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ON BLOCH WAVES FOR THE STOKES EQUATIONS

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Abstract. In this work, we study the Bloch wave decomposition for the Stokes equations in a periodic media in $\mathbb{R}^d$. We prove that, because of the incompressibility constraint, the Bloch eigenvalues, as functions of the Bloch frequency ξ, are not continuous at the origin. Nevertheless, when ξ goes to zero in a fixed direction, we exhibit a new limit spectral problem for which the eigenvalues are directionally differentiable. Finally, we present an analogous study for the Bloch wave decomposition for a periodic perforated domain.

1. **Introduction and main results.** The method of Bloch wave decomposition (or Floquet decomposition) is well known for reducing the problem of solving the Schrödinger equation in an infinite periodic medium to a family of simpler Schrödinger equations posed in a single periodicity cell and parametrized by the so-called Bloch frequency [9], [10], [17], [23]. We extend this method to the Stokes equations of incompressible fluid mechanics or equivalently to the equations of linear incompressible elasticity.

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A key ingredient in this study is the so-called Bloch cell spectral problem. Let $Y = [0, 2\pi[^d$ be the unit cell and $Y' = [0, 1[^d$ the dual cell. For any Bloch frequency $\xi \in Y'$, we consider the following Stokes system

$$\left\{ \begin{array}{lll} \text{Find } \lambda(\xi) \in \mathbb{R}, \phi \neq 0 \in H^1_\#(Y)^d,\ p \in L^2_\#(Y) & \text{such that} & \\ -D(\xi) \cdot (\mu(y) D(\xi)\phi) + \kappa(y)\phi + D(\xi)p & = \lambda(\xi)\phi & \text{in } Y, \\ D(\xi) \cdot \phi & = 0, & \text{in } Y, \end{array} \right. \quad (1)$$

where $D(\xi) = (\nabla + i\xi)$, $\mu \in L^\infty_\#(Y)$ is a given uniformly positive viscosity, and $\kappa \in L^\infty_\#(Y)$ is a given damping coefficient. We assume $\mu(y) \geq \mu_0 > 0$ and $\kappa(y) \geq 1$ a.e. in Y (without loss of generality since adding a constant to κ is equivalent to shifting the spectrum). All our results could be generalized to the case of κ being a symmetric matrix and μ a symmetric coercive fourth-order tensor. We also discuss the limit case $\kappa(y) = +\infty$ in some subset $T \subset Y$ which corresponds to a hole or obstacle T supporting a Dirichlet boundary condition.

We are mainly interested in the continuity and differentiability properties of the eigenvalues and eigenfunctions of (1). Indeed, it is well-known for the Schrödinger equation [7], [9], that the Hessian of the first eigenvalue at the origin $\xi = 0$ is the effective tensor for the corresponding homogenized problem. A first rigorous analysis of the homogenization process along these lines was given in [10]. Their results have been generalized to some extent for systems of diffusion equation [4] when the first eigenvalue is simple, and for the elasticity system [13] using merely directional derivability. Therefore, the differentiability structure of (1) is a problem of paramount importance for homogenization which is our main motivation here. The difficulty, in the case of Stokes equations, is that (1) is not posed in a fixed functional space when ξ varies because the incompressibility constraint $D(\xi) \cdot \phi = 0$ precisely depends on ξ.

Our first result (Proposition 2.8) is that, contrary to the above examples, the eigenvalues and eigenfunctions of (1) are usually not even continuous at the origin. The main reason for this strange phenomenon is that the limit of the incompressibility constraint $D(\xi) \cdot \phi = 0$, as ξ goes to 0 in a fixed direction e, is not only $D(0) \cdot \phi = 0$ but it also includes the additional constraint $\int_Y e \cdot \phi \, dy = 0$. A similar discontinuity of the Bloch eigenvalues at the origin was already obtained for a completely different model of fluid structure interaction in [5].

Our main result (Theorem 3.7) is to nevertheless prove that there exists a new family of spectral Stokes problem, featuring the additional constraint $\int_Y e \cdot \phi \, dy = 0$, which are the limits of (1) when ξ goes to 0 in a fixed direction e. We prove that eigenvalues and eigenfunctions are thus directionally analytic. Then, as a final result (Lemma 4.2), we partially recover the usual homogenized effective tensor of Stokes equations, whose entries are linked to the second-order derivatives of the first eigenvalues.

The content of the present paper is the following. In Section 2 we first study the simpler non-homogeneous Stokes problem, i.e. we replace the right hand side of (1) by a fixed force term. Already for this simpler problem we obtain a discontinuity result in Proposition 2.8. Section 3 is devoted to the analyticity properties of the Bloch waves and contain our main result, Theorem 3.7. In Section 4 we compute the derivatives of the first Bloch eigenvalue at $\xi = 0$ and we show that we can partially recover the homogenized tensor of the Stokes problem (see Lemma 4.2). In Section 5 we present the Bloch decomposition of the space of divergence-free vector fields on

$\mathbb{R}^d$. In Section 6, we study the Bloch decomposition and the regularity of the Bloch waves in the case of a periodically perforated domain, in a similar way as for non-perforated domains. Finally Section 7 contains some 2-d numerical computations which illustrate the discontinuity of the Bloch eigenvalues.

Notations. Let us make precise the definition of $D(\xi)$. If $\phi = (\phi_k)_{1\le k\le d}$ is a vector-valued function, then $D(\xi)\phi = \nabla\phi + i\phi\otimes\xi$ is a matrix of entries $(\partial\phi_k/\partial y_l + i\phi_k\xi_l)_{1\le k,l\le d}$. Let $L^2_\#(Y)$ denote the space of functions in $L^2_{loc}(\mathbb{R}^d)$ which are Y-periodic. A similar definition holds for the Sobolev space $H^1_\#(Y)$. Its dual is denoted by $H^{-1}_\#(Y)$. We denote by $L^2_{\#,0}(Y)$ the subspace of $L^2_\#(Y)$ made of functions with zero-average on Y. All these spaces are made of complex-valued functions.

2. **The non-homogeneous Stokes problem.** We first consider the Stokes equations with a source term $f\in L^2_\#(Y)^d$, namely,

$$\begin{cases} -D(\xi)\cdot(\mu(y)D(\xi)\phi)+\kappa(y)\phi+D(\xi)p &= f \qquad \text{in } Y\\ D(\xi)\cdot\phi &= 0 \qquad \text{in } Y\\ p,\phi \text{ are} & Y-\text{periodic}\end{cases} \tag{2}$$

Note that in the case $\xi = 0$, the well-posedness theory of (2) is well-known: for each function $f \in L^2_\#(Y)^d$, there exists a unique solution $(\phi,p)\in H^1_\#(Y)^d\times L^2_\#(Y)/\mathbb{C}$ (the pressure is defined up to an additive constant). As we shall see the existence theory in the case $\xi\neq 0$ is different but not more difficult. However, the surprising property that we shall establish is the lack of continuity of (2) as ξ goes to zero (see Proposition 2.8).

Proposition 2.1. *For $\xi\in Y'\setminus\{0\}$ and $f\in L^2_\#(Y)^d$, there exists a unique solution $(\phi,p)\in H^1_\#(Y)^d\times L^2_\#(Y)$ of (2), which satisfies*

$$\begin{aligned} &\|\phi\|_{1,Y}\le C\|f\|_{0,Y} \qquad && \|p-m(p)\|_{0,Y}\le C\|f\|_{0,Y}\\ &|\xi m(p)|\le C\|f\|_{0,Y}, && \end{aligned} \tag{3}$$

where the constant $C>0$ does not depend on ξ and m is the averaging operator defined by

$$m(p)=\frac{1}{|Y|}\int_Y p(y)\,dy.$$

(Note that, in this case, the pressure is uniquely defined.)

As usual, for $\xi=0$, there exists a unique solution $(\phi,p)\in H^1_\#(Y)^d\times L^2_\#(Y)/\mathbb{C}$ of (2).

Before proving Proposition 2.1 we need a series of lemma. We introduce the spaces of "generalized" divergence-free velocities

$$V_\xi = \{\phi\in H^1_\#(Y)^d \ :\ D(\xi)\cdot\phi = 0\}$$

$$H_\xi = \{\phi\in L^2_\#(Y)^d \ :\ D(\xi)\cdot\phi = 0\}$$

We first prove an adequate version of the De Rham's Theorem.

Lemma 2.2. *For $\xi\in Y'\setminus\{0\}$ we have*

$$H_\xi^\perp = \left\{D(\xi)\rho \ :\ \rho\in H^1_\#(Y)\right\} \quad \text{and} \quad V_\xi^\perp = \left\{D(\xi)\rho \ :\ \rho\in L^2_\#(Y)\right\},$$

with a unique representation.

Proof. Let $\xi \in Y' \setminus \{0\}$. First, take $\psi \in H_\xi^\perp$ (the orthogonal in the L^2 sense). By definition

$$\psi \in H_\xi^\perp \Longleftrightarrow \int_Y \varphi(y) \cdot \overline{\psi}(y) dy = 0, \qquad \forall \varphi \in H_\xi,$$

which, upon introducing the Fourier series $\varphi(y) = \sum_{k\in\mathbb{Z}^d} \hat{\varphi}(k) e^{ik\cdot y}$, implies

$$\hat{\varphi}(k) \cdot \overline{\hat{\psi}}(k) = 0, \qquad \forall k \in \mathbb{Z}^d.$$

Since $D(\xi) \cdot \varphi = 0$, the Fourier components $\hat{\varphi}(k)$ satisfy

$$i(k+\xi) \cdot \hat{\varphi}(k) = 0, \qquad \forall k \in \mathbb{Z}^d,$$

and we deduce that

$$\hat{\psi}(k) = i(k+\xi)\hat{p}(k), \qquad \forall k \in \mathbb{Z}^d \text{ with } \hat{p}(k) \in \mathbb{C},$$

which implies that $\psi = D(\xi)p$, with p a distribution. Moreover, since $\psi \in L^2_\#(Y)$ we obtain that

$$\sum_{k\in\mathbb{Z}^d} |k+\xi|^2 \, |\hat{p}(k)|^2 < \infty,$$

that is, $p \in H^1_\#(Y)$ (because $\xi \neq 0$), and we conclude that $H_\xi^\perp = \left\{ D(\xi)\rho : \rho \in H^1_\#(Y) \right\}$ as required.

Now, take $\psi \in V_\#^\perp$ (the orthogonal in the H^1 sense). By definition

$$\int_Y \nabla\varphi : \nabla\overline{\psi} + \int_Y \varphi \cdot \overline{\psi} = 0, \qquad \forall \varphi \in V_\xi,$$

that is, in terms of Fourier series,

$$(1+|k|^2)\hat{\varphi}(k) \cdot \overline{\hat{\psi}}(k) = 0, \qquad \forall k \in \mathbb{Z}^d,$$

and since $\hat{\varphi}(k)$ satisfies

$$i(k+\xi) \cdot \hat{\varphi}(k) = 0, \qquad \forall k \in \mathbb{Z}^d,$$

we deduce

$$\hat{\psi}(k) = i(k+\xi)\hat{p}(k), \qquad \forall k \in \mathbb{Z}^d \;\text{ with }\; \hat{p}(k) \in \mathbb{C},$$

and moreover

$$\sum_{k\in\mathbb{Z}^d} (1+|k|^2)^{-1} \left| \hat{\psi}(k) \right|^2 < \infty,$$

which implies

$$\sum_{k\in\mathbb{Z}^d} |\hat{p}(k)|^2 < \infty.$$

Therefore, we conclude that $p \in L^2_\#(Y)$, $V_\xi^\perp = \left\{ D(\xi)\rho : \rho \in L^2_\#(Y) \right\}$ and the representation is unique, which completes the proof. $\square$

On the other hand, we have the following regularity result for the pressure.

Lemma 2.3. *Let $g \in H^{-1}_\#(Y)^d$, such that $g = D(\xi)p$, with p a distribution. Then $p \in L^2_\#(Y)$. Furthermore, there exists a constant $C > 0$, independent of ξ, such that*

$$\|p - m(p)\|_{L^2_\#(Y)} \leq C \, \|g - m(g)\|_{H^{-1}_\#(Y)^d}$$

where m is the averaging operator defined by $m(f) = \frac{1}{|Y|}\int_Y f \, dy$.

