+j$ for $j \in [t]$ contains $t-j$ red cells that are not in the R-frame. Since GY-blocks are free of red, these cells must be in either the ath row translator or in column j of M_a. Note that the jth cell of the ath row translator is red if and only if $x_j^a = 0$. Therefore column j of M_a contains $n - j + x_j^a - 1$ red cells and no green cell. Similarly column $t+2+j$ of M_a contains $t - j - x_j^{a+1}$ green cells and no red cell.

This implies that M_a is a solution to the gadget defined by u, v, x, and y with $x = x^a$ and $y = x^{a+1}$. Then by Lemma 2.4 we obtain that $x^a \preccurlyeq x^{a+1}$. Since this holds for each $a = 0, 1 \ldots, N-1$, it follows that

$$x^0 \preccurlyeq x^1 \preccurlyeq \cdots \preccurlyeq x^N. \tag{2.2}$$

However, by Lemma 2.3 there are at most $k(t-k)+1$ different vectors in the sequence (2.2). Thus by the choice of N and the Pigeonhole principle, there exists an ℓ such that

$$x^\ell = x^{\ell+1} = \cdots = x^{\ell+\mu-1}.$$

Using the second part of Lemma 2.4, we have that $x_u^\ell + x_v^\ell = x_u^a + x_v^a \geqslant 1$ for each $uv = e_{a \bmod \mu}$ and $a \in \{\ell, \ldots, \ell+\mu-1\}$. Finally and since $E(H) = \{e_{\ell \bmod \mu}, \ldots, e_{\ell+\mu-1 \bmod \mu}\}$, we conclude that x^ℓ encodes a k-vertex cover of H. ∎

In Figure 2.6 we show a solution for the instance (r^R, r^G, s^R, s^G) of three-CTP associated to the complete graph on three vertices. Some other examples, as well as an applet that solves 3-CTP for small instances, can be found in [10].

THEOREM 2.6. *The decision problem associated to* 3-CTP *is NP-complete in the strong sense.*

Proof. Clearly a solution of the 3-CTP can be checked in polynomial time and then the associated decision problem lies in the class NP.

On the other hand, notice that $N = (k(t-k)+1)\mu + 1 = O(t^4)$ and then the number of rows and columns of the instance (r^R, r^G, s^R, s^G) are $m = N(t+1)+1 = O(t^5)$ and $n = N(t+2)+t = O(t^5)$, respectively. Then (r^R, r^G, s^R, s^G) consist of $2(m+n) = O(t^5)$ numbers, each of them bounded by $n = O(t^5)$. So the unary code

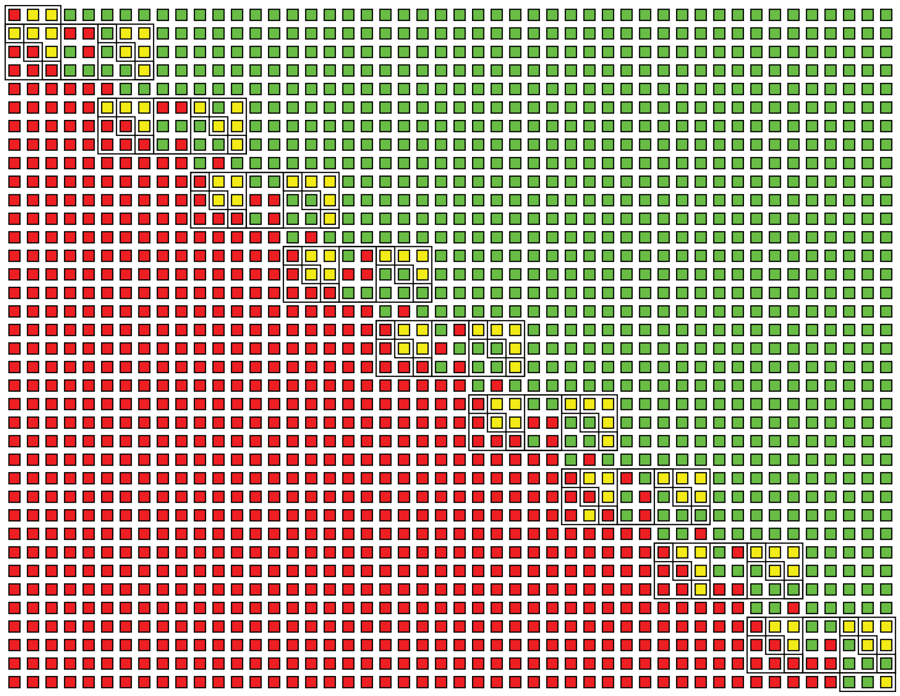


FIG. 2.6. *A solution of an instance of* 3-CTP *associated to the complete graph on three vertices. Edges are ordered as* $12, 13, 23$, $t = \mu = 3$, $k = 2$, $N = 9$, $N(t+1)+1 = 37$ *rows and* $N(t+2)+t = 48$ *columns.* $x^0 = x^1 = (0,1,1), x^2 = x^3 = x^4 = x^5 = x^6 = (1,0,1), x^7 = x^8 = x^9 = (1,1,0)$.

of the instance has size $O(t^{10})$. Then by Lemma 2.5 it turns out that the decision problem is strongly NP-complete. ∎

We recall that Theorem 2.6 solves the consistency part of Question 4.2 in [13], also known as the 2-atom problem in discrete tomography [7].

3. Simultaneous realization of graph factors. The degree sequence problem consists in finding a graph whose degree sequence equals a given nonincreasing sequence d of positive integers. When such a graph exists it is called a *realization* of d, and d is called a *graphical* sequence. There are several characterizations of graphical sequences leading to polynomial time algorithms for reconstructing a desired graph [11, 17, 16, 24]. Similarly, the graph factor problem[2] is the problem of finding a subset of edges F of a given graph $G = (V, E)$ such that the degree function of the graph $H = (V, F)$ is a given function $d : V \to \mathbb{N}$. The degree sequence problem is a particular case of the graph factor problem when G is the complete graph. It is known that the graph factor problem can be solved in polynomial time (see, for example, [2]).

A further generalization is the problem of finding two graphs $H = (V, F)$ and $G = (V, E)$ with $F \subseteq E$ and such that H and G are realizations of two given functions a and b from V to $\mathbb{N}$, respectively. In [19], Kundu proved that this problem always has a solution when b and the difference sequence $b - a$ are both graphical and a has *span* at most one. Here, the span of a sequence is the difference between its maximum and minimum values. Moreover, Kleitman and Wang gave a polynomial time algorithm which under the above conditions finds such realizations [18].

[2]Also known as the f-factor problem.

