Characterizing Q-Linear Transformations for Semidefinite Linear Complementarity Problems

Julio López [*] Rubén López[†] Héctor Ramírez[‡]

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Abstract

In this note we introduce a new class, called $\mathbf{F}$, of linear transformations defined from the space of real $n \times n$ symmetric matrices into itself. Within this new class, we show the equivalence between $\mathbf{Q}$- and $\mathbf{Q_b}$-transformations. We also provide conditions under which a linear transformation belongs to $\mathbf{F}$. Moreover, this class, when specialized to square matrices of size n, turns to be the largest class of matrices for which such equivalence holds true in the context of standard linear complementary problems.

Key words: Semidefinite complementarity problems, $\mathbf{Q}$-transformation, $\mathbf{Q_b}$-transformation.

1 Introduction

This paper is devoted to the study of the existence of solutions of linear complementarity problems over the cone S^n_+ of real $n \times n$ symmetric positive semidefinite matrices. The latter is usually called *semidefinite linear complementarity problem* (SDLCP). Recall that, given a linear transformation L, defined from the space of real $n \times n$ symmetric matrices S^n into itself (for short $L \in \mathcal{L}(S^n)$), and a matrix $Q \in S^n$, the SDLCP consists in finding a matrix $\bar{X}$ such that:

$$\bar{X} \in S^n_+, \quad \bar{Y} = L(\bar{X}) + Q \in S^n_+ \text{ and } \langle \bar{Y}, \bar{X} \rangle = 0. \tag{1.1}$$

where $\langle X, Y \rangle := \operatorname{tr}(XY) = \sum_{i,j=1}^n X_{ij} Y_{ij}$ denotes the trace of the (matrix) product XY. In the sequel, this problem will be denoted by $\mathrm{SDLCP}(L, S^n_+, Q)$, and its solution will be denoted by $\mathcal{S}(L, S^n_+, Q)$. Also, its feasible set is defined to be $\mathrm{Feas}(L, S^n_+, Q) := \{X \in S^n_+ : L(X) + Q \in S^n_+\}$.

The SDLCP was first introduced by Kojima et al. [17] and its applications include primal-dual semidefinite linear programs, control theory, linear and bilinear matrix inequalities, among others. This problem can be seen as a generalization of the (standard) *linear complementarity problem* LCP [4]. However, since the cone S^n_+ is nonpolyhedral, LCP theory cannot be trivially generalized to the SDLCP context. It is also a particular case of a

[*]Departamento de Ingeniería Matemática, FCFM, Universidad de Chile, Blanco Encalada 2120, Santiago, Chile (jclopez@dim.uchile.cl).

[†]Departamento de Matemática y Física Aplicadas, Universidad Católica de la Santísima Concepción, Alonso Ribera 2850, Concepción, Chile (rlopez@ucsc.cl).

[‡]Departamento de Ingeniería Matemática, Centro de Modelamiento Matemático (CNRS UMI 2807), FCFM, Universidad de Chile, Blanco Encalada 2120, Santiago, Chile (hramirez@dim.uchile.cl).

cone complementarity problem, which turns to be a particular case of a variational inequality problem [16]. Nevertheless, the direct application of existing results does not take advantage of its rich matrix structure. For more details, see Gowda and Soong [10], Sampagni [20] and the references therein.

On the other hand, when solving the LCP(M, q) (for some given $M \in \mathbb{R}^{n\times n}$ and $q \in \mathbb{R}^n$):

$$\text{find} \quad \bar{x} \in \mathbb{R}^n_+ \quad \text{such that} \quad \bar{y} = M\bar{x} + q \in \mathbb{R}^n_+ \quad \text{and} \quad \langle \bar{y}, \bar{x} \rangle = 0$$

by means of splitting methods, or when taking it as a basis for more sophisticated algorithms (see, for instance, [4, Chapter 5]), a specific class of matrices naturally emerges. This class, denoted by Q, contains all matrices $M \in \mathbb{R}^{n\times n}$ such that LCP(M, q) has solutions independently of q. Indeed, its study allows to verify when the mentioned algorithms are well-defined. This class also plays a relevant role in perturbation theory (see, for instance, [4, 8] and the references therein). These motivations explain the important effort made in order to characterize the class Q. In particular, it is usual to analyze when the class Q coincides with the smaller class $\mathrm{Q_b}$, where the latter consists of all matrices $M \in \mathbb{R}^{n\times n}$ such that the solution set of LCP(M, q) is not empty and bounded for all q. In this context, Flores and López [8] introduce a new class of matrices (called F_1 therein) and prove that $\mathrm{Q} = \mathrm{Q_b}$ holds true within that class. This result generalizes previous ones of the same kind (e.g. [19, Theorem 1.2]).

Here, our aim is to extend the class F_1, and some of the results of [8], to the SDLCP framework. Actually, we define a large class of linear transformation, called $\mathbf{F}$, for which it holds that $\mathbf{Q} = \mathbf{Q_b}$ (in the sense of linear transformations in $\mathcal{L}(S^n)$)[1] [2]. Then, we study its relations with LCP's, which motivates the definition of two subclasses; $\mathbf{F_1}$ and $\mathbf{F_2}$. This permits to show that class $\mathbf{F}$, when specialized to matrices, is actually larger than F_1. Hence, as a by-product of our analysis, characterization $\mathrm{Q} = \mathrm{Q_b}$ is now proved in a larger class than F_1, which constitutes a novelty and an improvement of former results in LCP theory. Then, we provide conditions under which a linear transformation belongs to these subclasses. Finally, we illustrate these conditions with some well-known linear transformations such as Lyapunov functions $L_A(X) := A^\top X + XA^\top$, among others.

This paper is organized as follows. Section 2 is dedicated to the preliminaries. It is split into two subsections; first one recalls some basic results on matrix analysis, while second one summarizes the most important classes of linear transformations in $\mathcal{L}(S^n)$ with their respective connections. In Section 3, we established our main results described in the paragraph above. For this, in a first subsection, we recall known characterizations of classes $\mathbf{Q}$ and $\mathbf{Q_b}$ obtained via a recession analysis.

2 Preliminaries

In this section we establish our preliminaries results. They are presented in two subsections; first one contains notations and some well-known matrix results needed in the sequel, while the second one recalls existing classes of linear transformations that are frequently used in the SDLCP theory.

[1] Recall that classes Q and $\mathrm{Q_b}$ are word-by-word extended to the SDLCP framework as follows: *A linear transformation $L \in \mathcal{L}(S^n)$ is said to be a* $\mathbf{Q}$-transformation *(*$\mathbf{Q}_b$-transformation*) if $\mathcal{S}(L, S^n_+, Q) \neq \emptyset$ (and bounded), for all $Q \in S^n$. For the sake of notation, we simply say that $L \in \mathbf{Q}$ ($L \in \mathbf{Q}_b$).*

[2] In order to avoid misunderstandings, bold letters (such as $\mathbf{Q}$) will denote classes of linear transformation in $\mathcal{L}(S^n)$, whereas roman-type letters (such as Q) will denote classes of matrices in $\mathbb{R}^{n\times n}$.