Proof. Let $g = D(\xi)p \in H^{-1}_{\#}(Y)^d$. Its Fourier coefficients satisfy for every $k \in \mathbb{Z}^d$

$$\hat{g}(k) = i(\xi + k)\hat{p}(k),$$

with $\hat{p}(k) \in \mathbb{C}$, moreover, it is easy to check that

$$m(g) = < g, 1 >_{H^{-1}_{\#} \times H^1_{\#}} = \hat{g}(0). \tag{4}$$

Let us assume that $m(g) = 0$, therefore multiplying by $-i(\xi + k)$ we obtain that

$$|\xi + k|^2 \hat{p}(k) = -i(\xi + k) \cdot \hat{g}(k),$$

or equivalently

$$\hat{p}(k) = \frac{-i(\xi + k)}{|\xi + k|^2} \cdot \hat{g}(k) \quad \text{for} \quad k \neq 0.$$

Furthermore, if $\xi \neq 0$ we have $\hat{p}(0) = 0$, while if $\xi = 0$ the value of $\hat{p}(0)$ is free. Thus, for $\xi \neq 0$, there exists a positive constant c, independent on k and ξ, such that, for every $k \in \mathbb{Z}^d$,

$$|\hat{p}(k)| \leq \frac{|\xi + k|}{|\xi + k|^2} |\hat{g}(k)| \leq c \frac{|\hat{g}(k)|}{\sqrt{1 + |k|^2}}. \tag{5}$$

When $\xi = 0$, inequality (5) holds true if we choose p such that $\hat{p}(0) = 0$. Therefore, for any value of ξ, summing up the squared inequalities (5) we conclude that $p \in L^2_{0,\#}(Y)$ and

$$\|p\|^2_{0,Y} \leq c^2 \|g\|^2_{-1,Y}. \tag{6}$$

Now, if we consider a general function $g = D(\xi)p \in H^{-1}_{\#}(Y)^d$, we check that

$$g - m(g) = D(\xi)(p - m(p))$$

and applying (6) we deduce that

$$\|p - m(p)\|_{0,Y} \leq c\|D(\xi)(p - m(p))\|_{-1,Y}.$$

and $p \in L^2_{\#}(Y)$. $\square$

Proof of Proposition 2.1 For $\xi = 0$ the result is well known [18]. For $\xi \neq 0$ we define the bilinear form on $H^1_{\#}(Y)$

$$a(\varphi, \psi) = \int_Y \mu D(\xi)\varphi : \overline{D(\xi)\psi} + \int_Y \kappa \varphi \overline{\psi}, \qquad \forall \varphi, \psi \in V_\xi,$$

which is easily seen to be symmetric, continuous and coercive since there exists a positive constant C such that (see [9])

$$C \left\{ \|\nabla\varphi\|_{0,Y} + |\xi| \, \|\varphi\|_{0,Y} \right\} \leq \|D(\xi)\varphi\|_{0,Y} \leq \left\{ \|\nabla\varphi\|_{0,Y} + |\xi| \, \|\varphi\|_{0,Y} \right\}.$$

By the Lax-Milgram's Lemma, the problem

$$a(\phi, \psi) = (f, \psi), \qquad \forall \psi \in V_\xi$$

has a unique solution $\phi \in V_\xi$ for any $f \in L^2_{\#}(Y)$. Since

$$\Big(D(\xi) \cdot (\mu D(\xi)\phi) - \kappa\phi + f \Big) \in V_\xi^\perp$$

by application of Lemma 2.2, there exists a unique pressure $p \in L^2_{\#}(Y)$ such that

$$D(\xi)p = D(\xi) \cdot (\mu D(\xi)\phi) - \kappa\phi + f$$

which proves the existence and uniqueness of the solution of (2).

Furthermore, since $\kappa(y) \geq 1$, the bilinear form is uniformly coercive, namely there exists a positive constant C, independent of ξ, such that

$$C\|\phi\|_{1,Y}^2 \leq a(\phi,\phi) = (f,\phi) \leq \|f\|_{0,Y}\|\phi\|_{1,Y},$$

which proves that

$$C\|\phi\|_{1,Y} \leq \|f\|_{0,Y}.$$

Introducing $g = D(\xi)\cdot(\mu D(\xi)\phi) - \kappa\phi + f)$, from Lemma 2.3, we deduce that

$$\|p - m(p)\|_{0,Y} \leq c\,\|g - m(g)\|_{-1,Y} \leq c_1\,\|f\|_{0,Y}$$

and

$$|\xi m(p)| = |m(g)| \leq c_2\,\|f\|_{0,Y}$$

which completes the proof. $\square$

Remark 2.4. The "generalized" incompressibility constraint $D(\xi)\cdot\phi = 0$ contains an additional implicit constraint. Indeed, any function $\phi \in H^1_\#(Y)^d$, such that $D(\xi)\cdot\phi = 0$, satisfies also the constraint

$$\xi \cdot \int_Y \phi(y)\,dy = 0,$$

which is simply obtained by integrating $D(\xi)\cdot\phi = 0$ over Y. Therefore, in the limit as ξ goes to 0 in a fixed direction e, we formally obtain two limit constraints: $\nabla\cdot\phi = 0$ and $e\cdot\int_Y \phi\,dy = 0$. This explains the introduction of a new Stokes problem (7) below.

To study the limit of the Stokes problem (2) when ξ goes to 0, we introduce a new family of Stokes problems, parametrized by a unit vector $e \in \mathbb{R}^d$ with $|e| = 1$,

$$\begin{cases} -\nabla\cdot(\mu\nabla u) + \kappa u + \nabla q + q_0 e &= f \qquad \text{in } Y \\ e\cdot\int_Y u(y)\,dy &= 0 \\ \nabla\cdot u &= 0 \qquad \text{in } Y \\ q, u \text{ are} & Y-\text{periodic} \end{cases} \tag{7}$$

Since there is an additional constraint on the velocity in (7), there is also an additional Lagrange multiplier $q_0 e$, where q_0 is a real constant. A possible physical interpretation is that adding the constraint $e\cdot\int_Y u = 0$ amounts to apply an affine pressure term $q_0\,e\cdot y$.

We introduce new spaces of "extended" divergence-free velocities: for each unit vector $e \in \mathbb{R}^d$, $|e| = 1$, we define

$$\mathbb{V}_e = \{\phi \in H^1_\#(Y)^d \ : \ \nabla\cdot\phi = 0,\ e\cdot\textstyle\int_Y \phi(y)\,dy = 0\}$$

$$\mathbb{H}_e = \{\phi \in L^2_\#(Y)^d \ : \ \nabla\cdot\phi = 0,\ e\cdot\textstyle\int_Y \phi(y)\,dy = 0\}$$

Lemma 2.5. *For a given unit vector $e \in \mathbb{R}^d$, $|e| = 1$, there exists a unique solution (u, q, q_0) in $H^1_\#(Y)^d \times L^2_{\#,0}(Y) \times \mathbb{R}$ of (7).*

Proof. Lax-Milgram's Lemma is easily applied to the variational formulation in $\mathbb{V}_e$

$$\int_Y \mu\nabla u : \nabla\overline{\psi} + \int_Y \kappa u\cdot\overline{\psi} = \int_Y f\cdot\overline{\psi} \qquad \forall\psi \in \mathbb{V}_e$$

which thus admits a unique solution $u \in \mathbb{V}_e$. To recover the pressure and the constant q_0, we use Lemma 2.6 below. A priori the pressure q is defined up to an additive constant, but forcing its average to be zero (as is the case in $L^2_{\#,0}(Y)$) uniquely determines q. $\square$

Lemma 2.6. *Let $e \in \mathbb{R}^d$ be a unit vector. We have*

$$\mathbb{H}_e^\perp = \left\{\nabla\rho + ce \ : \ \rho \in H^1_{\#,0}(Y),\ c \in \mathbb{R}\right\}$$

and

$$\mathbb{V}_e^\perp = \left\{\nabla\rho + ce \ : \ \rho \in L^2_{\#,0}(Y),\ c \in \mathbb{R}\right\},$$

with a unique representation.

Proof. Note that $\mathbb{V}_e = U \cap V$ where U and V are closed subspaces of $H^1_\#(Y)^d$

$$V = \left\{\varphi \in H^1_\#(Y)^d \ : \ \text{div } \varphi = 0 \text{ in } Y\right\}, \quad U = \left\{\varphi \in H^1_\#(Y)^d \ : \ e \cdot \int_Y \varphi = 0\right\}.$$

Thus $(U \cap V)^\perp = U^\perp + V^\perp$, which concludes the proof since $U^\perp = \{q_0 e \ : \ q_0 \in \mathbb{R}\}$ and $V^\perp = \left\{\nabla q \ : \ q \in L^2_{\#,0}(Y)\right\}$ by De Rham's Theorem. A similar proof applies to $\mathbb{H}_e$. □

Remark 2.7. In the spirit of Lemma 2.6, we can give a characterization of $V_\xi^\perp$, which is different from that of Lemma 2.2, namely,

$$V_\xi^\perp = \left\{D(\xi)q + ce \ : \ q \in L^2_{\#,0}(Y),\ c \in \mathbb{C}\right\},$$

with a unique representation.

We are now ready to prove that the family of Stokes equations (7) are the limits of (2) when ξ goes to 0. In other words the Stokes problem (2) is not continuous as ξ goes to 0.

Proposition 2.8. *For a given unit vector $e \in \mathbb{R}^d$, $|e| = 1$, we define $\xi = \varepsilon e$. Then, as ε tends to 0, the solution $(\phi(\xi), p(\xi))$ of (2) satisfies*

$$\begin{array}{rcl} \phi(\xi) & \to & u(e) \textit{ strongly in } H^1_\#(Y)^d \\ p(\xi) - m(p(\xi)) & \to & q(e) \textit{ strongly in } L^2_{\#,0}(Y) \\ i\xi m(p(\xi)) & \to & q_0(e)e, \end{array} \tag{8}$$

where $(u(e), q(e), q_0(e))$ is the unique solution of (7).

Proof. From the uniform bounds in (3), we deduce that there exist limits $(u(e), q(e), q_0) \in H^1_\#(Y)^d \times L^2_\#(Y) \times \mathbb{R}$ such that, up to a subsequence,

$$\begin{array}{rcl} \phi(\xi) & \rightharpoonup & u(e), \quad \text{weakly in } H^1_\#(Y)^d \\ p(\xi) - m(p(\xi)) & \rightharpoonup & q(e), \quad \text{weakly in } L^2_{\#,0}(Y) \\ i\xi m(p(\xi)) & \to & q_0(e)e. \end{array}$$

For any $\psi \in H^1_\#(Y)^d$ we have the variational formulation

$$\int_Y \mu D(\xi)\phi(\xi) : \overline{D(\xi)\psi} + \int_Y \kappa\phi(\xi) \cdot \overline{\psi} - \int_Y (p(\xi) - m(p(\xi)))\overline{D(\xi) \cdot \psi} + \int_Y m(p(\xi)) i\xi \cdot \overline{\psi} = \int_Y f \cdot \overline{\psi}$$

and passing to the limit as ε tends to zero, we obtain

$$\int_Y \mu\nabla u(e) : \nabla\overline{\psi} + \int_Y \kappa u(e) \cdot \overline{\psi} - \int_Y q(e)\nabla \cdot \overline{\psi} + q_0(e) \int_Y e \cdot \overline{\psi} = \int_Y f \cdot \overline{\psi}$$

which is precisely a variational formulation for (7). Thus $(u(e), q(e), q_0(e))$ is the unique solution of (7) and the entire sequence converges. On the other hand, we have

$$\int_Y \mu D(\xi)\phi(\xi) : \overline{D(\xi)\phi(\xi)} + \int_Y \kappa\phi(\xi)\cdot\overline{\phi(\xi)} = \int_Y f\cdot\overline{\phi(\xi)}$$

and

$$\int_Y \mu\nabla u(e) : \nabla\overline{u(e)} + \int_Y \kappa u(e)\cdot\overline{u(e)} = \int_Y f\cdot\overline{u(e)},$$

and since $\phi(\xi) \to u(e)$ strongly in $L^2_\#(Y)$, we can conclude that

$$\int_Y \mu\nabla\phi(\xi) : \nabla\overline{\phi(\xi)} \to \int_Y \mu\nabla u(e) : \nabla\overline{u(e)}$$

which implies the strong convergence of $\phi(\xi)$ in $H^1_\#(Y)^d$. A similar strong convergence for the pressure is finally obtained from Lemma 2.3. □

3. **Bloch Waves and Analyticity Properties.**

This section is devoted to the study of the Bloch spectral problem for the Stokes equation. In particular a detailed study of the Bloch waves as ξ tends to zero is performed since the Bloch waves are not smooth at the origin $\xi = 0$.

For $\xi \in Y'$ we consider the eigenvalue problem

$$\left\{\begin{array}{lll} \text{Find } \lambda(\xi)\in\mathbb{R}, \phi\neq 0\in H^1_\#(Y)^d,\ p\in L^2_\#(Y) & \text{such that} & \\ -D(\xi)\cdot(\mu(y)D(\xi)\phi)+\kappa\phi+D(\xi)p & =\lambda(\xi)\phi & \text{in } Y, \\ D(\xi)\cdot\phi & =0, & \text{in } Y. \end{array}\right. \quad (9)$$

We begin by stating a classical result of existence and regularity of the Bloch waves away from the origin $\xi = 0$.

Theorem 3.1. *For all fixed $\xi \in Y'\backslash\{0\}$, problem (9) admits a countable sequence of real positive eigenvalues each of which is of finite multiplicity. As usual, we arrange them in increasing order repeating each value as many times as its multiplicity:*

$$0 < \lambda_1(\xi) \leq \lambda_2(\xi) \leq \cdots \leq \lambda_n(\xi) \leq \cdots \to \infty.$$

The corresponding eigenfunctions denoted by $\{\phi_n(\xi)\}_{n\geq 1}$ forms a Hilbert basis in H_ξ.

Moreover, for all $n \geq 1$, the functions $\xi \in Y' \setminus \{0\} \to (\lambda_n(\xi), \phi_n(\xi), p_n(\xi))$ with values in $\mathbb{R}\times H^1_\#(Y)^d \times L^2_\#(Y)$ are Lipschitz continuous functions of ξ. □

We will refer to $\{\lambda_n(\xi)\}_{n\geq 1}$ as Bloch eigenvalues and to $\{\phi_n(\xi)\}_{n\geq 1}$ as Bloch eigenvectors or Bloch waves associated to the classical incompressible Stokes equations.