Lovász observed that this problem is polynomially equivalent to an edge coloring problem of the complete graph satisfying some degree constraints for each color [20]. In what follows we consider this version of the problem. Formally, the simultaneous realization problem (SRP) is the problem of finding two edge disjoint spanning subgraphs H^R and H^G of the complete graph whose degree functions are two given functions d^R and d^G. Hence, given functions d^R and d^G, a solution of the SRP is a partial edge coloring of the complete graph with colors red and green, where the set of red edges induces a graph with degree sequence d^R and the set of green edges induces a graph with degree sequence d^G. In [5], Chen gave a simple proof of Kundu's result which leads to a linear time algorithm for solving the SRP when the inputs are two graphical sequences d^R and d^G and the sequence $d^R + d^G$ has span at most one.

In [3], Brualdi and Ross studied the variant of the problem where the complete graph is replaced by a complete bipartite graph $K_{m,n}$ over sets of size m and n. We call *bigraphical sequences* the sequences that admit a realization as subgraphs of $K_{m,n}$. Later, Anstee showed that this variant has a solution whenever the input are two bigraphical sequences and the sum of them, restricted to one of the two independent sets of $K_{m,n}$, has span at most one. Under these restrictions a solution can be computed in polynomial time [4]. In [15], Guíñez, Matamala, and Thomassé obtained a necessary condition for the simultaneous realization of two bigraphical sequences, which turns out also to be sufficient when the sum of the sequences has span at most two in each independent set of $K_{m,n}$. This characterization allows us to obtain a polynomial time algorithm for constructing the simultaneous realization when the above conditions are satisfied.

As the reader may have already noticed, this variant of the problem is equivalent to the 3-CTP by means of the following identification between a grid of size $m \times n$ and a complete bipartite graph $K_{m,n}$: rows and columns are identified with vertices, and cells are identified with edges. Therefore, Theorem 2.6 implies NP-hardness of the analogous of SRP on complete bipartite graphs.

We will see that this hardness result also holds for complete graphs. Before proving this, we will show that the restriction of 3-CTP to symmetric instances is NP-hard.

SYMMETRIC 3-CTP.

- **Input:** an instance (r^R, r^G, s^R, s^G) of 3-CTP defined in the $n \times n$ grid such that $r^R = s^R$ and $r^G = s^G$.
- **Output:** a symmetric solution M of (r^R, r^G, s^R, s^G), that is, $M_{ij} = M_{ji}$, for each $i, j \in [n]$.

THEOREM 3.1. *The symmetric* 3*-CTP is NP-hard.*

Proof. The associated decision problem clearly belongs to the class NP. Thus, we construct a polynomial reduction from 3-CTP to symmetric 3-CTP as follows: Given an instance (r^R, r^G, s^R, s^G) of 3-CTP defined in an $m \times n$ grid, let $(\hat{r}^R, \hat{r}^G, \hat{s}^R, \hat{s}^G)$ be an instance of symmetric 3-CTP defined in an $(m+n) \times (m+n)$ grid, where for $i \in [m+n]$

$$\hat{r}_i^R = \hat{s}_i^R = \begin{cases} r_i^R + m & \text{if } i \in [m], \\ s_{i-m}^R & \text{otherwise}, \end{cases} \quad \text{and} \quad \hat{r}_i^G = \hat{s}_i^G = \begin{cases} r_i^G & \text{if } i \in [m], \\ s_{i-m}^G + n & \text{otherwise.} \end{cases}$$

By definition $(\hat{r}^R, \hat{r}^G, \hat{s}^R, \hat{s}^G)$ is symmetric and clearly it can be constructed in polynomial time in m and n. Then it remains to show that the instance (r^R, r^G, s^R, s^G) is feasible if and only if $(\hat{r}^R, \hat{r}^G, \hat{s}^R, \hat{s}^G)$ is feasible.

First, let $M \in \{R,G,Y\}^{m\times n}$ be a solution of (r^R, r^G, s^R, s^G). Then we define a coloring $\hat{M}$ of the $(m+n)\times(m+n)$ grid in which each cell in $[m]\times[m]$ is colored in red, each cell in $\overline{[m]}\times\overline{[m]}$ is colored in green, and the subgrids $[m]\times\overline{[m]}$ and $\overline{[m]}\times[m]$ are colored according to M and M^t, respectively. The reader can easily check that $\hat{M}$ is a solution for $(\hat{r}^R, \hat{r}^G, \hat{s}^R, \hat{s}^G)$.

On the other hand, consider a solution $\hat{M}$ of the instance $(\hat{r}^R, \hat{r}^G, \hat{s}^R, \hat{s}^G)$. We apply Lemma 2.2 for $(\hat{r}^R, \hat{s}^R)$ and intervals $I = J = [m]$. This shows that every cell in $[m]\times[m]$ is colored in red and that in $\overline{[m]}\times\overline{[m]}$ none cell is colored in red. Similarly, applying Lemma 2.2 for $(\hat{r}^G, \hat{s}^G)$ and intervals $I = J = \overline{[m]}$, we conclude that $\overline{[m]}\times\overline{[m]}$ is all colored in green. Let M be restriction of $\hat{M}$ to the subgrid $[m]\times\overline{[m]}$. Then the row projections of M are exactly r^R for color red and r^G for color green. Similarly, the columns projections are the vectors s^R and s^G for colors red and green, respectively. Hence M is a solution of (r^R, r^G, s^R, s^G). $\square$

From Theorem 3.1 it follows directly that the simultaneous realization problem is NP-hard for complete graphs with loops. However, from this observation we were not able to derive the same result for complete graphs without loops. Instead, we prove it directly.

THEOREM 3.2. *The simultaneous realization problem is NP-hard.*

Proof. We reduce it from the 3-CTP. Let (r^R, r^G, s^R, s^G) be an instance of the 3-CTP defined in a $m\times n$ grid. We set $V = [m+n]$, and the following degrees, for $i \in [m]$ and $j \in [n]$:

$$\begin{aligned} d_i^R &= r_i^R + m - 1, & d_i^G &= r_i^G, \\ d_{m+j}^R &= s_j^R, & d_{m+j}^G &= s_j^G + n - 1. \end{aligned}$$

Now we show that the instance (r^R, r^G, s^R, s^G) of 3-CTP is feasible if and only if the instance (d^R, d^G) of SRP is feasible. For one direction, assume that there is a solution M to the 3-CTP instance. We construct a solution, E^R, E^G, to the graph problem as follows. For any $i \in [m]$ and $j \in [n]$, the edge joining vertices i and $m+j$ belongs to E^R if $M_{ij} = R$ and to E^G if $M_{ij} = G$. Also, for any $i, i' \in [m]$ with $i \neq i'$, we have $ii' \in E^R$ and for any $j, j' \in [n]$ with $j \neq j'$, the edge that joins $m+j$ and $m+j'$ belongs to E^G. Clearly, E^R, E^G satisfy the required degrees.