2.1 Notation and basic results on matrix analysis

Some matrix operations are extensively used throughout this paper. For instance, we mention the trace and the diagonal of a square matrix $X = (X_{ij}) \in \mathbb{R}^{n\times n}$, defined by $\mathrm{tr}(X) := \sum_{i=1}^n X_{ii}$ and $\mathrm{diag}(X) := (X_{11}, X_{22}, ..., X_{nn})^\top$, respectively. The notion of a *submatrix* is also very useful in the sequel. For an $n\times n$ matrix $X = (X_{ij})$ and index sets α, $\beta \subseteq \{1, \ldots, n\}$, we write $X_{\alpha\beta}$ to denote the submatrix of X whose entries are X_{ij} with $i \in \alpha$ and $j \in \beta$. When $\alpha = \beta$, $X_{\alpha\alpha}$ is usually called the *principal submatrix of X corresponding to α*. In particular, when $\alpha = \{1, \ldots, k\}$ $(1 \le k \le n)$, $X_{\alpha\alpha}$ is called the *leading principal submatrix of X*.

Additionally, the Hadamard product has an important role in our approach. We recall that this operation is defined by $X \circ Y := (X_{ij}Y_{ij}) \in \mathbb{R}^{m\times n}$ for all $X = (X_{ij})$, $Y = (Y_{ij}) \in \mathbb{R}^{m\times n}$.

It is well-known that the set S^n of real $n \times n$ symmetric matrices is a finite dimensional real Hilbert space when it is equipped with the inner product $\langle X, Y\rangle = \mathrm{tr}(XY)$. As usual, this product defines a (Frobenius) norm $\|X\|_F := \sqrt{\langle X, X\rangle} = \sqrt{\sum_{i=1}^n \lambda_i(X)^2}$, where $\lambda_i(X)$ stands for the i-th eigenvalue (arranged in nonincreasing order) of X. Thus, $\|X\|_F = \|\lambda(X)\|$ for all $X \in S^n$, where $\|\cdot\|$ denotes the Euclidian norm in $\mathbb{R}^n$ and we have set $\lambda(X) := (\lambda_1(X), \ldots, \lambda_n(X))^\top$. Also, 0_n and I_n denote the zero and the identity matrices, respectively, of size n, but index n will be omitted if the size is clear from the context. Similarly, $\mathbb{1}_n$ (or $\mathbb{1}$) denotes the $n \times n$ matrix whose entries are all equal to 1.

Finally, for a vector $q \in \mathbb{R}^n$, we define $\mathrm{Diag}(q)$ as the diagonal matrix of size n whose diagonal entries are given by the entries of q.

We end this subsection by recalling matrix properties that we shall employ throughout this paper. Their proofs and more details can be found in [2, 14, 15].

Proposition 2.1. *The following results hold:*

(a) *For any $X \in \mathbb{R}^{n\times n}$ and any orthogonal matrix $U \in \mathbb{R}^{n\times n}$, it holds that* $\mathrm{tr}(X) = \mathrm{tr}(X^\top) = \mathrm{tr}(UXU^\top)$*. Moreover, when X is a symmetric matrix with the following block structure* $X = \begin{pmatrix} A & B \\ B^\top & C \end{pmatrix} \in S^n_+$*, then it holds that* $\mathrm{tr}(A)\mathrm{tr}(C) \ge \mathrm{tr}(BB^\top)$*;*

(b) (*Von Neumman-Theobald's inequality*) *For any $X, Y \in S^n_+$, it holds that* $\langle X, Y\rangle \le \mathrm{diag}(X)^\top\mathrm{diag}(Y)$*, with equality if and only if X and Y are simultaneously diagonalizable (that is, there exists an orthogonal matrix U such that* $X = U\,\mathrm{Diag}(\lambda(X))U^\top$ *and* $Y = U\,\mathrm{Diag}(\lambda(Y))U^\top$*);*

(c) (*Fejer's theorem*) *For any $X \in S^n$, it holds that $\langle X, Y\rangle \ge 0$ for all $Y \in S^n_+$ if and only if $X \in S^n_+$. Moreover, $\langle X, Y\rangle > 0$ for all $Y \in S^n_+ \setminus \{0\}$ if and only if $X \in S^n_{++}$, where S^n_{++} denotes the cone of real $n \times n$ symmetric positive definite matrices;*

(d) *Let $X, Y \in S^n_+$. If $\langle X, Y\rangle = 0$, then X and Y commute (that is $XY = YX$);*

(e) (*Simultaneous diagonalization*) *Let $X, Y \in S^n_+$. If X and Y commute, then X and Y are simultaneously diagonalizable;*

(f) *As a direct corollary of* **(d)** *and* **(e)***, if follows that for any $X, Y \in S^n_+$ such that $\langle X, Y\rangle = 0$, X and Y are simultaneously diagonalizable;*

(g) *If $X, Y \in S^n_+$, then $X \circ Y \in S^n_+$.*

2.2 Linear transformations review

The literature on SDLCP (see [3, 10, 11, 12, 18]) has already extended, from the LCP theory, most of the well-known classes of matrices used in that context. We list these classes here below. Let $L \in \mathcal{L}(S^n)$, we say that:

- L is a **R**-transformation with respect to D (where $D \in S^n_{++}$ is fixed), or simply $L \in \mathbf{R}(\mathbf{D})$, if $\mathcal{S}(L, S^n_+, \tau D) = \{0\}$ for all $\tau \geq 0$.
- L is regular or $L \in \mathbf{R}$ if there exists $D \in S^n_{++}$ such that L is a **R**-transformation with respect to D. Clearly $\mathbf{R} = \cup_{D \in S^n_{++}} \mathbf{R}(\mathbf{D})$.
- L is an $\mathbf{R_0}$-transformation if $\mathcal{S}(L, S^n_+, 0) = \{0\}$.
- L is copositive (resp. strictly copositive) if $\langle L(X), X\rangle \geq 0$ (resp. > 0) for all $X \in S^n_+$ (resp. for all $X \in S^n_+$, $X \neq 0$).
- L is monotone (resp. strongly or strictly monotone) if $\langle L(X), X\rangle \geq 0$ (resp. > 0) for all $X \in S^n$ (resp. for all $X \in S^n$, $X \neq 0$).
- L has the **P**-property if $[XL(X) = L(X)X \in -S^n_+ \Rightarrow X = 0]$.
- L has the $\mathbf{P_0}$-property if $L(\cdot) + \varepsilon I \in \mathbf{P}$, $\forall \varepsilon > 0$ where I denotes here the identity transformation[3] in $\mathcal{L}(S^n)$.
- L has the $\mathbf{P_2}$-property if $[X, Y \in S^n_+,\ (X-Y)L(X-Y)(X+Y) \in -S^n_+ \Rightarrow X = Y]$.
- L has the $\mathbf{P_2'}$-property if $[X \in S^n_+,\ XL(X)X \in -S^n_+ \Rightarrow X = 0]$.
- L is strictly semimonotone **SSM** or $L \in \mathbf{E}$ if
$$X \in S^n_+,\ XL(X) = L(X)X \in -S^n_+ \Rightarrow X = 0.$$
- L is semimonotone or $L \in \mathbf{E_0}$ if $L(\cdot) + \varepsilon I \in \mathbf{E}$ for all $\varepsilon > 0$.
- L is positive or $L \geq 0$ (resp. negative or $L \leq 0$) if $L(X) \in S^n_+$ (resp. $L(X) \in -S^n_+$) for all $X \in S^n_+$.
- L is nondegenerate if $[X\,L(X) = 0 \Rightarrow X = 0]$.
- L has the $\mathbf{Q_0}$-property if $[\mathrm{Feas}(L, S^n_+, Q) \neq \emptyset \Rightarrow \mathcal{S}(L, S^n_+, Q) \neq \emptyset]$.
- L has the globally uniquely solvable property or $L \in \mathbf{GUS}$ if $\mathcal{S}(L, S^n_+, Q)$ has a unique solution for all $Q \in S^n$.
- L has the **S**-property if there is $X \in S^n_+$ such that $L(X) \in S^n_{++}$, or equivalently, there is $X \in S^n_{++}$ such that $L(X) \in S^n_{++}$.
- L self-adjoint if $L^\top = L$, where $L^\top$ stands for the transpose (or adjoint) transformation of L.
- L is normal if L commutes with $L^\top$.