Remark 3.2. The continuity of the functions $\lambda_n : Y' \setminus \{0\} \to \mathbb{R}$ cannot be derived by using minimax principle as in [10], this is due to the dependence on the parameter $\xi \in Y' \setminus \{0\}$ of the spaces V_ξ. Thus the proof of Theorem 3.1 is similar to the proof of Theorem 3.7 (see Appendix) and the previous proofs in [20, 21, 22, 24] so we shall not repeat it here. The main idea is to consider, for any $\lambda > 0$ and $\xi \in Y' \setminus \{0\}$, the map

$$A : \left(H^1_\#(Y)\right)^d \times L^2_\#(Y) \longrightarrow \left(H^{-1}_\#(Y)\right)^d \times L^2_\#(Y)$$

defined by

$$A(\varphi,\pi) = \begin{pmatrix} -D(\xi)\cdot(\mu D(\xi)\varphi) + \kappa\varphi + D(\xi)\pi - \lambda\varphi \\ -D(\xi)\cdot\varphi \end{pmatrix}$$

and to use the Lyapunov-Schmidt method, which is well known in bifurcation theory. The point is that if $\lambda(\xi)$ is an eigenvalue of multiplicity h of (9) and $\phi_1(\xi),\dots,\phi_h(\xi)$ are the associated velocities and $p_1(\xi),\dots,p_h(\xi)$ the corresponding pressures, we have

$$\mathrm{Ker}(A) = \mathrm{span}\left\{(\phi_j(\xi), p_j(\xi)) : j = 1,\dots,h\right\}.$$

Then, the problem is reduced to the existence of the roots of a polynomial of degree h of the form

$$P(\xi,\alpha) = \alpha^h + a_{h-1}(\xi)\alpha^{h-1} + \ldots + a_1(\xi)\alpha + a_0(\xi)$$

where the coefficients $a_j(\xi)$ have an analytic dependence on the parameter ξ. Thus, since the parameter belongs to $\mathbb{R}^n$ we can conclude only the continuity of the roots $\alpha_j(\xi)$ of the polynomial and the eigenvalues have the form $\lambda(\xi) = \lambda + \alpha(\xi)$. We note that in the case of the scalar perturbation, the Weierstrass Preparation Theorem gives us the regularity of the eigenvalues.

The continuity of the functions (ϕ_n, p_n) is an easy consequence of the continuity of λ_n and the smoothness of the corresponding Green's operator.

Remark 3.3. The study of the analyticity properties of λ_n with respect to ξ requires special attention. A standard application of Rellich's theorem (see [16] p. 392) shows the existence of branches of eigenvalues of the Green's operator of problem (1) which are real analytic with respect to each individual variable ξ_l. However, perturbation theory is inadequate to study the real-analyticity of these branches with respect to all the variables ξ (see [16] p. 177). In the sequel, we will come back to study analyticity properties of λ_n, but following a completely different strategy which in fact merely allows to get a directional analyticity of these functions.

Remark 3.4. From Rellich's theory, one can check that if the Bloch eigenvalue at a given frequency ξ_0, $\xi_0 \neq 0$, is simple, the branch $\xi \to \lambda(\xi)$ is also simple in a neighborhood of ξ_0 and moreover, it is analytic in this same neighborhood. Hence the map $\xi \to \phi(\xi)$ can also be proved to be analytic in this neighborhood.

In the general case, if we consider $\xi = \xi_0 + \varepsilon e$, with $\varepsilon \in \mathbb{R}\setminus\{0\},\ \xi_0 \in Y'\setminus\{0\}$ and $e \in \mathbb{R}^d$, a unit vector, the classical results of Rellich [24] or Kato [16] show that the branches of eigenvalues and eigenvectors are analytic with respect to the real parameter ε in a neighborhood of 0. This allows to compute directional derivatives of the Bloch waves at ξ_0 in the direction e.

Theorem 3.1 has left apart the regularity of the spectrum of (9) at $\xi = 0$. Actually, the spectrum is not continuous at $\xi = 0$ because, as already mentioned in Remark 2.4, the incompressibility constraint $D(\xi)\cdot\phi = 0$ changes with ξ and its limit as ξ goes to 0 is not just $D(0)\cdot\phi = 0$. We shall establish below that the limits of (9) when ξ converges to 0 in the direction e are given by

$$\begin{cases}\text{Find } \ \nu(e)\in\mathbb{R}, u(e)\neq 0 \in H^1_{\#}(Y)^d,\ q(e)\in L^2_{\#,0}(Y),\ q_0\in\mathbb{R} \quad \text{such that} \\ \qquad -\nabla\cdot(\mu\nabla u(e)) + \kappa u(e) + \nabla q(e) + q_0 e = \nu(e)u(e) \quad \text{in } Y, \\ \qquad \nabla\cdot u(e) = 0, \quad \text{in } Y, \\ \qquad e\cdot\int_Y u(e) = 0. \end{cases} \tag{10}$$

To resolve this spectral problem, it is a classical technique to introduce the so-called Green's operator $G_e : L^2_{\#}(Y)^d \to L^2_{\#}(Y)^d$ which is defined as $G_e f = u$, where

$u \in \mathbb{V}_e$ is the unique weak solution of (7) given by Lemma 2.5. Applying next Rellich's compactness lemma it is straightforward to check that G_e is a compact operator. Furthermore, it is also selfadjoint and it is easily verified that $(\nu(e), u(e))$, $\nu(e) \neq 0$ satisfies (10) iff

$$G_e u(e) = \frac{1}{\nu(e)} u(e) \quad \text{and} \quad u(e) \not\equiv 0.$$

This means that $\nu(e)$ is a nonzero eigenvalue of problem (10) with corresponding eigenfunction $u(e)$ iff $\frac{1}{\nu(e)}$ is a nonzero eigenvalue of G_e with eigenvector $u(e)$. A standard application of the classical Hilbert-Schmidt theorem yields the following result about the spectrum of (10).

Lemma 3.5. *Problem* (10) *admits a countable sequence of real positive eigenvalues* $(\nu_n(e))_{n\geq 1}$ *converging to* $+\infty$ *with* n *(repeated with their multiplicity) and an Hilbert basis of associated eigenfunctions* $(u_n(e))_{n\geq 1}$.

To study the continuity of (9) when ξ converges to 0, we rewrite (9) in a slightly different form. For a given unit vector $e \in \mathbb{R}^d$, $|e| = 1$, we introduce a scalar parameter $\varepsilon \in \mathbb{R}$ and define

$$\xi = \varepsilon e.$$

The eigenvalue problem (9) is thus rewritten

$$\left\{ \begin{array}{lll} \text{Find } \lambda^\varepsilon \in \mathbb{R}, \phi^\varepsilon \neq 0 \in H^1_\#(Y)^d,\ p^\varepsilon \in L^2_\#(Y) & \text{such that} & \\ \quad -D(\varepsilon e)\cdot(\mu(y)D(\varepsilon e)\phi^\varepsilon) + \kappa\phi^\varepsilon + D(\varepsilon e)p^\varepsilon & = \lambda^\varepsilon\phi^\varepsilon & \text{in } Y, \\ \qquad\qquad D(\varepsilon e)\cdot\phi^\varepsilon & = 0, & \text{in } Y. \end{array} \right. \tag{11}$$

Introducing a new pressure variable $q^\varepsilon \in L^2_{\#,0}(Y)$ defined by

$$q^\varepsilon = p^\varepsilon - m(p^\varepsilon),$$

we obtain that

$$D(\varepsilon e)p^\varepsilon = D(\varepsilon e)q^\varepsilon + q_0^\varepsilon e \quad \text{with} \quad q_0^\varepsilon = i\varepsilon m(p^\varepsilon).$$

On the other hand, as already said in Remark 2.4, the condition $D(\varepsilon e)\cdot\phi^\varepsilon = 0$ implies that

$$e \cdot \int_Y \phi^\varepsilon(y)\,dy = 0.$$

Therefore, problem (11) is equivalent to

$$\left\{ \begin{array}{lll} \text{Find } \lambda^\varepsilon \in \mathbb{R}, \phi^\varepsilon \neq 0 \in H^1_\#(Y)^d,\ q^\varepsilon \in L^2_{\#,0}(Y),\ q_0^\varepsilon \in \mathbb{C} & \text{such that} & \\ \quad -D(\varepsilon e)\cdot(\mu(y)D(\varepsilon e)\phi^\varepsilon) + \kappa\phi^\varepsilon + D(\varepsilon e)q^\varepsilon + q_0^\varepsilon e & = \lambda^\varepsilon\phi^\varepsilon & \text{in } Y, \\ \qquad\qquad D(\varepsilon e)\cdot\phi^\varepsilon & = 0, & \text{in } Y, \\ \qquad\qquad e\cdot\int_Y \phi^\varepsilon(y)\,dy & = 0. & \end{array} \right. \tag{12}$$

The spectral problem (12) has a structure similar to (10), so we can expect the former is an analytic perturbation of the latter. The study of analytic properties of the solution of (12) with respect to ε will follow from the general analytic perturbation theory for solutions of operators depending on one real parameter (see [24]). This is the starting point to obtain the analyticity of the spectrum in a neighborhood of $\varepsilon = 0$.

Remark 3.6. As already noticed in Remark 2.7 the orthogonal of the space

$$V_{\varepsilon e} = \{\phi \in H^1_\#(Y)^d \ : \ D(\varepsilon e)\cdot\phi = 0\}$$

admits two representation, either

$$V_{\varepsilon e}^\perp = \left\{D(\varepsilon e)\rho \ : \ \rho \in L^2_\#(Y)\right\},$$

or equivalently

$$V_{\varepsilon e}^\perp = \left\{D(\varepsilon e)q + q_0 e \ : \ q \in L^2_{\#,0}(Y), \ q_0 \in \mathbb{C}\right\}.$$

We are now ready to prove a result on the regularity of $(\lambda^\varepsilon, \phi^\varepsilon, q^\varepsilon, q_0^\varepsilon) \in \mathbb{R} \times H^1_\#(Y)^N \times L^2_{\#,0}(Y) \times \mathbb{C}$ with respect to ε in a neighborhood of $\varepsilon = 0$. We shall follow the generalization proposed in [20] of Rellich's method. We remark that the main difficulty for applying Rellich's method is that the functional spaces are varying with ε.

Theorem 3.7. *Assume that ν is an eigenvalue of multiplicity h of the Stokes system* (10)*. Then there exist h analytic functions defined in a neighborhood of $\varepsilon = 0 \in \mathbb{R}$ with values in $\mathbb{R}$, $\varepsilon \to \lambda_j^\varepsilon$, and h analytic functions $\varepsilon \to (\phi_j^\varepsilon, q_j^\varepsilon, q_{0,j}^\varepsilon)$ defined in the same neighborhood of $\varepsilon = 0 \in \mathbb{R}$ with values in $H^1_\#(Y)^d \times L^2_{\#,0}(Y) \times \mathbb{C}$, $i = 1, \ldots, h$, such that*

1. $\lambda_j^\varepsilon|_{\varepsilon=0} = \nu, \qquad j = 1, \ldots, h,$
2. *for all ε small enough, $\left(\lambda_j^\varepsilon, \phi_j^\varepsilon, q_j^\varepsilon, q_{0,j}^\varepsilon\right)$ is a solution of (12),*
3. *for all ε small enough, the set $\{\phi_1^\varepsilon, \ldots, \phi_h^\varepsilon\}$ is orthonormal in $L^2_\#(Y)^d$,*
4. *for each interval $I \subset \mathbb{R}$ such that $\overline{I}$ contains only the eigenvalue ν, and for all ε small enough, there are exactly h eigenvalues (counting the multiplicity) $\lambda_1^\varepsilon, \ldots, \lambda_h^\varepsilon$ of* (11) *contained in I.*

The proof of Theorem 3.7 is given in the Appendix.

Remark 3.8. In the above theorem, if ν correspond to the $k+1$-th eigenvalue of (10) of multiplicity h, that is

$$\nu_1 \le \nu_2 \le \ldots \le \nu_k < \nu_{k+1} = \nu_{k+2} = \ldots = \nu_{k+h} < \nu_{k+h+1} \le \ldots$$

such that

$$\nu_{k+1} = \nu_{k+2} = \ldots = \nu_{k+h} = \nu,$$

then there exist h regular functions $\varepsilon \to \lambda_j^\varepsilon$, such that, λ_j^ε is an eigenvalue of (12) verifying $\lambda_j^\varepsilon|_{\varepsilon=0} = \nu$ and the branches of eigenvalues λ_j^ε correspond to the $k+1$-th to $k+h$-th eigenvalues of (12), but not necessarily ordered in an increasing order.

Remark 3.9. Since the Stokes problems (10) are the limits of the Bloch spectral problems (9) when ξ tends to 0, one can wonder if the purely periodic problem

$$\left\{\begin{array}{lll} \text{Find } \ \lambda \in \mathbb{R}, \phi \neq 0 \in H^1_\#(Y)^d, \ p \in L^2_{\#,0}(Y) & \text{such that} & \\ \quad -\nabla\cdot(\mu(y)\nabla\phi) + \kappa\phi + \nabla p & = \lambda\phi & \text{in } Y, \\ \quad \nabla\cdot\phi & = 0, & \text{in } Y, \end{array}\right. \tag{13}$$

obtained by taking $\xi = 0$ in (9), has any connections to (10). It turns out that any eigenvalue and eigenvector of (13) is also an eigenvalue and eigenvector of (10) for a well-chosen vector e (orthogonal to the average of the eigenvector) and for $q_0 = 0$. Therefore, (13) gives no new contribution in the spectrum of the Bloch problems.