For the converse, consider the quantity $\Phi = \sum_{i\in[m]} d_i^R - \sum_{j\in[n]} d_{m+j}^R$. By assumption (1.1), $\Phi = m(m-1)$. Since this value also equals

$$2(|E^R \cap [m]^2| - |E^R \cap \{m, \ldots, m+n-1\}^2|),$$

there is a red edge between every pair of vertices i, i' with $0 \leqslant i < i' < m$ and no red edge between any pair $m+j, m+j'$ with $0 \leqslant j < j' < n$. Similarly, we can show that there is a green edge between every pair of vertices $m+j, m+j'$ with $0 \leqslant j < j' < n$.

Now let M be the $m\times n$ grid with cell (i,j) colored in red or green if the edge joining i and $m+j$ belongs to E^R or E^G, respectively. By the degree requirements, M is a solution to the 3-CTP instance. $\square$

4. Tiling tomography. We generalize the k-CTP in the following manner: instead of assigning colors to each cell we assign colors to *tiles*, which are sets of connected cells. Throughout this section we consider only rectangular tiles which consist of all cells of $[p]\times[q]$ for some positive integers p, q. To ease the presentation we introduce some names for special kinds of tiles. A tile of dimension $p\times q$ is called a *unit tile* if $p = q = 1$, a *square* if $p = q = 2$, a *horizontal bar* of length p if $q = 1$, and a *vertical bar* of length q if $p = 1$. Bars of length 2 are called *dominoes*.

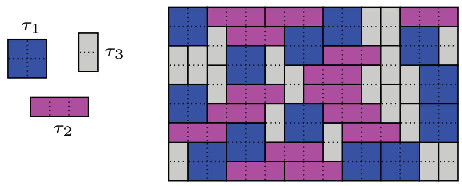


FIG. 4.1. *A tiling of the grid with copies of a 2×2-tile, a 1×3-tile, and a 2×1-tile.*

In this context let $\mathcal{T}$ be a set of tiles, each with a distinct color (but dimensions are not required to differ). A $\mathcal{T}$*-tiling* of an $m \times n$ grid is a partitioning of the grid into smaller colored rectangles, each being a translated copy of a tile in $\mathcal{T}$. In Figure 4.1 a $\mathcal{T}$-tiling of the grid 9×15 is given, where $\mathcal{T}$ consists of one square (τ_1), one horizontal bar of length 3 (τ_2), and one vertical domino (τ_3).

The ith horizontal projection of a tile $\tau \in \mathcal{T}$ of a $\mathcal{T}$-tiling of an $m \times n$ grid is the number of cells in the ith row of the grid being leftmost upper corners of copies of the tile τ. Vertical projections are defined similarly. In Figure 4.1 the horizontal and vertical projection of tiles τ_1 and τ_2 are

$$\begin{aligned} r^{\tau_1} &= (2, 2, 1, 1, 1, 2, 1, 3, 0), \\ r^{\tau_2} &= (3, 1, 1, 1, 2, 2, 2, 1, 2), \\ s^{\tau_1} &= (2, 1, 0, 2, 0, 0, 2, 0, 1, 1, 0, 1, 1, 2, 0), \\ s^{\tau_2} &= (1, 0, 2, 3, 0, 2, 1, 2, 2, 1, 0, 0, 1, 0, 0). \end{aligned}$$

We remark that another equivalent projection could have been defined, namely, the one that for each row i and tile $\tau \in \mathcal{T}$ counts the number of cells of the ith row being covered by some copy of τ. But it is easy to verify that a simple vector basis change relates these alternative projections to the projections defined above.

For a set of tiles $\mathcal{T}$, we consider the following problem.

($\mathcal{T}$-TTP).

- **Input:** vectors $r^\tau \in [n+1]^m$ and $s^\tau \in [m+1]^n$ for each $\tau \in \mathcal{T}$.
- **Output:** a $\mathcal{T}$-tiling T of the $m \times n$ grid with r^τ and s^τ and horizontal and vertical projections, respectively, for each $\tau \in \mathcal{T}$.

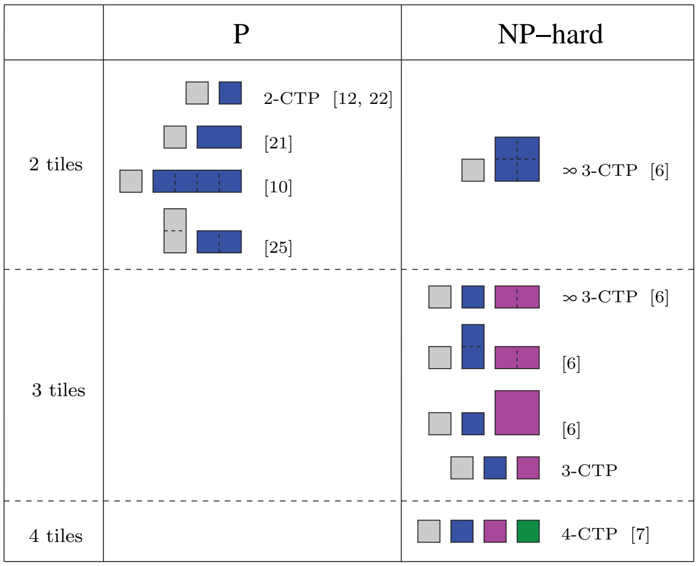
The complexity of $\mathcal{T}$-TTP has been studied for several sets of tiles (see Table 4.1). On one hand, a polynomial time algorithm exists when the set of tiles contains one unit tile and one bar of length q. The case $q = 1$ corresponds to 2-CTP and then can be solved by Ryser's algorithm [12, 22]. The case $q = 2$ was settled in [21] and then generalized to any q in [9]. Moreover, it has been proved to be polynomially solvable when the set of tiles is composed of one vertical domino and one horizontal domino [25].

On the other hand, if the set of tiles consists of one unit tile and one square, then the problem was proved to be as hard as 3-CTP [6]. This relation also holds for specific three-tile sets $\mathcal{T}$: when each tile is a unit tile, then the problem is equivalent to 3-CTP, and when it contains two unit tiles and one domino it is known to be as hard as 3-CTP [6]. Therefore from Theorem 2.6 we know that these three problems are NP-hard. The problem is also NP-hard when the set of tiles is composed of two unit tiles and one square, or one unit tile, one vertical domino, and one horizontal domino [6].