[3] Since the context is clear, symbol I is used as the identity matrix in S^n as well as the identity transformation in $\mathcal{L}(S^n)$ throughout this paper. Symbol 0 is similarly treated.

- L is a star-transformation if $[V \in \mathcal{S}(L, S^n_+, 0) \Rightarrow L^\top(V) \in -S^n_+]$.
- L has the **Z**-property if $[X, Y \in S^n_+,\ \langle X, Y\rangle = 0 \Rightarrow \langle L(X), Y\rangle \le 0]$.
- a Q-pseudomonotone (for a given $Q \in S^n$) if

$$X, Y \in S^n_+,\ \langle L(X) + Q, Y - X\rangle \ge 0 \Rightarrow \langle L(Y) + Q, Y - X\rangle \ge 0.$$

- L is Q-quasimonotone (for a given $Q \in S^n$) if

$$X, Y \in S^n_+,\ \langle L(X) + Q, Y - X\rangle > 0 \Rightarrow \langle L(Y) + Q, Y - X\rangle \ge 0.$$

The next proposition establishes the most important links between the classes mentioned above.

Proposition 2.2. *Let $L \in \mathcal{L}(S^n)$ and $Q \in S^n$ be given. The following relations hold:*

(a) *L is monotone $\Longrightarrow$ L is copositive;*

(b) *L is copositive and $L \in \mathbf{R_0} \Longrightarrow L \in \mathbf{R}(\mathbf{D})$ for all $D \in S^n_{++}$;*

(c) *L is nondegenerate $\Longrightarrow L \in \mathbf{R_0}$;*

(d) *L is strongly monotone or L is an isomorphism $\Longrightarrow L \in \mathbf{P} \Longrightarrow L \in \mathbf{E} \Longrightarrow L \in \mathbf{R_0}$;*

(e) *$L \in \mathbf{P_0}$ or L is copositive $\Longrightarrow L \in \mathbf{E_0}$;*

(f) *$L \in \mathbf{P_2} \Longrightarrow L \in \mathbf{P'_2} \Longrightarrow L \in \mathbf{E} \Longrightarrow L \in \mathbf{R}(\mathbf{I})$.*

(g) *L is strongly monotone $\Longrightarrow L \in \mathbf{P_2} \Longrightarrow L \in \mathbf{GUS} \Longrightarrow L \in \mathbf{P}$;*

(h) *L is monotone $\Longrightarrow$ L is Q-pseudomonotone $\Longrightarrow$ L is Q-quasimonotone for all $Q \in S^n$;*

(i) *L is Q-pseudomonotone and $\mathrm{Feas}(L, S^n_+, Q) \neq \emptyset \Longrightarrow$ L is copositive.*

(j) *L is Q-quasimonotone and $Q \neq 0 \Longrightarrow$ L is Q-pseudomonotone. Moreover, if L is Q-quasimonotone but is not monotone and there exists $X \in S^n_+$ such that $L(X) + Q \in S^n_+$ and $L(X) + Q \neq 0$, then L is copositive.*

(k) *$L \in \mathbf{S} \Longleftrightarrow \mathrm{Feas}(L, S^n_+, Q) \neq \emptyset$ for all $Q \in S^n$;*

(l) *L is 0-pseudomonotone (in particular, if L is monotone) or $L \le 0 \Longrightarrow$ L is a star-transformation;*

(m) *$\mathbf{Q} = \mathbf{Q_0} \cap \mathbf{S}$, and $\mathbf{GUS} \subseteq \mathbf{Q_b}$;*

(n) *$L \in \mathbf{E_0} \cap \mathbf{R_0}$ (in particular, if $L \in \mathbf{P}$ or $L \in \mathbf{E}$) $\Longrightarrow L \in \mathbf{Q_b}$;*

(o) *Let $L \in \mathbf{Z}$. Then*

$[L \in \mathbf{Q} \Longleftrightarrow \exists L^{-1}$ and $L^{-1}(S^n_+) \subseteq S^n_+$ (equivalently, $L^{-1}(S^n_{++}) \subseteq S^n_{++}$) $\Longleftrightarrow L \in \mathbf{S}$ $\Longleftrightarrow L^\top \in \mathbf{Q} \Longleftrightarrow \exists (L^\top)^{-1}$ and $(L^\top)^{-1}(S^n_+) \subseteq S^n_+$ (equivalently, $(L^\top)^{-1}(S^n_+) \subseteq S^n_+$) $\Longleftrightarrow L^\top \in \mathbf{S}]$;

Proof. Statements (a), (c) and (h) are direct from the definitions. Statement (d) is proven in [10]. Relation in (e) when $L \in \mathbf{P_0}$ is trivial. When L is copositive, the result can be found in [18, Theorem 5(i)]. The first implication of (g) is given in [20, Theorem 4], while the others appear in [10]. Statement (i) is shown in [7, Remark 4.1]. Statement (j) is demonstrated in [13, Propositions 4.1 and 5.4]. The first equality in (m) follows from (k) and the inclusion in (m) is trivial. Relation (n) can be seen [10, Theorem 4]. Finally, statement (o) has been proved in [12, Theorem 6]. The remaining relations need some adaptations of previous results. Thus, they are explained with more details here below.

(b): Let $D \in S^n_{++}$ be fixed. If $X \in \mathcal{S}(L, S^n_+, \tau D)$ for some $\tau > 0$, then $L(X)+\tau D \in S^n_+$ and $\langle L(X)+\tau D, X\rangle = 0$. By copositivity the latter can be written as $0 \leq \langle L(X), X\rangle = -\tau\langle D, X\rangle$. Due to Fejer's theorem (Proposition 2.1, Part (c)), this is a contradiction if $X \neq 0$. Therefore, $\mathcal{S}(L, S^n_+, \tau D) = \{0\}$. But since $L \in \mathbf{R_0}$, it follows that $L \in \mathbf{R(D)}$. A different proof of this relation is given in [18, Theorem 5(iii)] in a different framework.