4. **On the derivatives of the Bloch eigenpairs.** This section focus on the computation of the derivatives of the Bloch eigenvalues and eigenvectors near the origin $\xi = 0$. It is well known, in the case of the Laplace operator, that the first Bloch eigenvalue is analytic in a neighborhood of $\xi = 0$, which is a consequence of the simplicity of the first eigenvalue, and its Hessian matrix is just the usual homogenized tensor [9]. In the spirit of [12], we want to generalize this result for the Stokes equations. The main difficulty is that the first Bloch eigenvalue is not simple and not continuous. Fortunately we can compute directional derivatives due to the directional analyticity of the Bloch eigenvalues.

In view of the application that we have in mind (namely, the homogenization of the Stokes equations), we restrict ourselves to the special case

$$\kappa(y) \equiv 1 \quad \text{ in } Y. \tag{14}$$

For a unit vector $e \in \mathbb{R}^d$, we define $\xi = \varepsilon e$, for $\varepsilon \in \mathbb{R}$, and we study the directional derivatives of the first Bloch eigenvalues and eigenfunctions with respect to the real scalar parameter ε. Because of assumption (14), the first eigenvalue of (10) is $\nu_1(e) = 1$ which is of multiplicity $d-1$, i.e.,

$$\nu_1(e) = \dots = \nu_{d-1}(e) = 1.$$

The corresponding velocity eigenvectors are constant vectors orthogonal to e and the eigenpressures are 0. By Theorem 3.7 in the neighborhood of $\varepsilon = 0$, there exist $d-1$ (directionally) analytic branches of Bloch eigenvalues and Bloch eigenfunctions of (12)

$$\varepsilon \to \lambda_j^\varepsilon, \quad \varepsilon \to \phi_j^\varepsilon, \quad \varepsilon \to q_j^\varepsilon \quad \text{and} \quad \varepsilon \to q_{0,j}^\varepsilon, \quad j = 1, \dots, d-1,$$

verifying

$$\lambda_j^\varepsilon|_{\varepsilon=0} = \nu_j(e) = 1, \;\; \phi_j^\varepsilon|_{\varepsilon=0} = u_j(e), \;\; q_j^\varepsilon|_{\varepsilon=0} = q_j(e) = 0 \;\text{ and }\; q_{0,j}^\varepsilon|_{\varepsilon=0} = q_{0,j}(e) = 0, \tag{15}$$

where $\{u_j(e)\}_{1 \le j \le d-1}$ is an orthonormal family of vectors in $\mathbb{R}^d$ orthogonal to the chosen direction e. Note that the labeling of the above eigenvalues is not the usual one of increasing order and depends on the direction e. As usual, we normalize the eigenvectors as follows

$$\frac{1}{|Y|} \int_Y |\phi_j^\varepsilon|^2 dy = 1. \tag{16}$$

We differentiate the eigenvalue problem (12) with respect to ε or, equivalently, we differentiate problem (9) in the direction e to obtain

$$\begin{cases} -D(\varepsilon e) \cdot \left(\mu D(\varepsilon e) \phi_j'(\varepsilon)\right) + \phi_j'(\varepsilon) + D(\varepsilon e) q_j'(\varepsilon) + e q_{0,j}'(\varepsilon) - \lambda_j^\varepsilon \phi_j'(\varepsilon) = f(\varepsilon) \text{ in } Y, \\ \hfill D(\varepsilon e) \cdot \phi_j'(\varepsilon) = g(\varepsilon), \text{ in } Y, \end{cases} \tag{17}$$

where $\phi_j'(\varepsilon)$, $q_j'(\varepsilon)$, $q_{0,j}'(\varepsilon)$ and $\lambda_j'(\varepsilon)$ are the derivatives at ε of ϕ_j^ε, q_j^ε, $q_{0,j}^\varepsilon$ and λ_j^ε respectively, with

$$f(\varepsilon) = \lambda_j'(\varepsilon)\phi_j^\varepsilon - i q_j^\varepsilon e + ie \cdot \mu D(\varepsilon e) \phi_j^\varepsilon + i D(\varepsilon e) \cdot \left(\mu \phi_j^\varepsilon \otimes e\right)$$

and

$$g(\varepsilon) = -D'(\varepsilon e)\phi_j^\varepsilon = -ie \cdot \phi_j^\varepsilon,$$

where $D(\varepsilon e)\varphi = \nabla \varphi + i\varepsilon \varphi \otimes e$. To simplify the exposition, we choose as an orthonormal basis of $\mathbb{R}^d$ the family

$$e_1 = u_1(e), \; \dots \; , e_{d-1} = u_{d-1}(e), \; e_d = e. \tag{18}$$

In order to identify the derivatives at $\varepsilon = 0$ in (17) we recall the so-called cell problems [7] which are useful for the classical homogenization of Stokes equations

$$\begin{cases} -\mathrm{div}\,(\mu\nabla(w_{kl} + y_l e_k)) + \nabla\pi_{kl} & = 0 \quad \text{in } Y, \\ \mathrm{div}\, w_{kl} & = 0, \quad \text{in } Y, \end{cases} \tag{19}$$

which admits a unique solution $(w_{kl}, \pi_{kl}) \in (H^1_\#(Y)/R)^d \times L^2_\#(Y)/\mathbb{C}$ for $1 \leq k, l \leq d$. The homogenized tensor A^* is then defined by

$$A^*_{klpq} = \frac{1}{|Y|}\int_Y \mu\nabla(w_{kl} + y_l e_k) : \nabla(w_{pq} + y_q e_p)\, dy.$$

Lemma 4.1. *The first order derivatives at* $\varepsilon = 0$ *satisfy*

$$\lambda'_j(0) = 0,\ \phi'_j(0) = iw_{dj},\ q'_j(0) = i\pi_{dj} \text{ and } q'_{0,j}(0) = 0 \quad j = 1, \ldots, d-1.$$

Proof. By the Fredholm alternative $f(\varepsilon)$ must be orthogonal to ϕ^ε_j, that is,

$$\int_Y f(\varepsilon)\cdot\overline{\phi^\varepsilon_j} = 0,$$

which implies

$$\lambda'_j(\varepsilon)|Y| = i\int_Y q^\varepsilon_j e\cdot\overline{\phi^\varepsilon_j} - i\int_Y \left\{e\cdot\mu D(\varepsilon e)\phi^\varepsilon_j\right\}\cdot\overline{\phi^\varepsilon_j} - i\int_Y \left\{D(\varepsilon e)\cdot\left(\mu\phi^\varepsilon_j\otimes e\right)\right\}\cdot\overline{\phi^\varepsilon_j}$$

since the functions ϕ^ε_j satisfy the normalization condition (16). Taking $\varepsilon = 0$ and recalling (15) we obtain $\lambda'_j(0) = 0$. Moreover, $(\phi'_j(0), q'_j(0), q'_{0,j}(0))$ is solution of the problem

$$\begin{cases} -\mathrm{div}\ \left(\mu\nabla\phi'_j(0)\right) + \nabla q'_j(0) + eq'_{0,j}(0) & = i\mathrm{div}(\mu e_j\otimes e) \qquad \text{in } Y, \\ \mathrm{div}\ \phi'_j(0) & = 0, \qquad \text{in } Y, \\ e\cdot\int_Y \phi'_j(0)dx & = 0 \end{cases} \tag{20}$$

It is not difficult to check that (20) has a solution in $\mathbb{V}_e \times L^2_\#(Y)/\mathbb{C}\times\mathbb{C}$ which is unique up to the addition of a constant vector, orthogonal to e, to the velocity. Furthermore, $\phi'_j(0) = iw_{dj}, q'_j(0) = i\pi_{dj}, q'_{0,j}(0) = 0$ is such a solution since adding a suitable constant to w_{dj} makes its average orthogonal to e. □

We now compute the second order directional derivatives. Differentiating (17) with respect to ε yields

$$\begin{cases} -D(\varepsilon e)\cdot\left(\mu D(\varepsilon e)\phi''_j(\varepsilon)\right) + \phi''_j(\varepsilon) + D(\varepsilon e)q''_j(\varepsilon) + q''_{0,j}(\varepsilon)e - \lambda^\varepsilon_j\phi''_j(\varepsilon) = F(\varepsilon) & \text{in } Y, \\ D(\varepsilon e)\cdot\phi''_j(\varepsilon) = G(\varepsilon), & \text{in } Y, \end{cases}$$

where

$$F(\varepsilon) = -2\mu\phi^\varepsilon_j + 2ie\cdot\mu D(\varepsilon e)\phi'_j(\varepsilon) + 2iD(\varepsilon e)\cdot(\mu\phi'_j(\varepsilon)\otimes e) \\ - 2ieq'_j(\varepsilon) + \lambda''_j(\varepsilon)\phi^\varepsilon_j + 2\lambda'_j(\varepsilon)\phi'_j(\varepsilon) \tag{21}$$

and

$$G(\varepsilon) = -2D(\varepsilon e)'\phi'_j(\varepsilon) = -2ie\cdot\phi'_j(\varepsilon).$$

Lemma 4.2. *The second order derivative of the eigenvalue at $\varepsilon = 0$ satisfies*

$$\frac{1}{2}\lambda_j''(0)\delta_{jk} = A^*_{djdk} \quad j,k = 1,\dots,d-1, \tag{22}$$

where δ_{jk} is the Kronecker symbol and the homogenized tensor A^* is written in the basis (18).

Proof. By the Fredholm alternative $F(\varepsilon)$ must be orthogonal to ϕ_k^ε, which implies

$$\begin{aligned} \lambda_j''(\varepsilon)\delta_{jk}|Y| &= 2\int_Y \mu\phi_j^\varepsilon\overline{\phi_k^\varepsilon} - 2i\int_Y (e\cdot(\mu D(\varepsilon e)\phi_j'(\varepsilon)))\cdot\overline{\phi_k^\varepsilon} \\ &- 2i\int_Y D(\varepsilon e)(\mu\phi_j'(\varepsilon)\otimes e)\cdot\overline{\phi_k^\varepsilon} + 2i\int_Y q_j'(\varepsilon)e\cdot\overline{\phi_k^\varepsilon} - 2\lambda_j'(\varepsilon)\int_Y \phi_j'(\varepsilon)\cdot\overline{\phi_k^\varepsilon} \end{aligned} \tag{23}$$

For $\varepsilon = 0$, we obtain

$$\lambda_j''(0)\delta_{jk}|Y| = 2\int_Y \delta_{jk}\mu + 2\int_Y (e\cdot(\mu\nabla w_{dj}))\cdot e_k = 2\int_Y (\mu(e\otimes e_j + \nabla w_{dj})) : (e\otimes e_k).$$

Since multiplying equation (19) by w_{dk} yields

$$\int_Y (\mu(e\otimes e_j + \nabla w_{dj})) : \nabla w_{dk} = 0,$$

we deduce

$$\frac{1}{2}\lambda_j''(0)\delta_{jk} = A^*_{djdk}. \qquad \square$$

Remark 4.3. Formula (22) is similar to that obtained by Ganesh and Vanninathan in the elasticity case [13]. Recalling our choice (18) of the basis of $\mathbb{R}^d$, it is equivalent to say that $\frac{1}{2}\lambda_j''(0)$ is an eigenvalue and e_j is an eigenvector of the symmetric matrix $(A^*_{didk})_{i,k}$. There is however a notable change with the elasticity case in [13]: the indices j and k in (22) must be different from d, so the alluded matrix eigenvalue problem is of dimension $d-1$ instead of d as in [13]. In other words, the knowledge of only $d-1$ branches of eigenvalues λ_j yields less informations on the homogenized Stokes tensor A^* than in the elasticity case. For example, among others, the homogenized coefficient A^*_{dddd} is not characterized by (22).

In the elasticity case, Ganesh and Vanninathan were able to prove that the homogenized tensor A^* is completely recovered from the knowledge of the d derivatives $\lambda_j''(0)$ and d eigenvectors $e_j = u_j(e)$. One can not expect such a result for the Stokes equation. Indeed, one can check that adding a multiple of $I_2\otimes I_2$ (where I_2 is the identity matrix of order d) to A^* does not change formula (22) because it does not involve coefficients of the type A^*_{jjdd}. On the other hand, the underdeterminacy of A^* up to the addition of $I_2\otimes I_2$ does not matter in the homogenized Stokes equations since

$$\Big((A^*+cI_2\otimes I_2)\nabla u\Big)_{ij} = \sum_{k,l=1}^{d}(A^*_{ijkl}+c\delta_{ij}\delta_{kl})\frac{\partial u_k}{\partial x_l} = \Big(A^*\nabla u\Big)_{ij} + c\nabla\cdot u\delta_{ij} = \Big(A^*\nabla u\Big)_{ij}$$

because u is a divergence-free vector field. Eventually we conjecture that the sole knowledge of the $d-1$ derivatives $\lambda_j''(0)$ and the $d-1$ eigenvectors $e_j = u_j(e)$ completely characterizes the homogenized Stokes tensor, up to the addition of an unimportant $I_2\otimes I_2$ term. However, we have been unable to prove such a result so far.