In this section we clarify the complexity status for rectangular tiles. On the one hand, the problem is NP-hard when $|\mathcal{T}| \geqslant 3$ or $|\mathcal{T}| = 2$ and one of the tiles is not a bar. On the other hand, we show that the problem is polynomial time solvable when $\mathcal{T}$ consists of two horizontal bars (or by symmetry, of two vertical bars). The only

Table 4.1

Summary of previous known results of $\mathcal{T}$-TTP for rectangles. We also include the NP-hardness of 3-CTP (Theorem 2.6). We use "$\propto$3-CTP" to symbolize that the problem "admits a reduction from 3-CTP" (i.e., it is as hard as 3-CTP).



case that remains open is when $\mathcal{T}$ consists of one vertical bar of length p and one horizontal bar of length q. In this case, for $p, q \leqslant 2$ it is known that the problem can be solved in polynomial time [25], but its complexity is unknown otherwise, that is, when $p, q \geqslant 2$ and one of p, q is at least 3.

4.1. Polynomial time algorithm for two horizontal bars. In this section we exhibit a polynomial time algorithm for tiling tomography problem for two horizontal bars. Clearly, this also solves the case of two vertical bars. It generalizes the greedy algorithm from [9], which itself is a generalization of the greedy algorithm from [22] for 2-CTP.

To be more precise, we prove that a more general problem admits a polynomial time algorithm. For an integral nonnegative vector $h = (h_0, \ldots, h_{m-1})$, we denote by $\mathcal{H}(h)$ the finite collection of cells, arranged in left-aligned rows, such that row i contains exactly h_i cells. We say that $\mathcal{H}(h)$ is the *histogram*[3] of shape h. In particular, the $m \times n$ grid is the histogram of shape $h = (n, \ldots, n)$. We construct an algorithm that iteratively tiles the empty grid with horizontal bars from right to left. Thus, at each step the remaining set of cells to tile is a histogram.

Theorem 4.1. *The tiling tomography problem can be solved in polynomial time for two horizontal bars.*

Proof. Let q_1 and q_2 be the lengths of the two horizontal bars. Thus, we refer to them as a q_1-bar and a q_2-bar, respectively. As mentioned, the algorithm solves the more general problem, where the region to tile is a histogram $\mathcal{H}(h)$ for some integral vector h. Thus, at each stage of the algorithm its configuration is a tuple $(h; r^1, r^2, s^1, s^2)$, where (r^1, s^1) and (r^2, s^2) are the required projections for q_1-bars and q_2-bars, respectively.

[3]Notice that our use of the term "histogram" differs from the standard one used in graphical representations, where the bars are displayed vertically.

The algorithm is as follows:

Let T be the histogram $\mathcal{H}(h)$.
While $\max_j h_j > 0$, **do**
begin

- Let $\ell = \max_j h_j$. Let $j_1 = \ell - q_1$ and $j_2 = \ell - q_2$.
- If $s^1_{j_1} = s^2_{j_2} = 0$, abort and return "no solution". Otherwise let $c \in \{1, 2\}$ be such that $s^c_{j_c} > 0$.
- Let i be a row that maximizes r^c_i among all rows satisfying $h_i = \ell$.
- Place a q_c-bar at position (i, j_c) in T, decrease $s^c_{j_c}$ and r^c_i by one, and h_i by q_c.

end
Return T if all vectors h, r^1, r^2, s^1, s^2 are zero, and return "no solution" otherwise.

Clearly, if this algorithm produces a tiling, then it has the required projections. Now we have to show that if the instance has a solution, then the algorithm will actually find one. To this purpose we assume that the instance $I := (h, r^1, r^2, s^1, s^2)$ is feasible and show that the algorithm will not return "no-solution" and preserve feasibility of the intermediate instance.

Let S be a solution to I and $\ell = \max h_i$. If $\ell = 0$, then r^1, r^2, s^1, s^2 are all zero, since the instance is feasible and the algorithm produces S.

Otherwise, consider $j_1 = \ell - q_1$ and $j_2 = \ell - q_2$. For every row i satisfying $h_i = \ell$, we have that there is either a q_1-bar at position (i, j_1) or a q_2-bar at position (i, j_2) in S. Therefore one of $s^1_{j_1}, s^2_{j_2}$ must be nonzero. Let c, i be the values the algorithm chooses. Let I' be the instance obtained after the iteration of the algorithm, that is, $r^c_i, s^c_{j_c}$ are decreased by 1 and h_i by q_c.

If S has a q_c-bar at position (i, j_c), then the tiling S' obtained by eliminating that bar is a solution to I', which shows that feasibility is preserved. Now suppose that S does not have a q_c-bar at position (i, j_c). We will show that there is another solution S' which does have this property. Then we are in the case above, which completes the proof.

By $s^c_{j_c} > 0$, there must be another row k in S with $h_k = \ell$ containing a q_c-bar at position (k, j_c). By the choice of the algorithm, we know that $r^c_k \leqslant r^c_i$. From this inequality, there exists a column $j \in [\ell]$ such that the total number of q_c-bars at columns $j' \geqslant j$ is the same in both row i and row k. Note that j also satisfies that for $c' \neq c$, the total number of $q_{c'}$-bars at columns $j' \geqslant j$ in row i is the same as in the row k, since the length of both rows is ℓ. Take j as the largest column index satisfying this property. By the choice of j, there is a q_c-bar in S at position (i, j) and a $q_{c'}$-bar at position (k, j). Then exchanging the bars of rows i and k in S between columns j and ℓ does not change the projections of S, and hence we obtain the required property. In Figure 4.2 we exhibit an example of this exchange. ☐

4.2. NP-hardness results. In this section we present two NP-hardness proofs for the $\mathcal{T}$-TTP for the case when $\mathcal{T}$ consists of two rectangles where one is not a bar, and for the case when $\mathcal{T}$ consists of three rectangles. Both proofs are reductions from the 3-CTP and share a particular structure that we first outline.