(f): See [3, Theorem 2.2]. Here we have just added the implication $L \in \mathbf{E} \Rightarrow L \in \mathbf{R(I)}$ which is implicitly shown in the proof of the referenced theorem.

(k): ($\Leftarrow$) Let $D \in S^n_{++}$. By hypothesis $\text{Feas}(L, S^n_+, -D) \neq \emptyset$, that is, there exists $X \in S^n_+$ such that $Y = L(X) - D \in S^n_+$. From this we get $L(X) = Y + D \in S^n_{++}$. Hence $L \in \mathbf{S}$.
($\Rightarrow$) By hypothesis there is $X \in S^n_+$ such that $L(X) \in S^n_{++}$. Fix $Q \in S^n$. It is clear that for $t > 0$ large enough, the matrix $tL(X) + Q$ is symmetric positive definite. But, since $tL(X) = L(tX)$, the latter implies that $tX \in \text{Feas}(L, S^n_+, Q)$. The desired equivalence follows.

(l): Let $V \in \mathcal{S}(L, S^n_+, 0)$, that is, $V, L(V) \in S^n_+$ and $\langle L(V), V\rangle = 0$. It follows that $\langle L(V), tX - V\rangle \geq 0$ for every $X \in S^n_+$ and $t > 0$. Suppose that L is 0-pseudomonotone. We have that $\langle L(tX), tX - V\rangle \geq 0$ for every $X \in S^n_+$ and $t > 0$. From this, after dividing by t and taking limit $t \searrow 0$, we get $0 \geq \langle L(X), V\rangle$ for every $X \in S^n_+$. Thus, Fejer's theorem (Proposition 2.1, Part (c)) implies that $L^\top(V) \in -S^n_+$. Hence, L is a star-transformation. In the case when $L \leq 0$, the desired result is a direct consequence of Fejer's theorem. □

Proposition 2.2 shows the rich relations existing among the different classes defined in $\mathcal{L}(S^n)$. However, at this stage of the analysis, it is worth to point out that these relations are not necessarily the same we find for matrices in the LCP theory. We illustrate some differences here below through two enlightening examples. They can be found in [10] and [12], respectively.

Example 2.3. *It is known that a matrix M with the P-property ensures the existence and uniqueness of solutions of LCP(M,q), independently of vector q (see* [4, Theorem 3.3.7]*). However, this strong result is not longer true when we deal with linear transformation in $\mathcal{L}(S^n)$. Indeed, consider*

$$A = \begin{pmatrix} -1 & 2 \\ -2 & 2 \end{pmatrix} \qquad \textit{and} \qquad Q = \begin{pmatrix} 2 & 2 \\ 2 & 4 \end{pmatrix}.$$

*So, for the Lyapunov function $L_A = A^\top X + XA^\top$, we have that $L_A \in \mathbf{P}$. However, $\mathcal{S}(L_A, S^2_+, Q)$ is not longer a singleton because $D = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ as well as the null matrix are solutions. For this reason, **GUS**-property is studied separately from **Q** and $\mathbf{Q_b}$-transformations.*

Example 2.4. *When we work with matrices, we have $\mathrm{Z} \subseteq \mathrm{Q_0}$ (see* [4, Theorem 3.11.6]*). However, this inclusion is not longer true when we deal with linear transformation in $\mathcal{L}(S^n)$.*

Indeed, consider

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \textit{and} \qquad Q = \begin{pmatrix} 1 & 0.1 \\ 0.1 & 1 \end{pmatrix}.$$

Here we have that $L_A \in \mathbf{Z}$*,* $\mathrm{Feas}(L_A, S^2_+, Q) \neq \emptyset$*, and* $\mathcal{S}(L_A, S^2_+, Q) = \emptyset$*. Hence,* $L_A \notin \mathbf{Q_0}$*.*

3 Characterizations of Q and $\mathbf{Q_b}$-transformation

This section is devoted to the characterization of classes $\mathbf{Q}$ and $\mathbf{Q_b}$. In particular, we are interested in studying classes of functions in $\mathcal{L}(S^n)$ for which $\mathbf{Q_b}$ behaves similarly as $\mathbf{Q}$.

3.1 Known results based on recession analysis

In finite dimensional spaces, the notion of the asymptotic cone of a set becomes a fundamental tool in order to characterize its boundedness. For a nonempty set $A \subseteq \mathbb{R}^n$, this notion is defined as follows (e.g. [1]):

$$A^\infty := \left\{ v \in \mathbb{R}^n : \exists t_k \to +\infty \, \exists \{x^k\} \subseteq A \text{ such that } \frac{x^k}{t_k} \to v \right\}$$

(by convention, we set $\emptyset^\infty = \{0\}$). Indeed, it is well known that A is bounded if and only if $A^\infty = \{0\}$ (e.g. [1, Proposition 2.1.2]). So, the notion of asymptotic cones arises naturally when we deal with the class $\mathbf{Q_b}$. The following technical lemma illustrates this point.

Lemma 3.1. *Let* $L \in \mathcal{L}(S^n)$ *be given.*

(a) $\bigcup_{Q \in S^n} \mathcal{S}(L, S^n_+, Q)^\infty = \mathcal{S}(L, S^n_+, 0)$*;*

(b) *If* $L \in \mathbf{R_0}$*, then* $\mathcal{S}(L, S^n_+, Q)$ *is bounded (possibly empty) for all* $Q \in S^n$*;*

(c) $L \in \mathbf{R_0}$ *if and only if there exists a constant* $c > 0$ *such that*

$$\|X\|_F \leq c\|Q\|_F, \quad \textit{for all } Q \in S^n \textit{ and } X \in \mathcal{S}(L, S^n_+, Q).$$

Proof. See [6, Proposition 2.5.6]. □

As a direct consequence of Lemma 3.1 above, we establish the equivalence between the classes $\mathbf{Q}$ and $\mathbf{Q_b}$ within the class $\mathbf{R_0}$.

Corollary 3.2. *Let* $L \in \mathbf{R_0}$*. Then,* $L \in \mathbf{Q_b} \Longleftrightarrow L \in \mathbf{Q}$*.*

Our main goal is to prove the previous equivalence in a larger class of linear transformation in $\mathcal{L}(S^n)$. In order to do this, we need to recall a second lemma which will be useful in the sequel.

Lemma 3.3. $\mathbf{R} \subseteq \mathbf{Q_b} \subseteq \mathbf{R_0}$*. Consequently,* $\mathbf{Q_b} = \mathbf{Q} \cap \mathbf{R_0}$*.*

Proof. This is a particular case of [16, Theorem 3.1] where the desired inclusions is obtained for complementarity problems defined over general solid cones in finite dimensional spaces. The characterization of $\mathbf{Q_b}$ follows directly from previous results. □

3.2 The class of F-transformations and its subclasses

In the LCP context, Flores and López [8] introduce the following new class of matrices.