5. **Bloch wave decomposition of $H(\mathbb{R}^d)$.** Let $\mathcal{V}(\mathbb{R}^d)$ be the space (without topology)

$$\mathcal{V}(\mathbb{R}^d) = \left\{ u \in C_c^\infty(\mathbb{R}^d)^d \ : \ \nabla \cdot u = 0 \right\}$$

The closures of $\mathcal{V}(\mathbb{R}^d)$ in $L^2(\mathbb{R}^d)^d$ and in $H^1(\mathbb{R}^d)^d$ are denoted by $H(\mathbb{R}^d)$ and $V(\mathbb{R}^d)$ respectively.

We associate to every $v \in H(\mathbb{R}^d)$ and to every ξ the following function

$$\widetilde{v}(x,\xi) = \sum_{p \in \mathbb{Z}^d} v(x + 2\pi p) e^{-i\xi\cdot(x+2\pi p)}$$

It is easily seen that $\widetilde{v}(\cdot,\xi)$ belongs to $L^2_\#(Y)$. The right hand-side series converges and define an element of H_ξ as a function of x, satisfying the following properties:

(i) We recover v from $\widetilde{v}$ by the following formula:

$$v(x) = \int_{Y'} e^{i\xi\cdot x} \widetilde{v}(x,\xi) d\xi.$$

(ii) The norms of v and $\widetilde{v}(\cdot,\xi)$ are related by

$$||v||^2_{L^2(\mathbb{R}^d)^d} = \int_{Y'} ||\widetilde{v}(\cdot,\xi)||^2_{L^2_\#(Y)^d} d\xi$$

Since $\widetilde{v}(\xi) \in H_\xi$ and $(\phi_m(\xi))_{m\in\mathbb{N}}$ is a basis of H_ξ, we can write

$$\widetilde{v}(\xi) = \sum_{m=1}^{\infty} \left[\mathfrak{B}_m v(\xi)\right] \phi_m(\xi).$$

Theorem 5.1. *Let $v \in H(\mathbb{R}^d)$ be arbitrary.*

(i) *For $m \in \mathbb{N}$, the m^{th} Bloch coefficient is defined by*

$$\mathfrak{B}_m v(\xi) := \lim_{R\to\infty} \int_{|x|<R} v(x) \cdot e^{-i\xi\cdot x} \overline{\phi_m(\xi,x)} dx$$

where the limit is taken in $L^2(Y')$.

(ii) *Then the following inverse formula holds:*

$$v(x) = \lim_{k\to\infty} \int_{Y'} \sum_{m=1}^{k} \mathfrak{B}_m v(\xi) e^{i\xi\cdot x} \phi_m(\xi,x) d\xi$$

where the limit is taken in the space $H(\mathbb{R}^d)$.

(iii) *In particular, we have the following Parseval identity:*

$$\int_{\mathbb{R}^d} |v(x)|^2 dx = \int_{Y'} \sum_{m=1}^{\infty} |\mathfrak{B}_m v(\xi)|^2 d\xi.$$

(iv) *More generally, the following Plancherel Identity is also valid:*

$$\int_{\mathbb{R}^d} v(x) \cdot \overline{u}(x) dx = \int_{Y'} \sum_{m=1}^{\infty} \mathfrak{B}_m v(\xi) \overline{\mathfrak{B}_m u(\xi)} d\xi.$$

The proof of the above result is analogous to the one in [7] so we omit it.

6. **Bloch waves in a perforated domain.** In this section we briefly explain how the previous results can be extended to the case of perforated domains. This is an important issue since many models of flows in porous media, like Darcy's law, are obtained by homogenization of the Stokes equations in a periodically perforated domain. At least formally, the case of Stokes equations in a domain with holes, supporting a Dirichlet boundary condition, can be recovered by letting the damping coefficient κ goes to $+\infty$ inside the hole in equation (1). Nevertheless, in full mathematical rigor, some previous technical results need a specific proof for perforated domains that we now give.

Let us recall that $Y = [0, 2\pi[^d$ and let $T \subset Y$ be a smooth closed open subset. We assume that the hole T is isolated in the unit cell Y, i.e. the two holes of two adjacent cells do not touch (we denote this assumption by $T \subset\subset Y$). We can thus define:

$$Y^* = Y \setminus \overline{T} \quad \text{and} \quad \mathcal{O} = \mathbb{R}^d \setminus \cup_{p\in\mathbb{Z}^d} \{\overline{T + 2\pi p}\}$$

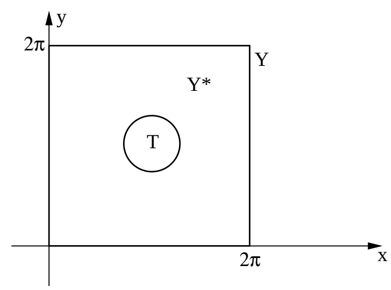

FIGURE 1. Periodic perforated cell

At first, we consider the Stokes equations with a source term $f \in L^2_\#(Y^*)$, namely

$$\left\{ \begin{array}{rll} -D(\xi)\cdot(\mu(y)D(\xi)\phi) + D(\xi)p & = f & \text{in } Y^* \\ D(\xi)\cdot\phi & = 0 & \text{in } Y^* \\ \phi & = 0 & \text{on } \partial T \\ p, \phi \text{ are} & Y - \text{periodic} & \end{array} \right. \tag{24}$$

We introduce the spaces of "generalized" divergence-free velocities

$$\begin{aligned} V_\xi(Y^*) &= \{\phi \in H^1_\#(Y^*)^d \;:\; D(\xi)\cdot\phi = 0 \text{ and } \phi = 0 \text{ on } \partial T\} \\ H_\xi(Y^*) &= \{\phi \in L^2_\#(Y^*)^d \;:\; D(\xi)\cdot\phi = 0 \text{ and } \phi\cdot\mathbf{n} = 0 \text{ on } \partial T\} \end{aligned} \tag{25}$$

In order to obtain uniform a priori estimate for the pressure in (25) we first need to generalize a result of Tartar [28] about a restriction operator on vector fields.

Lemma 6.1. *For each $\xi \in Y'$ there exists an operator*

$$R_\xi \;:\; H^1_\#(Y)^d \to H^1_\#(Y)^d,$$

such that $\|R_\xi(\varphi)\|_{H^1_\#(Y)^d} \leq C\|\varphi\|_{H^1_\#(Y)^d}$ $\forall\varphi \in H^1_\#(Y)^d$ where C does not depend on ξ and it satisfies:

(i) *If $D(\xi)\cdot\varphi = 0$, then $D(\xi)\cdot R_\xi(\varphi) = 0$.*
(ii) *For each $\varphi \in H^1_\#(Y)^d$, $R_\xi(\varphi) = 0$ in T.*
(iii) *If $\varphi = 0$ in T, then $R_\xi(\varphi) = \varphi$.*

Proof. From Lemma 4 pp 373 in [28], there exists an operator

$$R \ : \ H^1(Y)^d \to H^1(Y)^d$$

such that

$$||R\mathbf{w}||_{H^1(Y^d)} \leq C_1 ||\mathbf{w}||_{H^1(Y)^d}$$

for each $\mathbf{w} \in H^1(Y)^d$. This operator R satisfies:
(a) If $\nabla \cdot \mathbf{w} = 0$, then $\nabla \cdot R\mathbf{w} = 0$.
(b) For each $\mathbf{w} \in H^1(Y)^d$, $R\mathbf{w} = 0$ in $T \subset\subset Y$.
(c) If $\mathbf{w} = 0$ in T, then $R\mathbf{w} = \mathbf{w}$.

Define $R_\xi(\varphi) = e^{-ix\cdot\xi} R(e^{ix\cdot\xi}\varphi)$. The properties (ii) and (iii) of R_ξ follow directly from the properties (b) and (c) of R. Property (i) follows from property (a) and the equality $\nabla \cdot (e^{ix\cdot\xi}\varphi) = e^{ix\cdot\xi}(\nabla \cdot \varphi + i\xi \cdot \varphi)$, $\forall \varphi \in H^1_\#(Y)^d$. By the existence of constant C_1 for R, we can deduce the existence of constant C for R_ξ. $\square$

We establish an adequate version of De Rham's Theorem with a proof which is different from those of Lemmas 2.2 and 2.3. We denote by $H^1_{0,\#}(Y^*)$ the subspace of $H^1_\#(Y)$ made of functions vanishing on T, and by $H^{-1}_\#(Y^*)$ its dual.

Lemma 6.2. *If $f \in H^{-1}_\#(Y^*)$ is such that $< f, \phi >_{H^{-1}_\#, H^1_{0,\#}(Y^*)} = 0$ for any $\phi \in V_\xi(Y^*)$, then there exists $P \in L^2_\#(Y)$ such that, denoting by p the restriction $P|_{Y^*}$, we have $D(\xi)p = f$ and*

$$\|p - m(P)\|_{L^2(Y^*)} + \|D(\xi)P\|_{H^{-1}_\#(Y)^d} + |\xi m(P)| \leq C\|f\|_{H^{-1}_\#(Y^*)^d}.$$

In particular, if it is already known that $D(\xi)p = f$ with $p \in L^2_\#(Y^*)$, then the above $P \in L^2_\#(Y)$ is an extension of p.

Proof. Define $F \in H^{-1}_\#(Y)$ by $F = R^*_\xi(f)$, in other words:

$$< F, \psi >_{H^{-1}_\#, H^1_\#(Y)} = < f, R_\xi(\psi) >_{H^{-1}_\#, H^1_{0,\#}(Y^*)}, \ \forall \psi \in H^1_\#(Y)^d. \tag{26}$$

Since F satisfies

$$< F, \psi >_{H^{-1}_\#, H^1_\#(Y)} = 0, \ \forall \psi \in V_\xi(Y),$$

it follows from Lemma 2.2 that there exists a unique $P \in L^2_\#(Y)$ such that $D(\xi)P = F$. Taking $\psi \in H^1_{0\#}(Y^*)^d$ in (26) we get that $D(\xi)p = f$ since $R_\xi(\psi) = \psi$. From (26) we easily obtain

$$\|D(\xi)P\|_{H^{-1}_\#(Y)^d} \leq C\|D(\xi)p\|_{H^{-1}_\#(Y^*)^d}.$$

Since

$$\|p - m(P)\|_{L^2(Y^*)} \leq \|P - m(P)\|_{L^2(Y)},$$

the other estimates are then a consequence of Lemma 2.3. $\square$

Proposition 6.3. *For $\xi \in Y' \setminus \{0\}$ and $f \in L^2_\#(Y^*)^d$, there exists a unique solution $(\phi, p) \in H^1_\#(Y^*)^d \times L^2_\#(Y^*)$ of (24), which satisfies*

$$\begin{aligned} &\|\phi\|_{1,Y} \leq C\|f\|_{0,Y^*} \qquad\qquad \|p - m(P)\|_{0,Y^*} \leq C\|f\|_{0,Y^*} \\ &|\xi m(P)| \leq C\|f\|_{0,Y^*}, \end{aligned} \tag{27}$$

where the constant $C > 0$ does not depend on ξ and P is the extension of p given in Lemma 6.2. As usual, for $\xi = 0$, there exists a unique solution $(\phi, p) \in H^1_\#(Y^*)^d \times L^2_\#(Y^*)/\mathbb{C}$ of (24).

Proof. For $\xi = 0$ the result is well known [18]. For $\xi \neq 0$ we define the bilinear form on $H^1_{0,\#}(Y^*)$

$$a(\varphi, \psi) = \int_Y \mu D(\xi)\varphi : \overline{D(\xi)\psi}, \qquad \forall \varphi, \psi \in V_\xi(Y^*),$$

which is easily seen to be symmetric, continuous and coercive. Furthermore, since a Poincaré inequality is satisfied by the functions belonging to $H^1_{0,\#}(Y^*)^d$, the bilinear form is uniformly coercive. In the same manner as in Proposition 2.1, we can show that there exist $\varphi \in H^1_{0,\#}(Y^*)^d$ and $p \in L^2(Y^*)$ such that

$$D(\xi)p = D(\xi) \cdot (\mu D(\xi)\phi) + f$$

which proves the existence and uniqueness of the solution of (24).

Furthermore, since the bilinear form is uniformly coercive, we prove that

$$C\|\phi\|_{1,Y^*} \leq \|f\|_{0,Y^*}.$$

Introducing $g = D(\xi) \cdot (\mu D(\xi)\phi) + f) = D(\xi)p$, we have $\|g\|_{-1,Y^*} \leq C\|f\|_{Y^*}$. Lemma 6.2 then yields the estimates on the pressure. □

As in the case of the non-perforated domain, any function $\phi \in H^1_{0,\#}(Y)^d$, such that $D(\xi) \cdot \phi = 0$, satisfies also the constraint

$$\xi \cdot \int_Y \phi(y)\, dy = 0,$$

and given $e \in \mathbb{R}^d$ with $|e| = 1$, we define

$$\mathbb{V}_e(Y^*) = \{\phi \in H^1_{0,\#}(Y^*)^d \text{ such that } \nabla \cdot \phi = 0,\ e \cdot \textstyle\int_{Y^*} \phi(y)\, dy = 0 \text{ and } \phi = 0 \text{ on } \partial T\}$$

$$\mathbb{H}_e(Y^*) = \{\phi \in L^2_{\#}(Y^*)^d \text{ such that } \nabla \cdot \phi = 0,\ e \cdot \textstyle\int_{Y^*} \phi(y)\, dy = 0 \text{ and } \phi \cdot \mathbf{n} = 0 \text{ on } \partial T\}$$

It is easily seen that $\mathbb{V}_e(Y^*)^\perp = \{\nabla\rho + ce \mid \rho \in L^2_{\#}(Y^*),\ c \in \mathbb{R}\}$.