4.2.1. The general NP-hardness proof structure. Here is a general construction that reduces 3-CTP to $\mathcal{T}$-TTP.

Consider an instance (r^R, r^G, s^R, s^G) of the 3-CTP defined in an $m \times n$ grid. For the reduction we have to choose a constant-size $k \times \ell$ grid, which we call a *block*,

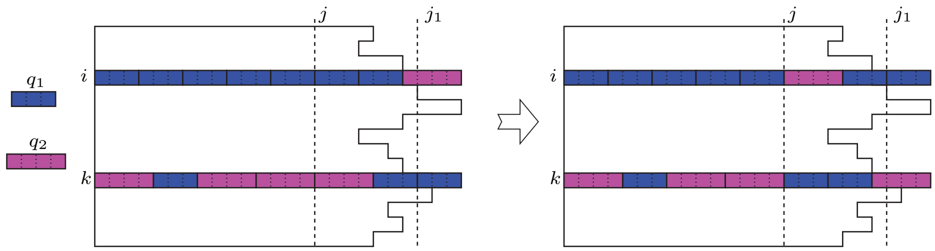


FIG. 4.2. *Transforming a solution when $c = 1$ and i are the choices of the algorithm.*

and three different $\mathcal{T}$-tilings of this block that we denote B^R, B^G, and B^Y. For each $c \in \{R, G, Y\}$ and $\tau \in \mathcal{T}$, let $r^{c,\tau}$ and $s^{c,\tau}$ be the τ-projections of the $\mathcal{T}$-tiling B^c. We write $r^{c,\mathcal{T}} = (r^{c,\tau})_{\tau\in\mathcal{T}}$ and $s^{c,\mathcal{T}} = (s^{c,\tau})_{\tau\in\mathcal{T}}$ for short. The reduction consists of an $mk \times n\ell$ grid, whose projections for each $\tau \in \mathcal{T}$ are defined as

$$
\begin{aligned}
r^\tau_{xk+i} &= r^R_x \cdot r^{R,\tau}_i + r^G_x \cdot r^{G,\tau}_i + r^Y_x \cdot r^{Y,\tau}_i, \\
s^\tau_{y\ell+j} &= s^R_y \cdot s^{R,\tau}_j + s^G_y \cdot s^{G,\tau}_j + s^Y_y \cdot s^{Y,\tau}_j
\end{aligned}
\tag{4.1}
$$

for each $i \in [k]$, $j \in [\ell]$, $x \in [m]$, and $y \in [n]$. We think of the $mk \times n\ell$ grid as being partitioned into $m \cdot n$ blocks of dimension $k \times \ell$. Thus we refer to the set of rows $xk, \ldots, xk + k - 1$ in the $mk \times n\ell$ grid as the *block row* x and to the set of columns $y\ell, \ldots, y\ell + \ell - 1$ as the *block column* y. Similarly, we refer to the block indexed by block row x and block column y as the (x, y) block. In the reduction each block represents one cell of the $m \times n$ grid, which is given by its corresponding block row and column indices. In order to prove the equivalence of both instances we need two requirements to be satisfied.

The first requirement is that $r^{R,\mathcal{T}}, r^{G,\mathcal{T}}$, and $r^{Y,\mathcal{T}}$ are affinely independent (this property is explained below). The same requirement must be satisfied by the column projections $s^{R,\mathcal{T}}, s^{G,\mathcal{T}}$, and $s^{Y,\mathcal{T}}$.

The second requirement is that in every solution T to the tiling instance defined by (4.1), all blocks of T are either B^R, B^G, B^Y or blocks that have equivalent projections.

We recall that matrices $A^1, \ldots, A^N \in \mathbb{R}^{p\times q}$ are *affinely independent* if the system

$$\sum_{i=1}^N \alpha_i A^i = 0, \quad \sum_{i=1}^N \alpha_i = 0$$

has a unique solution $\alpha_1 = \cdots = \alpha_N = 0$. It is not difficult to see that for affinely independent matrices $A^1, \ldots, A^N$, a matrix A, and an integer b, the system

$$\sum_{i=1}^N \alpha_i A^i = A, \quad \sum_{i=1}^N \alpha_i = b$$

can have at most one solution. Sufficient conditions for affine independence include, for example, (i) linear independence, and (ii) affine independence of the projection onto a lower dimensional space. For example, the row projection matrices $r^{R,\mathcal{T}}, r^{G,\mathcal{T}}, r^{Y,\mathcal{T}} \in \mathbb{N}^{k\times t}$ for $t = |\mathcal{T}|$ are affinely independent if the vectors $r^{R,\tau}, r^{G,\tau}, r^{Y,\tau} \in \mathbb{N}^k$ are affinely independent for some tile $\tau \in T$.

LEMMA 4.2. *Let $(r^{c,\mathcal{T}}, s^{c,\mathcal{T}})$ be the projections of block B^c for each $c \in \{R, G, Y\}$, and assume that they satisfy the two requirements. Then the instance $(r^\tau, s^\tau)_{\tau\in\mathcal{T}}$ of*

$\mathcal{T}$-TTP defined by (4.1) *has a solution if and only if the instance* (r^R, r^G, s^R, s^G) *of the* 3*-CTP has a solution.*

Proof. Let $M \in \{R, G, Y\}^{m \times n}$ be a solution for the 3-CTP. We transform it into a $\mathcal{T}$-tiling T of the $mk \times n\ell$ grid by replacing each cell (x, y) of M by the $k \times \ell$ block B^c for $c = M_{xy}$. We can check that by the definition of r^τ and s^τ in (4.1), this tiling is a solution to the problem.

For the converse, suppose that there is a solution T to the tiling problem. By the second requirement, every block (x, y) of T is B^R, B^G or B^Y or has projections that are equivalent to one of them. Therefore, we construct a $m \times n$ matrix M with entries in $\{R, G, Y\}$ by taking $M_{xy} = c$ according to whether the block (x, y) of T is B^c or something that has equivalent projections.

Fix some arbitrary $x \in [m]$. By the first requirement, the xth block row projection matrix $((r^\tau_{xk+i})_{i\in[k]})_{\tau\in\mathcal{T}}$ has a unique decomposition into $\alpha_1 r^{R,\mathcal{T}} + \alpha_2 r^{G,\mathcal{T}} + \alpha_3 r^{Y,\mathcal{T}}$ for values satisfying $\alpha_1 + \alpha_2 + \alpha_3 = n$. By the definitions of the projections, we have $\alpha_1 = r^R_x, \alpha_2 = r^G_x$ and $\alpha_3 = r^Y_x$. Then, in the block row x, the total number of blocks B^c (or equivalent projections) is r^c_x. By the definitions of the projections, we have that in xth row of M the total number of cells with entries equal to c is r^c_x. We proceed in the same manner for the columns, which shows that M is a solution to the 3-CTP instance. $\square$

4.2.2. Two rectangles where at least one is not a bar. Let $\mathcal{T} = \{\tau_1, \tau_2\}$ be such that τ_1 and τ_2 are rectangles of sizes $p_1 \times q_1$ and $p_2 \times q_2$, respectively. Observe that if p_1, p_2 are both multiples of some integer $a > 1$, then we can partition the rows into successive sets of a rows which, in any tiling, must be covered by the same tiles; then contracting these sets into single rows, and considering rectangles of sizes $(p_1/a) \times q_1$ and $(p_2/a) \times q_2$ instead, does not change the problem. Clearly the analogous holds for the columns if q_1, q_2 are both multiples of an integer greater than 1. Thus, without loss of generality, we assume that $\gcd(p_1, p_2) = \gcd(q_1, q_2) = 1$.