Definition 3.4. *A matrix $M \in \mathbb{R}^{n\times n}$ is said to be an* F_1*-matrix if, for every $v \in \mathcal{S}(M,0)\backslash\{0\}$, there exists a nonnegative diagonal matrix Γ such that $\Gamma v \neq 0$ and $M^\top \Gamma v \in -\mathbb{R}^n_+$. Here $\mathcal{S}(M,q)$ denotes, for given $M \in \mathbb{R}^{n\times n}$ and $q \in \mathbb{R}^n$, the solution of problem* $\mathrm{LCP}(M,q)$*.*

So, in [8], the equivalence of Corollary 3.2 is proven within the class F_1. This class turns to be larger than R_0, which makes the result interesting to be extended to our SDLCP framework. Inspired by that definition, we introduce the next new class of transformations in $\mathcal{L}(S^n)$.

Definition 3.5. *We say that $L \in \mathcal{L}(S^n)$ is an* **F***-transformation or $L \in \mathbf{F}$, if for each $V \in \mathcal{S}(L, S^n_+, 0)\backslash\{0\}$ there exists a linear transformation $\mathcal{T} : S^n \to \mathbb{R}^{n\times n}$ such that*

$$(\mathbf{i})\ \mathcal{T}(V) \in S^n_+ \qquad (\mathbf{ii})\ \langle \mathcal{T}(V), V\rangle > 0 \qquad (\mathbf{iii})\ L^\top(\mathcal{T}(V)) \in -S^n_+. \tag{3.1}$$

We now establish the main properties of the class $\mathbf{F}$. In particular, assertion (b) below extends Corollary 3.2 to the this larger class.

Theorem 3.6. *Let $L \in \mathcal{L}(S^n)$ be given.*

(a) *If $L \in \mathbf{F} \cap \mathbf{S}$, then $L \in \mathbf{R_0}$;*

(b) *Let $L \in \mathbf{F}$. Then, $L \in \mathbf{Q_b} \Longleftrightarrow L \in \mathbf{Q}$.*

Proof. (a): Let $L \in \mathbf{F} \cap \mathbf{S}$. We argue by contradiction. Suppose that $L \notin \mathbf{R_0}$, that is, there exist $V \in \mathcal{S}(L, S^n_+, 0)\backslash\{0\}$. Since $L \in \mathbf{F}$, there exists a linear transformation $\mathcal{T} : S^n \to \mathbb{R}^{n\times n}$ satisfying conditions ($\mathbf{i}$)-($\mathbf{iii}$) in Definition 3.5. This together with Fejer's theorem (Proposition 2.1, Part (c)) implies that $\langle L(X) - V, \mathcal{T}(V)\rangle < 0$ for all $X \in S^n_+$. Consequently, $L(X) - V \notin S^n_+$ for all $X \in S^n_+$. Therefore, $L \notin \mathbf{S}$ (otherwise we can always find $t > 0$ large enough such that $L(X) - V \in S^n_+$), obtaining a contradiction.

(b): Obviously $L \in \mathbf{Q_b}$ implies $L \in \mathbf{Q}$. If $L \in \mathbf{Q}$, then $L \in \mathbf{S}$ (because Proposition 2.2, Part (m)). Thus, $L \in \mathbf{F} \cap \mathbf{S}$. By item (a) above we conclude that $L \in \mathbf{R_0}$, and consequently $L \in \mathbf{Q} \cap \mathbf{R_0}$. We thus conclude that $L \in \mathbf{Q_b}$ thanks to equality $\mathbf{Q_b} = \mathbf{Q} \cap \mathbf{R_0}$ established in Proposition 3.3. □

Next example, adapted from [8], shows that the inclusion established in Part (a) of Theorem 3.6 above is strict.

Example 3.7. *Consider the matrix $M = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. For $\mathcal{M}$ given by (3.3), we have $\mathcal{S}(\mathcal{M}, S^2_+, 0) = \{0\}$. Thus, $\mathcal{M} \in \mathbf{R_0}$. However, it is easy to see that* $\mathrm{Feas}(\mathcal{M}, S^n_+, M) = \emptyset$*. This together with Proposition 2.2, Part (k), implies that $\mathcal{M} \notin \mathbf{S}$.*

To check whenever a linear transformation L belongs to $\mathbf{F}$ can be a difficult task. This is mainly because there is no a clear guide about how to chose, for a given $V \in \mathcal{S}(L, S^n_+, 0)\backslash\{0\}$, a linear transformation $\mathcal{T} : S^n \to \mathbb{R}^{n\times n}$ satisfying conditions ($\mathbf{i}$)–($\mathbf{iii}$) in Definition 3.5. For this, we focus now our analysis on the subclass of $\mathbf{F}$ for which the linear transformation $\mathcal{T}$ is chosen to be of the form:

$$\mathcal{T}(X) = \Lambda \circ X, \quad \text{for all } X \in S^n, \tag{3.2}$$

for some $\Lambda \in S^n_+$. Recall that symbol $\circ$ denotes the Hadamard product defined in Section 2.1. From now on, this subclass of $\mathbf{F}$ will be denoted by $\mathbf{F_1}$.

Remark 3.8. *Notice that, thanks to Part (g) of Proposition 2.1, condition* (**i**) *in Definition 3.5 becomes superfluous when* $L \in \mathbf{F_1}$.

The name $\mathbf{F_1}$ is justified by the close relation existing between this subclass and the original class F_1 defined in the LCP framework (see Definition 3.4 above). Indeed, it is easy to verify that

$$\begin{aligned} \bar{X} \in \mathcal{S}(\mathcal{M}, S^n_+, \mathrm{Diag}(q)) &\implies \bar{x} := \mathrm{diag}(\bar{X}) \in \mathcal{S}(M, q) \\ \bar{x} \in \mathcal{S}(M, q) &\implies \bar{X} := \mathrm{Diag}(\bar{x}) \in \mathcal{S}(\mathcal{M}, S^n_+, \mathrm{Diag}(q)), \end{aligned}$$

where the linear transformation $\mathcal{M} : S^n \to S^n$ is defined by

$$\mathcal{M}(X) := \mathrm{Diag}(M\mathrm{diag}(X)). \tag{3.3}$$

See, for instance, [21]. Thus, the mentioned relation is stated in the next proposition.

Proposition 3.9. *Let* $M \in \mathbb{R}^n$. *If* $\mathcal{M}$ *is given by* (3.3), *then*

$$M \in \mathrm{F}_1 \Longleftrightarrow \mathcal{M} \in \mathbf{F_1}.$$

Proof. We first point out that the transpose of $\mathcal{M}$ is given by $\mathcal{M}^\top(X) = \mathrm{Diag}(M^\top \mathrm{diag}(X))$.

($\Rightarrow$): Let $M \in \mathrm{F}_1$. If $V \in \mathcal{S}(\mathcal{M}, S^n_+, 0)\backslash\{0\}$, then $v = \mathrm{diag}(V) \in \mathcal{S}(M, 0)$. Clearly $v \neq 0$ (otherwise, since $V \in S^n_+$, V should be null). So, by hypothesis there exists a nonnegative diagonal matrix Γ such that $\Gamma v \neq 0$ and $M^\top \Gamma v \in -\mathbb{R}^n_+$. Thus, conditions (**i**)–(**iii**) of Definition 3.5 can be easily verified provided that $\Lambda = \Gamma \in S^n_+$. We then obtain that $\mathcal{M} \in \mathbf{F_1}$.