For a given $e \in \mathbb{R}^d$ with $|e| = 1$, there exists $(u, q, q_0) \in H^1_{0,\#}(Y^*)^d \times L^2_{\#}(Y^*) \times \mathbb{R}$ solution of the system

$$\begin{cases} -\nabla \cdot (\mu \nabla u) + \nabla q + q_0 e &= f \qquad \text{in } Y^* \\ e \cdot \displaystyle\int_{Y^*} u(y)\, dy &= 0 \\ \nabla \cdot u &= 0 \qquad \text{in } Y^* \\ u &= 0 \qquad \text{on } \partial T \\ q, u \text{ are } & Y^* - \text{periodic} \end{cases} \tag{28}$$

Proposition 6.4. *For a given unit vector $e \in \mathbb{R}^d$, $|e| = 1$, we define $\xi = \varepsilon e$. Then, as ε tends to 0, the solution $(\phi(\xi), p(\xi))$ of (24) satisfies*

$$\begin{array}{rcl} \phi(\xi) & \to & u(e) \text{ strongly in } H^1_{0,\#}(Y^*)^d \\ p(\xi) - m(P(\xi)) & \to & q(e) \text{ strongly in } L^2_{\#}(Y^*) \\ i\xi m(P(\xi)) & \to & q_0(e)e, \end{array}$$

where $(u(e), q(e), q_0(e))$ is solution of (28) and $P(\xi)$ is the extension of $p(\xi)$ given in Lemma 6.2.

For $\xi \in Y'$ we consider the eigenvalue problem

$$\begin{cases} \text{Find } \lambda(\xi) \in \mathbb{R}, \phi \neq 0 \in H^1_{0,\#}(Y^*)^d,\ p \in L^2_{\#}(Y^*) \quad \text{such that} \\ \qquad -D(\xi) \cdot (\mu(y) D(\xi)\phi) + D(\xi)p = \lambda(\xi)\phi \qquad \text{in } Y^*, \\ \qquad D(\xi) \cdot \phi = 0, \qquad \text{in } Y^*. \end{cases} \tag{29}$$

To study the continuity of (29) when ξ converges to 0, we rewrite (29) in a slightly different form. For a given unit vector $e \in \mathbb{R}^d$, $|e| = 1$, we introduce a scalar parameter $\varepsilon \in \mathbb{R}$ and define

$$\xi = \varepsilon e.$$

The eigenvalue problem (29) is thus rewritten

$$\left\{\begin{array}{lll} \text{Find } \lambda^\varepsilon \in \mathbb{R}, \phi^\varepsilon \neq 0 \in H^1_{0,\#}(Y^*)^d,\ p^\varepsilon \in L^2_\#(Y^*) \quad \text{such that} & & \\ -D(\varepsilon e)\cdot(\mu(y)D(\varepsilon e)\phi^\varepsilon) + D(\varepsilon e)p^\varepsilon & = \lambda^\varepsilon \phi^\varepsilon & \text{in } Y^*, \\ D(\varepsilon e)\cdot \phi^\varepsilon & = 0, & \text{in } Y^*. \end{array}\right. \tag{30}$$

Introducing a new pressure variable $q^\varepsilon \in L^2_\#(Y^*)$ defined by

$$q^\varepsilon = p^\varepsilon - m(P^\varepsilon),$$

we obtain that

$$D(\varepsilon e)p^\varepsilon = D(\varepsilon e)q^\varepsilon + q_0^\varepsilon e \quad \text{with} \quad q_0^\varepsilon = i\varepsilon\, m(P^\varepsilon).$$

On the other hand, as already said in Remark 2.4, the condition $D(\varepsilon e)\cdot\phi^\varepsilon = 0$ implies that

$$e \cdot \int_{Y^*} \phi^\varepsilon(y)\,dy = 0.$$

Therefore, problem (30) is equivalent to

$$\left\{\begin{array}{lll} \text{Find } \lambda^\varepsilon \in \mathbb{R}, \phi^\varepsilon \neq 0 \in H^1_{0,\#}(Y^*)^d,\ q^\varepsilon \in L^2_\#(Y),\ q_0^\varepsilon \in \mathbb{C} \quad \text{such that} & & \\ -D(\varepsilon e)\cdot(\mu(y)D(\varepsilon e)\phi^\varepsilon) + D(\varepsilon e)q^\varepsilon + q_0^\varepsilon e & = \lambda^\varepsilon\phi^\varepsilon & \text{in } Y^*, \\ D(\varepsilon e)\cdot\phi^\varepsilon & = 0, & \text{in } Y^*, \\ e\cdot\int_{Y^*}\phi^\varepsilon(y)\,dy & = 0. & \end{array}\right. \tag{31}$$

We shall establish that the limit spectral problem of (31) is

$$\left\{\begin{array}{lll} \text{Find } \nu(e) \in \mathbb{R}, u(e) \neq 0 \in H^1_{0,\#}(Y^*)^d,\ q(e) \in L^2_\#(Y^*), \quad q_0 \in \mathbb{R} \text{ such that} & & \\ -\nabla\cdot(\mu\nabla u(e)) + \nabla q(e) + q_0 e & = \nu(e)u(e) & \text{in } Y^*, \\ \nabla\cdot u(e) & = 0, & \text{in } Y^*, \\ e\cdot\int_{Y^*} u(e) & = 0. & \end{array}\right. \tag{32}$$

Theorem 6.5. *Assume that ν is an eigenvalue of multiplicity h of the Stokes system* (32)*. Then there exist h analytic functions defined in a neighborhood of $\varepsilon = 0$ with values in $\mathbb{R}, \varepsilon \to \lambda_j^\varepsilon$, and h analytic functions $\varepsilon \to (\phi_j^\varepsilon, q_j^\varepsilon, q_{0,j}^\varepsilon)$, with values in $H^1_{0,\#}(Y^*)^d \times L^2_\#(Y^*) \times \mathbb{C}$, $j = 1, \ldots, h$, defined in a neighborhood of $\varepsilon = 0$, such that*

1. $\lambda_j^\varepsilon|_{\varepsilon=0} = \nu, \qquad j = 1, \ldots, h,$
2. *for all ε small enough, $\left(\lambda_j^\varepsilon, \phi_j^\varepsilon, q_j^\varepsilon, q_{0,j}^\varepsilon\right)$ is a solution of (31),*
3. *for all ε small enough the set $\{\phi_1^\varepsilon, \ldots, \phi_h^\varepsilon\}$ is orthonormal in $L^2_\#(Y^*)$,*
4. *for each interval $I \subset \mathbb{R}$ such that $\overline{I}$ contains only the eigenvalue ν, and for all ε small enough, there are exactly h eigenvalues (counting the multiplicity) $\lambda_1^\varepsilon, \ldots, \lambda_h^\varepsilon$ of* (30) *contained in I.*

The proof of Theorem 6.5 follows as the proof of Theorem 3.7.

The above results allow us to compute directional derivatives of the first Bloch eigenvalue.

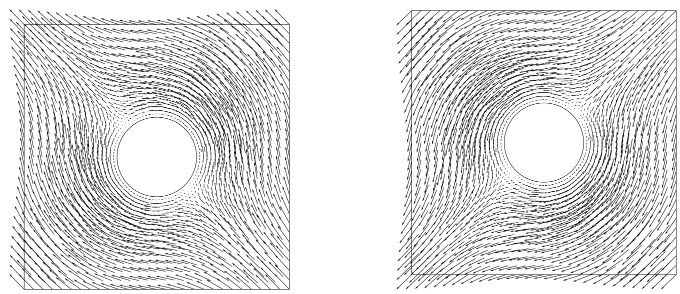

FIGURE 2. Two eigenvectors for the first (double) Bloch eigenvalue for a circular hole with $\theta = 0$ (periodic case)

7. **Numerical results.** To confirm our analysis and to check in which cases the Bloch eigenvalues are discontinuous at the origin, we perform numerical computations in dimension $d = 2$ with the finite element software FreeFem++ [14]. We compute the first and second eigenvalues and eigenvectors of (1) in a unit cell $(0,1)^2$ perforated by a hole T which is either a disk of radius 0.15, or an ellipsoid of principal axes aligned with the cell axes and of half sizes 0.1, 0.2. The holes support a Dirichlet boundary condition. The viscosity is uniform, $\mu(y) \equiv 1$, and there is no zero-order term, $\kappa(y) \equiv 0$. By using a rescaled dual variable $\xi = 2\pi\theta$ we still have $Y' = (0,1)^2$ as the dual cell for θ. The incompressibility constraint is obtained by penalization, i.e. instead of solving (1) we solve

$$\left\{ \begin{array}{llll} \text{Find } \ \lambda(\theta) \in \mathbb{R}, \phi \neq 0 \in H^1_\#((0,1)^2)^2 & \text{such that} & & \\ -D(\theta)\cdot(D(\theta)\phi) - \nu D(\theta)(D(\theta)\cdot\phi) & = \lambda(\theta)\phi & \text{in } (0,1)^2 \setminus T, & \\ \phi & = 0, & \text{on } \partial T, & \end{array} \right. \tag{33}$$

with $\nu = 10^5$ and $D(\theta) = (\nabla + 2i\pi\theta)$. We use P_2 finite elements and a triangular mesh with 7767 nodes for the circular hole and 7919 nodes for the ellipsoidal hole. We checked that our results are converged both with respect to mesh refinement and incompressibility penalization.

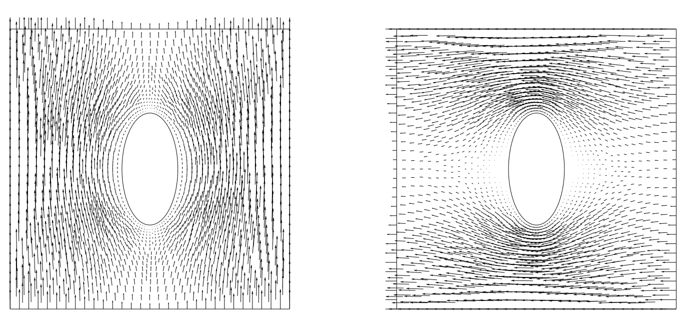

FIGURE 3. First (left) and second (right) Bloch eigenvectors for an ellipsoidal hole with $\theta = 0$ (periodic case)

The main difference between these two geometries is that, in the periodic case $\theta = 0$, the first eigenvalue is double for the circular hole (see Figure 2) and simple for the ellipsoidal hole (see Figure 3). We plot the functions $\theta \to \lambda_1(\theta), \lambda_2(\theta)$ on the dual cell $(-0.5, +0.5)^2$ with a zoom in the neighborhood $(-0.05, +0.05)^2$ of the origin (see Figures 4 and 5). In both cases we clearly see that the second eigenvalue is discontinuous at $\theta = 0$. However, only in the case of the ellipsoidal hole is the first eigenvalue discontinuous: we see on Figure 5 that $\lambda_1(\theta)$ is continuous in the x direction but discontinuous in the y direction. Actually, the limit of $\lambda_1(\theta)$ in the y direction is precisely equal to $\lambda_2(0)$. We checked that higher eigenvalues (typically the 3rd and 4th) are also discontinuous at the origin for both geometries.

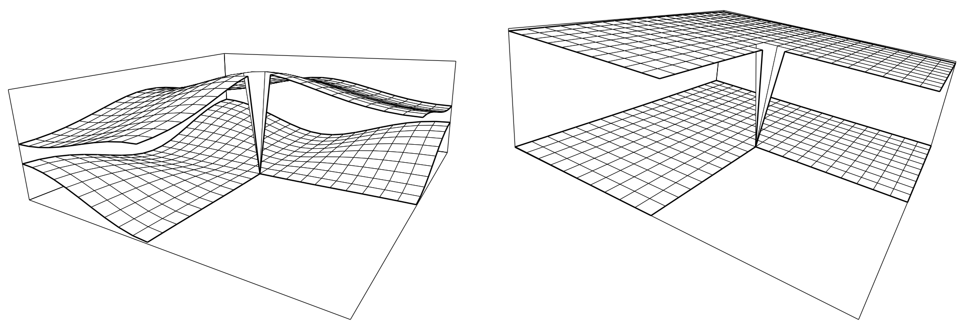

FIGURE 4. First and second Bloch eigenvalue in Y' for a circular hole: global picture (left), zoom around the origin (right)

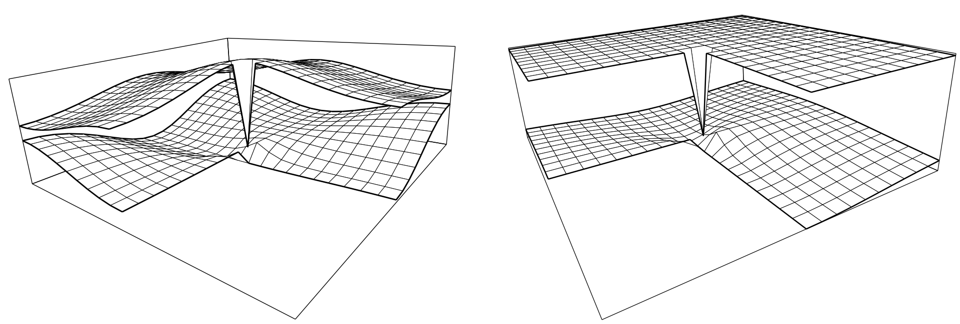

FIGURE 5. First and second Bloch eigenvalue in Y' for an ellipsoidal hole: global picture (left), zoom around the origin (right)

In view of these numerical simulations it seems that, for symmetric obstacles in a Stokes flow, the first eigenvalue is "smooth". More precisely, we conjecture that the first eigenvalue $\lambda_1(\epsilon, e)$ of the spectral problem (12) is differentiable (again for symmetric holes). Recall that (12) is equivalent to (33) for $\theta \neq 0$, with $2\pi\theta = \epsilon e$ with $\epsilon \in \mathbb{R}$ and e a unit vector in $\mathbb{R}^d$. If this were true, such a result would pave the way for the homogenization of the unsteady Stokes equations in a porous media by using methods proposed in [4]. Finally, let us mention that this discontinuity phenomenon for the Bloch eigenvalues at the origin was already numerically observed for a different model of fluid-structure interaction in [1].