THEOREM 4.3. *The tiling tomography problem is NP-hard for two rectangles if at least one of them is not a bar.*

Proof. If we denote by $p_1 \times q_1$ and $p_2 \times q_2$ the size of the rectangles, then $\min\{p_i, q_i\} \geqslant 2$ for at least one $i \in \{1, 2\}$. Without loss of generality, let us assume that $p_1, q_1 \geqslant 2$. We will apply Lemma 4.2 for $k = 2p_1p_2$ and $\ell = 2q_1q_2$. The three tilings of the $k \times \ell$ block are depicted in Figure 4.3 and defined formally as follows. The set of rows $I = [k]$ and the set of columns $J = [\ell]$ are partitioned into sets I_1, I_2, I_3, I_4 and J_1, J_2, J_3, J_4 defined as

$$\begin{aligned} I_1 &= \{0, \dots, p_2 - 1\}, & J_1 &= \{0, \dots, q_2 - 1\}, \\ I_2 &= \{p_2, \dots, p_1p_2 - 1\}, & J_2 &= \{q_2, \dots, q_1q_2 - 1\}, \\ I_3 &= \{p_1p_2, \dots, p_1p_2 + p_2 - 1\}, & J_3 &= \{q_1q_2, \dots, q_1q_2 + q_2 - 1\}, \\ I_4 &= \{p_1p_2 + p_2, \dots, 2p_1p_2 - 1\}, & J_4 &= \{q_1q_2 + q_2, \dots, 2q_1q_2 - 1\}. \end{aligned}$$

Then B^Y is defined as a tiling using only tiles of type τ_1, B^R is defined as the block tiling that covers $(I_1 \cup I_4) \times (J_3 \cup J_4)$ with τ_2-tiles and the rest with τ_1-tiles, while B^G is defined as the block tiling that covers $(I_3 \cup I_4) \times (J_1 \cup J_4)$ with τ_2-tiles and the rest with τ_1-tiles. These tilings are uniquely defined.

Let us show that the row τ_1-projections of the three tilings are affinely linear independent. For that, let us consider $\alpha_1 r^{R,\tau_1} + \alpha_2 r^{G,\tau_1} + \alpha_3 r^{Y,\tau_1} = 0$ such that $\alpha_1 +$

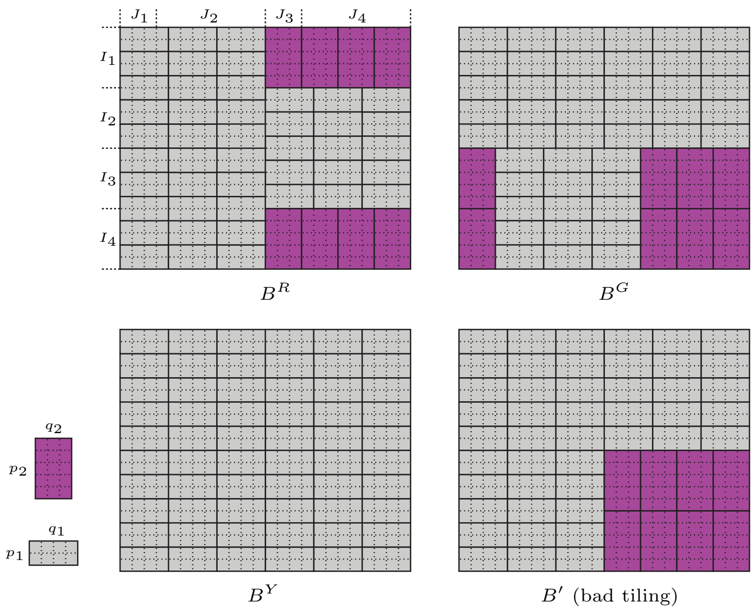


FIG. 4.3. *The three valid and the bad block tilings in the proof of Theorem* 4.3.

$\alpha_2+\alpha_3=0$. Observe that for row $i=p_2\in I_2$, we have $\alpha_1 r_i^{R,\tau_1}+\alpha_2 r_i^{G,\tau_1}+\alpha_3 r_i^{Y,\tau_1}=\alpha_1\cdot q_2+\alpha_2\cdot 0+\alpha_3\cdot 0$ and hence $\alpha_1=0$. Applying the same for $i=p_2p_1\in I_3$, we deduce that $\alpha_1+\alpha_2+2\alpha_3=0$. Clearly this system of linear equations has a unique solution $\alpha_1=\alpha_2=\alpha_3=0$. By the symmetry of τ_1-tiles with respect to rows and columns of $\{B^R,B^G,B^Y\}$, we can analogously deduce that the column τ_1-projections are affinely independent. Therefore the first requirement of the construction is satisfied.

The second requirement follows from a sequence of observations. Let T be the solution to the tiling instance, obtained by the reduction. First note that in the tiling B^R,B^G,B^Y, every tile is completely contained in the $k\times\ell$ block. Therefore the tiling instance has zero τ_1-projections at rows $xk+i$ such that $k-p_1<i<k$. Analogously, the τ_1-projections at columns $y\ell+j$ are equal to zero if $\ell-q_1<j<\ell$. As a result in T every τ_1-tile is completely contained in some $k\times\ell$ block. Of course, a similar observation holds for each τ_2-tile in T. In other words, every block of T is $\{\tau_1,\tau_2\}$-tiled.

Let B be a $k\times\ell$ block in T. Every row in I_2 is completely covered by τ_1-tiles in each block B^R,B^G, or B^Y. Therefore by the row projections, this also holds for B. The analogous holds for columns and hence each column $j\in J_2$ of B is fully covered by τ_1-tiles. Moreover, for rows others than those in I_2 we obtain a similar property. Notice that in any solution of $ap_1=p_2(2p_1-b)$ for integers a and b, since $\gcd(p_1,p_2)=1$, we have that a must be a multiple of p_2. Then $ap_1+bp_2=2p_1p_2$ with $a,b\geqslant 0$ implies that $(a,b)\in\{(0,2p_1),(p_2,p_1),(2p_2,0)\}$. Of course the same observation holds if we replace p_1,p_2 by q_1,q_2. This and the fact that B is a partition into τ_1- and τ_2-tiles imply the following property.