($\Leftarrow$): Let $\mathcal{M} \in \mathbf{F_1}$. If $v \in \mathcal{S}(M, 0) \setminus \{0\}$, then $V = \mathrm{Diag}(v) \in \mathcal{S}(\mathcal{M}, S^n_+, 0)$, and obviously $V \neq 0$. By hypothesis there exists a matrix $\Lambda \in S^n_+$ such that the linear transformation $\mathcal{T}$, given by (3.2), satisfies conditions (**i**)-(**iii**) in Definition 3.5. Take $\Gamma := \mathrm{Diag}(\mathrm{diag}(\Lambda))$. Clearly Γ is a diagonal matrix with nonnegative entries. Moreover, since $\Gamma v = \mathrm{diag}(\Lambda \circ V) \neq 0$ and $M^\top \Gamma v = \mathrm{diag}(\mathcal{M}^\top(\Lambda \circ V)) \in -\mathbb{R}^n_+$, it follows that $M \in \mathrm{F}_1$. $\square$

As a consequence of the analysis above we realize that we have also extended the class of matrices for which the equivalence between Q and $\mathrm{Q_b}$ (in the LCP framework) holds true. Indeed, former result in [8] only deals with the class F_1, which is smaller than the class $\mathbf{F}$ restricted to linear transformations $\mathcal{M}$ of the form (3.3). In other words, it is clear that $\mathcal{M} \in \mathbf{F}$ allows not only matrices $M \in \mathrm{F}_1$ (for instance, it allows to chose, in Definition 3.4, a matrix Γ which is not necessarily diagonal, provided that condition $\Gamma v \neq 0$ be replaced by the equivalent condition $v^\top \Gamma v > 0$; see example below). Thus, the equivalence established in Theorem 3.6 constitutes also an improvement of the existing LCP's theory.

Example 3.10. *In this example we show that* $\mathbf{F_1}$ *is properly contained in* $\mathbf{F}$. *Set*

$$M = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}.$$

it is not difficult to see that $M \notin \mathrm{F}_1$. *Consequently, by Proposition 3.9,* $\mathcal{M} \notin \mathbf{F_1}$. *However, the linear transformation*

$$\mathcal{T}(X) := \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}, \quad \textit{for all } X = \begin{pmatrix} x & y \\ y & z \end{pmatrix} \in S^2$$

satisfies, for every $V \in \mathcal{S}(\mathcal{M}, S^2_+, 0)\backslash\{0\}$, *conditions* (**i**)-(**iii**) *in Definition 3.4. Therefore,* $\mathcal{M} \in \mathbf{F}$.

Another different way to check whenever a linear transformation belongs to $\mathbf{F}$ is via the study of its block structure. The next proposition establishes a criterium based on this information.

Proposition 3.11. *Let $L \in \mathcal{L}(S^n)$. Suppose that for any orthogonal matrix $U \in \mathbb{R}^{n\times n}$ and for any index set $\alpha = \{1,\dots,k\}(1 \le k \le n)$, the existence of a solution $X \in S^n$ to the system*

$$X_{\alpha\alpha} \in S^{|\alpha|}_{++}, \quad X_{ij} = 0, \forall i,j \notin \alpha, \quad [\widehat{L}_U(X)]_{\alpha\alpha} = 0, \quad [\widehat{L}_U(X)]_{\alpha\bar{\alpha}} = 0, \quad [\widehat{L}_U(X)]_{\bar{\alpha}\bar{\alpha}} \in S^{|\bar{\alpha}|}_{+}, \tag{3.4}$$

where $\widehat{L}_U(X) := U^\top L(UXU^\top)U$ and $\bar{\alpha} = \{1,\dots,n\} \setminus \alpha$, implies that there is a nonzero matrix $Y \in S^n$ satisfyng

$$Y_{\alpha\alpha} \in S^{|\alpha|}_{+}, \quad Y_{ij} = 0, \forall i,j \notin \alpha, \quad [\widehat{L}^\top_U(Y)]_{\alpha\alpha} = 0, \quad [\widehat{L}^\top_U(Y)]_{\alpha\bar{\alpha}} = 0, \quad [\widehat{L}^\top_U(Y)]_{\bar{\alpha}\bar{\alpha}} \in -S^{|\bar{\alpha}|}_{+}. \tag{3.5}$$

Then, L is an $\mathbf{F}$-transformation.

Proof. Let V be a nonzero solution of SDLCP$(L, S^n_+, 0)$. Consider an orthonormal matrix $U \in \mathbb{R}^{n\times n}$ whose columns are eigenvectors of V. It follows that

$$U^\top V U = \begin{pmatrix} Z & 0 \\ 0 & 0 \end{pmatrix},$$

for some $Z \in S^k_{++}$ (actually Z is a diagonal matrix containing all positive eigenvalues of V) and $k \in \{1,...,n\}$. Set $\alpha := \{1,\dots,k\}$. We proceed to show that $X := U^\top V U$ is a solution of (3.4). First, X clearly satisfies first two conditions of (3.4). Also, $\widehat{L}_U(X) = U^\top L(V)U$. So, since $L(V) \in S^n_+$, it follows that $[\widehat{L}_U(X)]_{\bar{\alpha}\bar{\alpha}} \in S^n_+$. Moreover, condition $\langle L(V), V\rangle = 0$ implies that the columns of U can be chosen in order to be also a basis of orthonormal eigenvectors of $L(V)$ (cf. Propostion 2.1, Part (f)). This yields, on the one hand, to $[\widehat{L}_U(X)]_{\alpha\bar{\alpha}} = 0$ (because $\widehat{L}_U(X) = U^\top L(V)U$ is actually a diagonal matrix), and, on the other hand, to $[\widehat{L}_U(X)]_{\alpha\alpha} = 0$ (because of $\langle L(V), V\rangle = \langle \widehat{L}_U(X), X\rangle$). Hence, there exists a nonzero solution Y of (3.5).