8. **Appendix.** In this appendix we give the proof of Theorem 3.7, i.e. the regularity in each direction e. To prove this regularity of the eigenvalues and eigenfunctions we use the Lyapunov-Schmidt method (see [30], [8, pp. 30])

Lemma 8.1. [8, Lemma 4.1, pp. 31] *Suppose that X and Z are Hilbert spaces and $A : X \longrightarrow Z$ is a continuous linear operator. Let $U : X \longrightarrow N(A)$, $E : Z \longrightarrow R(A)$ be the orthogonal projection from X and Z on the kernel and range of A respectively.*

Then, there exists a bounded linear operator $K : R(A) \longrightarrow N(A)^{\perp}$ called the right inverse of A such that

$$AK = I : R(A) \longrightarrow R(A), \qquad KA = I - U : X \longrightarrow N(A)^{\perp}.$$

Let Λ be a closed subset of a Banach space, such that $Int\Lambda \neq \emptyset$. If $N : X \times \Lambda \longrightarrow Z$ is a continuous operator, then the problem

$$Ax - N(x, \lambda) = 0 \tag{34}$$

is equivalent to the equations:

$$z - KEN(y + z, \lambda) = 0 \tag{35}$$

$$(I - E)N(y + z, \lambda) = 0,$$

where $x = y + z$, $y \in N(A)$ and $z \in N(A)^{\perp}$.

Assume that the operator N verifies that

$$N(0, 0) = 0, \qquad \frac{\partial N}{\partial x}(0, 0) = 0$$

and consider the equation (35), for (x, λ) in a neighborhood of $(0, 0)$ in $X \times \Lambda$. Applying the Implicit Function Theorem to (35), we deduce the existence of a neighborhood $V \subset N(A) \times \Lambda$ of $(0, 0)$ and a function $z^* : V \longrightarrow N(A)^{\perp}$ with the same regularity of N providing the solution of (35). Therefore, if $\{y_1, \ldots, y_h\}$ is an orthonormal basis of $N(A)$, the solution $x(\lambda)$ of (35) satisfies

$$x(\lambda) = \sum_{i=1}^{h} c_i(\lambda) y_i + z^* \left(\sum_{i=1}^{h} c_i(\lambda) y_i, \lambda \right),$$

for suitable coefficients $c_1, \ldots, c_h$. Then, $(x, \lambda) \in V$ satisfy (34) iff

$$(I - E)\, N \left(\sum_{i=1}^{h} c_i(\lambda) y_i + z^* \left(\sum_{i=1}^{h} c_i(\lambda) y_i, \lambda \right), \lambda \right) = 0$$

which is a finite dimensional system of equations on the constants $c_1, \ldots, c_h$.

Let us define the operator S as

$$S : (-\varepsilon_0, \varepsilon_0) \longrightarrow \mathcal{L}\left(\left(H^1_{\#}(Y)\right)^d \times L^2_{\#,0}(Y) \times \mathbb{C}; \left(H^{-1}_{\#}(Y)\right)^d \times L^2_{\#,0}(Y) \times \mathbb{C} \right),$$

with

$$S(\varepsilon)(\varphi, \pi, r) = \begin{pmatrix} -D(\varepsilon e)\left(\mu D(\varepsilon e)\varphi\right) + \kappa\varphi + D(\varepsilon e)\pi + re \\ D(\varepsilon e) \cdot \varphi - \frac{1}{|Y|} \int_Y D(\varepsilon e) \cdot \varphi \\ e \cdot \int_Y \varphi \end{pmatrix}$$

Clearly, we have the following result.

Lemma 8.2. *The map S is analytic in a neighborhood of $\varepsilon = 0$ with values in* $\mathcal{L}\left(\left(H^1_\#(Y)\right)^d \times L^2_{\#,0}(Y) \times \mathbb{C}; \left(H^{-1}_\#(Y)\right)^d \times L^2_{\#,0}(Y) \times \mathbb{C}\right)$.

Now, we define two mappings T and A by

$$T : H^1_\#(Y)^d \times L^2_{\#,0}(Y) \times \mathbb{C} \to H^{-1}_\#(Y)^d \times L^2_{\#,0}(Y) \times \mathbb{C},$$

$$T(\varphi, \pi, r) = \begin{pmatrix} \varphi \\ 0 \\ 0 \end{pmatrix}$$

and $A = S(0) - \nu T$, with ν an eigenvalue of $S(0)$, that is,

$$A : H^1_\#(Y)^d \times L^2_{\#,0}(Y) \times \mathbb{C} \to H^{-1}_\#(Y)^d \times L^2_{\#,0}(Y) \times \mathbb{C},$$

$$A(\varphi, \pi, r) = \begin{pmatrix} -\mathrm{div}\,(\mu\nabla\varphi) + \kappa\varphi + \nabla\pi + re - \nu\phi \\ \nabla\cdot\varphi \\ e \cdot \displaystyle\int_Y \varphi. \end{pmatrix}$$

By definition A is self-adjoint and $R(A) = N(A)^\perp$ where $N(A)$ is the eigenspace associated to the eigenvalue ν of $S(0)$.

In order to prove Theorem 3.7 we first prove that, if ν is an eigenvalue of multiplicity h, we can find a first set of eigenvalue $\varepsilon \to \lambda^\varepsilon \in \mathbb{R}$, and eigenfunctions $\varepsilon \longrightarrow (\phi^\varepsilon, q^\varepsilon, q_0^\varepsilon) \in H^1_\#(Y)^d \times L^2_{\#,0}(Y) \times \mathbb{C}$ for (12). To find the other $h-1$ branches of eigenvalues and eigenfunctions, we shall apply later an iterative method.

Proposition 8.3. *Assume that ν is an eigenvalue of multiplicity h of the problem* (10)*. Then there exists at least one function $\varepsilon \longrightarrow (\lambda^\varepsilon, \phi^\varepsilon, q^\varepsilon, q_0^\varepsilon) \in \mathbb{R} \times H^1_\#(Y)^d \times L^2_{\#,0}(Y) \times \mathbb{C}$, which is analytic in a neighborhood of $\varepsilon = 0$, such that*

1. $\lambda^\varepsilon|_{(\varepsilon=0)} = \nu$,
2. $(\phi^\varepsilon, q^\varepsilon, q_0^\varepsilon)$ *is a solution of* (12) *for $\xi = \varepsilon e$, associated to the eigenvalue λ^ε.*

Proof of Proposition 8.3. Let ν be an eigenvalue of multiplicity h of the problem (10) and let $(u_j, q_j, q_{0,j}),\ j = 1, \ldots, h$, be the associated eigenfunctions in $H^1_\#(Y)^d \times L^2_{\#,0}(Y) \times \mathbb{C}$ such that $u_1, \ldots, u_h$ is orthonormal in $L^2_\#(Y)^d$. By definition of the operators $S(\varepsilon)$ and T, $(\phi^\varepsilon, q^\varepsilon, q_0^\varepsilon)$ is an eigenfunction associated to the eigenvalue λ^ε if

$$S(\varepsilon)\,(\phi^\varepsilon, q^\varepsilon, q_0^\varepsilon) - \lambda^\varepsilon T\,(\phi^\varepsilon, q^\varepsilon, q_0^\varepsilon) = 0,$$

or, introducing $R^\varepsilon = S(0) - S(\varepsilon)$, equivalently if

$$[S(0) - \nu T]\,(\phi^\varepsilon, q^\varepsilon, q_0^\varepsilon) = [R^\varepsilon + (\lambda^\varepsilon - \nu)T]\,(\phi^\varepsilon, q^\varepsilon, q_0^\varepsilon)\,. \tag{36}$$

From Lemma 8.1 we know that the map $A = S(0) - \nu T$ has a right inverse operator K. Thus, in view of (36) we obtain that

$$(\phi^\varepsilon, q^\varepsilon, q_0^\varepsilon) = [K\,(R^\varepsilon + (\lambda^\varepsilon - \nu)\,T)]\,(\phi^\varepsilon, q^\varepsilon, q_0^\varepsilon) + (\psi^\varepsilon, \pi^\varepsilon, \pi_0^\varepsilon), \tag{37}$$

where $(\psi^\varepsilon, \pi^\varepsilon, \pi_0^\varepsilon) \in N(A)$, that is,

$$(\psi^\varepsilon, \pi^\varepsilon, \pi_0^\varepsilon) = \sum_{l=1}^{h} c_l(\varepsilon)(u_l, q_l, q_{0,l}). \tag{38}$$

On the other hand, from (36)

$$[R^\varepsilon + (\lambda^\varepsilon - \nu)T]\,(\phi^\varepsilon, q^\varepsilon, q_0^\varepsilon) \in R(A) = N(A)^\perp.$$

Thus, introducing $Q^\varepsilon = R^\varepsilon + (\lambda^\varepsilon - \nu)T$, we have that

$$\begin{aligned}
0 &= \langle Q^\varepsilon(\phi^\varepsilon, q^\varepsilon, q_0^\varepsilon), (u_j, q_j, q_{0,j})\rangle \\
&= \left\langle Q^\varepsilon \left[I - KQ^\varepsilon\right]^{-1} (\psi^\varepsilon, \pi^\varepsilon, \pi_0^\varepsilon), (u_j, q_j, q_{0,j})\right\rangle \\
&= \left\langle Q^\varepsilon \left[I - KQ^\varepsilon\right]^{-1} \sum_{l=1}^{h} c_l(\varepsilon)(u_l, q_l, q_{0,l}), (u_j, q_j, q_{0,j})\right\rangle \qquad (39)\\
&= \sum_{l=1}^{h} c_l(\varepsilon) \left\langle Q^\varepsilon \left[I - KQ^\varepsilon\right]^{-1} (u_l, q_l, q_{0,l}), (u_j, q_j, q_{0,j})\right\rangle,
\end{aligned}$$

for all $j = 1, ..., h$ which is a linear system of equations on the unknowns $c_l(\varepsilon)$. This system has a non trivial solution if and only if

$$\det\left(\left\langle Q^\varepsilon \left[I - KQ^\varepsilon\right]^{-1} (u_l, q_l, q_{0,l}), (u_j, q_j, q_{0,j})\right\rangle\right) = 0.$$

We replace $\lambda^\varepsilon - \nu$ by α and we define $\widehat{R}(\varepsilon, \alpha) = R^\varepsilon + \alpha T$,

$$f_{lj}(\varepsilon, \alpha) = \left\langle \left[\widehat{R}(\varepsilon, \alpha)\right] \left[I - K\left(\widehat{R}(\varepsilon, \alpha)\right)\right]^{-1} (u_l, q_l, q_{0,l}), (u_j, q_j, q_{0,j})\right\rangle, \qquad (40)$$

and

$$F(\varepsilon, \alpha) = \det\left(f_{lj}(\varepsilon, \alpha)\right). \qquad (41)$$

For ε small enough, the map $\varepsilon \longrightarrow \left[I - K\widehat{R}(\varepsilon, \alpha)\right]^{-1}$ is well defined. Indeed for $\alpha = 0$ and $\varepsilon = 0$ we have that $\left[I - K\widehat{R}(0, 0)\right] = I$ and the map is analytic in a neighborhood of $\varepsilon = 0$. On the other hand, as we mentioned above, if $F(\varepsilon, \alpha) = 0$, system (39) has a non trivial solution $c_1(\varepsilon), \ldots, c_h(\varepsilon)$, and then $\lambda^\varepsilon = \nu + \alpha$ is an eigenvalue of (10). Moreover from (37) and (38) we deduce that

$$(\phi^\varepsilon, q^\varepsilon, q_0^\varepsilon) = \sum_{l=1}^{h} c_l(\varepsilon) \left[I - K\left(R^\varepsilon + (\lambda^\varepsilon - \lambda)T\right)\right]^{-1} (u_l, q_l, q_{0,l}) \qquad (42)$$

is an eigenfunction of (10) for $\xi = \varepsilon e$, associated to the eigenvalue λ^ε.