PROPERTY 1. *Every row or column of B is either covered completely by τ_1-tiles or covered half by τ_1-tiles and half by τ_2-tiles.*

The trickiest observation of the proof is that in B, the region $I_1\times J_1$ is fully covered by τ_1-tiles. For the sake of contradiction, let us suppose that it is partially covered by

τ_2-tiles and, in fact, by a single tile of type τ_2 since $|I_1 \times J_1| = |p_2 \times q_2| = |\tau_2|$ and the rows in I_2 and the columns in J_2 are covered only with τ_1-tiles. As $\gcd(q_1, q_2) = 1$, the cell (p_2, q_2) is covered by a tile τ_1 at position (p_2, j) for some column $j < q_2$. By the same argument, the cell (p_2, q_2) is *also* covered by a tile of type τ_1 at position (q_2, i) for some row $i < p_2$. Therefore these two tiles overlap in (p_2, q_2), which contradicts that T is a (valid) $\mathcal{T}$-tiling. We remark that this result and a previous observation show that the whole sub-block $(I_1 \cup I_2) \times (J_1 \cup J_2)$ of B is fully covered with τ_1-tiles.

Let us first assume that row 0 of B is partly covered by τ_2-tiles. We will show that B is indeed equal to B^R. By the previous property, the τ_2-tiles must cover half the columns of J in row 1. Since these columns cannot be those in $J_1 \cup J_2$ and we also have $|I_1| = p_2$, it is straightforward that $I_1 \times (J_3 \cup J_4)$ is fully covered by τ_2-tiles. Again, Property 1 states that for every column each cell in a row in I_2 is covered by a τ_1-tile. In particular, this shows that for $j \in J_3 \cup J_4$, column j must be half covered by τ_1-tiles. Let us show that these p_1p_2 cells (i, j) are exactly those such that $i \in I_2 \cup I_3$. Clearly this holds if $i \in I_2$. On the other hand, observe that $|I_2| = p_2(p_1 - 1)$ is not zero (here we use $p_1 \geqslant 2$) and it is not multiple of p_1. Therefore the cell (p_1p_2, j) is covered by a τ_1-tile. But by the definition of the blocks, $r_i^{c,\tau_2} \neq 0$ for a row $i \in I_3$ if and only if $c = G$ and $i = p_1p_2$.

We conclude that $(I_2 \cup I_3) \times (J_3 \cup J_4)$ is fully covered by τ_2-tiles and $I_4 \times (J_3 \cup J_4)$ is fully covered by τ_1-tiles. It remains to show that $(I_3 \cup I_4) \times J_1$ contains only tiles of type τ_1. This is straightforward by Property 1 and the fact that each column in J_2 is fully covered by τ_1-tiles.

In a similar fashion, we conclude that if column 0 is covered partly by τ_2-tiles, then B is exactly B^G. Otherwise, that is, if row 0 and column 0 are covered only with τ_1-tiles, we have that $(I_1 \cup I_2) \times J$ and $I \times (J_1 \cup J_2)$ are covered completely by τ_1-tiles. As a result in B $(I_3 \cup I_4) \times (J_3 \cup J_4)$ is either fully covered with τ_1-tiles or fully covered with τ_2-tiles by Property 1. These two possibilities correspond to the tiling B^Y and another tiling B', which we call *bad*, that is also depicted in Figure 4.3.

We now show that no bad tiling appears in T. Let N_R be the number of blocks in T that are tiled as B^R. Similarly, let N' be number of bad blocks in T. Note that the row projection of a bad tiling is equal to the row projection of B^G and that its column projection corresponds with that in B^R. Therefore by the projections we have the equalities

$$N_R = \sum_{x \in [m]} r_x^R, \qquad\qquad N_R + N' = \sum_{y \in [n]} s_y^R.$$

Since by assumption $\sum_{x \in [m]} r_x^R = \sum_{y \in [n]} s_y^R$, we conclude that $N' = 0$. Hence the second requirement of our construction is satisfied, and by Lemma 4.2 this ends the proof of Theorem 4.3. ◻

4.2.3. Three rectangles. Clearly $\mathcal{T}$-TTP seems to be more general than 3-CTP when $|\mathcal{T}| \geqslant 3$, but for a fixed tile set $\mathcal{T}$ the generalization is not straightforward. In this section we show that 3-CTP reduces to any $\mathcal{T}$-TTP where $\mathcal{T}$ consists of three rectangles.

THEOREM 4.4. *The tiling tomography problem is NP-hard for any three rectangular tiles.*

Proof. Let $p_1 \times q_1$, $p_2 \times q_2$, and $p_3 \times q_3$ be the respective sizes of the three arbitrary rectangles τ_1, τ_2, and τ_3.

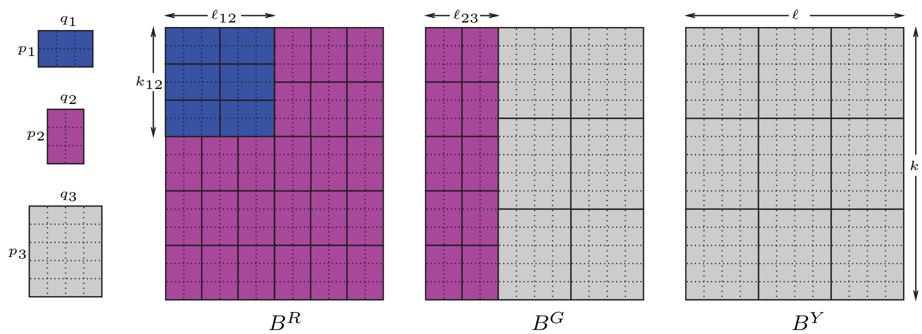


FIG. 4.4. *The three block tilings used in the proof of Theorem* 4.4.

We apply the general proof scheme from section 4.2.1 with the three tilings B^R, B^G, and B^Y that we define shortly. First, let us consider the following numbers:

$$\begin{aligned} k_{12} &= \operatorname{lcm}(p_1, p_2), & \ell_{12} &= \operatorname{lcm}(q_1, q_2), \\ k_{13} &= \operatorname{lcm}(p_1, p_3), & \ell_{13} &= \operatorname{lcm}(q_1, q_3), \\ k_{23} &= \operatorname{lcm}(p_2, p_3), & \ell_{23} &= \operatorname{lcm}(q_2, q_3). \end{aligned}$$

Without loss of generality, we assume that

$$\begin{aligned} &k_{12} \leqslant k_{13} \leqslant k_{23} \text{ and} \\ &k_{12} = k_{13} \Rightarrow \ell_{12} \leqslant \ell_{13}; \end{aligned} \tag{4.2}$$

otherwise we could rename the tiles. We apply Lemma 4.2 for $k = k_{23}$ and ℓ being the smallest integer greater than or equal to ℓ_{12} that is both a multiple of q_2 and a multiple of q_3. The reader can easily check that $k - k_{12}$ is a multiple of p_2, $\ell - \ell_{12}$ is a multiple of q_2, and $\ell - \ell_{23}$ is a multiple of q_3 (being zero if $\ell_{23} \geqslant \ell_{12}$).