We claim that the linear transformation $\mathcal{T} : S^n \to \mathbb{R}^{n\times n}$ defined as

$$\mathcal{T}(W) := U \begin{pmatrix} Y_{\alpha\alpha} Z^{-1} & 0 \\ 0 & 0 \end{pmatrix} U^\top W, \quad \forall W \in S^n,$$

satisfies conditions (**i**)–(**iii**) in (3.1). Indeed, since $\mathcal{T}(V) = UYU^\top$, it clearly follows that $\mathcal{T}(V) \in S^n_+$. So, due to positive definiteness of Z and Fejer's theorem (see Proposition 2.1, Part (c)), we obtain that $\langle \mathcal{T}(V), V\rangle = \langle Y_{\alpha\alpha}, Z\rangle > 0$. Finally, since $\widehat{L}^\top_U(Y) = U^\top L^\top(\mathcal{T}(V))U$ and $\widehat{L}^\top_U(Y) \in -S^n_+$ (consequence of (3.5)), it follows that $L^\top(\mathcal{T}(V)) \in -S^n_+$. We have thus deduced that L is an $\mathbf{F}$-transformation. □

Remark 3.12. *In the implication stated in Proposition 3.11 above, we can chose an index set $\alpha \subseteq \{1,...,n\}$ not necessarily of the form $\{1,...,k\}$ for some $k \in \{1,...,n\}$. Indeed, given an orthonormal matrix $U \in \mathbb{R}^{n\times n}$ and an arbitrary index set $\alpha \subseteq \{1,...,n\}$, let $\tilde{X}$ be a matrix satisfying* (3.4)*. We write α_i to denote the i-th component of α. So, we define $P \in \mathbb{R}^{n\times n}$ as the permutation matrix such that the position α_i is switched with position i, for all $i \in \{1,...,|\alpha|\}$ (that is, if $\tilde{x} = Px$, then $x_i = x_{\alpha_i}$). Set $k := |\alpha|$. Since any permutation matrix is orthonormal, it follows that $\tilde{U} := UP^\top$ is orthonormal. Then, it is easy to note that $X := P\tilde{X}P^\top$ satisfies* (3.4) *when α is replaced by $\{1,...,k\}$ and U is replaced by $\tilde{U}$. Thus, if* *the implication stated in Proposition 3.11 holds, we obtain the existence of* $Y \in S^n$ *solution of* (3.5) *for the same data (i.e.* $\{1, ..., k\}$ *and* $\tilde{U}$*). Finally, it suffices to note* $\tilde{Y} := P^\top Y P$ *satisfies* (3.5) *for the original* α *and* U*.*

From now on, the class of transformations L such that the implication stated in Proposition 3.11 holds true will be denoted by $\mathbf{F_2}$. Clearly, Proposition 3.11 above shows that $\mathbf{F_2}$ is a subclass of $\mathbf{F}$.

Once again, this subclass is closely related to the class F_1, defined in the LCP framework (see Definition 3.4). Indeed, it is easy to see that that a matrix $M \in \mathbb{R}^{n\times n}$ is an F_1-matrix if and only if, for any nonempty set $\alpha \subseteq \{1, \ldots, n\}$, the existence of a vector $x_\alpha \in \mathbb{R}^{|\alpha|}$ satisfying

$$x_\alpha > 0, \quad M_{\alpha\alpha} x_\alpha = 0 \quad \text{and} \quad M_{\bar{\alpha}\alpha} x_\alpha \geq 0, \tag{3.6}$$

implies that there exists a nonzero vector $w_\alpha \in \mathbb{R}^{|\alpha|}_+$ such that

$$w_\alpha^\top M_{\alpha\alpha} = 0 \quad \text{and} \quad w_\alpha^\top M_{\alpha\bar{\alpha}} \leq 0. \tag{3.7}$$

So, keeping this characterization in mind, we establish the mentioned relation here below.

Proposition 3.13. *Let* $M \in \mathbb{R}^n$ *and consider* $\mathcal{M}$ *defined in* (3.3)*. If* $\mathcal{M} \in \mathbf{F_2}$*, then* $M \in \mathrm{F}_1$*.*

Proof. Thanks to Remark 3.12, we can consider any arbitrary index set $\alpha \subseteq \{1, \ldots, n\}$ in the definition of $\mathbf{F_2}$. Then, it suffices to note that systems (3.4) and (3.5) coincide with (3.6) and (3.7), respectively, when we consider $U = I$ (the identity matrix), $X_{\alpha\alpha} = \mathrm{Diag}(x_\alpha)$, $X_{ij} = 0$ for all i, $j \notin \alpha$, and $Y_{\alpha\alpha} = \mathrm{Diag}(w_\alpha)$, $Y_{ij} = 0$ for all i, $j \notin \alpha$, . □

In the following proposition we list various classes of linear transformations that are contained in the classes $\mathbf{F_1}$ and $\mathbf{F_2}$.

Proposition 3.14. $L \in \mathbf{F_1} \cap \mathbf{F_2}$ *if any of the following conditions is satisfied:*

(a) L *is a star-transformation*;

(b) $L \in \mathbf{Z}$ *and*

  (i) $-L$ *is copositive or*

  (ii) L *is normal*;

(c) $L \in \mathbf{R_0}$.

Proof. In order to prove items (a) and (b), we split the proof into two parts; in the first one we prove that $L \in \mathbf{F_1}$ while in the second one we show that $L \in \mathbf{F_2}$. For both classes, item (c) is trivially verified by vacuity.

$L \in \mathbf{F_1}$:

(a): Let $V \in \mathcal{S}(L, S^n_+, 0)\backslash\{0\}$. Since L is a star-transformation, we have $L^\top(V) \in -S^n_+$. Then, conditions **(i)**–**(iii)** of Definition 3.5 can be easily checked provided that $\mathcal{T}$ is of the form (3.2) with $\Lambda = \mathbb{1}$ (note that $\mathbb{1} \in S^n_+$). The result follows.

(b): Let $V \in \mathcal{S}(L, S^n_+, 0)\backslash\{0\}$, that is, $V, L(V) \in S^n_+$ and $\langle L(V), V\rangle = 0$. Since $L \in \mathbf{Z}$, we get $\langle L(V), L(V)\rangle \leq 0$, and consequently $L(V) = 0$. We proceed to prove both cases.

(i): If $-L$ is copositive, then $\langle L(tX+V), tX+V\rangle \leq 0$ for all $X \in S^n_+$ and for all $t > 0$. From this, after dividing by t we get $t\langle L(X), X\rangle + \langle L(X), V\rangle \leq 0$ for all $t > 0$. Taking limit $t \searrow 0$ we obtain $\langle X, L^\top(V)\rangle \leq 0$ for all $X \in S^n_+$. From Fejer's theorem (Theorem 2.2, Part (c)), we conclude that $L^\top(V) \in -S^n_+$, that is, L is a star-transformation. The desired result follows from (a).
(ii): Since L is normal and $L(V) = 0$, we obtain that

$$\|L^\top(V)\|_F^2 = \langle V, L(L^\top(V))\rangle = \langle V, L^\top(L(V))\rangle = 0.$$

That is, $L^\top(V) = 0$, which is in a particular a matrix in $-S^n_+$. Thus, the desired result follows again from (a).