According to our previous discussion, for these values of $\alpha(\varepsilon)$ and setting $\lambda^\varepsilon = \nu + \alpha(\varepsilon)$, system (39) admits a solution $c_1(\varepsilon), ..., c_h(\varepsilon)$, not all the components being zero. We have that

$$f_{lj}(0, \alpha) = \left\langle [\alpha T] \left[I - \alpha KT\right]^{-1} (u_l, q_l, q_{0,l}), (u_j, q_j, q_{0,j})\right\rangle.$$

For α sufficiently small, the operator $I - \alpha KT$ is invertible and, moreover,

$$[I - \alpha KT]^{-1} = \sum_{n\geq 0} (\alpha KT)^n = I + \sum_{n\geq 1} \alpha^n (KT)^n.$$

Therefore, for all $l, j = 1, ..., h$ we have

$$\begin{aligned}
f_{lj}(0, \alpha) &= \left\langle [\alpha T] \left[I - \alpha KT\right]^{-1} (u_l, q_l, q_{0,l}), (u_j, q_j, q_{0,j})\right\rangle \\
&= \alpha \left\langle T(u_l, q_l, q_{0,l}), (u_j, q_j, q_{0,j})\right\rangle + \alpha \sum_{n\geq 1} \alpha^n \left\langle T(KT)^n(u_l, q_l, q_{0,l}), (u_j, q_j, q_{0,j})\right\rangle \\
&= \alpha\delta_{lj} + \sum_{n\geq 1} \alpha^{n+1} \left\langle T(KT)^n(u_l, q_l, q_{0,l}), (u_j, q_j, q_{0,j})\right\rangle. \qquad (43)
\end{aligned}$$

Thus

$$F(0,\alpha)=\alpha^h+\sum_{n\geq 1}s_n\alpha^{n+h}$$

for suitable coefficients s_n, and F satisfies

$$\frac{\partial^r F}{\partial\alpha^r}(0,0)=0,\ r=0,...,h-1;\quad \frac{\partial^h F}{\partial\alpha^h}(0,0)\neq 0.$$

Applying Weierstrass Preparation Theorem we deduce that

$$F(\varepsilon,\alpha)=\left(\alpha^h+a_1(\varepsilon)\alpha^{h-1}+\ldots+a_h(\varepsilon)\right)E(\varepsilon,\alpha)$$

with $E(\varepsilon,\alpha)\neq 0$ in a neighborhood of $(0,0)$. Then for (ε,α) small enough we have that $E(\varepsilon,\alpha)\neq 0$ and the functions $a_j(\varepsilon)$ are analytic in a neighborhood of $\varepsilon=0$. Consequently, $F(\varepsilon,\alpha)=0$ if and only if

$$\alpha^h+a_1(\varepsilon)\alpha^{h-1}+\ldots+a_h(\varepsilon)=0. \tag{44}$$

Let $\alpha_j(\varepsilon)$, $j=1,\ldots,h$, be the complex roots of (44). According to (42) $\lambda^\varepsilon=\nu+\alpha_1(\varepsilon)$ is an eigenvalue of (10).

Note that if $c_j(\varepsilon)$ is complex, it is enough to consider the real part $\mathcal{R}c_j(\varepsilon)$ to get a real eigenfunction. Since the Stokes operator is self-adjoint we have that $\alpha_j(\varepsilon)$ is real, which completes the proof of Proposition 8.3. $\square$

Remark 8.4. Proposition 8.3 yields the existence of one branch of eigenpairs associated to the root $\alpha(\varepsilon)$ of (44). We do not use the eigenpairs associated to the other roots α_j by now since, so far, we do not know whether they coincide or not with the eigenpair associated to $\alpha_1(\varepsilon)$.

We are now equiped to finish the proof of Theorem 3.7.

Proof of Theorem 3.7. By using an iterative argument on h we prove the existence of the h analytic functions $\varepsilon\to\left(\lambda_j^\varepsilon,\phi_j^\varepsilon,q_j^\varepsilon,q_{0,j}^\varepsilon\right)$, $j=1,\ldots,h$, which are the eigenvalues and eigenfunctions of (10). From Proposition 8.3 there exists at least one analytic function $\varepsilon\to\left(\lambda_1^\varepsilon,\phi_1^\varepsilon,q_1^\varepsilon,q_{0,1}^\varepsilon\right)$ defined in a neighborhood of $\varepsilon=0$ with values in $\mathbb{R}\times H_\#^1(Y)^d\times L_{\#,0}^2(Y)\times\mathbb{R}$, λ_1^ε being an eigenvalue of the Stokes system, $(\phi_1^\varepsilon,q_1^\varepsilon,q_{0,1}^\varepsilon)$ the corresponding eigenfunction. Therefore, Theorem 3.7 holds for $h=1$. We must prove it for $h\geq 2$.

Let $\Pi_1(\varepsilon):V_\varepsilon\longrightarrow V_\varepsilon$ be the orthogonal projection on the eigenspace generated by ϕ_1^ε. Then we define the map

$$B(\varepsilon)=P(\varepsilon)-\Pi_1(\varepsilon),$$

where $P(\varepsilon)$ is the composition of the Stokes operator with the projection operator from $L_\#^2(Y)^d$ into $H_{\varepsilon e}$. Then

$$B(0)u_j=\left(P(0)-\Pi_1(0)\right)u_j=\nu u_j-\delta_{1j}u_j,$$

that is,

$$B(0)u_j=\nu u_j,\quad j=2,\ldots,h,$$

and

$$B(0)u_1=(\nu-1)u_1.$$

In other words, ν is an eigenvalue of multiplicity $h-1$ of the operator $B=B(0)$ with eigenfunctions $u_2,\ldots,u_h$. There are no other linearly independent eigenfunctions of B associated to ν. Indeed, if u is another eigenfunction of B associated to the eigenvalue ν such that $\langle u, u_j \rangle = 0$, $j = 2, \ldots, h$, then $\langle u, u_1 \rangle = 0$ (because u_1 is an eigenfunction associated to the eigenvalue $\nu - 1$) and $Bu = \nu u$. Then

$$Pu = Bu + \Pi_1 u = Bu + \langle u, u_1 \rangle u_1 = \nu u,$$

that is, u is an eigenfunction of P associated to ν and thus, ν is an eigenvalue of multiplicity $h + 1$, which is impossible because the multiplicity of ν is h.

It is not difficult to see that $B(\varepsilon)$ satisfies the same conditions of the Stokes problem (12) to apply the Lyapunov-Schmidt Method used in the proof of Proposition 8.3. Applying this method in an iterative form we obtain $h-1$ analytic functions in a neighborhood of $\varepsilon = 0$, $\varepsilon \longrightarrow \lambda_j^\varepsilon$ and $\varepsilon \longrightarrow \left(\phi_j^\varepsilon, q_j^\varepsilon, q_{0,j}^\varepsilon\right)$, with $j = 2, \ldots, h$ such that

$$B(\varepsilon)\phi_j^\varepsilon = \lambda_j^\varepsilon \phi_j^\varepsilon.$$

Moreover, the functions $\phi_2^\varepsilon, \ldots, \phi_h^\varepsilon$ form an orthonormal set in $H_{\varepsilon e}$. This shows us the existence of the h branches of eigenpairs.

We now prove the last part of the theorem. Since the eigenvalues $\varepsilon \rightarrow \lambda_j^\varepsilon$ are analytic in a neighborhood of $\varepsilon = 0$, there exist constants c_i such that

$$\left| \lambda_j^\varepsilon - \lambda_j^{\varepsilon'} \right| \leq c_j \left| \varepsilon - \varepsilon' \right|.$$

Let $\nu_1 \leq \nu_2 \leq \ldots \leq \nu_n \ldots$ be the eigenvalues of the Stokes problem at $\varepsilon = 0$ and assume that

$$\ldots \leq \nu_{n-1} < \nu = \nu_n = \ldots = \nu_{n+h-1} < \nu_{n+h} \leq \ldots.$$

Let $I \subset \mathbb{R}$ be an interval such that ν is the unique eigenvalue contained in $\overline{I}$. Then there exists $\delta > 0$ such that

$$I \subset (\nu_{n-1} + \delta, \nu_{n+h} - \delta).$$

Let $\varepsilon \in B\left(0, \dfrac{\delta}{c}\right)$, with $c = \max\{c_j \,:\, j = 1, \ldots, n+h\}$. Then

$$\left|\lambda_{n-1}^\varepsilon - \nu_{n-1}\right| \leq c_{n-1}|\varepsilon| < c_{n-1}\frac{\delta}{c} \leq \delta,$$

and

$$\left|\lambda_{n+h}^\varepsilon - \nu_{n+h}\right| \leq c_{n+h}|\varepsilon| < c_{n+h}\frac{\delta}{c} \leq \delta.$$

Therefore $\lambda_{n-1}^\varepsilon \notin \overline{I}$ and $\lambda_{n+h}^\varepsilon \notin \overline{I}$, that is, (11) has at most h eigenvalues contained in $\overline{I}$ counting multiplicity. This completes the proof of Theorem 3.7. □

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REFERENCES

[1] F. Aguirre, C. Conca, *Eigenfrequencies of a tube bundle immersed in a fluid*, Appl. Math. Optim., **18** (1988), 1–38.

[2] J. H. Albert, *Genericity of simple eigenvalues for elliptic pde's*, Proc. AMS, **48** (1975), no. 2, 413–418.

[3] J. H. Albert, *Topology of the nodal and critical points sets for eigenfunctions of elliptic operators.* Ph. D. Thesis, M. I. T., Boston MA, (1971).

[4] G. Allaire, Y. Capdeboscq, A. Piatnitski, V. Siess and M. Vanninathan, *Homogenization of periodic systems with large potentials*, Arch. Rational Mech. Anal., **174** (2004), 179–220.

[5] G. Allaire, C. Conca, *Bloch wave homogenization for a spectral problem in fluid-solid structures*, Arch. Rat. Mech. Anal., **135** (1996), 197–257.

[6] J. A. Bello, E. Fernández-Cara, J. Lemoine and J. Simon. *The differentiability of the drag with respect to the variations of a Lipschitz domain in a Navier-Stokes flow.* SIAM J. Control Optim., **35** (1997), no. 2, 626–640.

[7] A. Bensoussan, J.-L. Lions and G. Papanicolaou, "Asymptotic Analysis for Periodic Structures", North-Holland: Amsterdam, 1978.

[8] S. Chow, J. Hale, "Methods of Bifurcation Theory", Springer-Verlag, New York, 1982.

[9] C. Conca, J. Planchard and M. Vanninathan, "Fluids and Periodic Structures", RMA **38**, J. Wiley & Masson, Paris, 1995.

[10] C. Conca, M. Vanninathan, *Homogenization of periodic structures via Bloch decomposition*, SIAM J. Appl. Math., **57** (1997), no. 6, 1639–1656.

[11] R. Courant, D. Hilbert, "Methods of Mathematical Physics", Interscience Publishers, Inc., New York, Vol. I, 1953.

[12] S. S. Ganesh, M. Vanninathan, *Bloch wave homogenization of scalar elliptic operators*, Asymptot. Anal., **39** (2004), 15–44.

[13] S. S. Ganesh, M. Vanninathan, *Bloch wave homogenization of linear elasticity system,* ESAIM Control Optim. Calc. Var., **11** (2005), no. 4, 542–573.

[14] F. Hecht, O. Pironneau, A. Le Hyaric and K. Ohtsuka, FreeFem++ Manual, downloadable at `http://www.freefem.org`, 2005.

[15] V. Jikov, S. Kozlov and O. Oleinik, "Homogenization of Differential Operators and Integral Functionals", Springer, Berlin, 1995.

[16] T. Kato, "Perturbation Theory for Linear Operators", Springer-Verlag, Berlin, Heidelberg, 1980.

[17] P. Kuchment, *Floquet theory for partial differential equations*, Operator Theory: Advances and Applications, **60**, Birkhäuser Verlag, Basel, 1993.

[18] O. Ladyzhenskaya "The Mathematical Theory of Viscous Incompressible Flows", Gordon and Breach, New York, 1968.

[19] A. M. Micheletti. *Perturbazione dello spettro di un operatore ellittico di tipo variazionale, in relazione ad una variazione del campo.* Ann. Scuola Norm. Sup. Pisa, **26** (1972), no. 3, 151–169.

[20] J. H. Ortega, E. Zuazua, *Generic simplicity of the eigenvalues of the Stokes system in two spaces dimensions*, Advances in Differential Equations, **6** (2001), no. 8, 987–1023.

[21] J. H. Ortega, E. Zuazua, *Generic simplicity of the spectrum and stabilization for a plate equation*, SIAM J. Control Optim., **39** (2001), no. 5, 1585–1614.

[22] J. H. Ortega, E. Zuazua, *Addendum to Generic simplicity of the spectrum and stabilization for a plate equation*, SIAM J. Contol Optim., **42** (2003), no. 5, 1905–1910.

[23] M. Reed, B. Simon, "Methods of Modern Mathematical Physics", Academic Press, New York, 1978.

[24] F. Rellich. "Perturbation theory of eigenvalue problems". Gordon & Breach Science Publishers Inc., New York, 1969.

[25] E. Sánchez-Palencia, "Non-Homogeneous Media and Vibration Theory", Lectures Notes in Physics 127, Springer-Verlag, Berlin, 1980.

[26] J. Simon *Démonstration constructive d'un théorème de G. de Rham.* C. R. Acad. Sci. Paris, Série I, **316** (1993), 1167–1172.

[27] J. Simon *Differentiation with respect to the domain in boundary value problems.* Numer. Func. Anal. Optim., **2** (1980), 649–687.

[28] L. Tartar, "Convergence of the Homogenization Process". Appendix of [25].

[29] K. Uhlenbeck, *Generic properties of eigenfunctions,* American Journal of Mathematics, **98** (1976), no. 4, 1059–1078.

[30] M. M. Vainberg, V. A. Trenogin, "Theory of Branching of Solutions of Nonlinear Equations", Noordhoff International Publishing, Leyden, 1974.

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