The three tilings of the $k \times \ell$ grid are depicted in Figure 4.4, and they are formally defined as follows. The block B^Y is completely tiled with copies of τ_3. In B^R, the $[k_{12}] \times [\ell_{12}]$ sub-block is completely tiled with τ_1-tiles and the remainder with τ_2-tiles. Finally, B^G is the block having the $[k] \times [\ell_{23}]$ sub-block completely tiled with copies of τ_2 and the remainder with τ_3-tiles.

Let us first show that the projections of these tiles satisfy the first requirement for the general proof structure. For that, let $\alpha_1, \alpha_2, \alpha_3$ be such that $\alpha_1 r^{R,\mathcal{T}} + \alpha_2 r^{G,\mathcal{T}} + \alpha_3 r^{Y,\mathcal{T}} = 0$ and $\alpha_1 + \alpha_2 + \alpha_3 = 0$. In particular, for the first row we obtain $\alpha_1 r_0^{R,\mathcal{T}} + \alpha_2 r_0^{G,\mathcal{T}} + \alpha_3 r_0^{Y,\mathcal{T}} = 0$. Notice that r_0^{c,τ_1} is nonzero only for $c = R$, since B^R is the only block that contains τ_1-tiles. This shows that $\alpha_1 = 0$. Similarly, there is no τ_2-tile in B^Y but at least one in B^G, and hence $r_0^{Y,\tau_2} = 0$ and $r_0^{G,\tau_2} \neq 0$. This and the fact that $\alpha_2 + \alpha_3 = 0$ are sufficient to conclude that $\alpha_2 = \alpha_3 = 0$, and hence $r^{R,\mathcal{T}}, r^{G,\mathcal{T}}, r^{Y,\mathcal{T}}$ are affinely independent. In order to prove that $s^{R,\mathcal{T}}, s^{G,\mathcal{T}}, s^{Y,\mathcal{T}}$ are affinely independent we can use the same argument for the first column.

Before proving that the second requirement is satisfied, there are some observations to make. Observe that each B^Y, B^R, or B^G, satisfies the following property:

$$\tau_a\text{-tile is at position } (i,j) \Rightarrow i \bmod p_a = 0 \text{ and } j \bmod q_a = 0. \tag{4.3}$$

This means that the τ_a-projections r_i^{c,τ_a} and s_j^{c,τ_a} are zero if $i \bmod p_a \neq 0$ and $j \bmod q_a \neq 0$, respectively, for all $c \in \{R, G, Y\}$. Hence in any solution T to the tiling instance resulting from the reduction, property (4.3) must hold for all blocks as well. This observation is crucial for the proof.

In particular it implies the following fact. Fix some solution T to the tiling instance resulting from the reduction. Consider a block B in T. We say that cell (i,j) is a τ_a-τ_b *row separation* if $i > 0$ and either $(i-1,j)$ or (i,j) is covered by a copy of τ_a and the other by a copy of τ_b. Then by (4.3) we obtain

$$(i,j) \text{ is a } \tau_a\text{-}\tau_b \text{ row separation} \Rightarrow i \bmod p_a = 0 \text{ and } i \bmod p_b = 0. \tag{4.4}$$

Obviously, the same observation holds for *column separations*, which are defined similarly. We use this observation to show that the construction also satisfies the second requirement.

Fix some solution T to the tiling instance resulting from the reduction. We distinguish three types of blocks: (B1) blocks that contain a copy of τ_1, (B2) blocks that do not contain any copy of τ_1 but contain at least one copy of τ_2, and (B3) blocks that are completely tiled with copies of τ_3. Thus blocks of type (B3) are exactly B^Y. It remains to show that blocks of type (B1) are exactly B^R and blocks of type (B2) are B^G.

Consider a block B of type (B1). Then by the projections all copies of τ_1 must be contained in the region $[k_{12}] \times [\ell_{12}]$, and by observation (4.4) and assumption (4.2), this region cannot contain any copy of τ_2.

For the sake of contradiction, let us suppose that this region contains some copy of τ_3. Then somewhere in this region, a copy of τ_1 must be side by side with a copy of τ_3. There cannot be a τ_1-τ_3 row separation at (i,j) for some $i < k_{12}$. Otherwise i would be both a multiple of p_1 and p_3, implying $k_{13} < k_{12}$ and hence contradicting assumption (4.2).

Then there must be a column separation at some cell (i,j) for $j < \ell_{12}$. This implies $\ell_{13} < \ell_{12}$ and by (4.2), $k_{12} < k_{13}$. Assume that $(i,j-1)$ and (i,j) are covered by a copy of τ_1 and τ_3, respectively. The other case is symmetric. Then since there is no τ_1-τ_3 row separation, $(k_{12}-1, j-1)$ is covered by a copy of τ_1 and $(k_{12}-1, j)$ by a copy of τ_3. Since $k_{12} < k_{13}$, the cell $(k_{12}, j-1)$ cannot be covered by a copy of τ_3. The only possibility that remains is that it is covered by a copy of τ_2.

Let us see what happens with cell (k_{12}, j). If it is covered by a copy of τ_3, then we have a τ_2-τ_3 column separation at (k_{12}, j), implying in particular that j is a multiple of q_2. Since j is already a multiple of q_1, we obtain a contradiction to the definition of ℓ_{12}. On the other hand, if it is covered by a copy of τ_2, then k_{12} is a multiple of p_1, p_2, and p_3, contradicting $k_{12} < k_{13}$. This implies that the region $[k_{12}] \times [\ell_{12}]$ is completely tiled with copies of τ_1.

We have to show that the rest of the block is completely tiled with τ_2-tiles. By definition of k_{23}, there cannot be a τ_2-τ_3 row separation. If there is a τ_2-τ_3 column separation at some cell (i,j), then by definition of ℓ it must be for a column $j < \ell_{12}$. But then we can repeat the argument above, which leads to a contradiction. Therefore we conclude that every type (B1) block is of the form B^R.

We now use a counting argument to show that all the blocks of type (B2) must be B^G. First, observe that the total number of (B1) blocks equals the projection r^R in the original instance of the 3-color tomography problem. Then we conclude that in every (B2) block there is no copy of τ_2 after column ℓ_{23}. Since there is no τ_2-τ_3 separation either before column ℓ_{23} or before row k_{23}, every (B2) type block is exactly of the form B^G. Then the construction satisfies the second requirement for Lemma 4.2, and we are done. ∎

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