$L \in \mathbf{F_2}$:

(a): Let U be an orthonormal matrix of size n, $\alpha = \{1, \ldots, k\}$ $(1 \leq k \leq n)$ be a nonempty index set, and $X \in S^n$ a solution of the system (3.4). It is easy to show that

$$V = UXU^\top = U \begin{pmatrix} X_{\alpha\alpha} & 0 \\ 0 & 0 \end{pmatrix} U^\top$$

is a nonzero solution of SDLCP$(L, S^n_+, 0)$. Indeed, $V = UXU^\top$, $L(V) = U\widehat{L}_U(X)U^\top \in S^n_+$ (because $X, \widehat{L}_U(X) \in S^n_+$) and $\langle L(V), V\rangle = \langle \widehat{L}_U(X), X\rangle = 0$. Since L is a star-transformation, we have that $L^\top(V) \in -S^n_+$. Consequently, $\widehat{L}_U^\top(X) = U^\top L^\top(V)U \in -S^n_+$. On the other hand, since $X_{\alpha\alpha} \in S^{|\alpha|}_{++}$ and $[\widehat{L}_U^\top(X)]_{\alpha\alpha} \in -S^{|\alpha|}_+$ and the equality

$$\langle [\widehat{L}_U^\top(X)]_{\alpha\alpha}, X_{\alpha\alpha}\rangle = \langle \widehat{L}_U^\top(X), X\rangle = \langle X, \widehat{L}_U(X)\rangle = 0,$$

it follows that $[\widehat{L}_U^\top(X)]_{\alpha\alpha} = 0$. This together with condition $-\widehat{L}_U^\top(X) \in S^n_+$ implies that $[\widehat{L}_U^\top(X)]_{\alpha\bar{\alpha}} = [\widehat{L}_U^\top(X)]_{\bar{\alpha}\alpha} = 0$ (because of Proposition 2.2, Part (a)). Hence, $Y = X$ solves (3.5). We have thus conclude that $L \in \mathbf{F_2}$

(b): Let U be an orthonormal matrix of size n, $\alpha = \{1, \ldots, k\}$ $(1 \leq k \leq n)$ be a nonempty index set, and $X \in S^n$ a solution of the system (3.4). As before, $V = UXU^\top$ is a nonzero solution of SDLCP$(L, S^n_+, 0)$. Since $L \in \mathbf{Z}$, we get $\langle L(V), L(V)\rangle \leq 0$ and consequently $L(V) = 0$. Hence, $\widehat{L}_U(X) = U^\top L(V)U = 0$. Moreover, $-\widehat{L}_U$ is copositive when $-L$ is copositive and $\widehat{L}_U$ is normal when L is normal. Thus, the arguments given in order to prove that $L \in \mathbf{F_1}$, but applied to $\widehat{L}_U$ instead of L, imply that $L \in \mathbf{F_2}$. □

Remark 3.15. *Proposition 2.2, Part (l), provides classes included in the class of star transformations. Part (i) of the same proposition do the same for the class of copositive transformations. On the other hand, self-adjoint transformations are examples of normal transformations. Finally, Proposition 2.2, Part (d), provides situations when a linear function is an* $\mathbf{R_0}$*-transformation.*

3.3 Examples: Lyapunov, multiplicative, and Stein transformations

Some linear transformations in $\mathcal{L}(S^n)$ arises naturally in matrix theory and its applications. This is the case of Lyapunov, multiplicative and Stein transformations, that are defined, for a given $A \in \mathbb{R}^{n\times n}$, as follows:

- $L_A(X) = AX + XA^\top$,

- $M_A(X) = AXA^\top$,
- $S_A(X) = X - AXA^\top$.

We recall some properties of these transformations.

Proposition 3.16. *Let $A \in \mathbb{R}^{n\times n}$ be given.*

(a) $L_A^\top = L_{A^\top}$, $M_A^\top = M_{A^\top}$, *and* $S_A^\top = S_{A^\top}$;

(b) *If A is normal (i.e. $AA^\top = A^\top A$) if and only if L_A, M_A and S_A are normal;*

(c) *If A is symmetric, then L_A, M_A and S_A are self-adjoint. If A is skew-symmetric, then M_A and S_A are self-adjoint;*

(d) $L_A,\ S_A \in \mathbf{Z}$ *for all* $A \in \mathbb{R}^{n\times n}$.

Proof. See [11, 12]. □

We want to study conditions on A that ensure that previous transformations belong to class $\mathbf{F_1}$. For this, we need to recall the following well-known equivalences.

Theorem 3.17. *Let $A \in \mathbb{R}^{n\times n}$ be given.*

(a) *A is positive definite (i.e. $\langle Ax, x\rangle > 0$ for all nonzero $x \in \mathbb{R}^n$) $\iff L_A$ is strongly monotone $\iff L_A \in \mathbf{P_2} \iff L_A \in \mathbf{P'_2}$;*

(b) *A is positive stable (i.e. all the eigenvalues of A have a positive real part) $\iff L_A \in \mathbf{P}$ $\iff L_A \in \mathbf{E} \iff L_A \in \mathbf{E_0} \cap \mathbf{R_0} \iff L_A \in \mathbf{Q} \iff S^n_{++} \subseteq L_A(S^n_{++}) \iff L_A(S^n_{++}) \cap S^n_{++} \neq \emptyset$;*

(c) *A is positive definite or negative definite $\iff M_A \in \mathbf{P_2} \iff M_A \in \mathbf{GUS} \iff M_A \in \mathbf{P}$ $\iff M_A \in \mathbf{R_0} \iff M_A \in \mathbf{P'_2}$;*

(d) *A is Schur stable (i.e. all the eigenvalues of A lie in the open unit disk) $\iff S_A \in \mathbf{P}$ $\iff S_A \in \mathbf{GUS} \iff S^n_{++} \subseteq S_A(S^n_{++}) \iff S_A(S^n_{++}) \cap S^n_{++} \neq \emptyset$.*

Proof. Statement (a) is proven in [20, Theorem 5] and [3, Theorem 3.3]. Statement (b) is demonstrated in [10, Theorem 5]. Statement (c) is proven [3, Theorem 4.2] and [21]. Finally, statement (d) is shown in [9, Theorem 11 and Remark 4]. □

As a consequence of Proposition 3.14, Proposition 3.16, and Theorem 3.17 we obtain the next result.

Corollary 3.18. *Let $A \in \mathbb{R}^{n\times n}$ be given.*

(a) *If A is normal, then $L_A, S_A \in \mathbf{F_1} \cap \mathbf{F_2}$;*

(b) *If A is positive definite or positive stable, then $L_A \in \mathbf{F_1} \cap \mathbf{F_2}$;*

(c) *If A is positive definite or negative definite, then $M_A \in \mathbf{F_1} \cap \mathbf{F_2}$;*

(d) *If A is Schur stable, then $S_A \in \mathbf{F_1} \cap \mathbf{F_2}$;*

Proof. (a): By Proposition 3.16 we have that L_A and S_A are normal and $\mathbf{Z}$-transformations. The result follows from Proposition 3.14, Part (b).

(b): Let A be positive definite. Theorem 3.17, Part (a), says that this is equivalent to L_A being strongly monotone, which in turn by Proposition 2.2, Part (d), implies that $L_A \in \mathbf{R_0}$. Let A be positive stable. Theorem 3.17, Part (b), implies that $L_A \in \mathbf{R_0}$. In both cases, the result follows from Proposition 3.14, Part (c).

(c): By Theorem 3.17, Part (c), the hypothesis is equivalent to $M_A \in \mathbf{R_0}$. The result follows from Proposition 3.14, Part (c).

(d): By Theorem 3.17, Part (d), the hypothesis is equivalent to $S_A \in \mathbf{P}$. Thus, Proposition 2.2, Part (d), implies that $S_A \in \mathbf{R_0}$. The result also follows from Proposition 3.14, Part (c). □

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